LECTURE 30
MODULE 59

POWER SERIES FUNCTIONS

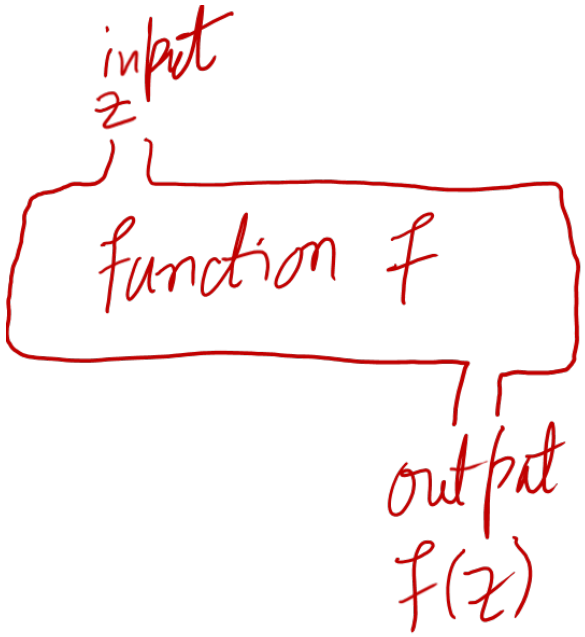
PART I

The purpose of discussing the power series in this course is to construct the complex valued functions with the help of power series. The power series also help to explore new structure of the complex valued functions.

Before constructing the complex valued function from the infinite series, let us look at what a function is.

What is a function?

A function is just like a machine in which we give input and then we get output. We give in a complex number z and obtain a complex number f(z).



How can a series be used to define a function?



First, we look at what an infinite series is.

An infinite series an expression of the form

$$\sum_{n=1}^{\infty} a_n$$ MCQ

Consider an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (z-\alpha)^n$$ MCQ

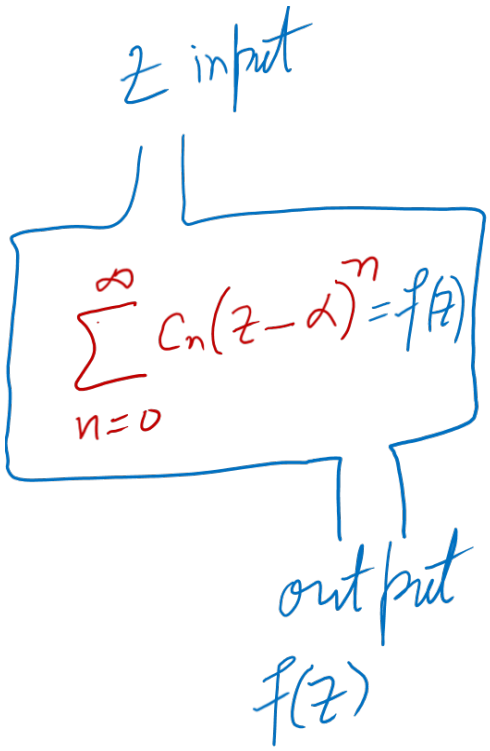
where c_n and α are fixed complex numbers and z is a complex variable.

WARNING

The series

$$\sum_{n=0}^{\infty} c_n (z-\alpha)^n$$ MCQ

defines a function only if it converges.



POWER SERIES FUNCTION

The power series function is given by

$$f(z) = \sum_{n=0}^{\infty} c_n\,(z-\alpha)^n$$

For properly defining a function with the help of power series, we need to define its domain. Since a function on an infinite series can be made if the infinite series is convergent. If an infinite series is divergent, then it is useless to define a function on it.

agar series divergent hy to function define karna useless hy

agar infinite series convergent hy to function define ho ga

Therefore, the Domain of the power series function must consists of the collection of all points z for which the series converges.

EXAMPLE

$$\alpha = 3,\ c_n = n!$$

$$f(z) = \sum_{n=0}^{\infty} n!\ (z-3)^n$$ MCQ

EXAMPLE

$\alpha = 0\ c_n = 1$ MCQ

$f(z) = \sum_{n=0}^{\infty} (z)^n$ geometric series

This is the geometric series.

NOTATION

$$\sum_{n=0}^{\infty} c_n\ (z-\alpha)^n$$

$$= c_0(z-\alpha)^0 + c_1(z-\alpha)^1 + \ldots + c_n(z-\alpha)^n + \ldots$$

But there is problem with the first term when $z = \alpha$ and $n = 0$, then the first term becomes 0^0 which is is undefined. That is why we agree on the following convention or notation that the first term is always c_0.

$$\sum_{n=0}^{\infty} c_n\ (z-\alpha)^n$$

$$= c_0 + c_1(z-\alpha)^1 + \ldots + c_n(z-\alpha)^n + \ldots$$

LECTURE 30
MODULE 60

POWER SERIES FUNCTIONS

PART II

Power Series Function is given by

$$f(z) = \sum_{n=0}^{\infty} c_n \, (z-\alpha)^n$$

Domain= collection of all points z for which the series converges.

NOTATION

We agree on the following convention

$$\sum_{n=0}^{\infty} c_n \, (z-\alpha)^n$$

$$= c_0 + c_1(z-\alpha)^1 + \ldots + c_n(z-\alpha)^n + \ldots$$

THEOREM: Suppose that $f(z) = \sum_{n=0}^{\infty} c_n \, (z-\alpha)^n$. Then the set of points z for which the series converges is one of the following:

1. the single point $z = \alpha$
2. the disk $D_\rho(\alpha) = \{z: |z-\alpha| < \rho\}$, along with part (either none, some or all) of the circle $C_\rho = \{z: |z-\alpha| = \rho\}$
3. the entire complex plane

PROOF:

We know that if the series $\sum_{n=0}^{\infty} \zeta_n$ has $\limsup_{n\to\infty} |\zeta_n|^{\frac{1}{n}} = L$

Then the series

- absolutely convergent if $L < 1$
- divergent if $L > 1$
- No conclusion if $L = 1$

Now for the series

$$\sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

we need to calculate the following limit

$$\limsup_{n\to\infty} |\zeta_n|^{\frac{1}{n}} .$$

$$\text{where } \zeta_n = c_n (z-\alpha)^n$$

Hence the limit is

$$\limsup_{n\to\infty} |c_n (z-\alpha)^n|^{\frac{1}{n}}$$

$$= |z-\alpha| \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} \quad \text{_____} (1)$$

There are three possibilities to consider for the values of

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$$

$L=0$ ① $\quad 0<L<\infty$ ② , ∞ ③

1. **The first possibility**; if

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = \infty$$

then the series $\sum_{n=0}^{\infty} c_n\,(z-\alpha)^n$ diverges for all $z \neq \alpha$.

$z \neq \alpha \quad |z-\alpha| \lim_{n\to\infty} \sup |c_n|^{1/n} > 1$

But the inequality

$$|z-\alpha| \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} < 1$$

holds for $z = \alpha$.

$\sum_{n=0}^{\infty} c_n (z-\alpha)^n$, For $z=\alpha$

$c_0 + 0 + 0 + 0$

This gives the first case, that is, **"the single point $z = \alpha$ "**

2. **The second possibility**; if

$$0 < \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} < \infty$$

According to the root test, the series $\sum_{n=0}^{\infty} c_n\,(z-\alpha)^n$ converges for all z satisfying

$$|z-\alpha| \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} < 1$$

Since $\boldsymbol{\limsup_{n\to\infty}} |\boldsymbol{c_n}|^{\frac{1}{n}}$ is a finite number, so we get the following condition on z

$$|z-\alpha| < \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}$$

Hence the series $\sum_{n=0}^{\infty} c_n\,(z-\alpha)^n$ converges for all z satisfying

$$|z-\alpha| < \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}$$



OR

For all $z \in D_\rho(\alpha)$, where

$$D_\rho(\alpha) = \{z: |z-\alpha| < \rho\}$$

and

$$\rho = \frac{1}{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}}}\,.$$

Since the root test does not say anything about the points of C_ρ, where

$$C_\rho = \{z: |z - \alpha| = \rho\}$$

So the series may converge for none, some, or for all points of C_ρ.

This gives the second case, that is,

"the disk $D_\rho(\alpha) = \{z: |z - \alpha| < \rho\}$, along with part (either none, some or all) of the circle $C_\rho = \{z: |z - \alpha| = \rho\}$"

3. **The third possibility**; if

$$\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 0$$

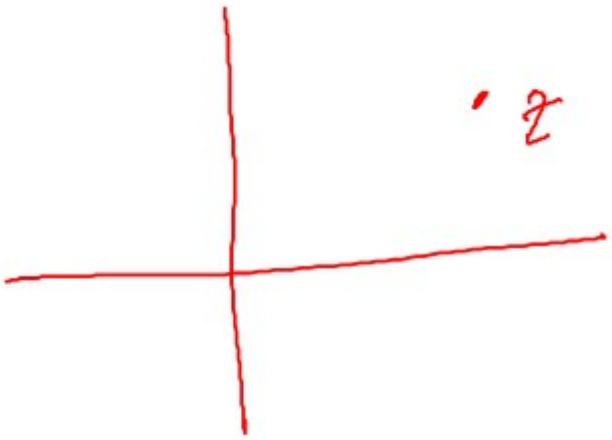
According to the root test the series $\sum_{n=0}^{\infty} c_n (z - \alpha)^n$ converges for all z satisfying

$$|z - \alpha| \limsup_{n\to\infty} |c_n|^{\frac{1}{n}} < 1$$

Since $\boldsymbol{\limsup_{n\to\infty} |c_n|^{\frac{1}{n}} = 0}$ so the following is satisfied for all complex numbers

$$|z - \alpha| = 0 < 1$$

This proves the third case, that is, **"the entire complex plane"**



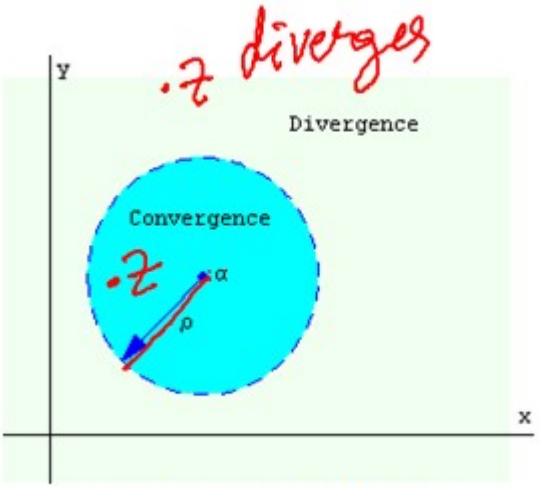
DEFINITION The number ρ such that the series

$$\sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

converges for all z satisfying $|z-\alpha| < \rho$

and diverges for all z satisfying $|z-\alpha| > \rho$

is called radius of convergence.



Source: Complex analysis by Mathews

REMARK

If the series

$$\sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

converges for **"the single point $z = \alpha$"** the radius of convergence is 0.

REMARK

If the series

$$\sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

converges for **"the disk** $\boldsymbol{D_\rho(\alpha) = \{z: |z - \alpha| < \rho\}}$**",** the radius of convergence is ρ.

REMARK

If the series

$$\sum_{n=0}^{\infty} c_n \, (z - \alpha)^n$$

converges for **"the entire complex plane ",** the radius of convergence is ∞.

LECTURE 30
MODULE 61

POWER SERIES FUNCTIONS

PART III

Power Series Function is given by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

Domain= collection of all points z for which the series converges.

THEOREM: Suppose that $f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$. Then the set of points z for which the series converges is one of the following:

1. the single point $z = \alpha$
2. the disk $D_\rho(\alpha) = \{z: |z-\alpha| < \rho\}$, along with part (either none, some or all) of the circle $C_\rho = \{z: |z-\alpha| = \rho\}$
3. the entire complex plane

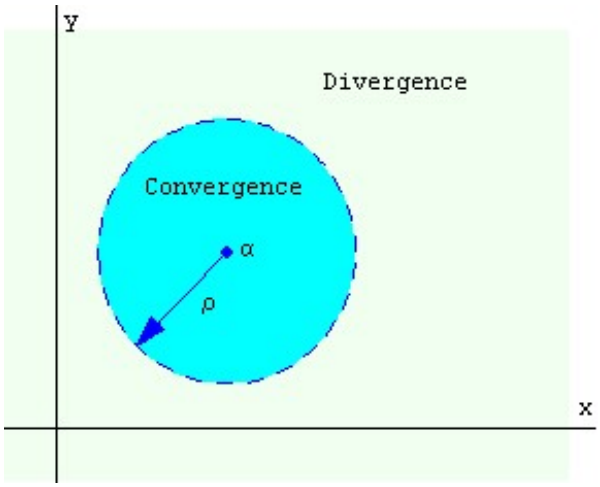
DEFINITION: The number ρ such that the series

$$\sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

converges for all z satisfying $|z-\alpha| < \rho$

and diverges for all z satisfying $|z-\alpha| > \rho$

is called **radius of convergence**.



Question: How to find radius of convergence?



THEOREM: For the power series function

$$f(z) = \sum_{n=0}^{\infty} c_n \, (z - \alpha)^n$$

We can find its radius of convergence, denoted by, ρ using any of the following methods

1. Cauchy's root test

$$\rho = \frac{1}{\lim\limits_{n\to\infty} |c_n|^{\frac{1}{n}}}$$

(provided the limit exist)

THEOREM: For the power series function

$$f(z) = \sum_{n=0}^{\infty} c_n \, (z - \alpha)^n$$

We can find its radius of convergence, denoted by, ρ using any of the following methods

2. Cauchy-Hadamard root test

$$\rho = \frac{1}{\lim\limits_{n\to\infty} sup \, |c_n|^{\frac{1}{n}}}$$

(the limit always exist)

THEOREM: For the power series function

$$f(z) = \sum_{n=0}^{\infty} c_n \, (z - \alpha)^n$$

We can find its radius of convergence, denoted by, ρ using any of the following methods

2. d'Alembert's ratio test

$$\rho = \frac{1}{\lim\limits_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|}$$

(provided the limit exist)

EXAMPLE: Find radius of convergence of the following series

$$\sum_{n=0}^{\infty}\left(\frac{n+2}{3n+1}\right)^{n}(z-4)^{n}$$

SOLUTION: According to Cauchy's root test, the radius of convergence is

$$\rho=\frac{1}{\lim\limits_{n\to\infty}|c_n|^{\frac{1}{n}}}$$

So we calculate the following limit

$$\lim_{n\to\infty}|c_n|^{\frac{1}{n}}$$

where $c_n=\left(\frac{n+2}{3n+1}\right)^{n}$

$$\lim_{n\to\infty}|c_n|^{\frac{1}{n}}$$

$$=\lim_{n\to\infty}\left(\frac{n+2}{3n+1}\right)^{n}$$

$$=\lim_{n\to\infty}\frac{1+\frac{2}{n}}{3+\frac{1}{n}}$$ (yahan oper or nichy sy n ko common lia gia hy)

$$=\frac{1}{3}$$

Hence radius of convergence is

$$\rho=\frac{1}{1/3}=3$$

EXAMPLE: Find radius of convergence of the following series

$$\sum_{n=1}^{\infty}c_nz^n=4z+5^2z^2+4^3z^3+5^4z^4+\dots$$

SOLUTION:

In this case,

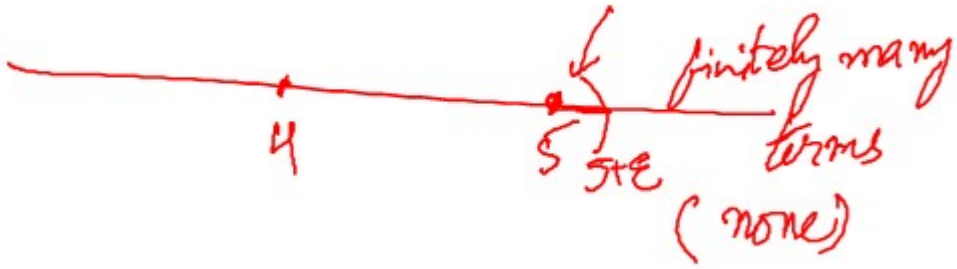
$$\left\{|c_n|^{\frac{1}{n}}\right\} = \{4,5,4,5,4,5,4, \ldots\}$$

So the limit $\lim_{n\to\infty} |c_n|^{\frac{1}{n}}$ does not exist.

Hence we use Cauchy-Hadamard root test.

In this case

$$\left\{|c_n|^{\frac{1}{n}}\right\} = \{4,5,4,5,4,5,4, \ldots\}$$



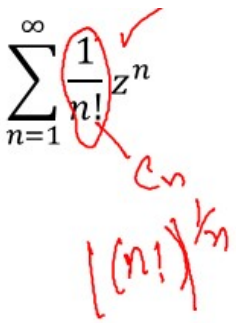
So the limit $\lim_{n\to\infty} \sup |c_n|^{\frac{1}{n}} = 5$.

Hence radius of convergence is $\rho = \frac{1}{5}$.

EXAMPLE: Find radius of convergence of the following series

$$\sum_{n=1}^{\infty} \frac{1}{n!} z^n$$

SOLUTION:



According to d'Alembert's ratio test, the radius of convergence is

$$\rho = \frac{1}{\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|}$$

where, in this example,

$$c_n = \frac{1}{n!}$$

We calculate the following limit

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1/(n+1)!}{1/n!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{1}{n+1}\right|$$

$$= 0$$

So radius of convergence is

$$\rho = \infty$$

LECTURE 30
MODULE 62

POWER SERIES FUNCTIONS

Term by term differentiation

PART I

In this module, we will discuss differentiation and integration of these power series functions. Here in this module, we will mainly focus on differentiation of the power series functions.

So, now first we recall definitions of power series function and the radius of convergence.

Power Series Function is given by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

Domain= collection of all points z for which the series converges.

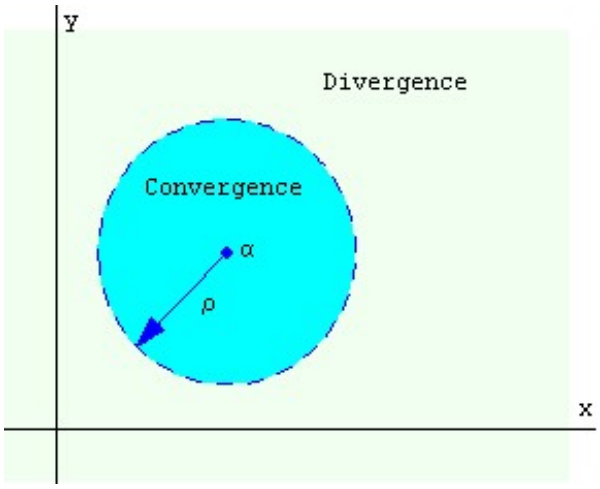
DEFINITION: The number ρ such that the series

$$\sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

converges for all z satisfying $|z-\alpha| < \rho$

and diverges for all z satisfying $|z-\alpha| > \rho$

is called **radius of convergence**.



QUESTIONS

When can we differentiate the power series function?

What is the derivative, if exists?

We will focus on these two questions in the rest of our discussion in this module.

First, we see some notations which we will use.

NOTATION

For a function $f(z)$, we denote by $f^{(k)}$ the kth derivative of $f(z)$.

When $k = 0$, $f^{(k)}$ denotes the function itself, that is,

$$f^{(0)}(z) = f(z)$$

$$f^{(1)}(z) = f'(z)$$

$$f^{(2)}(z) = f''(z)$$

.

.

.

THEOREM: Suppose that the function

$$f(z) = \sum_{n=0}^{\infty} c_n \, (z-\alpha)^n$$

has radius of convergence $\rho > 0$. Then

(i). f(z) is infinitely differentiable for all $z \in D_\rho(\alpha)$.

(ii). In fact, for all k

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1) \, c_n \, (z-\alpha)^{n-k}$$

and

(iii). $c_k = \frac{f^{(k)}(\alpha)}{k!}$

PROOF: We first prove the second part (ii), that is,

> **" for all k**
>
> $\boldsymbol{f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1) \, c_n \, (z-\alpha)^{n-k}}$ **"**

In the **first step,** we prove the above statement for $k = 1$.

In the **second step** we prove it for any k.

Proof (ii): **First step $k = 1$**

We define

$$g(z) = \sum_{n=1}^{\infty} n\, c_n (z-\alpha)^{n-1}$$

$$f(z) = \sum_{n=1}^{\infty} c_n (z-\alpha)^n$$

Now first we divide $g(z)$ into two parts $S_j(z)$ and R_j:

$$S_j(z) = \sum_{n=0}^{j} c_n (z-\alpha)^n \quad R_j = \sum_{n=j+1}^{\infty} c_n (z-\alpha)^n$$

Here

- $S_j(z)$ is simply the $(j+1)$st partial sum of the series $f(z)$,
- $R_j(z)$ is the sum of the remaining terms of that series.

Recall

Theorem: For the power series function

$$f(z) = \sum_{n=0}^{\infty} c_n\, (z-\alpha)^n$$

We can find its radius of convergence, denoted by, ρ using d'Alembert's ratio test as follows

$$\rho = \frac{1}{\lim\limits_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|}$$

(provided the limit exist)

The radius of convergence of $g(z) = \sum_{n=1}^{\infty} n\, c_n (z-\alpha)^{n-1}$ is ρ.

Given, radius of convergence of $\sum_{n=0}^{\infty} c_n (z-\alpha)^n$ is ρ, that is

$$\rho = \frac{1}{\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|}$$

For $g(z)$, $\lim_{n\to\infty} \left|\frac{(n+1)c_{n+1}}{n\, c_n}\right| = \lim_{n\to\infty}\left(\frac{n+1}{n}\right) \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$

$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right) \lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right|$

$= 1 \cdot 1/\rho = 1/\rho$

For fixed $z_0 \in D_\rho(\alpha)$ we show that

$$f'(z_0) = g(z_0)$$

That is, we want to show that

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = g(z_0)$$

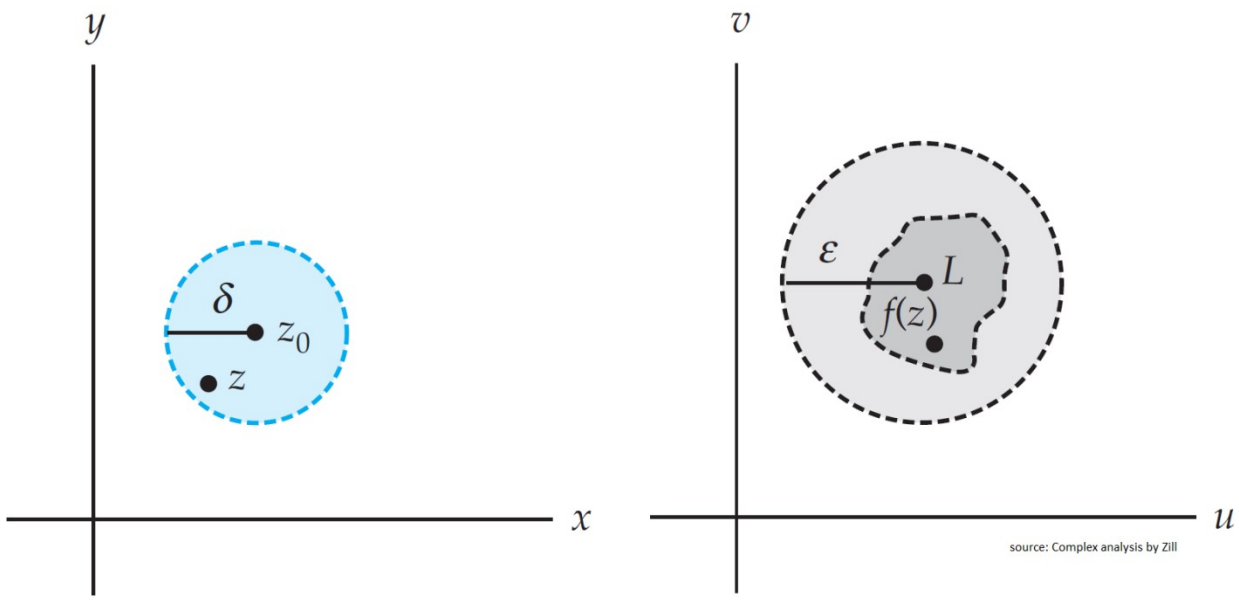
RECALL the definition of limit

$$\lim_{z\to z_0} f(z) = L$$

is

"For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

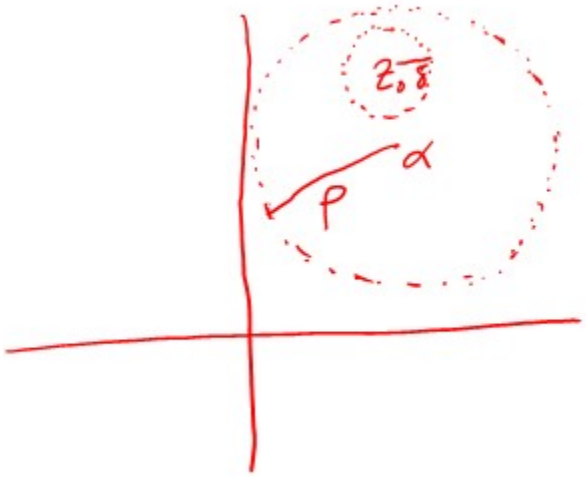
$$|f(z) - L| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta."$$



Hence we need to show the following:

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - g(z_0)\right| < \varepsilon \quad \text{whenever } 0 < |z - z_0| < \delta \text{ and } z \in D_\rho(\alpha)$$

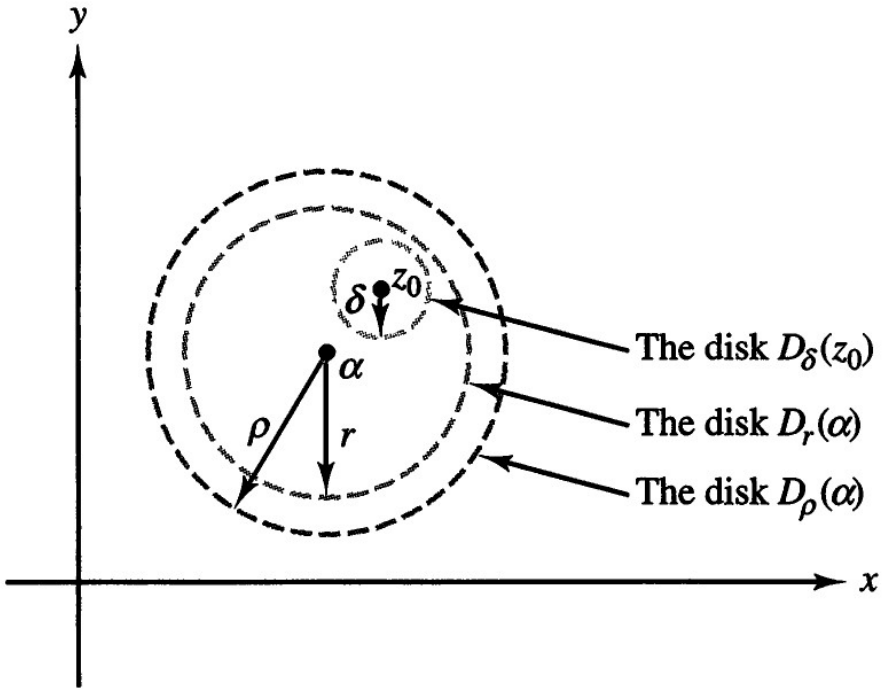


Let $\varepsilon > 0$ and $z_0 \in D_\rho(\alpha)$.

Choose $r < \rho$ such that $z_0 \in D_r(\alpha)$. We want to find δ small enough such that

$$D_\delta(z_0) \subset D_r(\alpha) \subset D_\rho(\alpha)$$

and other conditions that we impose during the proof.



Since $f(z) = S_j(z) + R_j(z)$, so

$$\left[\frac{f(z)-f(z_0)}{z-z_0} - g(z_0)\right]$$

$$= \left[\frac{S_j(z) + R_j(z) - (S_j(z_0) + R_j(z_0))}{z-z_0} - g(z_0)\right]$$

$$= \left[\frac{S_j(z) - S_j(z_0)}{z-z_0} + \frac{R_j(z) - R_j(z_0)}{z-z_0} - g(z_0)\right]$$

$$= \left[\frac{S_j(z) - S_j(z_0)}{z-z_0} - S_j'(z_0)\right] + \left[S_j'(z_0) - g(z_0)\right] + \left[\frac{R_j(z) - R_j(z_0)}{z-z_0}\right]$$

$S_j'(z)$ is added and subtracted. Then we take modulus on both sides.

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-g(z_0)\right|$$

$$=\left|\left(\frac{S_j(z)-S_j(z_0)}{z-z_0}-S_j'(z_0)\right)+\left(S_j'(z_0)-g(z_0)\right)+\left(\frac{R_j(z)-R_j(z_0)}{z-z_0}\right)\right|$$

$$\leq\left|\frac{S_j(z)-S_j(z_0)}{z-z_0}-S_j'(z_0)\right|+\left|S_j'(z_0)-g(z_0)\right|+\left|\frac{R_j(z)-R_j(z_0)}{z-z_0}\right|$$

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-g(z_0)\right| < \varepsilon$$

$$\leq\underbrace{\left|\frac{S_j(z)-S_j(z_0)}{z-z_0}-S_j'(z_0)\right|}_{|B|}+\underbrace{\left|S_j'(z_0)-g(z_0)\right|}_{|C|}+\underbrace{\left|\frac{R_j(z)-R_j(z_0)}{z-z_0}\right|}_{|D|}$$

$$\wedge\ \varepsilon/3 \quad + \quad \wedge\ \varepsilon/3 \quad + \quad \wedge\ \varepsilon/3=\varepsilon$$

$$|D|=\left|\frac{R_j(z)-R_j(z_0)}{z-z_0}\right|$$

$$=\left|\frac{1}{z-z_0}\sum_{n=j+1}^{\infty}c_n(z-\alpha)^n-\sum_{n=j+1}^{\infty}c_n(z_0-\alpha)^n\right|$$

$$=\left|\frac{1}{z-z_0}\sum_{n=j+1}^{\infty}c_n(z-\alpha)^n-c_n(z_0-\alpha)^n\right|$$

$$\leq\sum_{n=j+1}^{\infty}|c_n|\left|\frac{(z-\alpha)^n-(z_0-\alpha)^n}{z-z_0}\right|$$

$$\left|\frac{(z-\alpha)^n-(z_0-\alpha)^n}{z-z_0}\right| < nR^{n-1}$$ ✓

Hint

$$\left|\frac{p^n-q^n}{p-q}\right| = |p^{n-1}+p^{n-2}q+p^{n-3}q^2+\cdots+pq^{n-2}+q^{n-1}|$$

$$\leq |p^{n-1}|+|p^{n-2}q|+\cdots+|pq^{n-2}|+|q^{n-1}|$$

(R^{n-1}; ✓; n terms)

$p=z-\alpha$, $q=(z_0-\alpha)$, $\Rightarrow p-q=z-z_0$

Also $|z-\alpha|<R$, $|z_0-\alpha|<R$

$$\left|\frac{(z-\alpha)^n-(z_0-\alpha)^n}{z-z_0}\right| < nR^{n-1}$$

$$\Rightarrow |D| = \left|\frac{R_j(z)-R_j(z_0)}{z-z_0}\right|$$

$$\leq \sum_{n=j+1}^{\infty} |C_n| \left|\frac{(z-\alpha)^n-(z_0-\alpha)^n}{z-z_0}\right|$$

$$< \sum_{n=j+1}^{\infty} |C_n|\, nR^{n-1}$$ Converges ?

$\sum_{n=j+1}^{\infty} |C_n| \, n \, r^{n-1}$ Converges

Here $\eta_{n+1} = |C_{n+1}| (n+1) r^n$, $\eta_n = C_n n r^{n-1}$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{\eta_{n+1}}{\eta_n}\right| = \lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n} \frac{(n+1)}{n} \frac{r^n}{r^{n-1}}\right|$$

$$= r \lim_{n\to\infty} \left|\frac{C_{n+1}}{C_n}\right| \lim_{n\to\infty} \frac{n+1}{n}$$

$$= r \frac{1}{\rho} < 1 \quad \because r < \rho \Rightarrow \frac{r}{\rho} < 1$$

$$|D| = \left|\frac{R_j(z) - R_j(z_0)}{z - z_0}\right| < \sum_{n=j+1}^{\infty} |C_n| \, n \, r^{n-1}$$

Since $\sum_{n=j+1}^{\infty} |C_n| n r^{n-1}$ Converges, so we can find N_1 such that

$$\sum_{n=j+1}^{\infty} |C_n| \, n \, r^{n-1} < \frac{\varepsilon}{3} \quad \forall \, j \geq N_1$$

$$\Rightarrow \left|\frac{R_j(z) - R_j(z_0)}{z - z_0}\right| < \frac{\varepsilon}{3} \quad \forall \, j \geq N_1$$

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-g(z_0)\right|$$

$$\leq \underbrace{\left|\frac{S_j(z)-S_j(z_0)}{z-z_0}-S_j'(z_0)\right|}_{|B|} + \underbrace{\left|S_j'(z_0)-g(z_0)\right|}_{|C|} + \underbrace{\left|\frac{R_j(z)-R_j(z_0)}{z-z_0}\right|}_{|D|}$$

$|B| < \varepsilon/3 \quad \forall j \geq N_1$

$|C|$ ✓

$|D|$ ✓

The remaining part of this proof will be done in the next module.

LECTURE 30
MODULE 63

POWER SERIES FUNCTIONS

Term by term differentiation

PART II

This module is the continuation of the previous module in which we were proving the following theorem:

THEOREM: Suppose that the function

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

has radius of convergence $\rho > 0$. Then

(i). f(z) is infinitely differentiable for all $z \in D_\rho(\alpha)$.

(ii). In fact, for all k

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1)\, c_n (z-\alpha)^{n-k}$$

and

(iii). $c_k = \frac{f^{(k)}(\alpha)}{k!}$

PROOF (Continued from previous Module)

For fixed $z_0 \in D_\rho(\alpha)$ we show that

$$f'(z_0) = g(z_0)$$

That is, we want to show that

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = g(z_0)$$

Hence we need to show the following:

For every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left|\frac{f(z)-f(z_0)}{z-z_0} - g(z_0)\right| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta \text{ and } z \in D_\rho(\alpha)$$

$$\left|\frac{f(z) - f(z_0)}{z - z_0} - g(z_0)\right|$$

$$\leq \underbrace{\left|\frac{S_j(z) - S_j(z_0)}{z - z_0} - S_j'(z_0)\right|}_{|B|} + \underbrace{\left|S_j'(z_0) - g(z_0)\right|}_{|C|} + \underbrace{\left|\frac{R_j(z) - R_j(z_0)}{z - z_0}\right|}_{|D|}$$

$$|C| = \left|S_j'(z_0) - g(z_0)\right|$$

Given

$$g(z) = \sum_{n=1}^{\infty} n\, c_n (z - \alpha)^{n-1} \textbf{ and } S_j(z) = \sum_{n=0}^{j} c_n (z - \alpha)^n$$

Since $S_j(z)$ is a polynomial, so

$$S_j^{'}(z) = \sum_{n=1}^{j} n\, c_n (z-\alpha)^{n-1}$$

Hence

$$\lim_{j\to\infty} S_j^{'}(z) = g(z)$$

which implies

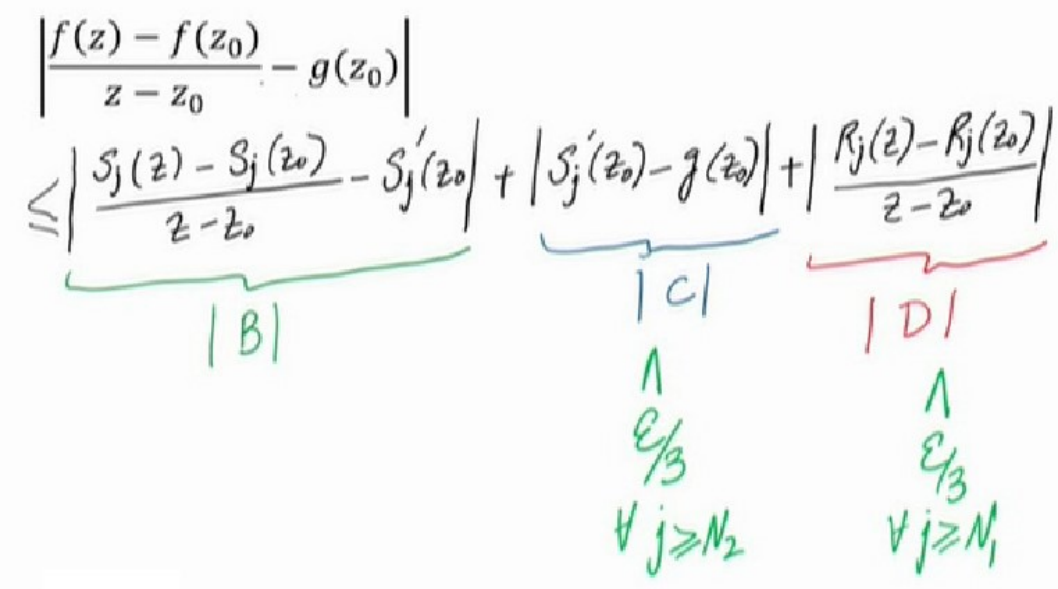
$$\lim_{j\to\infty} S_j^{'}(z_0) = g(z_0)$$

Using definition of limit for the following

$$\lim_{j\to\infty} S_j^{'}(z_0) = g(z_0)$$

we get

$$\exists\ N_2 \text{ such that } \left|S_j(z_0) - g(z_0)\right| < \tfrac{\varepsilon}{3} \text{ for all } j \geq N_2$$



$$|B| = \left| \frac{S_j(z) - S_j(z_0)}{z - z_0} - S_j'(z_0) \right|$$

Since $S_j(z)$ is a polynomial, so it is differentiable.

Hence for $\frac{\varepsilon}{3} > 0$ there exist $\delta > 0$ such that

$$|B| = \left| \frac{S_j(z) - S_j(z_0)}{z - z_0} - S_j'(z_0) \right| < \frac{\varepsilon}{3}$$

whenever $z \in D_\rho(\alpha)$ & $0 < |z - z_0| < \delta$

$\forall\ j \geq \max\{N_1, N_2\}$

$$\rightarrow \left| \frac{R_j(z) - R_j(z_0)}{z - z_0} \right| < \frac{\varepsilon}{3} \quad \forall\ j \geq N_1$$

$\rightarrow \exists\ N_2$ such that $|S_j(z_0) - g(z_0)| < \frac{\varepsilon}{3}$ for all $j \geq N_2$

$$\rightarrow |B| = \left| \frac{S_j(z) - S_j(z_0)}{z - z_0} - S_j'(z_0) \right| < \frac{\varepsilon}{3}$$

whenever $z \in D_\rho(\alpha)$ & $0 < |z - z_0| < \delta$

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-g(z_0)\right|$$

$$\leq \left|\frac{S_j(z)-S_j(z_0)}{z-z_0}-S_j'(z_0)\right|+\left|S_j'(z_0)-g(z_0)\right|+\left|\frac{R_j(z)-R_j(z_0)}{z-z_0}\right|$$

$$\leq \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} \quad \forall \; j \geq \max\{N_1, N_2\}$$

$$\text{and} \quad z \in D_\delta(z_0) \subseteq D_\rho(\alpha)$$

So the following holds for $k = 1$

" **for all** $\boldsymbol{k}$

$$\boldsymbol{f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1)\, c_n\, (z-\alpha)^{n-k}}$$ "

In the **second step** we prove it for any k.

$$\boldsymbol{f^{(1)}(z) = \sum_{n=1}^{\infty} n\, c_n\, (z-\alpha)^{n-1}}$$

$$\boldsymbol{g(z) = \sum_{n=2}^{\infty} n(n-1)\, c_n\, (z-\alpha)^{n-2}}$$

Keeping in view $f^{(1)}(z)$ and $g(z)$, we can prove it for k = n + 1 or for any k.

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1)\, c_n\, (z-\alpha)^{n-k} \quad \text{_____(1)}$$

For proving $c_k = \frac{f^{(k)}(\alpha)}{k!}$, we put $z = \alpha$ in equation (1)

LECTURE 30
MODULE 64

POWER SERIES FUNCTIONS

Term by term differentiation

PART II

Power Series Function

is given by

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

Domain= collection of all points z for which the series converges.

NOTATION

For a function $f(z)$ we denote by $f^{(k)}$ the kth derivative of $f(z)$

When $k = 0$, $f^{(k)}$ denotes the function itself, that is,

$$f^{(0)}(z) = f(z)$$

THEOREM: Suppose that the function

$$f(z) = \sum_{n=0}^{\infty} c_n (z-\alpha)^n$$

has radius of convergence $\rho > 0$. Then

(i). f(z) is infinitely differentiable for all $z \in D_\rho(\alpha)$. In fact

(ii). for all k

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)(n-2)\ldots(n-k+1)\, c_n (z-\alpha)^{n-k}$$

and

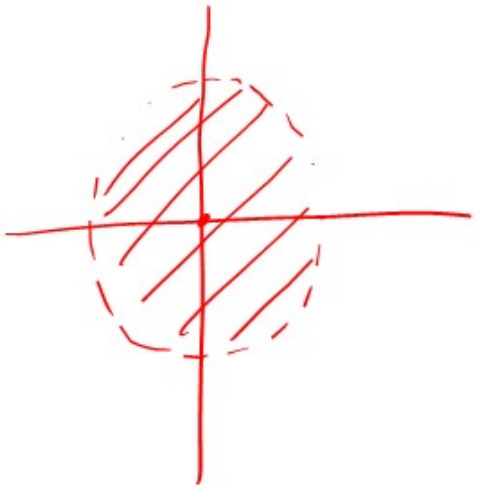
(iii). $c_k = \frac{f^{(k)}(\alpha)}{k!}$

EXAMPLE: Show that

$$\sum_{n=0}^{\infty}(n+1)z^n = \frac{1}{(1-z)^2}$$

for all $z \in D_1(0)$.

SOLUTION:



Recall

Theorem: If $|z| < 1$, the series $\sum_{n=0}^{\infty} z^n$ converges to

$f(z) = \frac{1}{1-z}$. That is, if $|z| < 1$, then

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots z^n + \ldots = \frac{1}{1-z}$$

If $|z| \geq 1$ then the series diverges.

Taking derivative of the following

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + \ldots + z^n + \ldots = \frac{1}{1-z}$$

We get

$$\sum_{n=1}^{\infty} n\, z^{n-1} = \ 1 + 2z + 3z^2 + \ldots + nz^{n-1} + \ldots = \frac{1}{(1-z)^2}$$

$$\sum_{n=1}^{\infty} n\, z^{n-1} = \ 1 + 2z + 3z^2 + \ldots + nz^{n-1} + \ldots = \frac{1}{(1-z)^2}$$

Can be written as

$$\sum_{n=0}^{\infty} (n+1)\, z^n = \ 1 + 2z + 3z^2 + \ldots + nz^{n-1} + \ldots = \frac{1}{(1-z)^2}$$

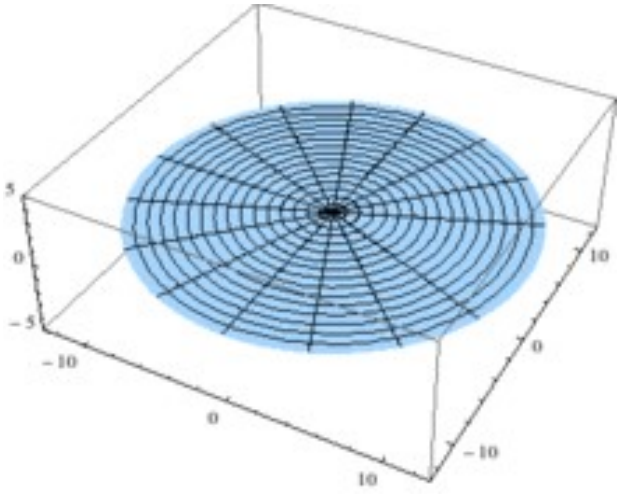
Hence

$$\sum_{n=0}^{\infty} (n+1) z^n = \frac{1}{(1-z)^2}$$

EXAMPLE: The Bessel function of order zero is defined as

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n} .$$

Find the derivative $J_0(z)$.



SOLUTION:

First we need to find radius of convergence of $J_0(z)$.

$$\zeta_n = \frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n} \quad \text{and so } \zeta_{n+1} = \frac{(-1)^{n+1}}{\left((n+1)!\right)^2}\left(\frac{z}{2}\right)^{2(n+1)}$$

We use ratio test to find radius of convergence.

$$\lim_{n\to\infty}\left|\frac{\zeta_{n+1}}{\zeta_n}\right| = \lim_{n\to\infty}\left|\frac{\left[\frac{(-1)^{n+1}}{\left((n+1)!\right)^2}\left(\frac{z}{2}\right)^{2(n+1)}\right]}{\left[\frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}\right]}\right|$$

$$= \lim_{n\to\infty}\left|\left[\frac{(-1)^{n+1}}{\left((n+1)!\right)^2}\left(\frac{z}{2}\right)^{2(n+1)}\right]\left[\frac{(n!)^2}{(-1)^n}\left(\frac{2}{z}\right)^{2n}\right]\right|$$

$$= \lim_{n\to\infty}\left|(-1)\frac{z^2}{2^2}\right| \quad \lim_{n\to\infty}\frac{1}{(n+1)^2} = 0$$

Hence the radius of convergence of

$$J_0(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}$$

is ∞. So we can perform term by term differentiation for any $z \in \mathbb{C}$.

Hence

$$J_0(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n} = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2\,4^2} - \frac{z^6}{2^2 4^2 6^2} + \cdots$$

becomes

$$J_0'(z) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!\,(n)!}\left(\frac{2n}{2}\right)\left(\frac{z}{2}\right)^{2n-1} = -\frac{z}{2} + \frac{1}{1!\,2!}\left(\frac{z}{2}\right)^3 - \frac{1}{2!\,3!}\left(\frac{z}{2}\right)^5 + \cdots$$

Changing the index, we get

$$J_0'(z) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n!\,(n+1)!}\left(\frac{z}{2}\right)^{2n+1} = -\frac{z}{2} + \frac{1}{1!\,2!}\left(\frac{z}{2}\right)^3 - \frac{1}{2!\,3!}\left(\frac{z}{2}\right)^5 + \cdots$$

The Bessel function of order zero, given by

$$J_0(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n!)^2}\left(\frac{z}{2}\right)^{2n}.$$

has derivative

$$J_0^{'}(z) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!\,(n+1)!} \left(\frac{z}{2}\right)^{2n+1}$$

Also $j_0{}'(z)$ has ∞ as radius convergence.

The Bessel function of order one, denoted by $J_1(z)$ can be found by solving the following differential equation

$$J_1(z) = -J_0^{'}(z).$$

LECTURE 31
MODULE 65

COMPLEX EXPONENTIAL FUNCTIONS

PART I

In this module, we will construct elementary functions with the help of infinite power series. By elementary functions, we mean the trigonometric functions, logarithmic functions, exponential functions, hyperbolic functions etc. etc.

Elementary functions

e^x, $\cos x$, $\sin x$

$\log x$, $\tan x$

z^c, $\tanh z$

$\cosh z$, $\sinh z$ . . .

AIM

We want to define "complex version" of elementary functions such as e^x, $\sin x$, $\cos x$,

$\log x$, $\tan x$, etc

<u>Criteria</u>

1. $\mathbb{R} \subset \mathbb{C}$. So the restriction of these functions to $\mathbb{R}$ must give the corresponding real-valued function on $\mathbb{R}$. If the complex number z is restricted to get the real values only, then the elementary functions would give the corresponding real values.

2. The properties of these new functions must correspond to their real counterparts. It means that the newly constructed complex valued functions must have the same properties as their corresponding real valued functions have.

$\cos^2 x + \sin^2 x = 1$

$\frac{d}{dx}\cos x = -\sin x$

$\frac{d}{dx} e^x = e^x$

$e^{x_1 + x_2} = e^{x_1} \cdot e^{x_2}$ etc.

STRATEGY

Use expansion of real-valued functions as power series.

In particular, if

$$f(x) = \sum_{n}^{\infty} a_n (x - a)^n$$

where a, a_n are real numbers and x is real variable. We can extend the domain to $\mathbb{C}$ by defining

$$f(z) = \sum_{n}^{\infty} c_n (z - \alpha)^n$$

Here c_n, α are complex numbers and z is complex variable.

So we simply replace the real variable x by complex variable z and a_n by c_n.

EXPONENTIAL FUNCTIONS

We know that real exponential function e^x has power series expansion

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}\, x^n$$

Thus we define complex exponential function e^z (also denoted by $\exp(z)$)

as

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!}\, z^n$$

Now our next aim is to prove different properties of complex exponential functions.

Two key properties which are associated to the associated real exponential functions are:

1. $\frac{d}{dx} e^x = e^x$
2. $e^{x_1+x_2} = e^{x_1} e^{x_2}$

We want to see if these properties hold for complex exponential functions or not.

Also we give a proof of Euler formula

$$e^{i\theta} = \cos\theta + i \sin\theta$$

THEOREM: The function e^z is an entire function satisfying the following conditions:

1. $\frac{d}{dz} e^z = e^z$
2. $e^{z_1+z_2} = e^{z_1} e^{z_2}$

PROOF:

Proof ($\frac{d}{dz} e^z = e^z$) : We know that

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

For finding its derivative, we would apply term by term differentiation on the series. For this, first we find its radius of convergence. In this case

$$c_n = \frac{1}{n!}$$

So by the ratio test we have

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{1}{(n+1)!}}{\frac{1}{n!}}\right| = \lim_{n\to\infty} \left|\frac{n!}{(n+1)!}\right| = \lim_{n\to\infty} \frac{1}{n+1} = \frac{1}{\infty} = 0$$

Hence

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = 0$$

So this series is absolutely convergent. In this limit, there is no condition on z.

Hence

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

has infinite radius of convergence.

Hence

$$e^z = \exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

has infinite radius of convergence. So

$$\frac{d}{dz} e^z = \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{n-1} = \sum_{n=0}^{\infty} \frac{1}{n!} z^n = e^z$$

Hence the proof.

PROOF($e^{z_1+z_2} = e^{z_1}e^{z_2}$) : We define

$$g(z) = e^z\, e^{\zeta - z}, \quad \text{where } \zeta \text{ is any complex number}$$

Using product rule, chain rule, we get

$$g'(z) = e^z\, e^{\zeta - z} + e^z\, (-\, e^{\zeta - z})$$

$$= e^z\, e^{\zeta - z} - e^z\, e^{\zeta - z}$$

$$= 0.$$

So $g(z)$ is a constant function. Hence

$$g(z) = g(0) = e^{\zeta}$$

Now using $z = z_1$ and $\zeta = z_1 + z_2$ in

$$e^{\zeta} = g(z) = e^z\, e^{\zeta - z}$$

$$e^{z_1+z_2} = g(z) = e^{z_1}\, e^{z_1+z_2-z_1}$$

$$e^{z_1+z_2} = e^{z_1}\, e^{z_2}$$

THEOREM (Euler Formula): If θ is a real number, then

$$e^{i\theta} = \cos\theta + i\sin\theta$$

PROOF: We know that

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{1}{n!}\,(i\theta)^n$$

$$= \sum_{n=0}^{\infty} \left[\left[\frac{1}{(2n)!}\,(i\theta)^{2n} \right] + \left[\frac{1}{(2n+1)!}\,(i\theta)^{2n+1} \right] \right]$$

$$= \sum_{n=0}^{\infty} \left[\left[\frac{(i)^{2n}}{(2n)!}\,(\theta)^{2n} \right] + \left[\frac{i\,(i^{2n})}{(2n+1)!}\,(\theta)^{2n+1} \right] \right] \qquad \text{since} \quad i^{2n} = (i^2)^n = (-1)^n$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n)!}\,(\theta)^{2n} \right] + i \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n+1)!}\,(\theta)^{2n+1} \right]$$

$$= \cos\theta + i\sin\theta$$

$$\text{Since } \cos\theta = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(2n)!}(\theta)^{2n}\right],\ \sin\theta = \sum_{n=0}^{\infty}\left[\frac{(-1)^n}{(2n+1)!}(\theta)^{2n+1}\right]$$

REMARK

Using Euler formula and other properties we can now prove DeMoivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

$$\begin{aligned}\text{L. H. S.} &= (\cos\theta + i\sin\theta)^n \\ &= \left(e^{i\theta}\right)^n \\ &= e^{in\theta} \\ &= \cos n\theta + i\sin n\theta \\ &= R.H.S.\end{aligned}$$

REMARK

If $z = x + iy,$ then

$$\begin{aligned}e^z &= e^{x+iy} \\ &= e^x\, e^{iy} \\ &= e^x(\cos y + i\sin y)\end{aligned}$$

Hence

$$e^z = e^x(\cos y + i\sin y)\,.$$

Using this expression of e^z we can easily discuss continuity and differentiability of exponential function.

If $z = x + i0,$ then $\ e^z = e^x(\cos 0 + i\sin 0)\ = e^x(1 + i0) = e^x$

LECTURE 31
MODULE 67

COMPLEX EXPONENTIAL FUNCTIONS

PART III

In this module, we will discuss the range of the exponential function. We are also interested in knowing whether the exponential function is one-one or onto, or not.

The exponential function is not onto. The simple reason is that $e^z = e^x(\cos y + i \sin y)$ has e^x which cannot be 0 for any value of x. Similarly, $\cos y$ and $\sin y$ cannot be 0 at the same value of y.

The exponential function is not one-to-one. We have seen an example in the previous module that many values of e^z are mapped on a single point. That is why the exponential function e^z is not one-to-one function.

Now we will explore further why the exponential function e^z is not one-to-one and onto. We will also explore more properties of the exponential function.

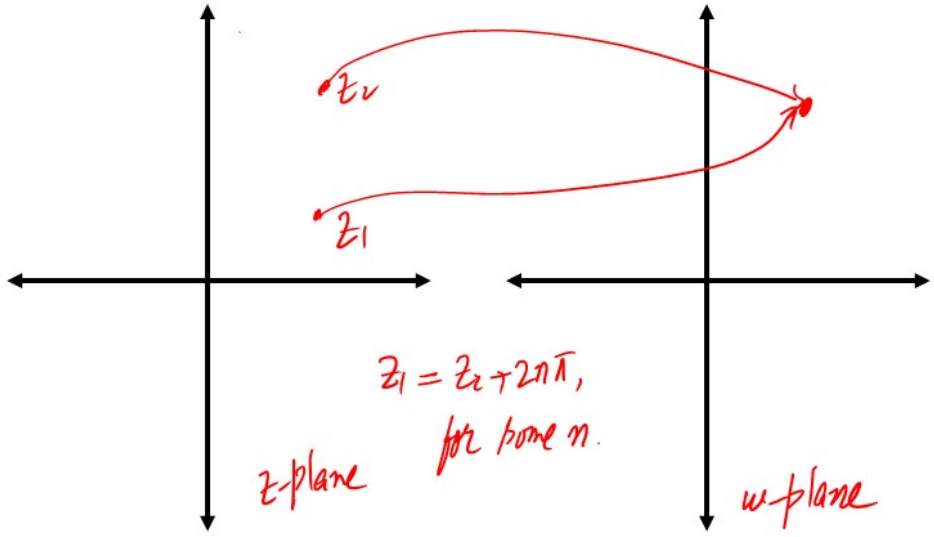
RANGE OF EXPONENTIAL FUNCTION

First, we answer the question whether the exponential function

$$\exp(z) = e^x(\cos y + i \sin y)$$

is One-to-one and Onto, or not.

The Exponential function $w = e^z$ has the same value in w-plane for two different values of z_1 and z_2.



Let $z = x + iy$, and $w = e^z = e^{x+iy}$

The equation $w = e^z$ becomes

$$w = e^x e^{iy} = \rho e^{i\phi}$$

We know that

$$e^{z_1} = e^{z_2} \text{ if and only if } z_2 = z_1 + i\,2n\pi \text{ for some integer } n$$

$$\text{So, } \rho = e^x, \quad iy = i\phi + i2n\pi$$

$$\ln\rho = x, \quad y = \phi + 2n\pi$$

$$x = \ln\rho, \quad y = \phi + 2n\pi$$

$$x = \ln|w|, \quad y = \phi + 2n\pi$$

Now, we put these values of x and y in $z = x + iy$, so we get

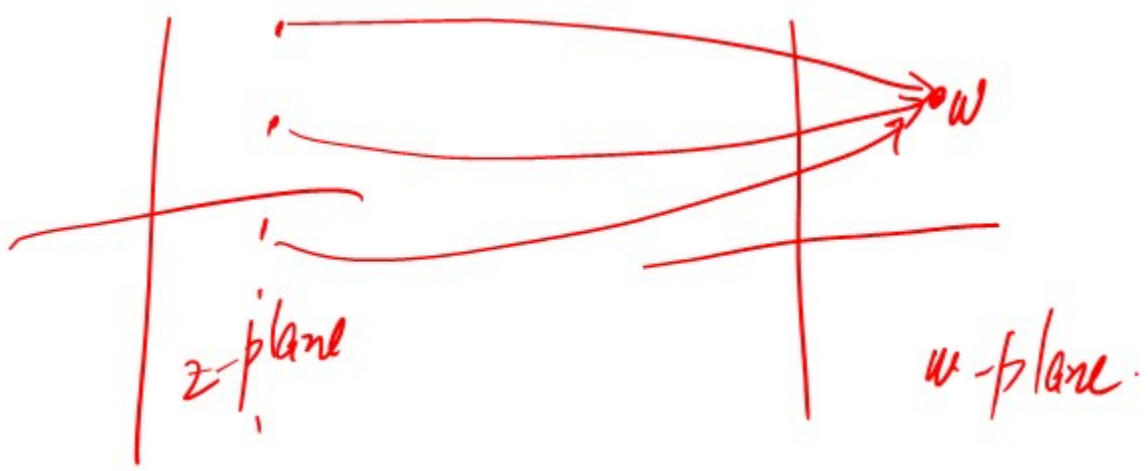
$$z = \ln|w| + i(\phi + 2n\pi)$$

$$z = \ln|w| + i(Arg(w) + 2n\pi)$$

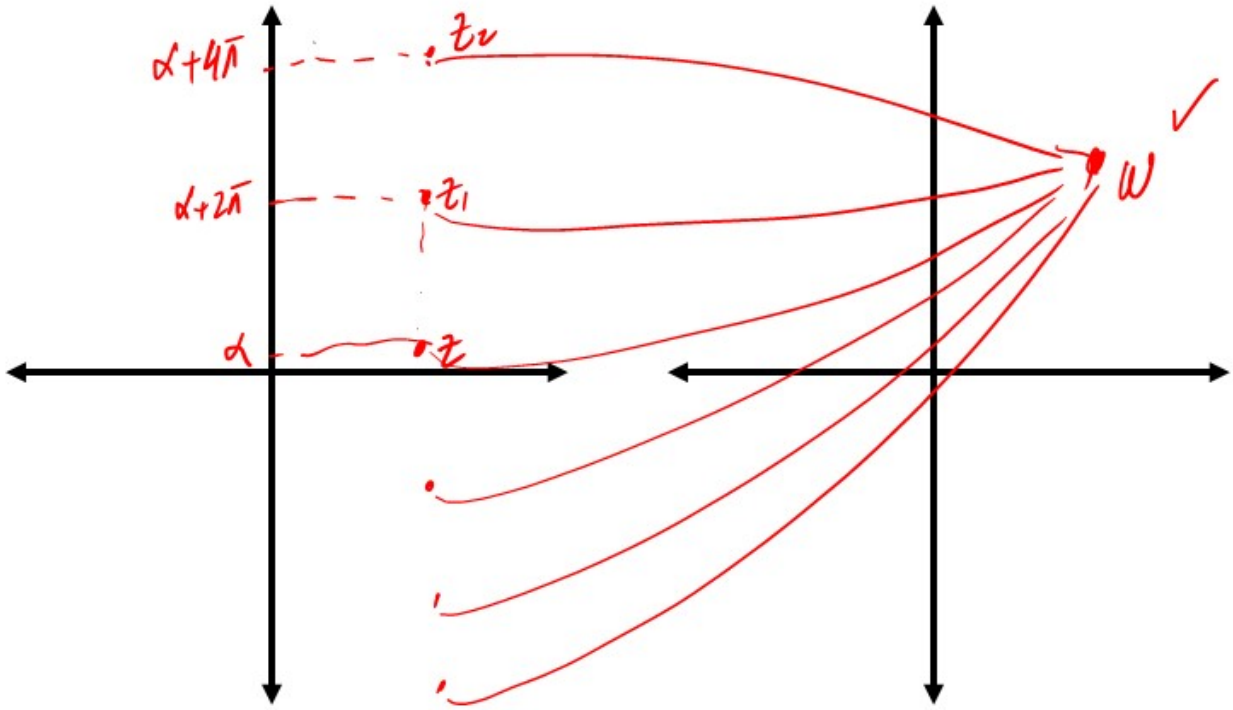
Hence, the equation $e^z = w$ becomes

$$\exp[\ln|w| + i(Arg(w) + 2n\pi)] = w$$

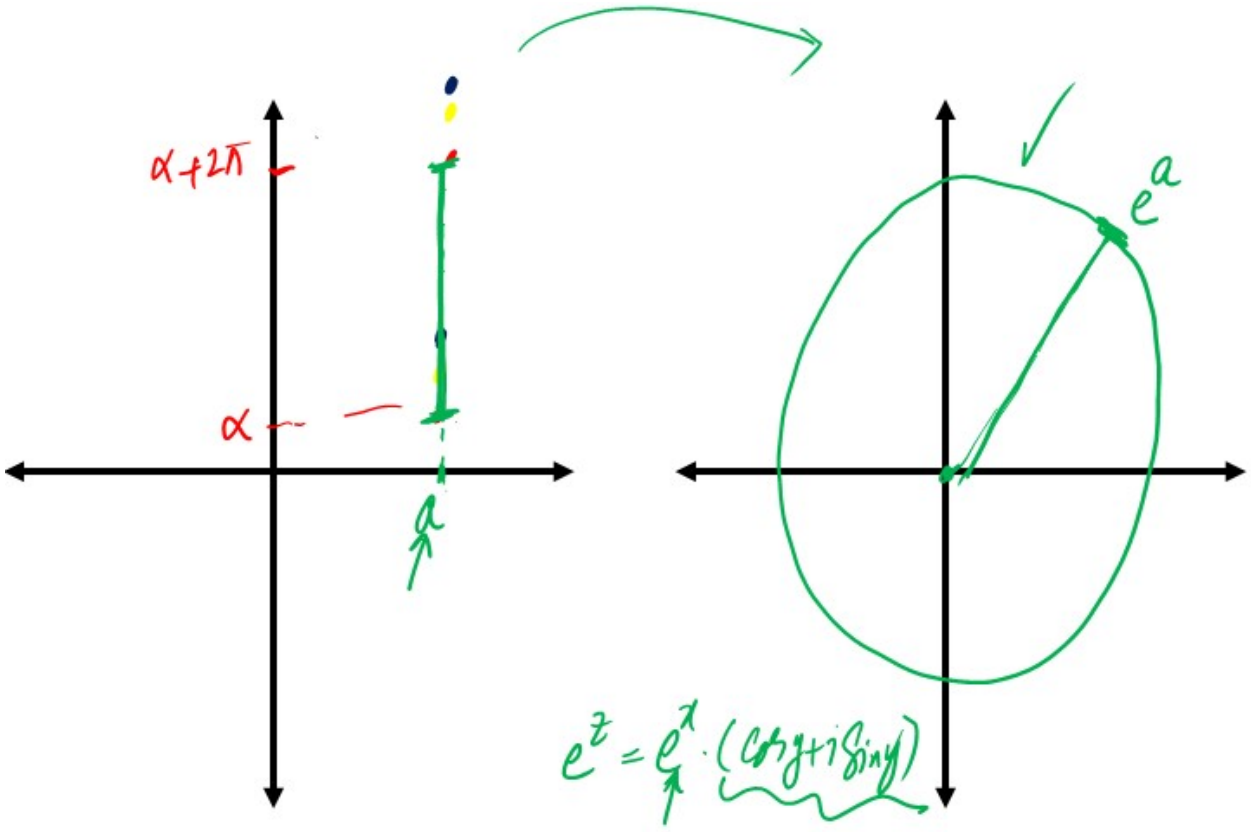
So the complex exponential map e^z is not one-to-one map.



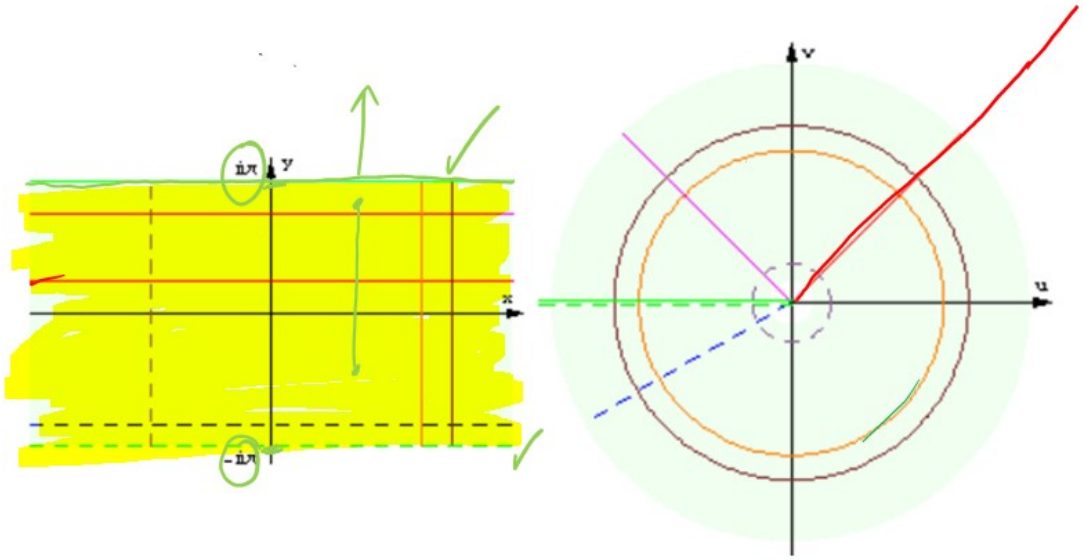
There are many values of z in z-plane, that are mapped on a single value in w-plane under the exponential function.



The real part of z is fixed at point a, then the periodic values of z in z-plane will remain at a distance of a from y-axis. And the values of w will remain at the distance of e^a from the origin in w-plane.



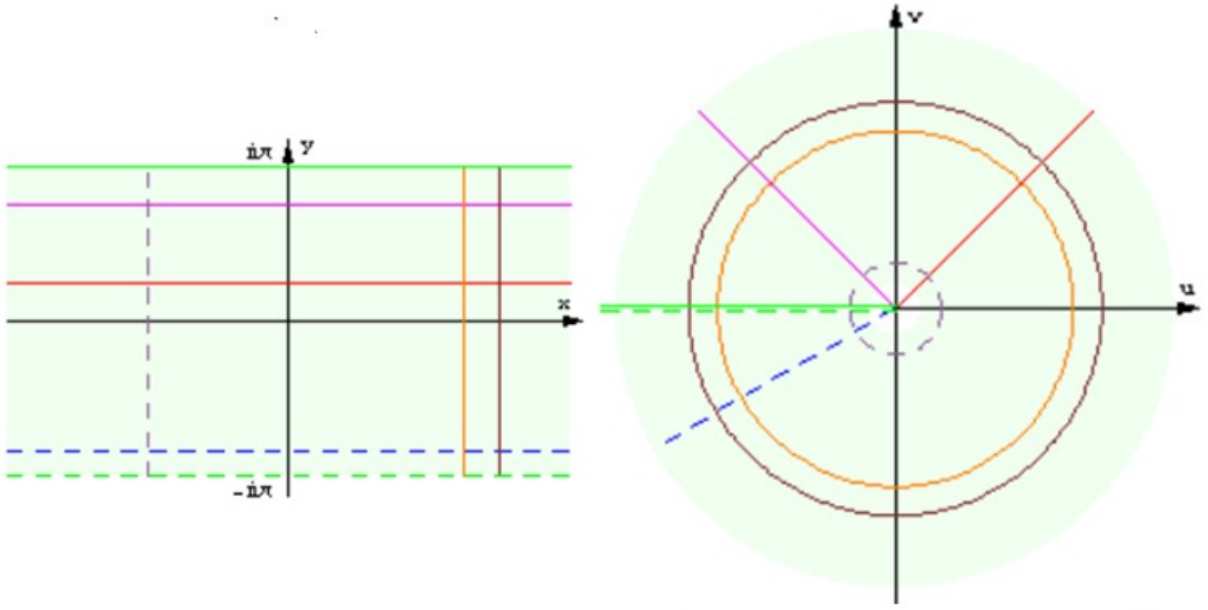
If we fix the real part and keep the imaginary part changing, then we get the vertical lines in z-plane for each fixed value of real part. So you can see that the vertical brown line in z-plane is mapped on brown circle in w-plane. The vertical orange line in z-plane is mapped on orange circle in w-plane.



Source: Complex variables and applications by Brown and Churchill

On the other way around, if we fix the imaginary part and change the real part of z, then we get horizontal lines in z-plane. Then this horizontal line in z-plane is mapped on the radial lines coming from origin, but it does not contain origin because $e^z \neq 0$.

The distance between solid green line and dotted green line is 2π. but both are mapped on the same radial line in w-plane. So the trip between $-\pi$ and π in z-plane, we observe no repetition of graph of radial lines - each radial line is unique in w-plane. The repetition is observed above and below this stripe. This is called Fundamental Period Strip.



Source: Complex variables and applications by Brown and Churchill

FUNDAMENTAL PERIOD STRIP

If in the solution of $w = e^z = e^{x+iy}$, given by

$$\exp[\ln|w| + i(Arg(w) + 2n\pi)] = w$$

we only consider principal argument

$$-\pi < Arg(w) \leq \pi$$

then the map of $w = e^z$ is one-to-one.

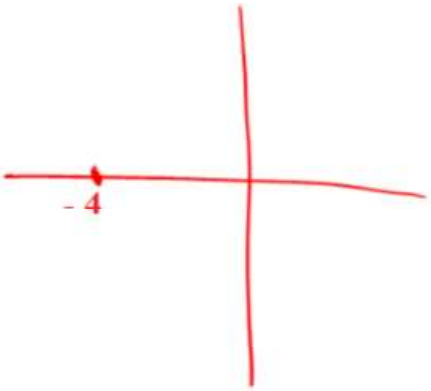
EXAMPLE : Find all values of z for which the following equation hold

$$e^z = -4$$

$e^z = -4 = w$

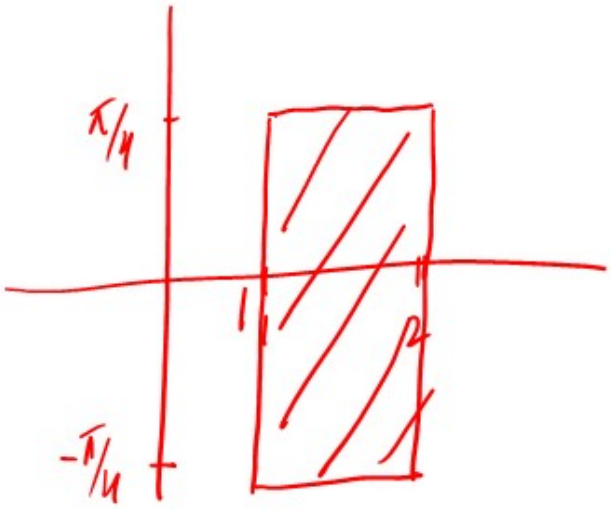
$z = \ln|w| + i(\text{Arg}(w) + 2n\pi)$ ✓

$z = \ln(4) + i\,2n\pi$



EXAMPLE: Find image under the complex exponential function of all complex numbers in A

$$A = \left\{x + iy : 1 \leq x \leq 2,\ -\frac{\pi}{4} \leq y \leq \frac{\pi}{4}\right\}$$



z-plane

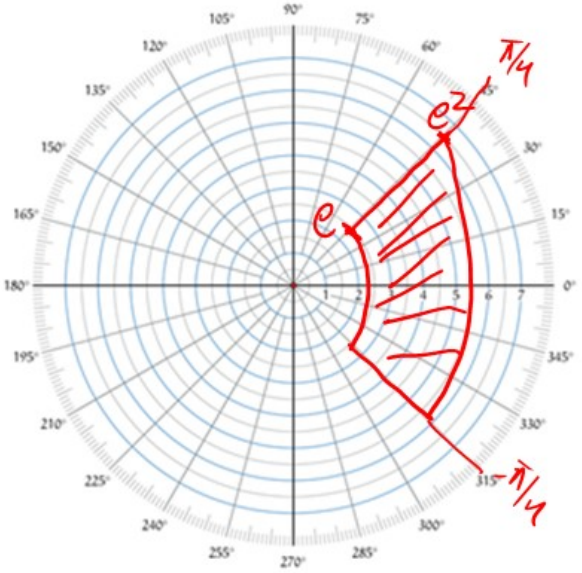
$e^1 \le e^x \le e^2, \quad 1 \le x \le 2$

$-\pi/4 = \text{Arg}\left(e^{x-i\pi/4}\right)$

$\le \text{Arg}\left(e^{x+iy}\right)$

$\le \text{Arg}\left(e^{x+i\pi/4}\right)$

$= \pi/4$



w-plane

output b/w pi/4 & -pi/4

LECTURE 31
MODULE 66

COMPLEX EXPONENTIAL FUNCTIONS

PART II

In this module, we will further explore the exponential function.

PROPERTIES OF EXPONENTIAL FUNCTION

We know that

$$\exp(z) = e^x(\cos y + i \sin y)$$

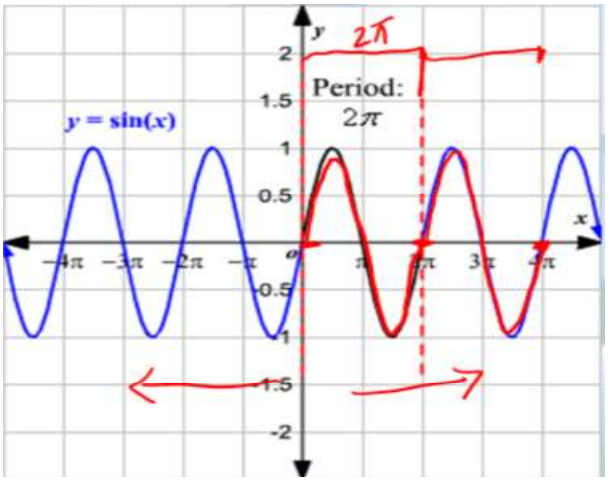
OR

$$e^z = e^x(cos\, y + i\, sin\, y)$$

In this definition of e^z, we can see the trigonometric functions $cos\, y$ and $sin\, y$, where y is a real number.

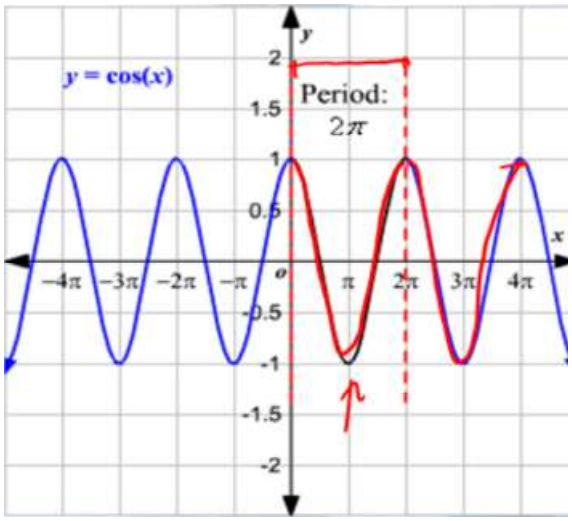
We know that $\sin x$ is a periodic function which repeats itself after a specific interval 2π.

$$\sin(x + 2n\pi) = \sin x$$



The similar observation holds for cos x. We know that cos x is a periodic function.It repeats itself after a specific interval 2π.

$$\cos\,(x\,+\,2n\pi)\;=\;\cos x$$



$sin\,x$ and $cos\,x$ both are periodic functions with a slightly different graphs.

For $z = x + iy$, we have

$$e^z = \exp(z) = e^x(\cos y + i\sin y)$$

$$= e^x(\cos(2n\pi + y) + i\sin(2n\pi + y))$$

$$= e^x e^{i(2n\pi+y)}$$

$$= e^{x+iy}\; e^{i\,2n\pi}$$

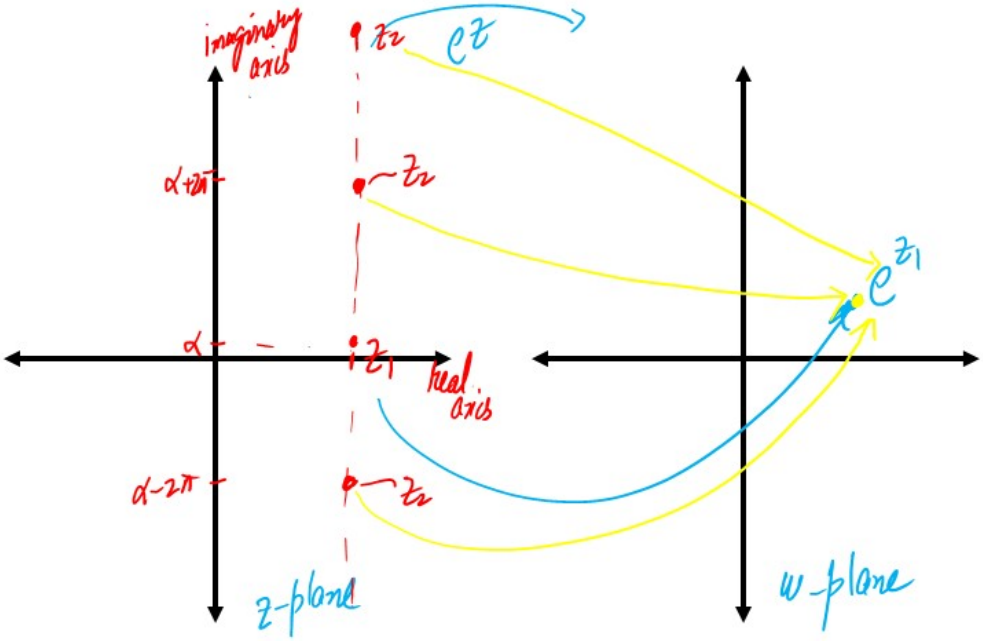
$$= e^z\; e^{i\,2n\pi}$$

$$= e^{z+i2n\pi}$$

Hence $e^{z+i2n\pi} = e^z$ for all z, provided n is an integer.

CONSEQUENCES OF $e^{z+i2n\pi} = e^z$

1. $e^z = 1$ if and only if $z = i\,2n\,\pi$ for some integer n.
2. $e^{z_1} = e^{z_2}$ if and only if $z_2 = z_1 + i\,2n\pi$ for some integer n

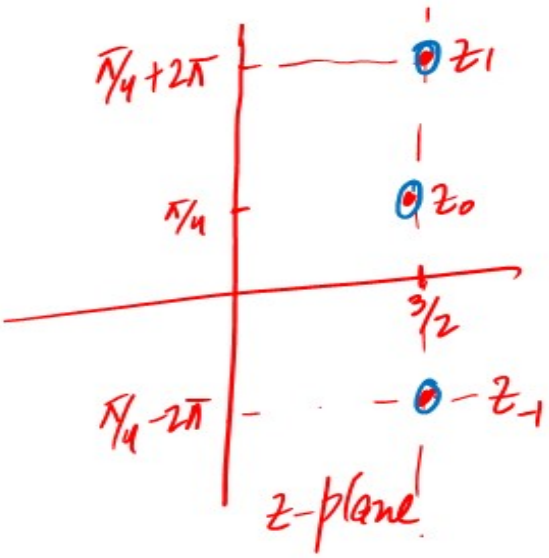


EXAMPLE: For any integer n, the points

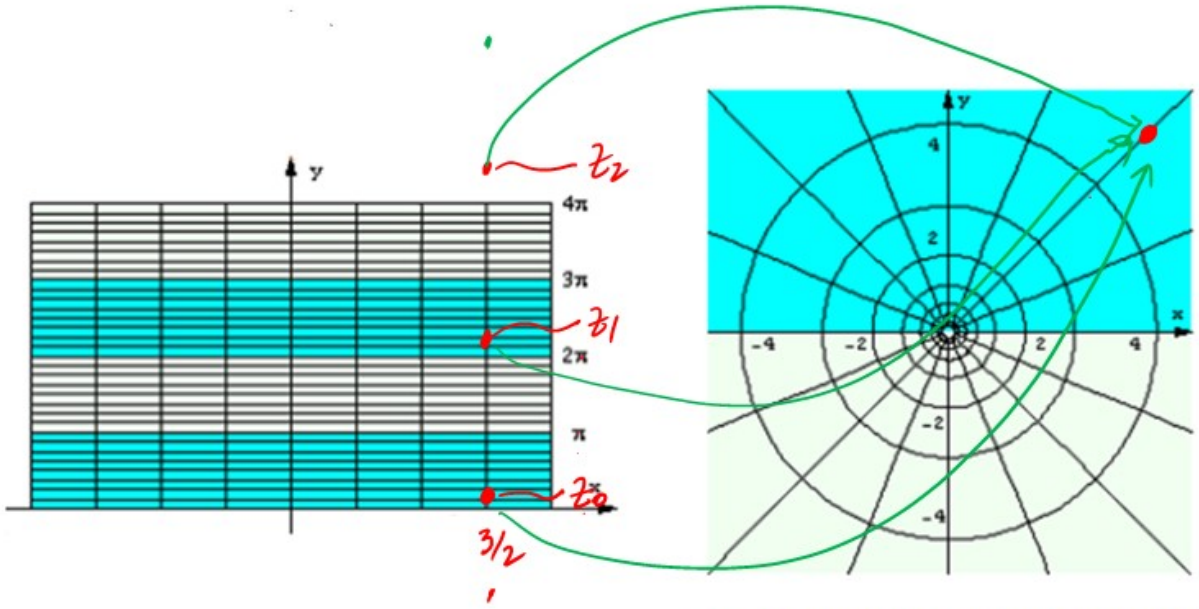
$$z_n = \frac{3}{2} + i\left(\frac{\pi}{4} + 2n\pi\right)$$

in the z-plane. These points are mapped in the w-plane to the one single point.

In z_n, the real part remain the same, i.e. $\frac{3}{2}$, so there is a line parallel to y-axis and passing through ($\frac{3}{2}$, 0). Only the imaginary part varies. So these are infinite points which a mapped on the single point $3.1 + i3.1$ in w-plane.



So these all infinite many points $\frac{3}{2} + i\left(\frac{\pi}{4} + 2n\pi\right)$ in z-plane are mapped to $3.1 + i\,3.1$ in w-plane.



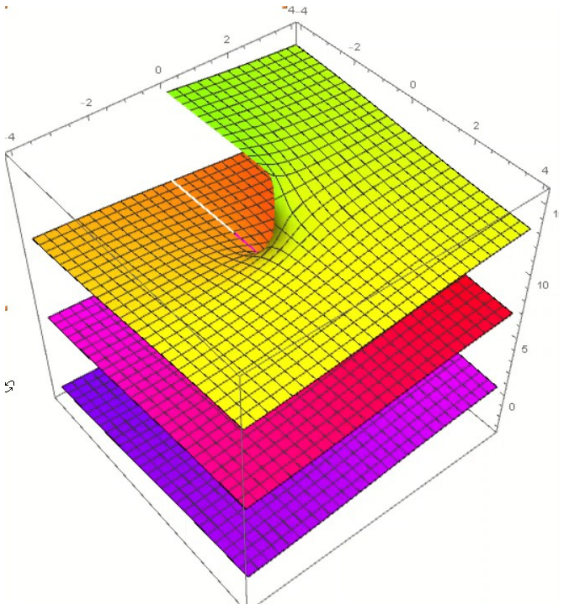
Source: Complex variables and applications by Brown and Churchill

So the exponential function is not one-to-one function. We explore this property of the exponent function further in the next module.

LECTURE 31
MODULE 68

THE COMPLEX LOGARITHM

PART I



In the previous module, we discussed an elementary complex exponential function which is complex version of real valued exponential function. In this module, we will discuss another elementary function which is logarithmic function.

From the Real Analysis, we know that there I a relation between exponential function and logarithmic function. In fact, they are inverses of each other.

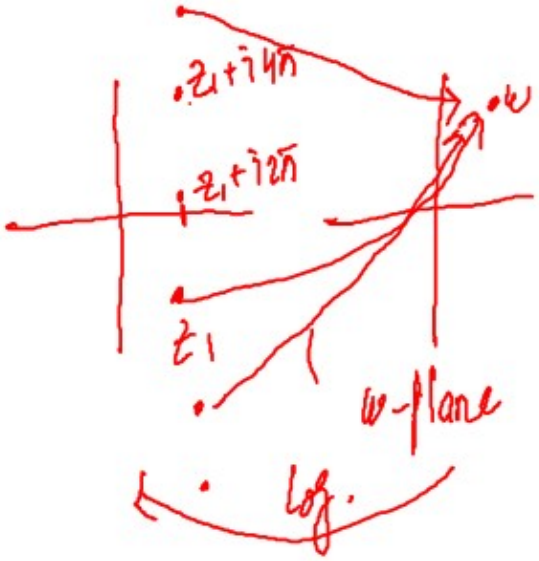
Now a question is, 'How to define logarithm function?'

Moreover, can we define logarithmic function as an Inverse of exponential function?

As we know that a function has an inverse function if it is one-to-one and onto function.

We have a Problem

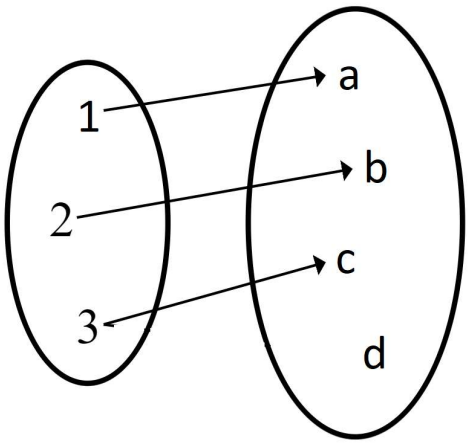
Complex exponential function is not one-to-one function



Here you can see that there are infinite many values of $z = z_1 + i\,2n\pi$ which are mapped on single value $w_1 = e^{z_1}$ in w-plane. So the exponential function is not one-to-one. If we take inverse of exponential function and go back from w-plane to z-plane, we do not have a unique image. Now we see what definition of a function says:

What is a function $f: A \to B$?

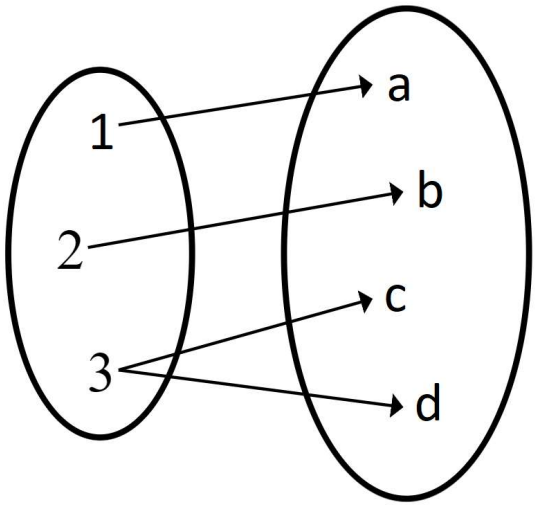
A function is a rule which assigns a unique element of B to every element of A.



So if we go back from w-plane to z-plane, every element w has infinite many pre-images in z-plane. So there is no unique pre-image in z-plane.

To solve this problem, we introduce a multi-valued function.

Multi-valued function



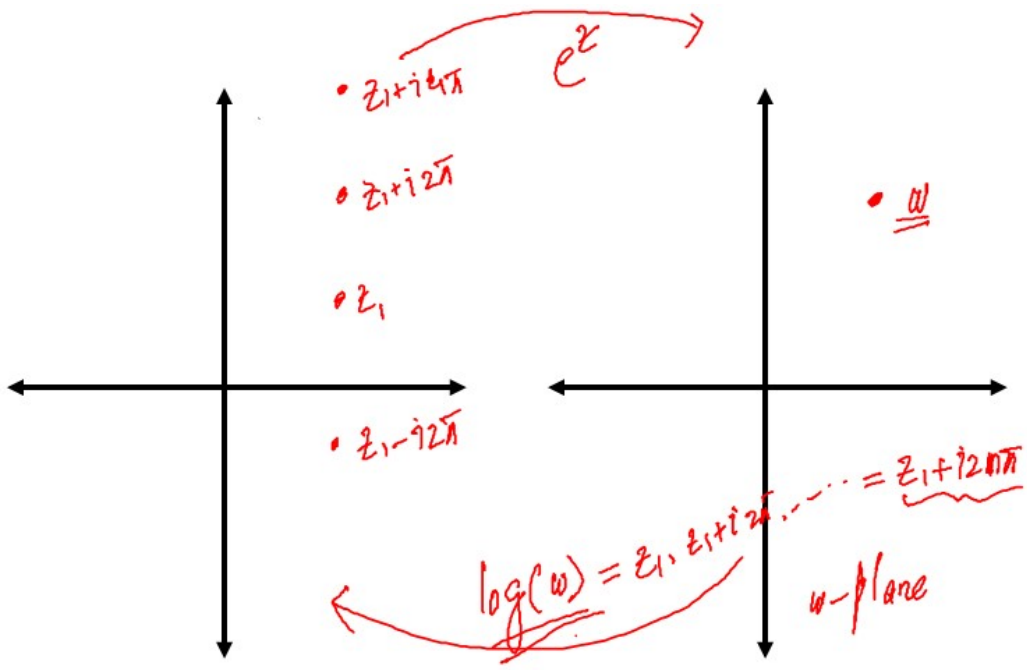
Multi-valued function is a function whose elements have more than one image, so there is no unique image. Multi-valued function is the generalization of the ordinary function.

In the above function, 1 has ne image, 2 has one image and 3 has 2 images. So it is a multi-valued function.

Multi-valued complex logarithm

For $z \neq 0$, we define the multi-valued function log as the inverse of the exponential function; that is, $\log(z) = w$ iff $\exp(w) = z$

Multi-valued logarithm is an inverse of Exponential function



Let us find the exact image of the logarithmic function.

We know from our previous discussion:

For $z = x + iy$, and

$$w = e^z = e^{x+iy}$$

we have

$$w = \exp[\ln|w| + i(Arg(w) + 2n\pi)]$$

By applying log on both sides, we get

$$\log(w) = \ln|w| + i(Arg(w) + 2n\pi)$$

if we denote the domain elements by z then

$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$

$$\log(z) = \ln|z| + i\arg(z)$$

The multi-valued complex logarithmic function can be converted into single valued function by taking principal value the logarithm.

Principal value of the logarithm

For $z \neq 0$, we define Log, as a single valued function by taking the principal value of the logarithm by

$$\text{Log}\,(z) = \ln|z| + iArg(z)$$

Domain: Set of all nonzero complex numbers.

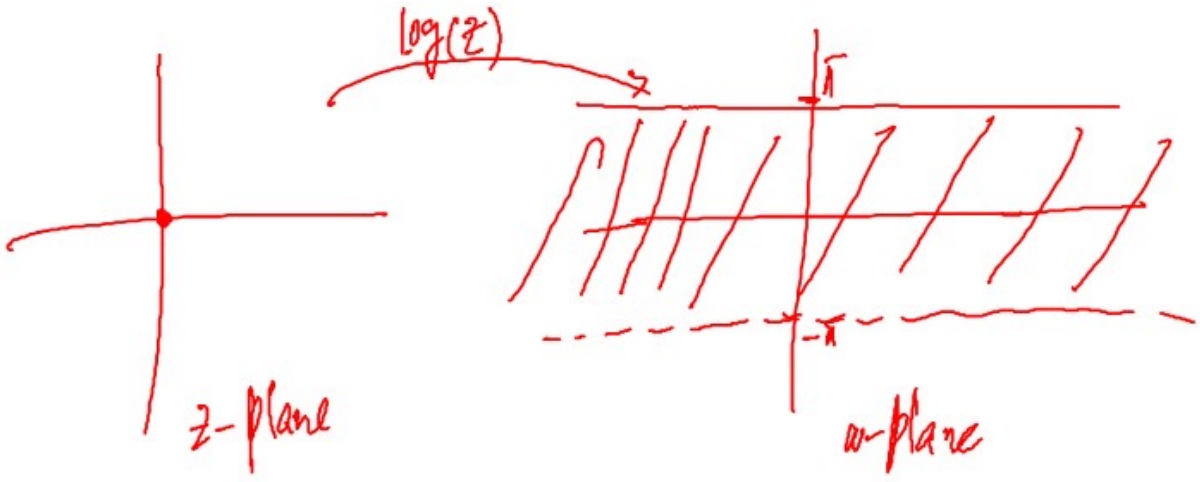
Range of $\text{Log}\,(z) = \ln|z| + iArg(z)$

As

$$-\pi < Arg\,(z) \leq \pi$$

So range of $w = Log(z)$ is

$$\{\, w: -\pi < Im\,(w) \leq \pi \,\}$$



$\text{Log}(z)$ is a single value function from nonzero complex numbers to the fundamental period strip.

FUNDAMENTAL PERIOD STRIP

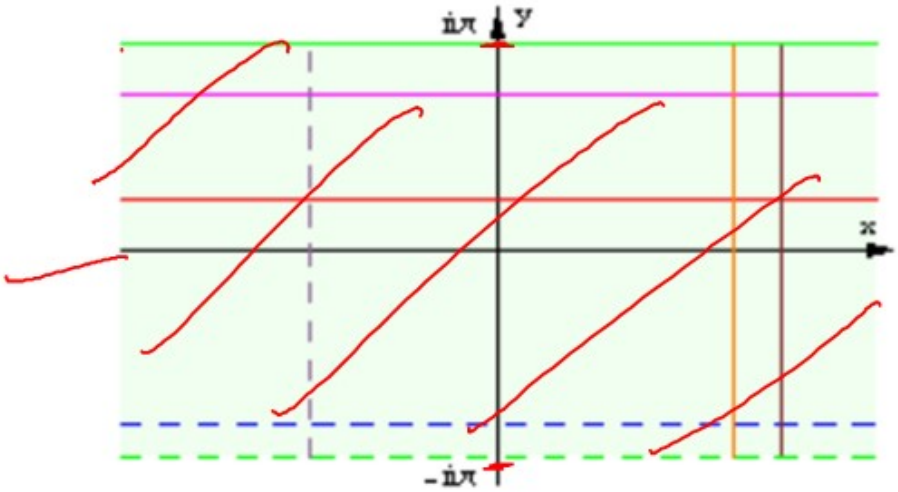
If in the solution of $w = e^z = e^{x+iy}$, given by

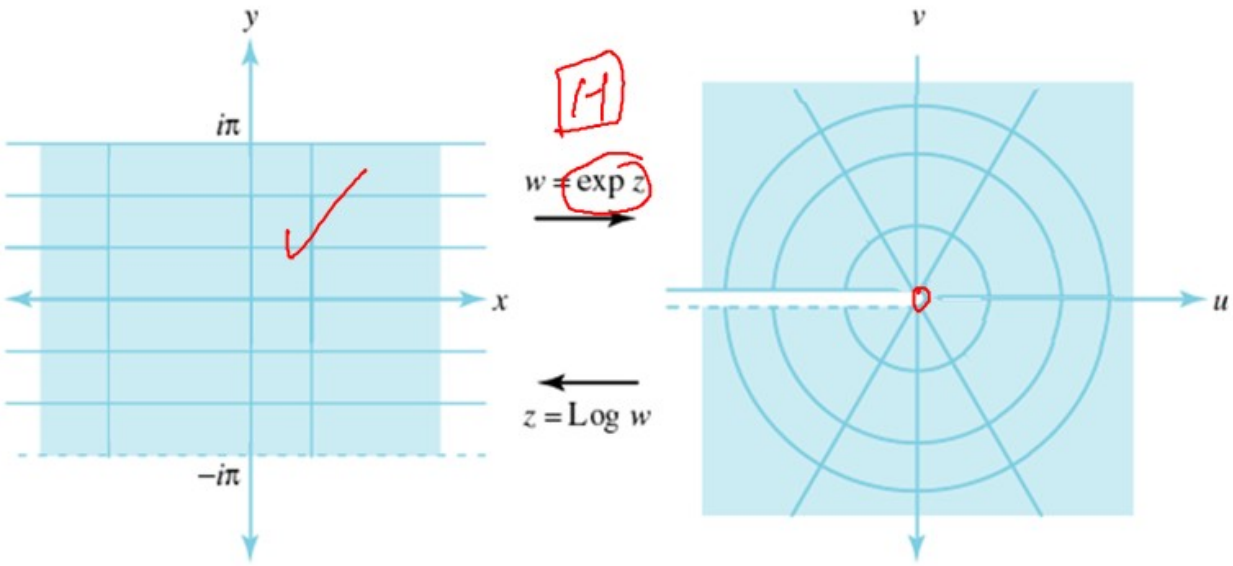
$$\exp\left[\ln|w| + i(Arg(w) + 2n\pi)\right] = w$$

we only consider principal argument

$$-\pi < Arg(w) \leq \pi$$

then the map is one-to-one.





The map $\exp : FPS \to \mathbb{C} \setminus \{0\}$ is inverse of $\text{Log}: \mathbb{C} \setminus \{0\} \to FPS$

That is

$$\exp\big(\text{Log}(z)\big) = z$$

and

$$\text{Log}(\exp(z)) = z$$

$$\exp\big(\text{Log}(z)\big) = z$$

We know that

$$\text{Log}(z) = \ln|z| + iArg(z)$$

Also

$$\exp[\ln|w| + i(Arg(w) + 2n\pi)] = w$$

So $$\exp\big(\text{Log}(z)\big) = \exp[\ln|z| + iArg(z)] = z$$

Similarly we can show the following:

$$\text{Log}(\exp(z)) = z$$

The map $\exp : FPS \to \mathbb{C} \setminus \{0\}$ is inverse of $\text{Log}: \mathbb{C} \setminus \{0\} \to FPS$

That is

$$\exp\big(\text{Log}(z)\big) = z$$

and

$$\text{Log}(\exp(z)) = z$$

EXAMPLE: Find the values of $\log(2 + i2)$ and $\mathrm{Log}(2 + i2)$.

SOLUTION: we know that

$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$

$$\mathrm{Log}(z) = \ln|z| + iArg(z)$$

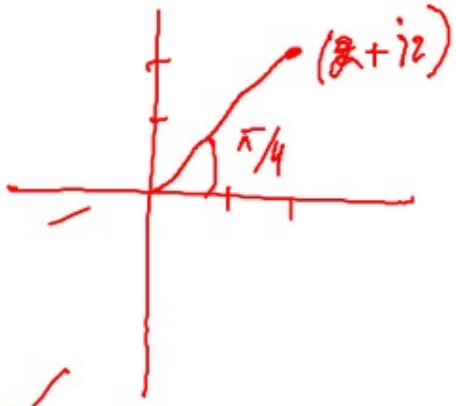
$z = 2 + i2$

$|z| = \sqrt{4+4} = \sqrt{8} = 2\sqrt{2}$

$Arg(z) = \pi/4$

$\mathrm{Log}(z) = \mathrm{Log}(2+i2) = \ln 2\sqrt{2} + i\,\pi/4$

$\log(2+i2) = \ln 2\sqrt{2} + i(\pi/4 + 2n\pi)$ ✓



LECTURE 31
MODULE 69

THE COMPLEX LOGARITHM

PART II

MULTIVALUED LOGARITHM

For $z \neq 0$, we define the multi-valued function log as the inverse of the exponential function; that is,

$$\log(z) = w \text{ iff } \exp(w) = z$$

OR

$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi), \quad n \in Z$$

It is a multi-valued function because it has different values for same value of z and different values of n.

PRINCIPAL VALUE OF LOGARITHM

For $z \neq 0$, we define Log, the principal value of the logarithm, by

$$\text{Log}(z) = \ln|z| + iArg(z)$$

Domain: Set of all nonzero complex numbers.

REMARK

1. $\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$ is a multi-valued function.
2. But $\text{Log}(z) = \ln|z| + iArg(z)$ is a single valued function

We define the continuity and differentiability for the single valued function only, so now on, we will discuss only the principal value of logarithmic function. We will see at what points the logarithmic function is continuous, at what points, logarithmic function is differentiable.

Properties of Log(z)

- Continuity
- Differentiability

THEOREM

The principal logarithm $Log(z)$ becomes $\ln(x)$ when it is restricted to real numbers.

PROOF

For $z = x + iy$, where $x > 0$

$$\text{Log}(z) = \ln|z| + iArg(z)$$

Hence for $z = x + i0$ we get

$$\text{Log}(z) = \ln|x + i\,0| + iArg(x + i0)$$

$$\text{Log}(z) = \ln|x| + i\,tan^{-1}\left(\frac{0}{x}\right)$$

$$\text{Log}(z) = \ln|x| + 0$$

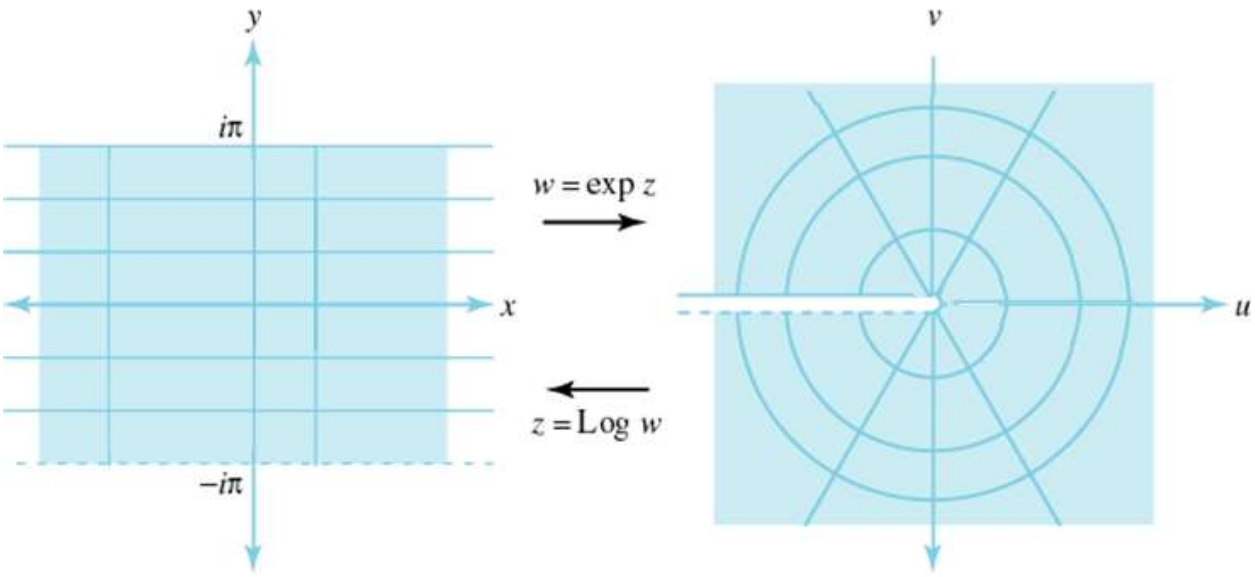
Hence

$$\text{Log}(z) = \ln|x|$$

Continuity of Log(z)

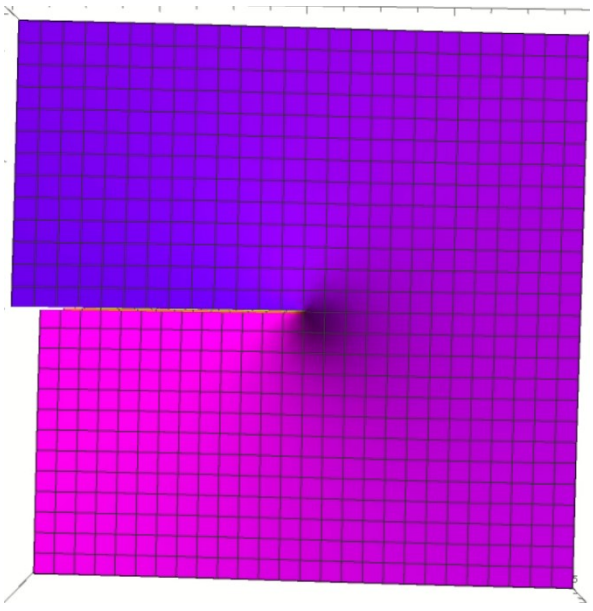
$$\text{Log}(z) = \ln|z| + iArg(z)$$

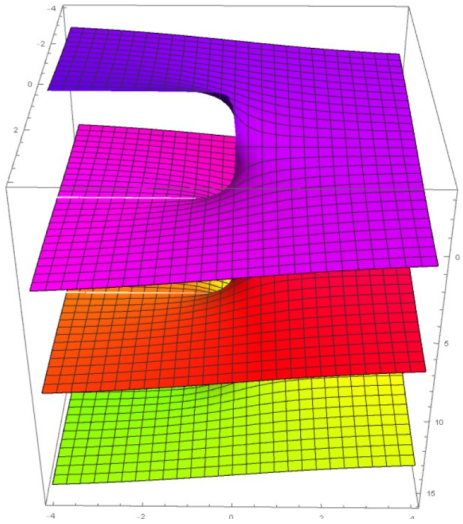
The image of $Log(z)$ is one sheet of complex numbers.



The graph of $\log(z)$ looks like many sheets, one for each n

$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$





If $w = f(z)$ is a multi-valued function. A branch of f is any single-valued function f_0 that is continuous is some domain (except perhaps at the boundary). At each point z in the domain it assigns one of the values of $f(z)$.

The multi-valued function

$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$

has branches - infinite branches, one for each integer value of n. Let

$$\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots$$

be the branches of $\log(z)$.

The negative real axis is the branch cut for the functions

$$\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots$$

Each point on the branch cut is a point of discontinuity for all branches of $\log(z)$.

The principal value of logarithm

$$\text{Log}(z) = \ln|z| + iArg(z)$$

corresponds to just one branch, that is, f_0.

So we can say that the function $Log(z)$ is discontinuous at all the points of negative real axis.

LECTURE 32
MODULE 70

THE COMPLEX LOGARITHM

PART III

Multivalued algorithm

For $z \neq 0$, we define the multivalued function log as the inverse of the exponential function; that is,

$$\log(z) = w \text{ iff } \exp(w) = z$$

OR

$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$

Principal value of the logarithm

For $z \neq 0$, we define Log, the principal value of the logarithm, by

$$\mathrm{Log}(z) = \ln|z| + iArg(z)$$

Domain: Set of all nonzero complex numbers.

Properties of $\mathrm{Log}(z)$

- Continuity
- Differentiability

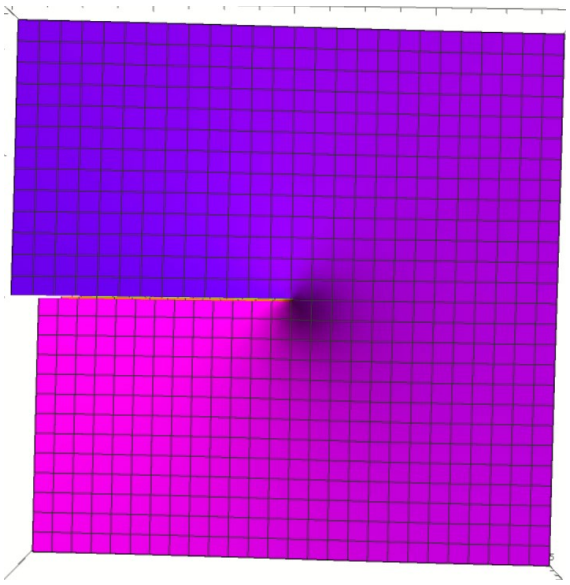
The multivalued function

$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$

has branches infinite branches, one for each integer value of n. Let

$$\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots$$

be the branches of $\log(z)$.



The negative real axis is the branch cut for the functions

$$\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots$$

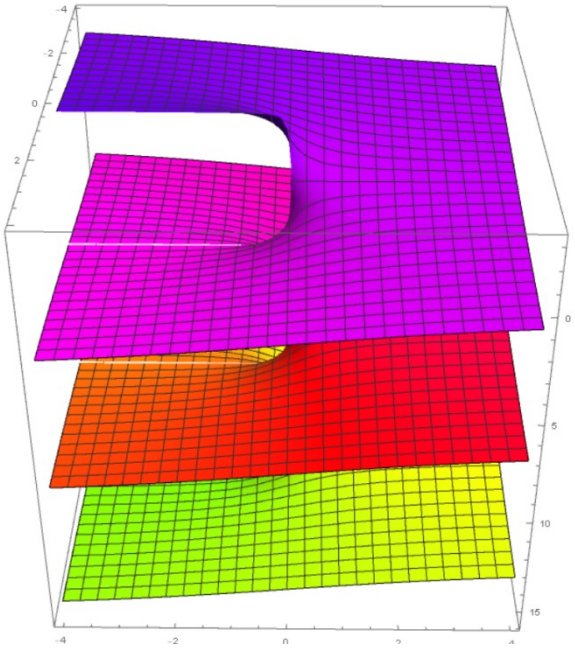
Each point on the branch cut is a point of discontinuity for all branches of $\log(z)$.

The principal value of logarithm

$$\mathrm{Log}(z) = \ln|z| + iArg(z)$$

Corresponds to just one branch, that is, f_0.

So we can say that the function $Log(z)$ is discontinuous at all the points of negative real axis.



Differentiability of Log(z)

For $z = re^{i\theta}$

$$\text{Log}\,(z) = \ln|z| + iArg(z)$$

becomes

$$\text{Log}\left(re^{i\theta}\right) = \ln r + i\theta$$

Hence

$$\text{Log}\,(z) = U(r,\theta) + iV(r,\theta)$$

where

$$U(r,\theta) = \ln r\,,\; V(r,\theta) = \theta$$

Consider

$$\text{Log}\,(z) = U(r,\theta) + iV(r,\theta) = \ln r + i\theta$$

where

$$U(r,\theta) = \ln r\,,\; V(r,\theta) = \theta$$

in domain

$$-\pi < Arg(z) < \pi$$

$$U_r(r,\theta) = \frac{1}{r},\;\; U_\theta(r,\theta) = 0$$

and

$$V_r(r,\theta) = 0\,,\;\; V_\theta(r,\theta) = 1$$

Also

$$r\,U_r = V_\theta \qquad and \qquad -rV_r = U_\theta$$

Hence

$$\frac{d}{dz}Log\,(z) = e^{-i\theta}(\,U_r + i\,V_r)$$

$$\frac{d}{dz}Log\,(z) = e^{-i\theta}\left(\frac{1}{r} + i\,0\right)$$

$$\frac{d}{dz}Log\,(z) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Provided $r > 0$ and $-\pi < \theta < \pi$.

LECTURE 32
MODULE 71

THE COMPLEX LOGARITHM

PART IV

Multivalued algorithm

For $z \neq 0$, we define the multivalued function log as the inverse of the exponential function; that is,

$$\log(z) = w \text{ iff } \exp(w) = z$$

OR

$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$

Principal value of the logarithm

For $z \neq 0$, we define Log, the principal value of the logarithm, by

$$\text{Log}(z) = \ln|z| + iArg(z)$$

Domain: Set of all nonzero complex numbers.

So far we discussed Continuity & Differentiability Of Log (z)

Further Properties of Log(z) and log (z)

THEOREM: Let z_1 and z_2 be nonzero complex numbers. The multivalued function log obeys the familiar properties of logarithms.

1. $\log(z_1 z_2) = \log(z_1) + \log(z_2)$

$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$$

Proof ($\log(z_1 z_2) = \log(z_1) + \log(z_2)$):

We know that

$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$

Or

$$\log(z) = \ln|z| + i(\arg(z))$$

PROOF$\left(\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)\right)$:

We know that

$$\log(z) = \ln|z| + i(\arg(z))$$

Does the following property hold for $Log(z)$?

$$\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$$

Does the following property hold for $Log(z)$?

$$\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$$

Answer: No

EXAMPLE: Let $z_1 = -1$ and $z_2 = -2$. Check whether the following holds

$$\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)\ .$$

SOLUTION: We know that

$$\text{Log}(z) = \ln|z| + iArg(z)$$

THEOREM: The identity

$$\text{Log}(z_1 z_2) = \text{Log}(z_1) + \text{Log}(z_2)$$

holds true iff

$$-\pi < Arg(z_1) + Arg(z_2) \leq \pi$$

LECTURE 32
MODULE 72

THE COMPLEX LOGARITHM

PART V

Branches of The complex Logarithm

Multivalued logarithm

For $z \neq 0$, we define the multivalued function log as the inverse of the exponential function; that is,

$$\log(z) = w \text{ iff } \exp(w) = z$$

OR

$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$

Principal value of the logarithm

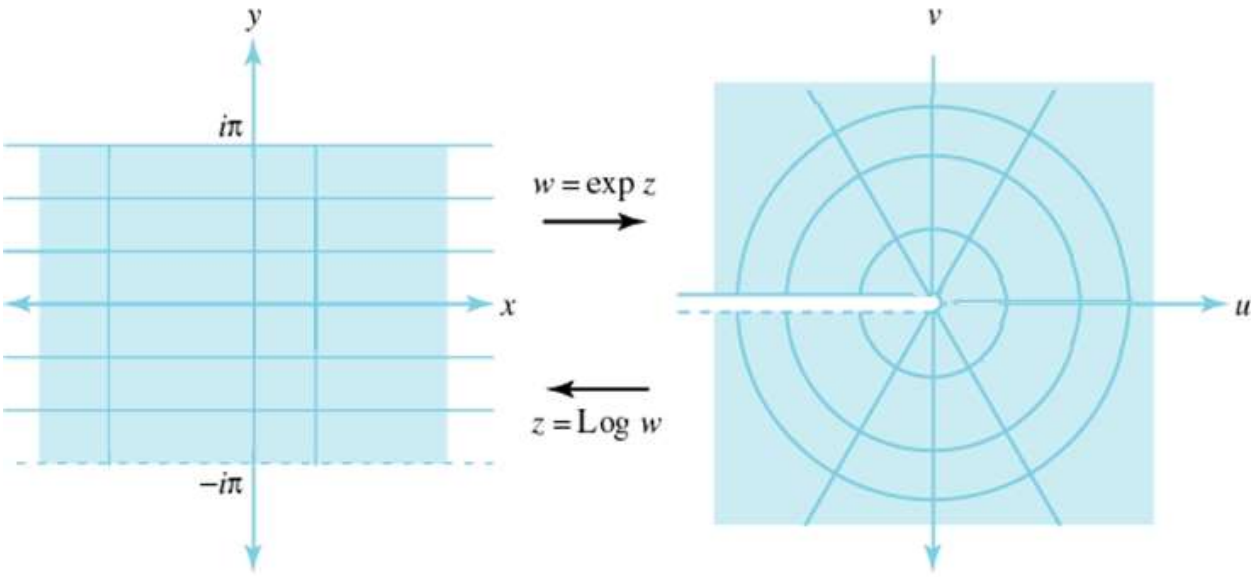
For $z \neq 0$, we define Log, the principal value of the logarithm, by

$$\text{Log}(z) = \ln|z| + iArg(z)$$

Domain: Set of all nonzero complex numbers.

$$\text{Log}(z) = \ln|z| + iArg(z)$$

The image of Log(z) is one complex sheet/strip of non-zero complex numbers.



The multivalued function

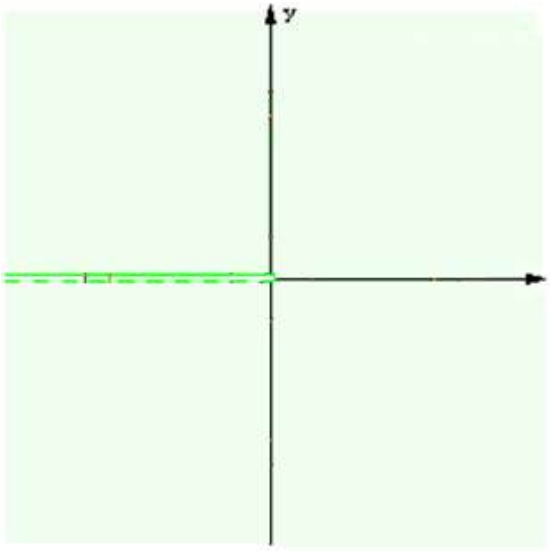
$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$

has infinite branches, one for each integer value of n. Let

$$\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots$$

be the branches of $\log(z)$.

The negative real axis is the branch cut.



The principal value of logarithm

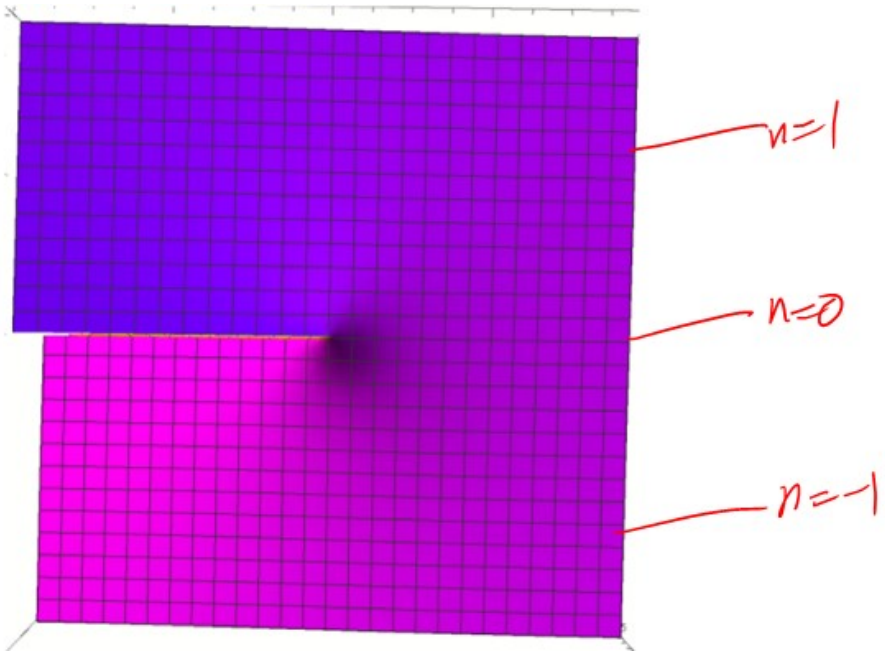
$$\text{Log}\,(z) = \ln|z| + iArg(z)$$

corresponds to just one branch, that is, f_0.

So we can say that the function $\text{Log}(z)$ is discontinuous at all the points of negative real axis or branch cut.

The graph of looks like many sheets, one for each n

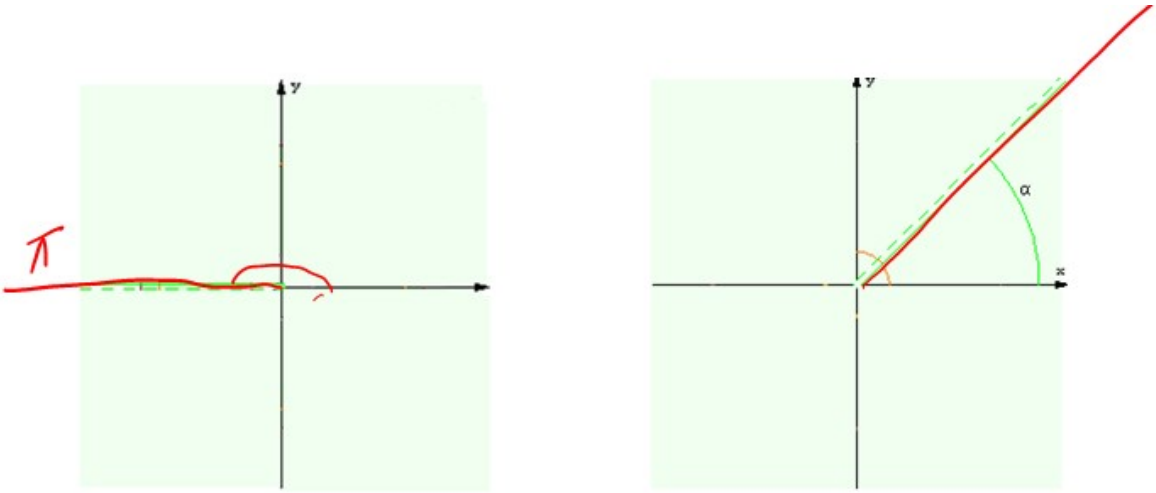
$$\log(z) = \ln|z| + i(Arg(z) + 2n\pi)$$



If $w = f(z)$ is a multivalued function. A branch of f is any single-valued function f_0 that is continuous is some domain (except perhaps at the boundary). At each point z in the domain it assigns one of the valued of $f(\text{z})$.

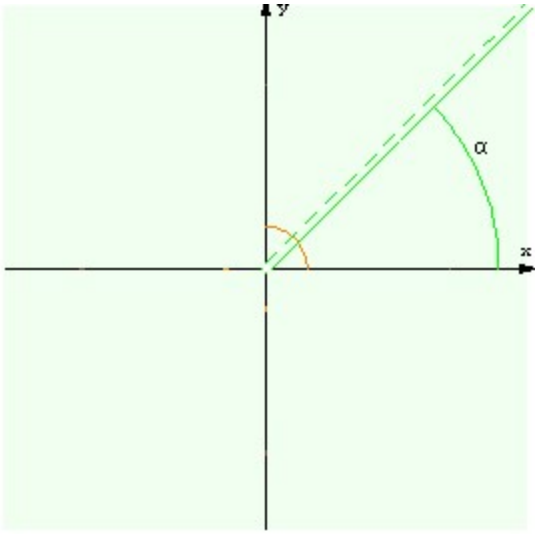
We can choose branch cut to be any preassigned array $\{r\,e^{i\alpha}: r > 0\}$

(where α is a fixed real number)



We can choose branch cut to be any preassigned array $\{r\, e^{i\alpha}: r > 0\}$

So that the branches of the multivalued logarithm function are continuous and differentiable except at the points of this array.



We define, for $z = r\, e^{i\theta}$

$$\log_\alpha z = \ln r + i\,\theta$$

$$where\ \ \theta \in \arg z\ \ such\ that\ \alpha < \theta \leq \alpha + 2\pi$$

to be a single-valued branch of the complex logarithm. The branch cut in this case is the following ray

$$\{r\, e^{i\alpha}: r > 0\}$$

We define, for $z = r\,e^{i\theta}$

$$\log_\alpha z = \ln r + i\,\theta$$

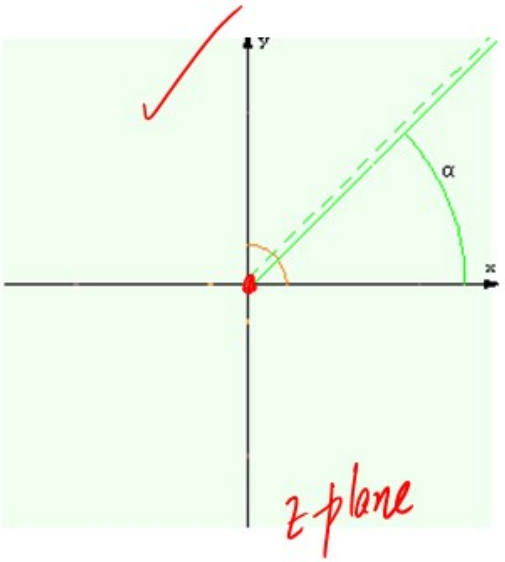
$$where\ \theta \in \arg z\ such\ that\ \alpha < \theta \leq \alpha + 2\pi$$

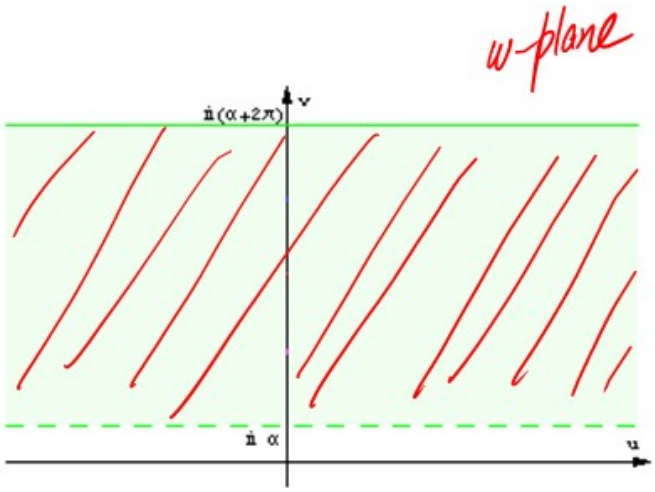
to be a single-valued branch of the complex logarithm. The branch cut in this case is the following ray

$$\{r\,e^{i\alpha} : r > 0\}$$

The mapping $\log_\alpha z$ is one-to-one mapping from $|z| > 0$ to the horizontal strip

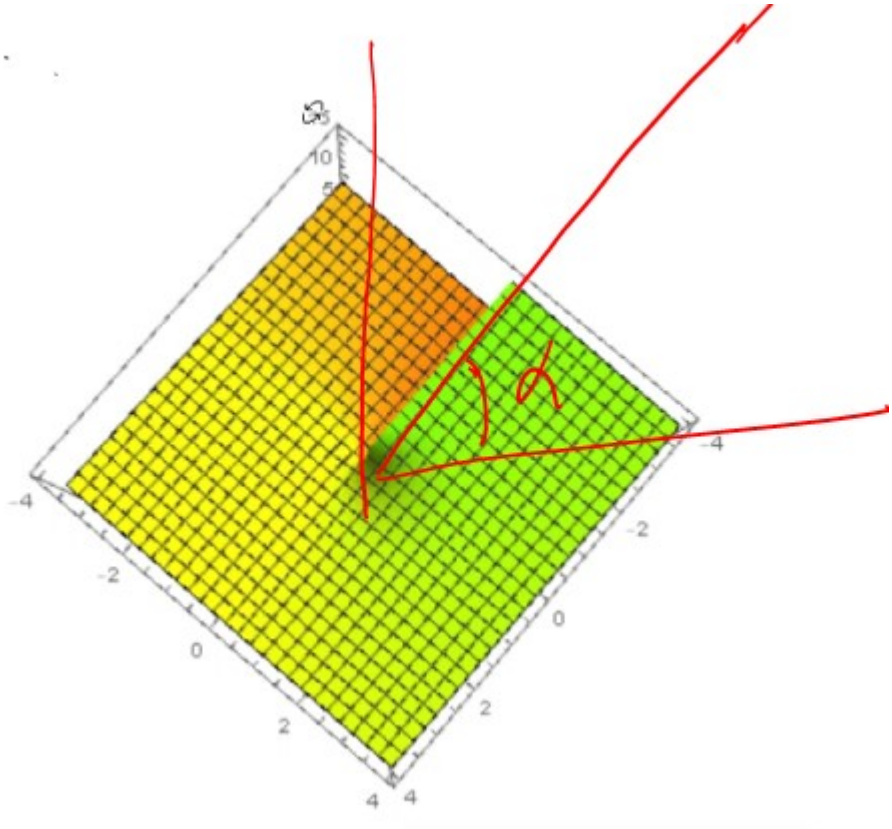
$$\omega = \{\omega : \alpha < \mathrm{Im}\,(\omega) \leq \alpha + 2\pi\}$$



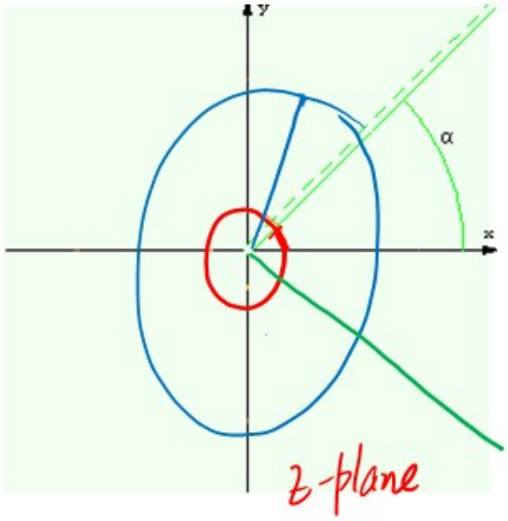


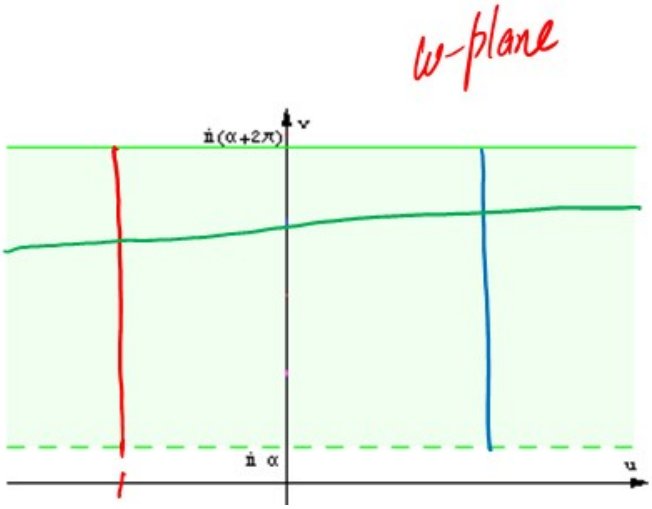
So different branches are separated by the following branch cut

$$\{r\,e^{i\alpha} : r > 0\}$$



The mapping $\log_\alpha z$ maps different subsets of the z-plane in the following way



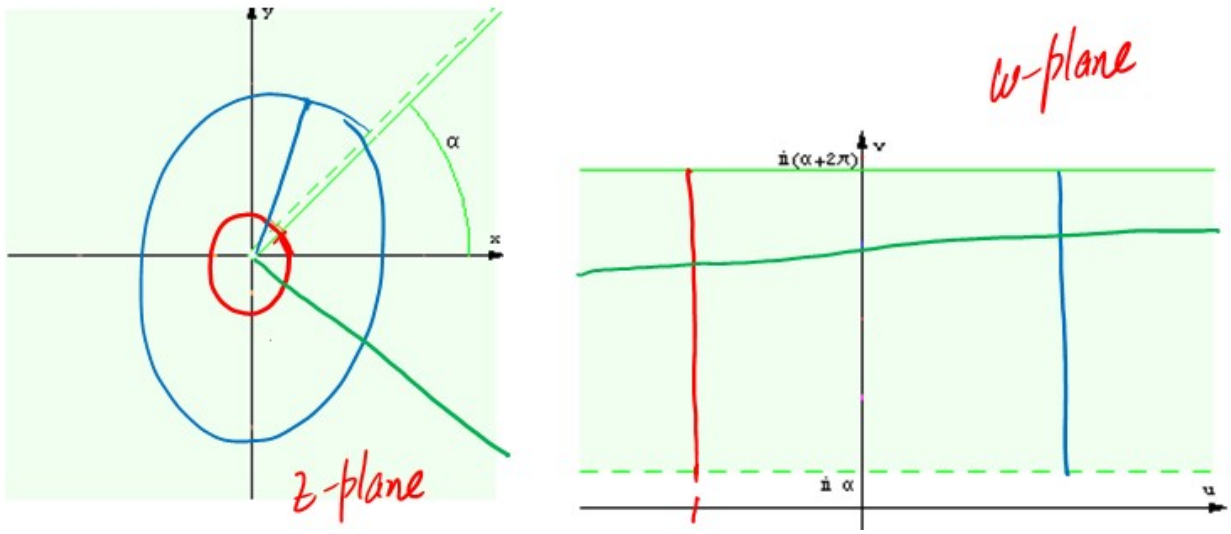


The region

$$\{re^{i\theta}: a < r < b, \quad c < \theta < d\}$$

is mapped to

$$\{u + iv : \ln a < u < \ln b,\ c < v < d\}$$

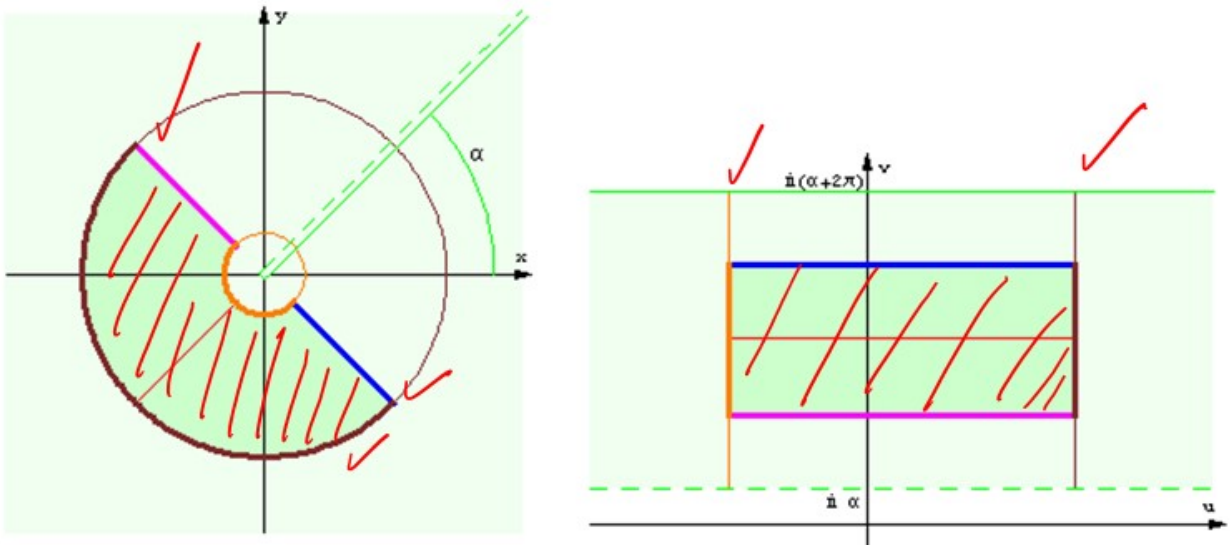


The region

$$\{re^{i\theta} : a < r < b,\quad c < \theta < d\}$$

is mapped to

$$\{u + iv : \ln a < u < \ln b,\ c < v < d\}$$



Since the mapping $\log_\alpha z$ is a single valued function so we can talk about its derivative.

EXAMPLE

Show that

$$\frac{d}{dz}\log_\alpha z = \frac{1}{z},\ for\ z = re^{i\theta},\ \alpha < \theta < \alpha + 2\pi$$

$$1 = \frac{d}{dz} z = \frac{d}{dz} \exp\left[\log_\alpha(z)\right]$$

$$= \exp\left[\log_\alpha(z)\right] \frac{d}{dz} \log_\alpha(z)$$

$$1 = z \cdot \frac{d}{dz} \log_\alpha(z) \Rightarrow \boxed{\frac{d}{dz} \log_\alpha(z) = \frac{1}{z}}$$

LECTURE 32
MODULE 73

THE COMPLEX COMPONENTS

PART I

In this module, we are going to discuss complex exponents. Previously we have calculated, for example, 2^2 or $\sqrt{9}$ or π^3 or other exponents of real numbers. Now we are going to see, what is i^i that we have not discussed so far. Similarly, π^i. So how will we calculate them? Let us explore how we will define them.

We can define the real exponents, like 2^2, $\pi^{\frac{2}{3}}$, $\sqrt{87}$ etc.

Now what happens if the power is a complex number, like i^i, $\sqrt{1+i}$, 2^{1+i} etc.

Main idea for solving the complex exponents is that we use our knowledge about the complex exponential function and complex logarithm to define **complex exponents**.

Let us see how we are going to do that.

We know that

$$\exp[\log(z)] = z, \quad \text{for} \quad z \neq 0$$

Also,

$$\log(z^n) = n \log(z)$$

where n is any natural number.

Hence

$$\exp[\log(z^n)] = \exp[n \log(z)] = z^n, \quad \text{for} \quad z \neq 0$$

$$z^n = \exp[n \log(z)]$$

DEFINITION – Complex Exponent

Let $z \neq 0$ and c be a complex number. We define z^c as

$$z^c = \exp[c \log(z)]$$

EXAMPLE 1: Evaluate $4^{\frac{1}{2}}$.

SOLUTION:

Here $z = 4, \quad c = \frac{1}{2}$

Now

$$4^{\frac{1}{2}} = \exp\left[\frac{1}{2} \log(4)\right]$$

$$= \exp\left[\frac{1}{2} \left(\log|4| + i\,(Arg(4) + 2n\pi)\right)\right]$$

$$= \exp\left[\frac{1}{2} \left(\log|4| + i\,(0 + 2n\pi)\right)\right]$$

$$= \exp\left[\frac{1}{2} \log|4| + i\frac{1}{2}\,(2n\pi)\right]$$

$$= \exp\left[\log|4|^{\frac{1}{2}} + i\frac{1}{2}\,(2n\pi)\right]$$

$$= \exp[\log 2 + i\,n\pi]$$

For n = 0, $4^{\frac{1}{2}}$: $\exp[\log 2 + i\,0\pi] = \exp[\log 2] = 2$

For n = 1, $4^{\frac{1}{2}}$: $\exp[\log 2 + i\,1\pi] = \exp[\log 2 + i\pi]$

$$= e^{\log 2 + i\pi}$$

$$= e^{\log 2} e^{i\pi}$$

$$= 2\,e^{i\pi}$$

$$= 2[\cos\pi + i\sin\pi]$$

$$= 2[-1 + i0]$$

$$= -2$$

Therefore, $4^{\frac{1}{2}} = \{2, -2\}$

But in the real case, $\sqrt{4}$ has the positive value only, $\sqrt{4} = 2$.

Since $\log(z)$ is a multivalued function, so in general, z^c is a multivalued function.

So we can define the principal branch of the multivalued function of z^c as

$$z^c = \exp[c\,\mathrm{Log}(z)]$$

Sometimes, it is denoted by

$$\mathrm{P.V.}\, z^c = \exp[c\,\mathrm{Log}(z)]$$

EXAMPLE 2: Evaluate P.V. i^i.

SOLUTION:

P.V. i^i means principal value of i^i.

Here $z = i,\ \ c = i$

Now

$$i^i = \exp[i\log(i)]$$

$$= \exp\left[i\left(\log|i| + i\left(Arg(i)\right)\right)\right]$$

$$= \exp\left[i\left(\log\left(\sqrt{0^2+1^2}\right) + i\left(tan^{-1}\left(\frac{1}{0}\right)\right)\right)\right]$$

$$= \exp\left[i\left(\log(1) + i\left(\frac{\pi}{2}\right)\right)\right]$$

$$= \exp\left[i\left(0 + i\left(\frac{\pi}{2}\right)\right)\right]$$

$$= \exp\left[i\left(i\left(\frac{\pi}{2}\right)\right)\right]$$

$$= \exp\left(-\frac{\pi}{2}\right) = 0.20788$$

LECTURE 32
MODULE 74

THE COMPLEX COMPONENTS

PART II

So far, we have defined the complex components.

Let $z \neq 0$ be complex number and c be also a complex number. We define z^c as

$$z^c = \exp[c \log(z)]$$

i.e. complex exponent is defined in terms of composition of exponential function and logarithmic function. Since log(z) is a multivalued function, so z^c is also a multivalued function.

Now we focus on the particular cases.

CASE 1: When c is a rational number, $c \in \mathbb{Q}$.

Let $z = r\, e^{r\theta}$, $c = \frac{j}{k}$ where $k \neq 0$.

$$z^{\frac{j}{k}} = \exp\left[\frac{j}{k} \log(z)\right]$$

$$= \exp\left[\frac{j}{k} \log\left(r\, e^{r\theta}\right)\right]$$

$$= \exp\left[\frac{j}{k} \left(\ln r + i\,(\theta + 2n\pi)\right)\right]$$

$$= \exp\left[\left(\ln r^{\frac{j}{k}} + i\,\frac{(\theta+2n\pi)j}{k}\right)\right]$$

$$= \exp\left[\ln r^{\frac{j}{k}}\right] \exp\left[i\,\frac{(\theta+2n\pi)j}{k}\right] \qquad \because\ e^{a+b} = e^a e^b$$

$$= r^{\frac{j}{k}}\ \exp\left[i\,\frac{(\theta+2n\pi)j}{k}\right]$$

$$= r^{\frac{j}{k}}\ \left(\cos\left(\frac{(\theta+2n\pi)j}{k}\right) + i \sin\frac{(\theta+2n\pi)j}{k}\right)$$

$$\text{Hence,} \quad z^{\frac{j}{k}} = r^{\frac{j}{k}} \left(\cos\left(\frac{\theta}{k} + \frac{2n\pi}{k}\right) j + i \sin \frac{(\theta + 2n\pi)j}{k}\right)$$

There are k distinct values of $z^{\frac{j}{k}}$ corresponding to the values $n = 0, 1, 2, \ldots, k-1$.

After the value of k, the values of sin and cos will start to repeat.

Now we discuss the subcases of c as a rational number.

SUBCASE (a): If $j = mk$, then

$$\begin{aligned} z^{\frac{j}{k}} &= z^{\frac{mk}{k}} \\ &= z^m \\ &= r^m \left(\cos(\theta + 2n\pi)m + i \sin(\theta + 2n\pi)m\right) \\ &= r^m \left(\cos(m\theta + 2n\pi m) + i \sin(m\theta + 2n\pi m)\right) \\ &= r^m \left(\cos(m\theta) + i \sin(m\theta)\right) \\ z^m &= r^m \left(\cos(m\theta) + i \sin(m\theta)\right) \end{aligned}$$

This is the single valued function which is mth power of z.

SUBCASE (b): If $j = 1$, then

$$\begin{aligned} z^{\frac{j}{k}} &= z^{\frac{1}{k}} \\ z^{\frac{j}{k}} &= r^{\frac{j}{k}} \left(\cos\left(\frac{(\theta + 2n\pi)j}{k}\right) + i \sin \frac{(\theta + 2n\pi)j}{k}\right) \\ z^{\frac{j}{k}} &= r^{\frac{1}{k}} \left(\cos\left(\frac{(\theta + 2n\pi)}{k}\right) + i \sin \frac{(\theta + 2n\pi)}{k}\right) \end{aligned}$$

There are k distinct values of $z^{\frac{j}{k}}$ corresponding to the values $n = 0, 1, 2, \ldots, k-1$.

So, $z^{\frac{1}{k}}$ is the multivalued kth root function, i.e. $z^{\frac{1}{k}}$ has k outputs.

CASE 2: When c is not a rational number, $c \notin \mathbb{Q}$.

In this case, there are infinitely many values of z^c if $z \neq 0$.

Now take a look to see how it happens.

$$2^{\frac{1}{9}+\frac{i}{50}}$$

$$= \exp\left[\left(\frac{1}{9}+\frac{i}{50}\right)\log(2)\right]$$

$$= \exp\left[\left(\frac{1}{9}+\frac{i}{50}\right)(\ln 2 + i\,2n\pi)\right]$$

$$= \exp\left[\left(\frac{\ln 2}{9}-\frac{n\pi}{25}\right)+i\left(\frac{\ln 2}{50}+\frac{2n\pi}{9}\right)\right]$$

$$= \exp\left[\frac{\ln 2}{9}\right]\ \exp\left[-\frac{n\pi}{25}\right]\exp\left[i\left(\frac{\ln 2}{50}+\frac{2n\pi}{9}\right)\right]$$

$$= \exp\left[\ln 2^{\frac{1}{9}}\right]\quad \exp\left[-\frac{n\pi}{25}\right]\quad \exp\left[i\left(\frac{\ln 2}{50}+\frac{2n\pi}{9}\right)\right]$$

$$= 2^{\frac{1}{9}}\quad e^{-\frac{n\pi}{25}}\quad \exp\left[i\left(\frac{\ln 2}{50}+\frac{2n\pi}{9}\right)\right]$$

$$= 2^{\frac{1}{9}}\quad e^{-\frac{n\pi}{25}}\quad \left[\cos\left(\frac{\ln 2}{50}+\frac{2n\pi}{9}\right)+i\sin\left(\frac{\ln 2}{50}+\frac{2n\pi}{9}\right)\right]$$

Hence,

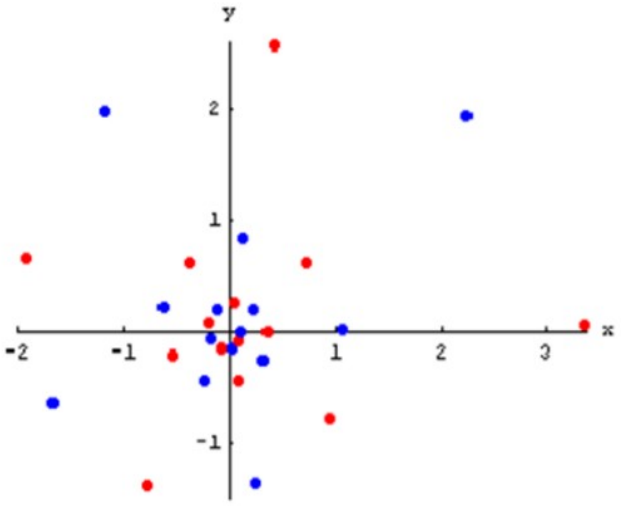
$$2^{\frac{1}{9}+\frac{i}{50}} = 2^{\frac{1}{9}}\quad e^{-\frac{n\pi}{25}}\quad \left[\cos\left(\frac{\ln 2}{50}+\frac{2n\pi}{9}\right)+i\sin\left(\frac{\ln 2}{50}+\frac{2n\pi}{9}\right)\right]$$

There are distinct values of $2^{\frac{1}{9}+\frac{i}{50}}$ corresponding to the following values of n;

$$n = \ldots, -3, -2, -1, 0, 1, 2, \ldots$$

So there are infinitely many values of $2^{\frac{1}{9}+\frac{i}{50}}$ for n.

The outputs of this function $f(z) = 2^{\frac{1}{9}+\frac{i}{50}}$ for n are shown below:



Source: Complex analysis by Mathews and Russell

You can see a pattern of spiral in the graph.

REMARK

If we take a function $f(z) = 2^{\frac{1}{9}+\frac{i}{50}}$, then we get output from 1 to infinity.

RULES FOR EXPONENTS

1. $\frac{1}{z^c} = z^{-c}$
2. $z^c z^d = z^{c+d}$
3. $\frac{z^c}{z^d} = z^{c-d}$
4. $(z^c)^n = z^{nc}$, only if n is integer.

These rules hold for the real values of c and d. Here we check whether these rules are true for the complex values of c and d.

PROOF (1):

$$L.H.S. = \frac{1}{z^c} = \frac{1}{\exp[c\log(z)]} = (\exp[c\log(z)])^{-1}$$
$$= \left(e^{c\log(z)}\right)^{-1} = e^{-c\log(z)}$$

$$= \exp[-\mathrm{c}\log(\mathrm{z})]$$
$$= z^{-c} = R.H.S.$$

PROOF (2):

$$L.H.S. = z^c z^d$$
$$= \left[\exp[\mathrm{c}\log(\mathrm{z})]\right]\left[\exp[\mathrm{d}\log(\mathrm{z})]\right]$$
$$= \left(\mathrm{e}^{\mathrm{c}\log(\mathrm{z})}\right)\left(\mathrm{e}^{\mathrm{d}\log(\mathrm{z})}\right)$$
$$= \mathrm{e}^{\mathrm{c}\log(\mathrm{z})+\mathrm{d}\log(\mathrm{z})}$$
$$= \left(\mathrm{e}^{(c+d)\log(\mathrm{z})}\right)$$
$$= z^{c+d} = R.H.S.$$

PROOF (3):

$$L.H.S. = \frac{z^c}{z^d}$$
$$= \left[\exp[\mathrm{c}\log(\mathrm{z})]\right] \Big/ \left[\exp[\mathrm{d}\log(\mathrm{z})]\right]$$
$$= \frac{\left(\mathrm{e}^{\mathrm{c}\log(\mathrm{z})}\right)}{\left(\mathrm{e}^{\mathrm{d}\log(\mathrm{z})}\right)}$$
$$= \mathrm{e}^{\mathrm{c}\log(\mathrm{z})-\mathrm{d}\log(\mathrm{z})}$$
$$= \left(\mathrm{e}^{(c-d)\log(\mathrm{z})}\right)$$
$$= z^{c-d} = R.H.S.$$

PROOF (4):

$$L.H.S. = (z^c)^n$$
$$= \left[\exp[\mathrm{c}\log(\mathrm{z})]\right]^n$$
$$= \left(\mathrm{e}^{\mathrm{c}\log(\mathrm{z})}\right)^n$$
$$= \mathrm{e}^{\mathrm{nc}\log(\mathrm{z})}$$
$$= \exp[\mathrm{nc}\log(\mathrm{z})] = z^{nc} = R.H.S.$$

REMARK

The following rule does not hold if n is not an integer.

$$(z^c)^n = z^{nc}$$

We can see it through the following example.

EXAMPLE

Evaluate and compare the values of $(i^2)^i$ and i^{2i}.

SOLUTION:

$$(i^2)^i = (-1)^i = \exp[\mathrm{i}\log(-1)]$$

$$= \exp\left[i\left(\ln|-1| + i\,arg(-1)\right)\right]$$

$$= \exp\left[i\left(\ln(1) + i\,(\pi + 2n\pi)\right)\right]$$

$$= \exp\left[i\left(0 + i\,(\pi + 2n\pi)\right)\right]$$

$$= \exp[i^2\,\pi(1 + 2n)]$$

$$= \exp[-\pi(1 + 2n)] \qquad \text{_____(1)}$$

$$i^{2i} = \exp[2i\log(i)]$$

$$= \exp\left[2i\left(\ln|i| + i\,arg(i)\right)\right]$$

$$= \exp\left[2i\left(\ln(1) + i\left(\frac{\pi}{2} + 2n\pi\right)\right)\right]$$

$$= \exp\left[2i\left(0 + i\left(\frac{\pi}{2} + 2n\pi\right)\right)\right]$$

$$= \exp\left[2i^2\left(\frac{\pi}{2} + 2n\pi\right)\right]$$

$$= \exp[-\pi\,(1 + 4n)] \qquad \text{_____(2)}$$

By (1) and (2), $(i^2)^i \neq i^{2i}$.

REMARK

The following rule does not hold in general.

$$(z_1\, z_2)^c \;=\; {z_1}^c\, {z_2}^c$$

EXERCISE

Let $z_1 = 1 + i,\ z_2 = 1 - i,\ z_3 = -1 - i$.

Compare the values of $(z_1\, z_2)^i$ and ${z_1}^i\, {z_2}^i$.

Compare the values of $(z_2\, z_3)^i$ and ${z_2}^i\, {z_3}^i$.

LECTURE 32
MODULE 75

THE COMPLEX COMPONENTS

PART III

RECALL

Let $z \neq 0$ be complex number and c be also a complex number. We define z^c as

$$z^c = \exp[c \log(z)]$$

Since $\log(z)$ is a multivalued function, so z^c is also a multivalued function.

Its principal value is defined as

$$\text{P.V. } z^c = \exp[c \log(z)]$$

Its principal value of z^c is a single valued function.

DERIVATIVE OF z^c

Since z^c is a multivalued function, so we consider the derivative of principal branch of z^c.

$$\frac{d}{dz}(z^c) = \frac{d}{dz}\left[\exp[c \log(z)]\right]$$

$$\frac{d}{dz}(z^c) = \frac{d}{dz}\left[\left(e^{c \log(z)}\right)\right]$$

$$= \left(e^{c \log(z)}\right) \frac{d}{dz}\left(c \log(z)\right)$$

$$= \left(e^{c \log(z)}\right) \frac{c}{z}$$

$$= \frac{c}{z} e^{c \log(z)}$$

$$= \frac{c}{z} z^c$$

$$= c\, z^{c-1}$$

$$\frac{d}{dz}(z^c) = c\ z^{c-1}$$

So far we have defined the exponential complex function e^z, which has base e.

Now we define the exponential complex function with base any complex number b.

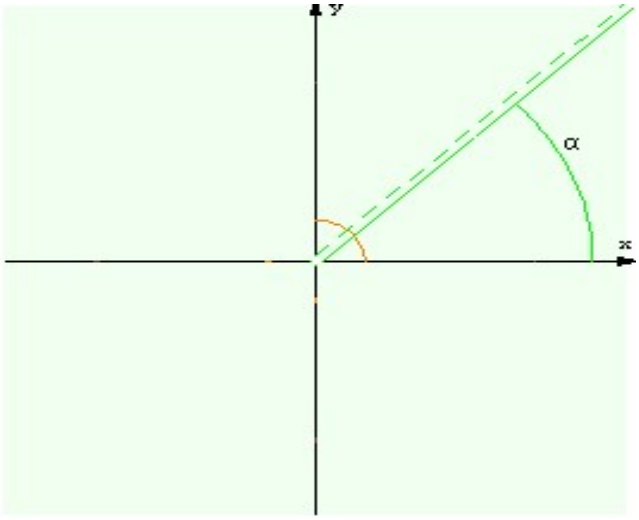
The exponential complex function with base $b \neq 0$ is defined as

$$b^z = \exp[z \log(b)]$$

Now we can explore the properties of $b^z = \exp[z \log(b)]$ by using the properties of exponential complex e^z.

DERIVATIVE OF b^z

The function b^z is multivalued, so we can calculate derivative of a branch. We choose a branch cut depending on α in such a way that b does not lie on the branch cut because on the branch cut there is discontinuity. So it will be a single valued function and we can take its derivative.



$$\frac{d}{dz}(b^z) = \frac{d}{dz}\left[\exp[z \log_\alpha(b)]\right]$$

$$\frac{d}{dz}(b^z) = \frac{d}{dz}\left[\left(e^{z \log_\alpha(b)}\right)\right]$$

$$= \mathrm{e}^{\mathrm{z}\log_{\alpha}(\mathrm{b})}\ \frac{d}{dz}\left(\mathrm{z}\log_{\alpha}(\mathrm{b})\right)$$

$$= \mathrm{e}^{\mathrm{z}\log_{\alpha}(\mathrm{b})}\left(\log_{\alpha}(\mathrm{b})\right)\frac{d}{dz}\ (\mathrm{z})$$

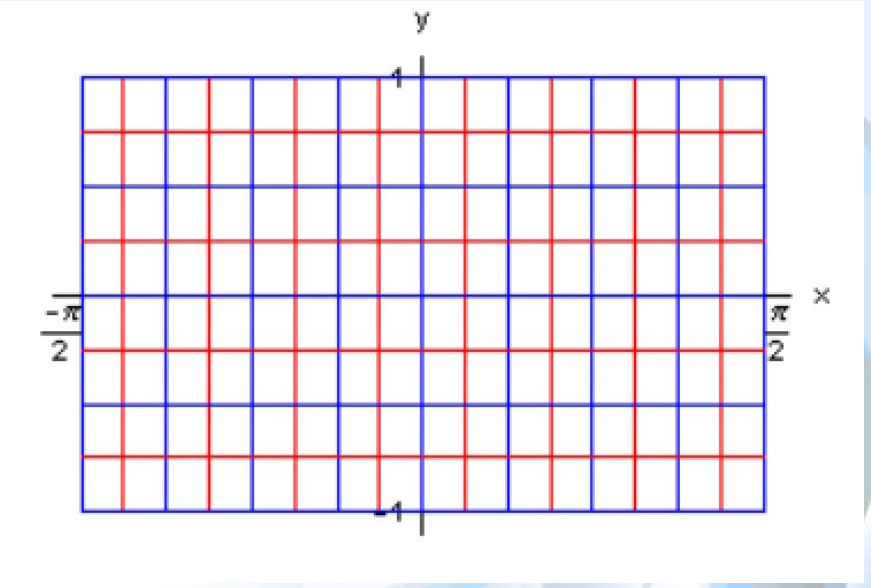
$$= \mathrm{e}^{\mathrm{z}\log_{\alpha}(\mathrm{b})}\left(\log_{\alpha}(\mathrm{b})\right) \times 1$$

$$= b^{z}\log_{\alpha}(\mathrm{b})$$

$$\frac{d}{dz}(b^{z}) = b^{z}\log_{\alpha}(\mathrm{b})$$

Trigonometric Functions

Trigonometric functions
Part I



Source: Complex Analysis by Mathews

Trigonometric Functions

We know from real analysis

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

Trigonometric Functions

We define

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Trigonometric Functions

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$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Trigonometric Functions

The other trigonometric functions are

$$\tan z = \frac{\sin z}{\cos z}, \quad \cot z = \frac{\cos z}{\sin z}$$

$$\sec z = \frac{1}{\cos z}, \quad \csc z = \frac{1}{\sin z}$$

Trigonometric Functions

What happens when z is restricted to real numbers ?

Trigonometric Functions

Properties of trigonometric functions

Trigonometric Functions

$$\sin(-z) = -\sin(z)$$

$$\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Trigonometric Functions

$$\cos(-z) = \cos(z)$$

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

Trigonometric Functions

$$\cos^2 z + \sin^2 z = 1$$

Trigonometric Functions

We can define trigonometric functions in another way using the following fact

$$e^{iz} = \cos z + i\sin z$$

Trigonometric Functions

Theorem: If z is a complex number, then

$$e^{iz} = \cos z + i\sin z$$

$e^{i\theta} = \cos\theta + i\sin\theta$

Trigonometric Functions

Theorem: If z is a complex number, then

$$e^{iz} = \cos z + i\sin z$$

Proof: We know that $e^{iz} = \sum_{n=0}^{\infty} \frac{1}{n!} (iz)^n$

pichy ham ny parha tha e^z= summition n=0 to infinity 1/n!(z)^n

Trigonometric Functions

Theorem: If z is a complex number, then

$$e^{iz} = \cos z + i \sin z$$

Proof: We know that $e^{iz} = \sum_{n=0}^{\infty} \frac{1}{n!} (iz)^n$

$$= \sum_{n=0}^{\infty} \left[\left[\frac{1}{(2n)!} (iz)^{2n} \right] + \left[\frac{1}{(2n+1)!} (iz)^{2n+1} \right] \right]$$

0, 2, 4, ...

1, 3, 5, ...

Trigonometric Functions

Theorem: If z is a complex number, then

$$e^{iz} = \cos z + i\sin z$$

Proof: We know that $e^{iz} = \sum_{n=0}^{\infty} \frac{1}{n!} (iz)^n$

$$= \sum_{n=0}^{\infty} \left[\left[\frac{1}{(2n)!} (iz)^{2n} \right] + \left[\frac{1}{(2n+1)!} (iz)^{2n+1} \right] \right]$$

$$= \sum_{n=0}^{\infty} \left[\left[\frac{(i)^{2n}}{(2n)!} (z)^{2n} \right] + \left[\frac{i(i^{2n})}{(2n+1)!} (z)^{2n+1} \right] \right]$$

Trigonometric Functions

Theorem: If z is a complex number, then

$$e^{iz} = \cos z + i \sin z$$

Proof: We know that $e^{iz} = \sum_{n=0}^{\infty} \frac{1}{n!} (iz)^n$

$$= \sum_{n=0}^{\infty} \left[\left[\frac{1}{(2n)!} (iz)^{2n} \right] + \left[\frac{1}{(2n+1)!} (iz)^{2n+1} \right] \right]$$

$$= \sum_{n=0}^{\infty} \left[\left[\frac{(i)^{2n}}{(2n)!} (z)^{2n} \right] + \left[\frac{i\,(i^{2n})}{(2n+1)!} (z)^{2n+1} \right] \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n)!} (z)^{2n} \right] + i \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n+1)!} (z)^{2n+1} \right]$$

$\cos z + i \sin z$

Trigonometric Functions

Theorem: If z is a complex number, then

$$e^{iz} = \cos z + i\sin z$$

Proof: We know that $e^{iz} = \sum_{n=0}^{\infty} \frac{1}{n!} (iz)^n$

$$= \sum_{n=0}^{\infty} \left[\left[\frac{1}{(2n)!} (iz)^{2n} \right] + \left[\frac{1}{(2n+1)!} (iz)^{2n+1} \right] \right]$$

$$= \sum_{n=0}^{\infty} \left[\left[\frac{(i)^{2n}}{(2n)!} (z)^{2n} \right] + \left[\frac{i\,(i^{2n})}{(2n+1)!} (z)^{2n+1} \right] \right]$$

$$= \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n)!} (z)^{2n} \right] + i \sum_{n=0}^{\infty} \left[\frac{(-1)^n}{(2n+1)!} (z)^{2n+1} \right]$$

$$= \cos z + i\sin z$$

Trigonometric Functions

$$e^{iz} = \cos z + i \sin z$$

Trigonometric Functions

$$e^{iz} = \cos z + i\sin z$$

and

$$e^{-iz} = \cos z - i\sin z$$

Trigonometric Functions

$$e^{iz} = \cos z + i\sin z$$

and

$$e^{-iz} = \cos z - i\sin z$$

implies

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

Trigonometric Functions

On the same lines

$$e^{iz} = \cos z + i\sin z$$

and

$$e^{-iz} = \cos z - i\sin z$$

implies

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

Trigonometric Functions

Similarly the following trigonometric functions can be defined

$\tan z$,$\cot z$,$\csc z$,$\sec z$

Trigonometric Functions

Consequences

Trigonometric Functions

Theorem: For $z = x + iy$, we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Trigonometric Functions

Theorem: For $z = x + iy$, we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Proof:

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

Trigonometric Functions

Theorem: For $z = x + iy$, we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Proof:

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

$$= \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right)$$

Trigonometric Functions

Theorem: For $z = x + iy$, we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Proof:

$$\begin{aligned}
\sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
&= \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right) \\
&= \frac{1}{2i}\left(e^{-y+ix} - e^{y-ix}\right)
\end{aligned}$$

Trigonometric Functions

Theorem: For $z = x + iy$, we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Proof:

$$\begin{aligned}
\sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
&= \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right) \\
&= \frac{1}{2i}\left(e^{-y+ix} - e^{y-ix}\right) \\
&= \frac{1}{2i}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right)
\end{aligned}$$

Trigonometric Functions

Theorem: For $z = x + iy$, we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Proof:

$$\begin{aligned}
\sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
&= \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right) \\
&= \frac{1}{2i}\left(e^{-y+ix} - e^{y-ix}\right) \\
&= \frac{1}{2i}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right) \\
&= \sin x\left(\frac{e^{y} + e^{-y}}{2}\right) + i\cos x\left(\frac{e^{y} - e^{-y}}{2}\right)
\end{aligned}$$

Trigonometric Functions

Theorem: For $z = x + iy$, we have

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Proof:

$$\begin{aligned}
\sin z &= \frac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
&= \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right) \\
&= \frac{1}{2i}\left(e^{-y+ix} - e^{y-ix}\right) \\
&= \frac{1}{2i}\left(e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)\right) \\
&= \sin x \ \frac{e^{y} + e^{-y}}{2} + i\cos x \frac{e^{y} - e^{-y}}{2} \\
&= \sin x \cosh y + i \cos x \sinh y
\end{aligned}$$

Trigonometric Functions

On the same lines we can show the following

Theorem: For $z = x + iy$, we have

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

Trigonometric Functions

Theorem: For all complex numbers $z = x + iy$, we have

$$\sin(z + 2\pi) = \sin z,$$
$$\cos(z + 2\pi) = \cos z,$$
$$\sin(z + \pi) = -\sin z,$$
$$\cos(z + \pi) = -\cos z,$$
$$\tan(z + \pi) = \tan z,$$
$$\cot(z + \pi) = \cot z.$$

Trigonometric Functions

Proof $(\sin(z + 2\pi) = \sin z)$

Trigonometric Functions

Proof $(\sin(z + 2\pi) = \sin z)$ $z = x + iy$

$$\sin(z + 2\pi)$$

$$= \sin(x + iy + 2\pi)$$

Trigonometric Functions

Proof $(\sin(z + 2\pi) = \sin z)$

$$\sin(z + 2\pi)$$
$$= \sin(x + iy + 2\pi)$$
$$= \sin((x + 2\pi) + iy)$$
$$= \sin(x+2\pi)\cosh y + i\cos(x+2\pi)\sinh y$$

Recall : $\sin z = \sin x \cosh y + i \cos x \sinh y$

Trigonometric Functions

Proof $(\sin(z+2\pi)=\sin z)$

$$\sin(z+2\pi)$$
$$=\sin(x+iy+2\pi)$$
$$=\sin((x+2\pi)+iy)$$
$$=\sin(x+2\pi)\cosh y+i\cos(x+2\pi)\sinh y$$

Recall : $\sin z=\sin x\cosh y+i\cos x\sinh y$

Trigonometric Functions

Proof $(\sin(z + 2\pi) = \sin z)$

$$\sin(z + 2\pi)$$
$$= \sin(x + iy + 2\pi)$$
$$= \sin((x + 2\pi) + iy)$$
$$= \sin(x + 2\pi)\cosh y + i\cos(x + 2\pi)\sinh y$$
$$= \sin x \cosh y + i\cos x \sinh y$$

$$= \sin(x + iy) = \sin z$$

$$\Rightarrow \quad \sin(z + 2\pi) = \sin(z)$$

Trigonometric Functions

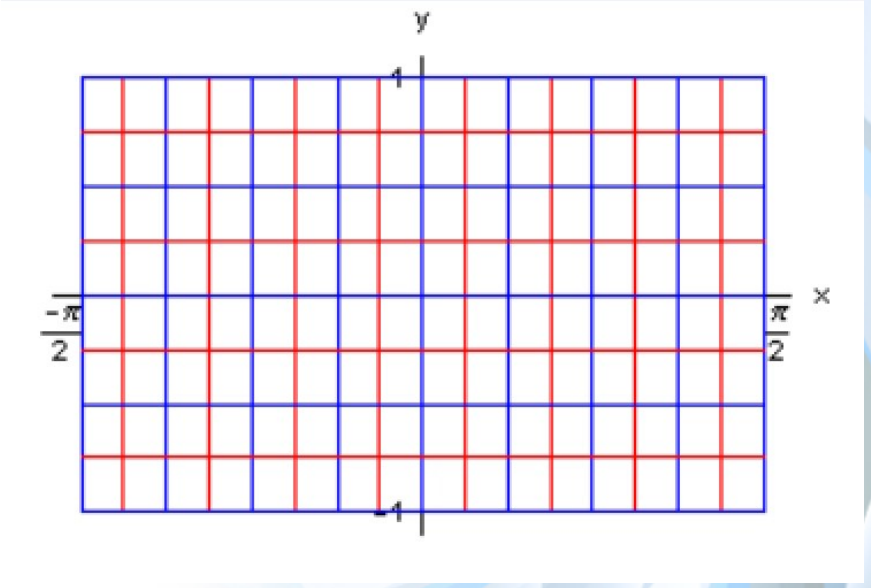
Theorem: For all complex numbers $z = x + iy$, we have

$$\sin(z + 2\pi) = \sin z,$$
$$\cos(z + 2\pi) = \cos z,$$
$$\sin(z + \pi) = -\sin z,$$
$$\cos(z + \pi) = -\cos z,$$
$$\tan(z + \pi) = \tan z,$$
$$\cot(z + \pi) = \cot z.$$

Proof: Similar to first part.

Trigonometric Functions

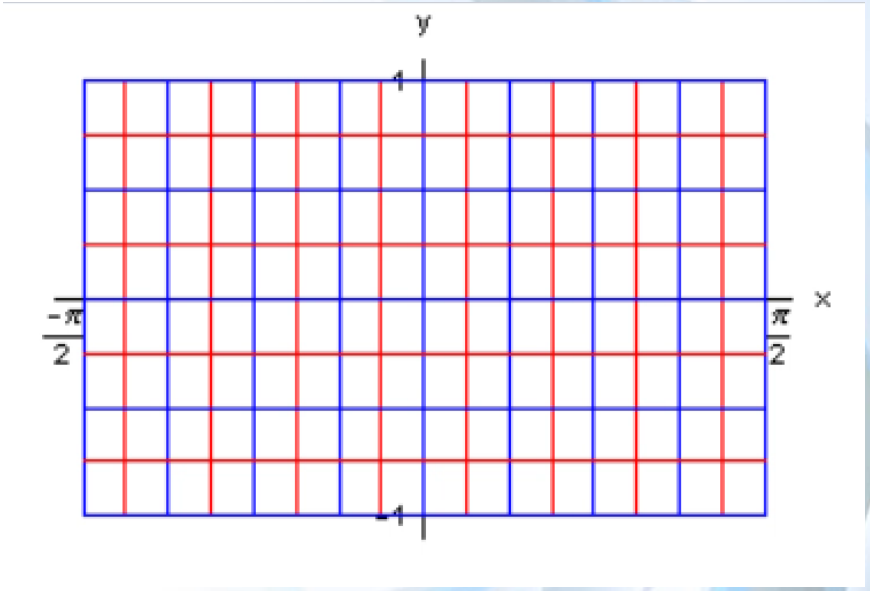
Trigonometric functions
Part I



Source: Complex Analysis by Mathews

Trigonometric Functions

Trigonometric functions
Part II



Source: Complex Analysis by Mathews

Trigonometric Functions

Further Properties of trigonometric functions

Trigonometric Functions

If z_1 and z_2 are two complex numbers, then

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos z_1 \sin z_2,$$

Trigonometric Functions

Theorem: If z_1 and z_2 are two complex numbers, then

$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin z_1 \sin z_2$$

and

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos z_1 \sin z_2,$$

Trigonometric Functions

Theorem: If z_1 and z_2 are two complex numbers, then

$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin z_1 \sin z_2$$

and

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos z_1 \sin z_2,$$

Proof: Observe that

$$\cos z_1 \cos z_2 = \frac{1}{4}\left(e^{i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{i(z_2-z_1)} + e^{-i(z_1+z_2)}\right)$$

$\cos z_1 = \left(\frac{e^{iz_1} + e^{-iz_1}}{2}\right)$, $\cos z_2 = \frac{e^{iz_2} + e^{-iz_2}}{2}$

Trigonometric Functions

Theorem: If z_1 and z_2 are two complex numbers, then

$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin z_1 \sin z_2$$

and

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos z_1 \sin z_2,$$

Proof: Observe that

$$\cos z_1 \cos z_2 = \frac{1}{4}\left(e^{i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{i(z_2-z_1)} + e^{-i(z_1+z_2)}\right)$$

and

$$-\sin z_1 \sin z_2 = \frac{1}{4}\left(e^{i(z_1+z_2)} - e^{i(z_1-z_2)} - e^{i(z_2-z_1)} + e^{-i(z_1+z_2)}\right)$$

$\sin(z_1) = \frac{1}{2}\left(e^{iz_1} - e^{-iz_1}\right)$, $\sin(z_2) = \frac{1}{2}\left(e^{iz_2} - e^{-iz_2}\right)$

Trigonometric Functions

Theorem: If z_1 and z_2 are two complex numbers, then

$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin z_1 \sin z_2$$

and

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos z_1 \sin z_2,$$

Proof: Observe that

$$\cos z_1 \cos z_2 = \frac{1}{4}\left(e^{i(z_1+z_2)} + e^{i(z_1-z_2)} + e^{i(z_2-z_1)} + e^{-i(z_1+z_2)}\right)$$

and

$$-\sin z_1 \sin z_2 = \frac{1}{4}\left(e^{i(z_1+z_2)} - e^{i(z_1-z_2)} - e^{i(z_2-z_1)} + e^{-i(z_1+z_2)}\right)$$

Hence, we get

$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin z_1 \sin z_2$$

Trigonometric Functions

Theorem: If z_1 and z_2 are two complex numbers, then

$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin z_1 \sin z_2$$

and

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos z_1 \sin z_2,$$

Proof: Similarly we can prove the following

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos z_1 \sin z_2$$

I II

Trigonometric Functions

If z_1 and z_2 are two complex numbers, then

$$\sin(z_1 + z_2) = \sin(z_1)\cos(z_2) + \cos z_1 \sin z_2,$$

and

$$\cos(z_1 + z_2) = \cos(z_1)\cos(z_2) - \sin z_1 \sin z_2$$

Trigonometric Functions

As a consequence we get

$$\sin(2z) = 2\sin z \cos z,$$

Trigonometric Functions

As a consequence we get

$$\sin(2z) = 2\sin z \cos z,$$
$$\cos(2z) = (\cos z)^2 - (\sin z)^2,$$

Trigonometric Functions

$$\sin\left(\frac{\pi}{2}+z\right)=\sin\left(\frac{\pi}{2}-z\right)=\cos z.$$

$\sin(z_1+z_2)=\sin z_1\cos z_2+\cos z_1\sin z_2$

$z_1=\pi/2,\ z_2=z$

$\sin\left(\frac{\pi}{2}+z\right)=\sin\left(\pi/2\right)\cos z+\cos\frac{\pi}{2}\sin z$

1 0

$\sin\left(\frac{\pi}{2}+z\right)=\cos z$

Trigonometric Functions

Zeros of
$\sin z$ and $\cos z$

Trigonometric Functions

$$\sin z = 0$$

if and only if

$$z = n\pi,\ n \in \mathbb{Z}$$

Trigonometric Functions

Fact: $\sin z = 0$ if and only if $z = n\pi, n \in \mathbb{Z}$.

$z = x+iy \Rightarrow \sin(x+iy) = \sin x \cosh y + i \cos x \sinh y = 0$

$\Leftrightarrow \sin x \cosh y = 0$ & $\cos x \sinh y = 0$

$\Leftrightarrow \cosh y \neq 0$ $\Leftrightarrow \cos(n\pi) \sinh y = 0$

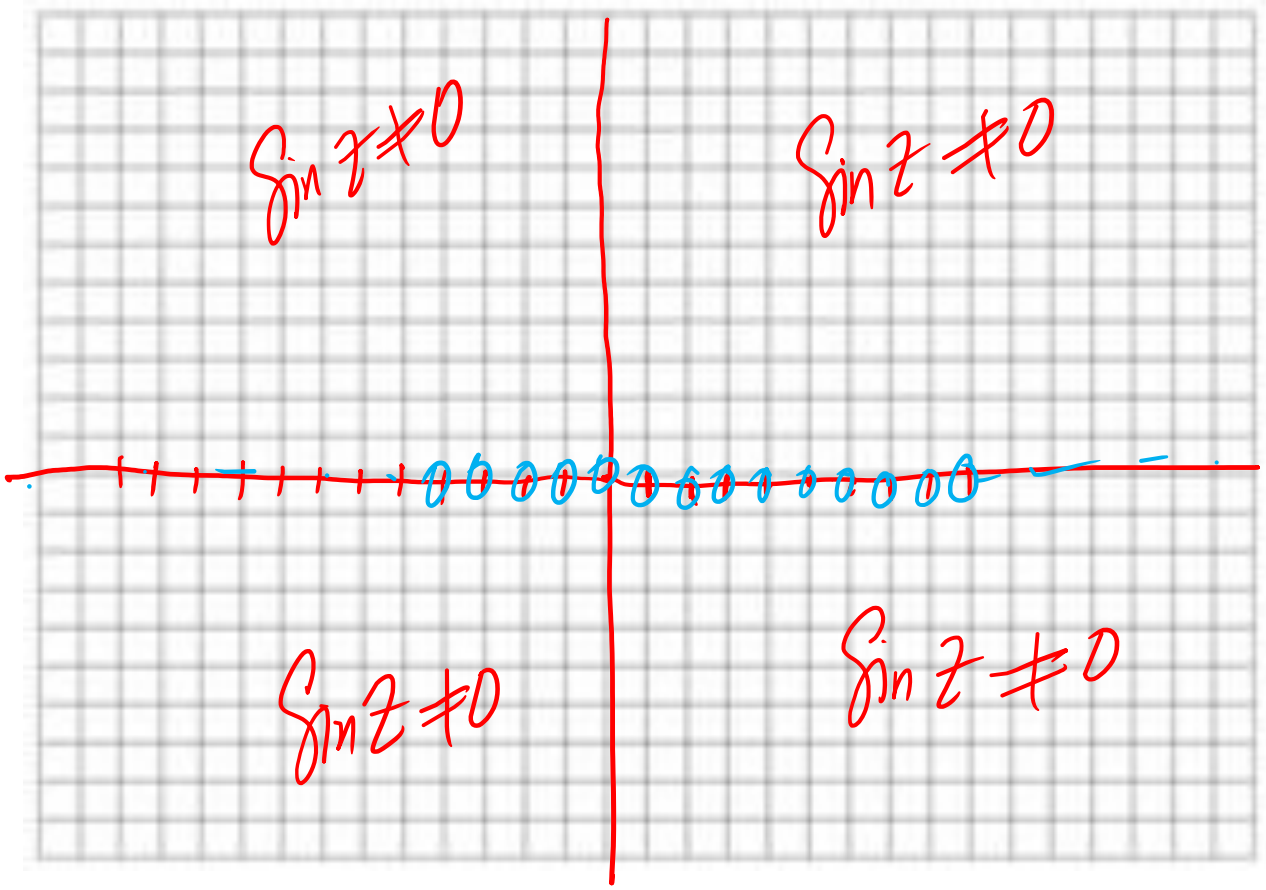
$\Leftrightarrow \sin x = 0 \Leftrightarrow x = n\pi$ $\Leftrightarrow (-1)^n \sinh y = 0$

$\Leftrightarrow \sinh y = 0 \Rightarrow y = 0$

$\Rightarrow x+iy = n\pi + i(0) \Leftrightarrow z = n\pi$

Trigonometric Functions

Fact: $\sin z = 0$ if and only if $z = n\pi$



Trigonometric Functions

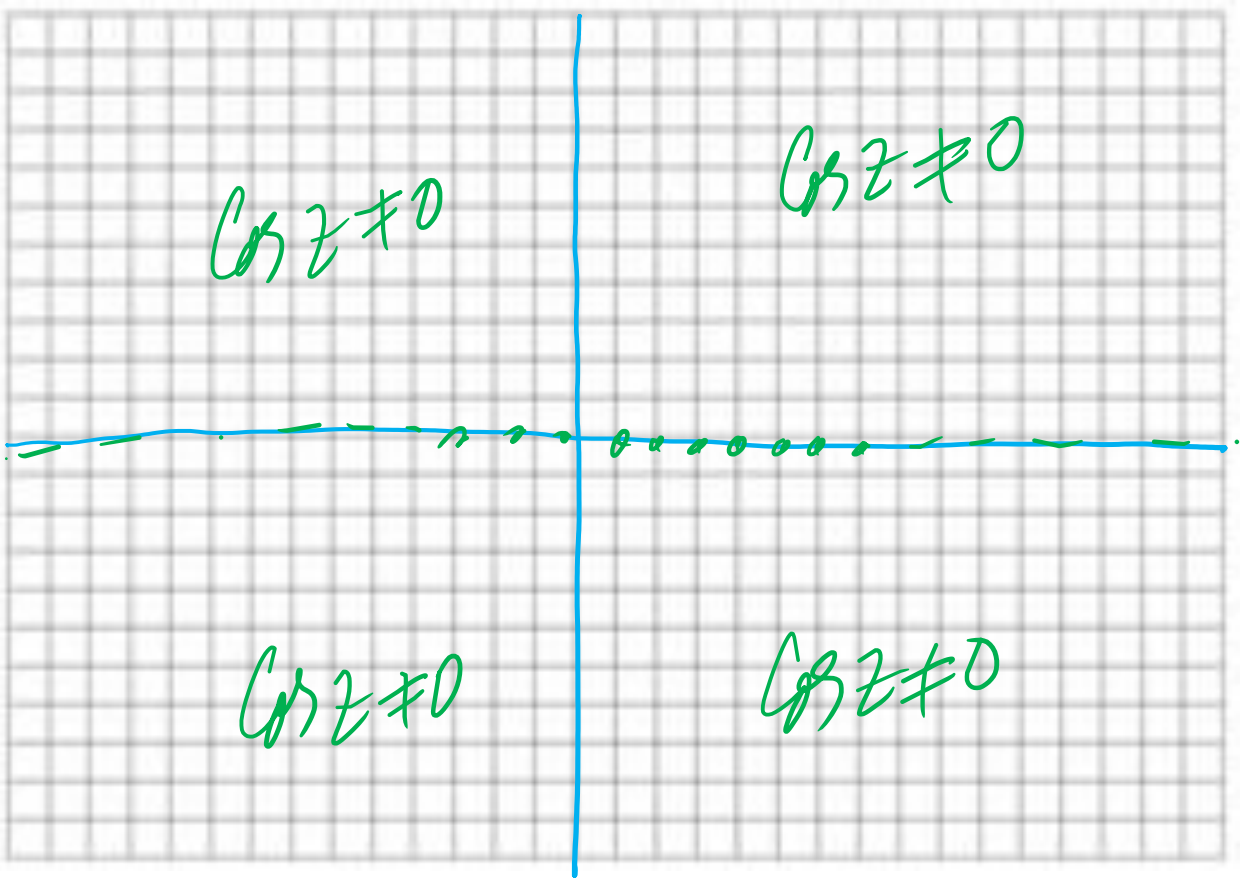
$$\cos z = 0$$

if and only if

$$z = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$$

Trigonometric Functions

Fact: $\cos z = 0$ if and only if $z = \left(n + \frac{1}{2}\right)\pi, n \in \mathbb{Z}$



Trigonometric Functions

There is one big difference between real and complex trigonometric functions

Trigonometric Functions

For real variable x

$|\sin x| \leq 1$

But for complex variable z

$\sin z$ is unbounded

Trigonometric Functions

Similarly for cosine function

For real variable x

$|\cos x| \leq 1$

But for complex variable z

$\cos z$ is unbounded

Trigonometric Functions

Fact: For complex variable z, $\sin z$ is unbounded.

$z = x + iy$

$$|\sin z|^2 = |\sin(x + iy)|^2$$

Trigonometric Functions

Fact: For complex variable z, $\sin z$ is unbounded.

$$
\begin{aligned}
|\sin z|^2 &= |\sin(x+iy)|^2 \\
&= |\sin x \cosh y + i \cos x \sinh y|^2
\end{aligned}
$$

Trigonometric Functions

Fact: For complex variable z, $\sin z$ is unbounded.

$$
\begin{aligned}
|\sin z|^2 &= |\sin(x+iy)|^2 \\
&= |\sin x \cosh y + i \cos x \sinh y|^2 \\
&= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y
\end{aligned}
$$

Trigonometric Functions

Fact: For complex variable z, $\sin z$ is unbounded.

$$
\begin{aligned}
|\sin z|^2 &= |\sin(x+iy)|^2 \\
&= |\sin x \cosh y + i \cos x \sinh y|^2 \\
&= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y(\cos^2 x + \sin^2 x)
\end{aligned}
$$

Trigonometric Functions

Fact: For complex variable z, $\sin z$ is unbounded.

$$
\begin{aligned}
|\sin z|^2 &= |\sin(x+iy)|^2 \\
&= |\sin x \cosh y + i \cos x \sinh y|^2 \\
&= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x(\cosh^2 y - \sinh^2 y) + \sinh^2 y(\cos^2 x + \sin^2 x)
\end{aligned}
$$

Using $\cosh^2 y - \sinh^2 y = 1$ and $\cos^2 x + \sin^2 x = 1$, we get

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

Trigonometric Functions

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

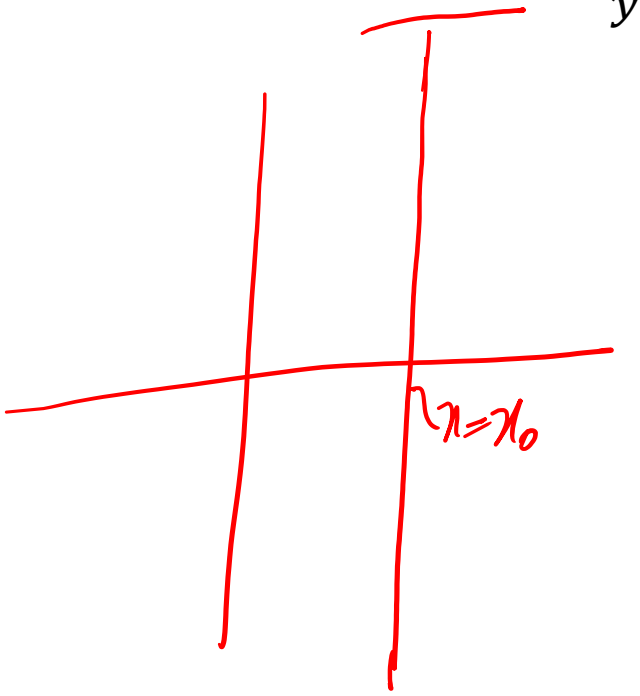
$$\lim_{y \to \infty} |\sin(x_0 + iy)|^2$$

Trigonometric Functions

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

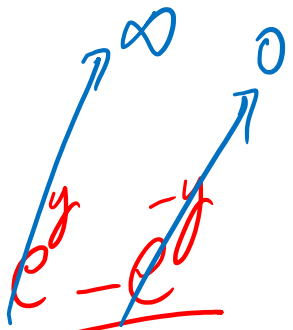
$$\lim_{y\to\infty} |\sin(x_0 + iy)|^2$$

$$= \sin^2 x_0 + \lim_{y\to\infty} \sinh^2 y = \boxed{\infty}$$



$$\sinh y = \frac{e^y - e^{-y}}{2}$$

$$\lim_{y\to\infty} \sinh y = \lim_{y\to\infty} \frac{e^y - e^{-y}}{2}$$



Trigonometric Functions

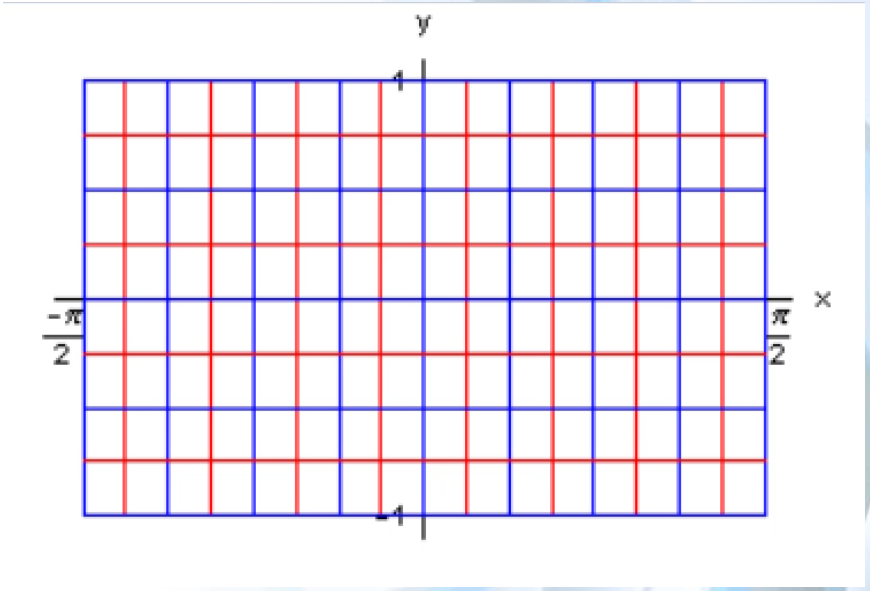
Fact: For complex variable z, $\cos z$ is unbounded.

Similarly we can prove the following:

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

Trigonometric Functions

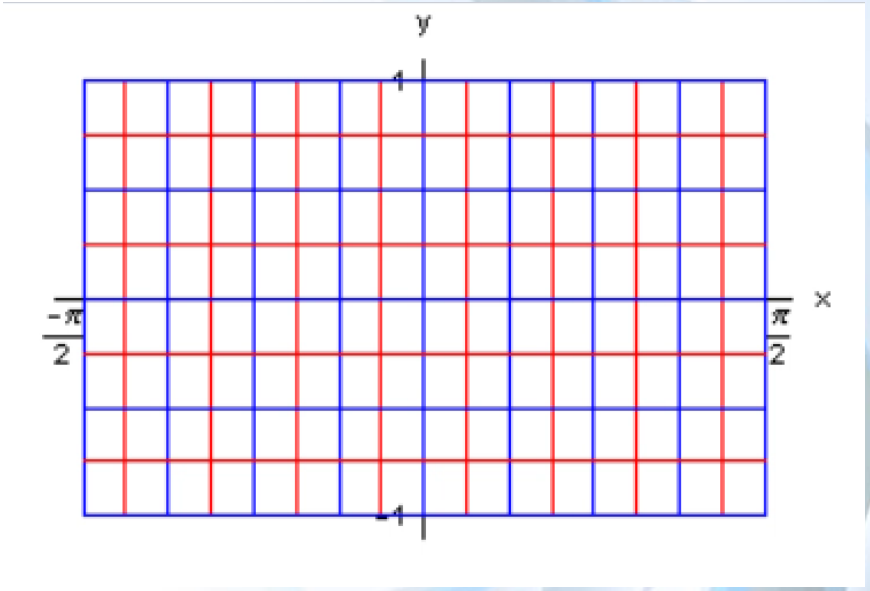
Trigonometric functions
Part II



Source: Complex Analysis by Mathews

Trigonometric Functions

Trigonometric functions
Part III



Source: Complex Analysis by Mathews

The mapping $w = sin\, z$

Mapping

$w = \sin z$

The mapping $w = \sin z$

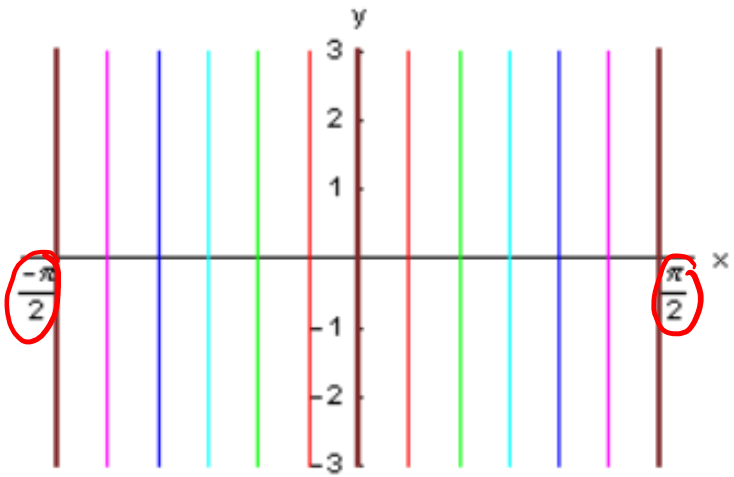
To understand the mapping $w = \sin z$, we study images of line segments from z-plane into w-plane

The mapping $w = \sin z$

To understand the mapping $w = \sin z$, we study images of line segments from z-plane into w-plane. We consider line segments parallel to y-axis with x-coordinates given by

$$x = -\frac{\pi}{2} + \frac{k\pi}{12}, k = 0,1, \dots, 12$$

and y vary continuously in the interval $-3 \leq y \leq 3$.



Z-plane

?

W-plane

Source: Complex Analysis by mathews

The mapping $w = \sin z$

The line segments are parallel to y-axis so the x-coordinate is constant on each line segment. So line segments are given by the equation $x = a$, where

$$-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$$

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Source: Complex Analysis by mathews

The mapping $w = \sin z$

Let $z = x + iy$ be coordinates in the z-plane; let

$$w = \sin z = u + iv$$

be the coordinates in the w-plane.

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We know that

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We first consider the line segments $x = a$, where

$$-\frac{\pi}{2} < a < \frac{\pi}{2}, a \neq 0$$

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The mapping $w = \sin z$

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$$u = \sin a \cosh y, \qquad v = \cos a \sinh y$$

Since $a \neq 0, \frac{\pi}{2}, -\frac{\pi}{2}$ so $\sin a \neq 0$ and $\cos a \neq 0$.

The mapping $w = \sin z$

So

$$u = \sin a \cosh y, \qquad v = \cos a \sinh y$$

Since $a \neq 0, \frac{\pi}{2}, -\frac{\pi}{2}$ so $\sin a \neq 0$ and $\cos a \neq 0$. Hence we get

$$\left(\frac{u}{\sin a}\right)^2 - \left(\frac{v}{\cos a}\right)^2 = 1,$$

which is equation of hyperbola with vertices $(\pm \sin a, 0)$.

$\frac{u}{\sin a} = \cosh y, \quad \frac{v}{\cos a} = \sinh y$

$\cosh^2 y - \sinh^2 y = 1$

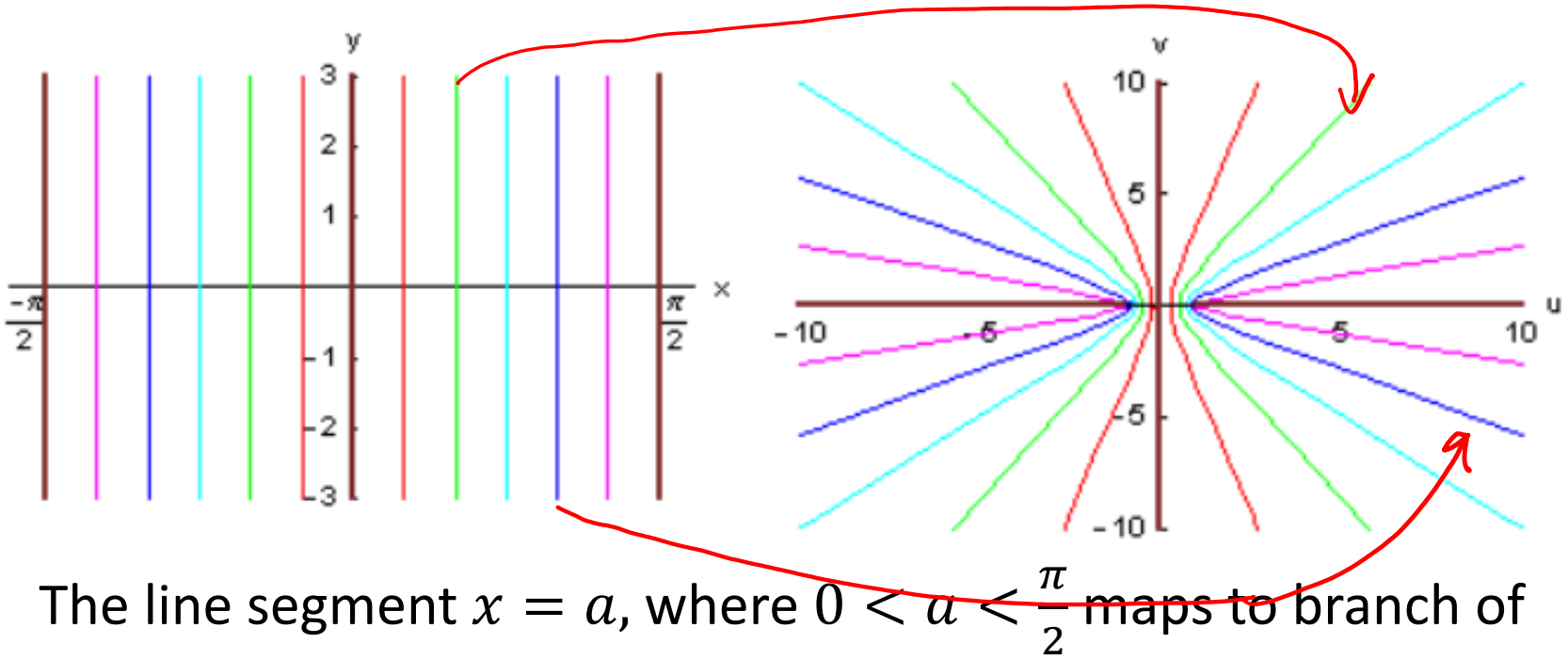
Vertices.

$v = 0$

$\frac{u^2}{\sin^2 a} = 1$

$\Rightarrow u = \pm \sin a.$

The mapping $w = \sin z$

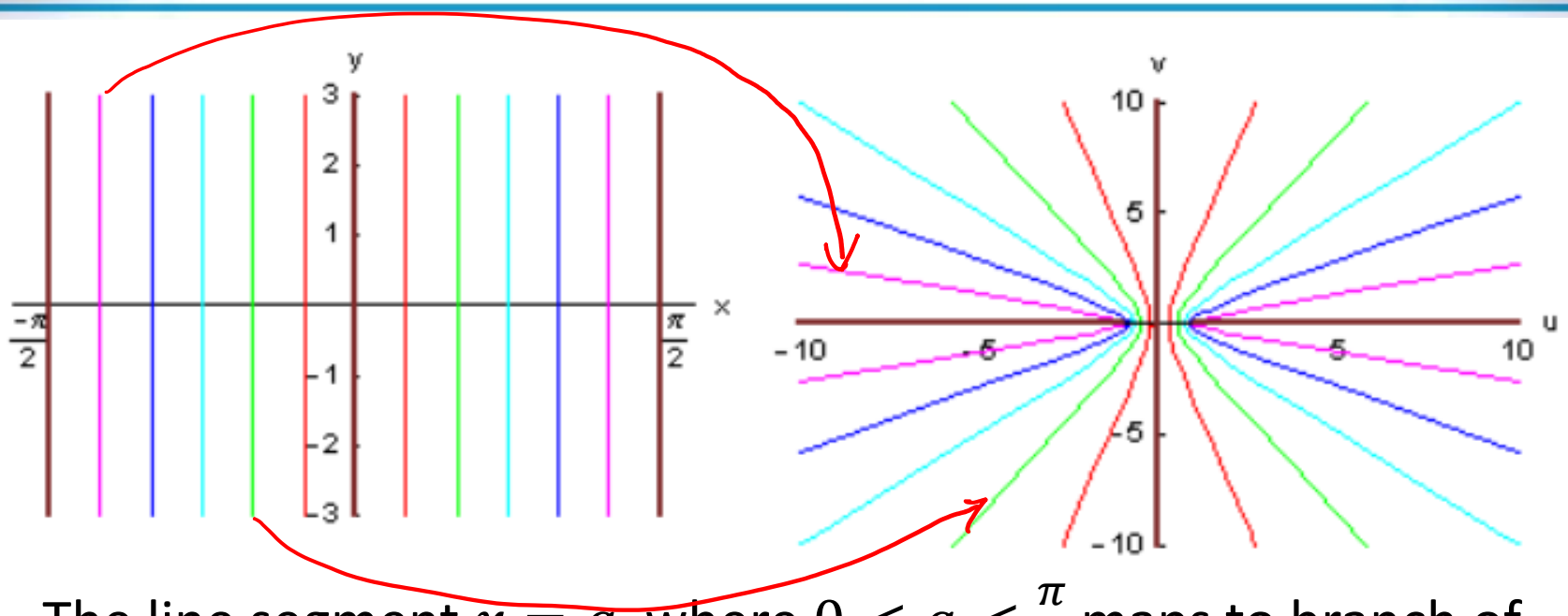


The line segment $x = a$, where $0 < a < \frac{\pi}{2}$ maps to branch of hyperbola on the right of v-axis.

$u = \sin a \cosh y$ ✓

Source: Complex Analysis by mathews

The mapping $w = \sin z$

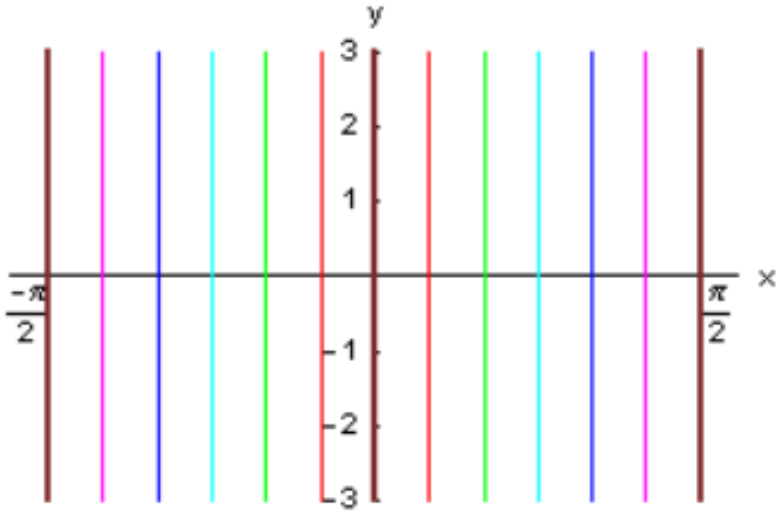


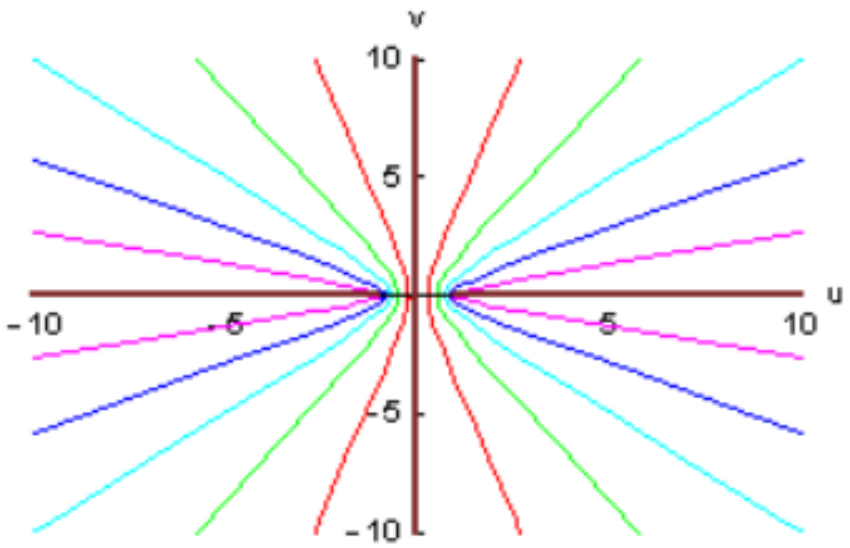
The line segment $x = a$, where $0 < a < \frac{\pi}{2}$ maps to branch of hyperbola on the right of v-axis. Similarly $x = -a$, where $0 < a < \frac{\pi}{2}$ maps to branch of hyperbola on the left of v-axis.

$u = \sin a \cosh y$

Source: Complex Analysis by mathews

The mapping $w = \sin z$



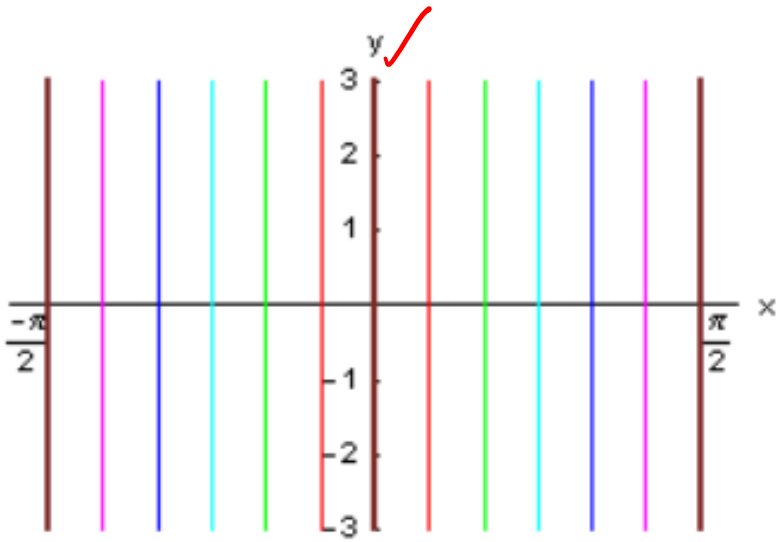


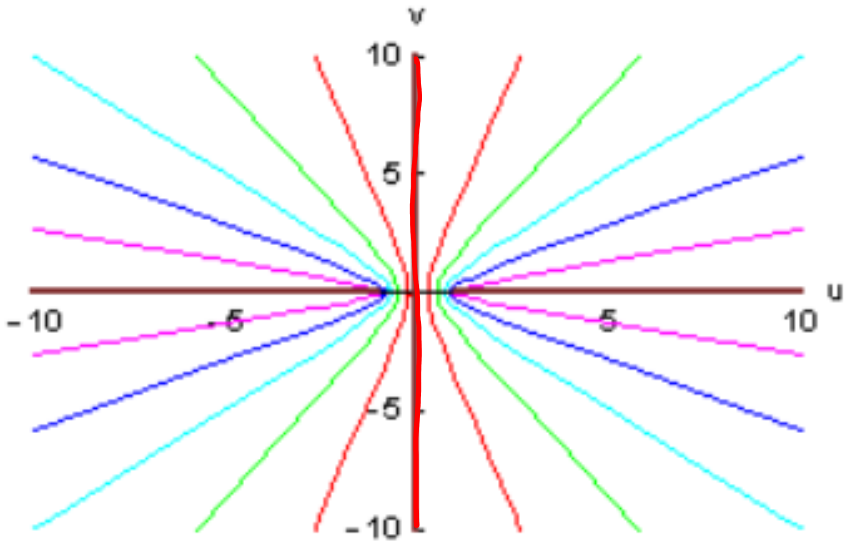
The line segment $x = a$, where $0 < a < \frac{\pi}{2}$ maps to branch of hyperbola on the right of v-axis. Similarly $x = -a$, where $0 < a < \frac{\pi}{2}$ maps to branch of hyperbola on the left of v-axis.

So pair of lines $x = \pm a$, where $0 < a < \frac{\pi}{2}$, mapped to one hyperbola given by $\left(\frac{u}{\sin a}\right)^2 - \left(\frac{v}{\cos a}\right)^2 = 1$.

Source: Complex Analysis by mathews

The mapping $w = \sin z$



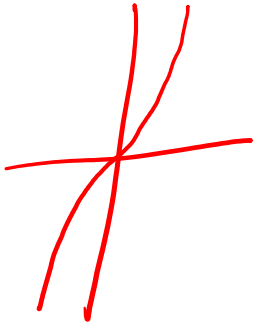


Now consider $a = 0$, that is, put $x = 0$ in the following

$$u = \sin x \cosh y\,, \qquad v = \cos x \sinh y$$

We get

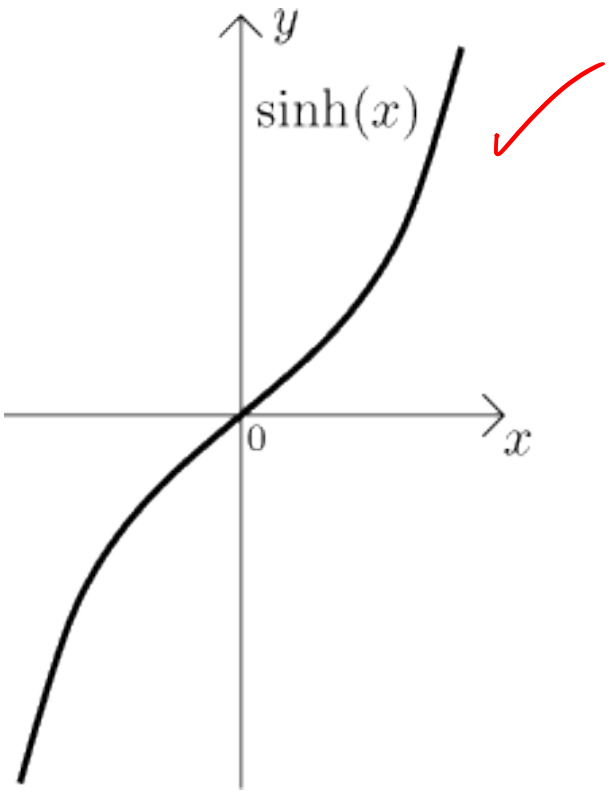
$$u = 0, v = \sinh y\,, \qquad -3 \le y \le 3$$



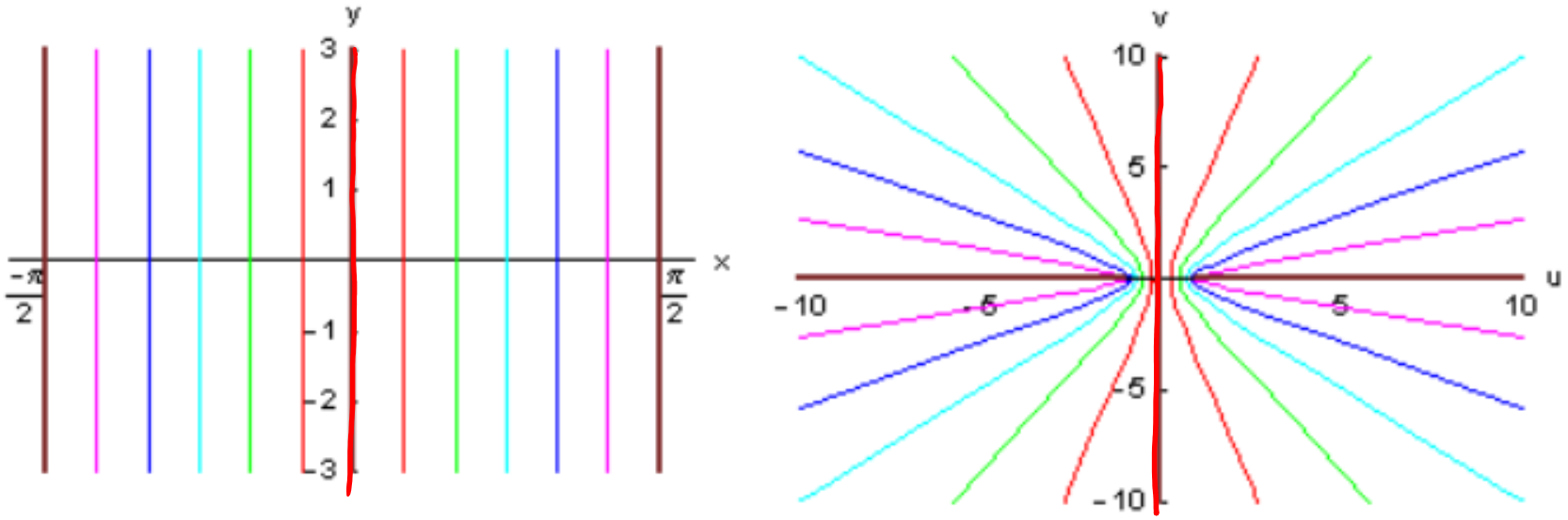
Source: Complex Analysis by mathews

The mapping $w = sin\ z$

Graph of $\sinh x$:



The mapping $w = \sin z$



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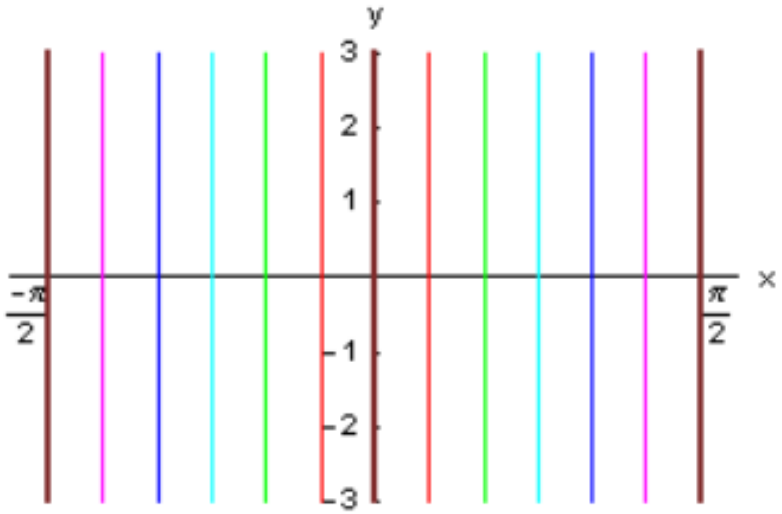
$$u = \sin x \cosh y\,, \qquad v = \cos x \sinh y$$

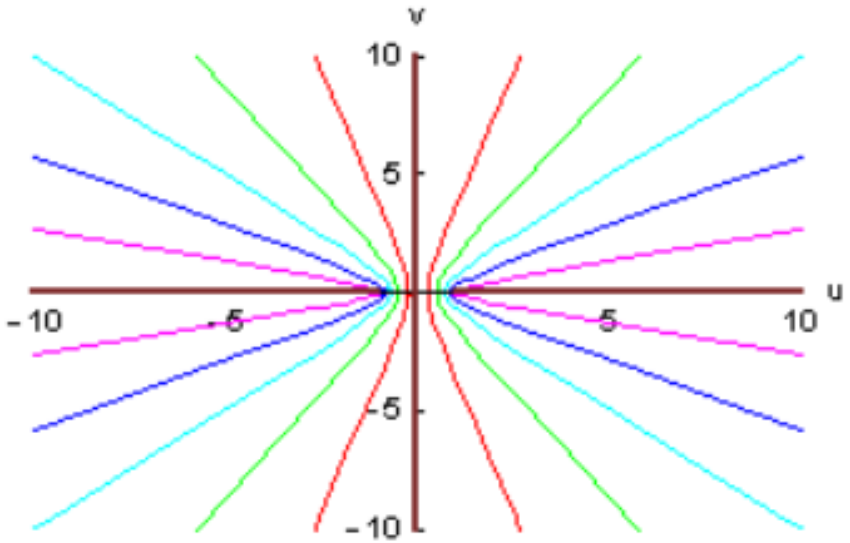
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$$u = 0, v = \sinh y\,, \qquad -3 \le y \le 3$$

Source: Complex Analysis by mathews

The mapping $w = sin\ z$





Now consider $a = \pm\frac{\pi}{2}$, that is, put $x = \pm\frac{\pi}{2}$ in the following

$$u = \sin x \cosh y\,, \qquad v = \cos x \sinh y$$

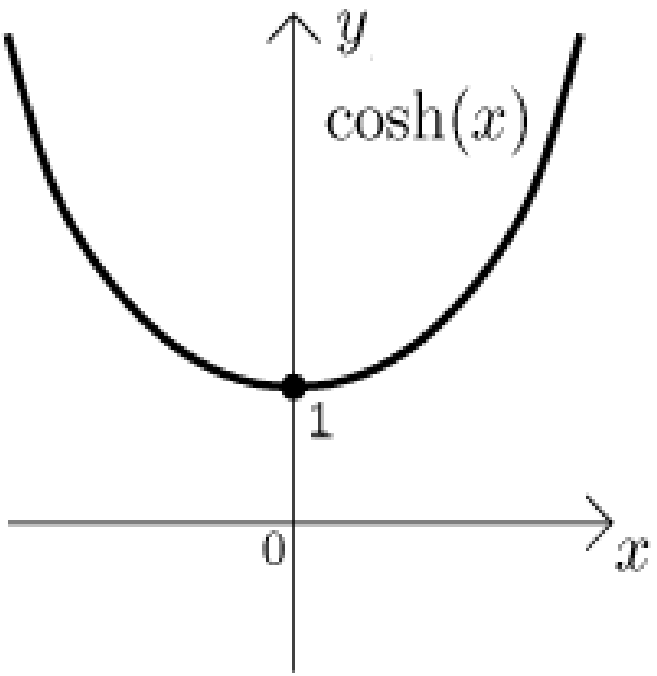
We get

$$u = \pm\cosh y\,, v = 0, \qquad -3 \le y \le 3$$

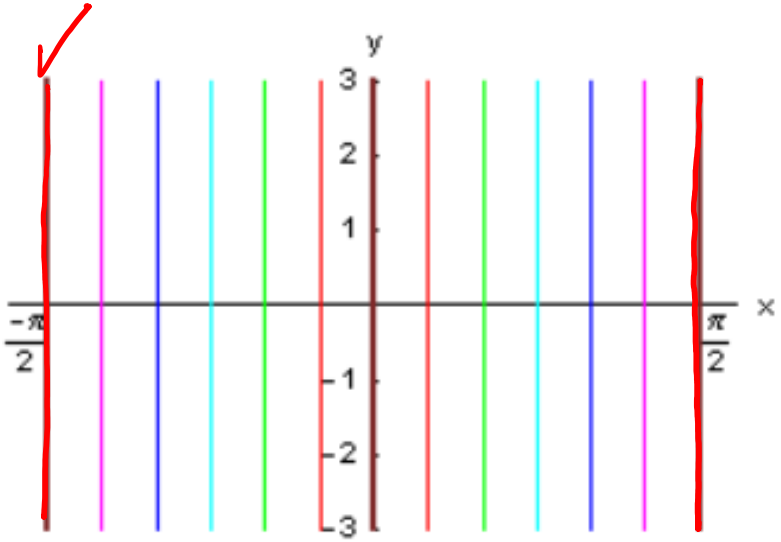
Source: Complex Analysis by mathews

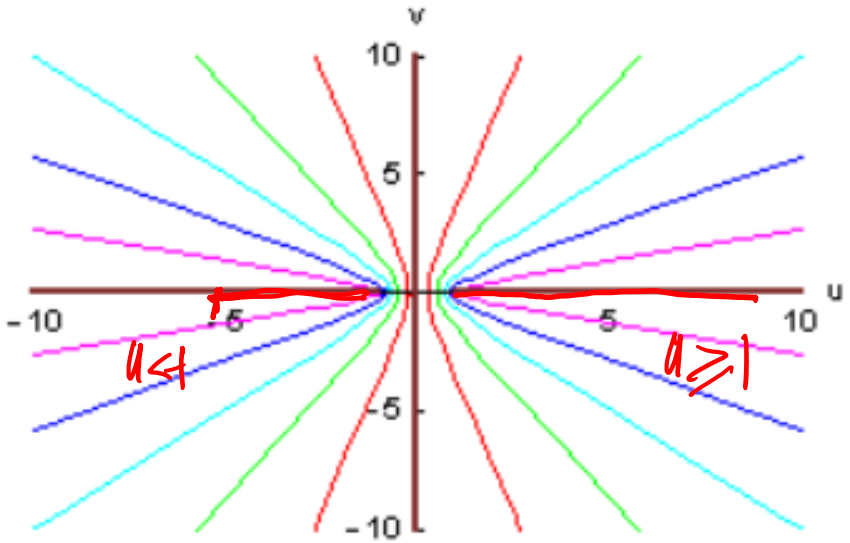
The mapping $w = sin\ z$

Graph of $\cosh x$:



The mapping $w = \sin z$





Now consider $a = \pm\frac{\pi}{2}$, that is, put $x = \pm\frac{\pi}{2}$ in the following

$$u = \sin x \cosh y\,, \qquad v = \cos x \sinh y$$

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$$u = \pm\cosh y\,, v = 0, \qquad -3 \le y \le 3$$

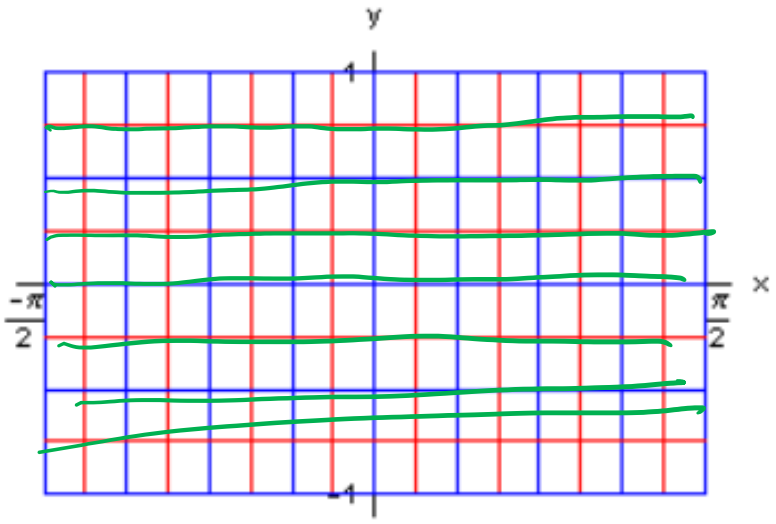
Source: Complex Analysis by mathews

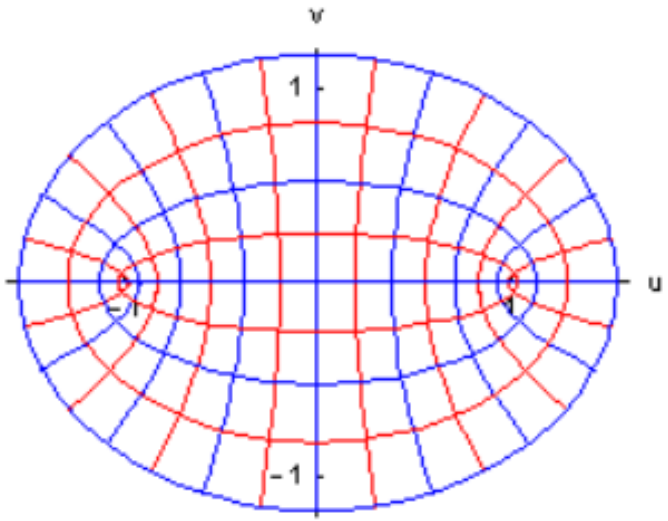
The mapping $w = \sin z$

On the same lines we can show that image of horizontal lines

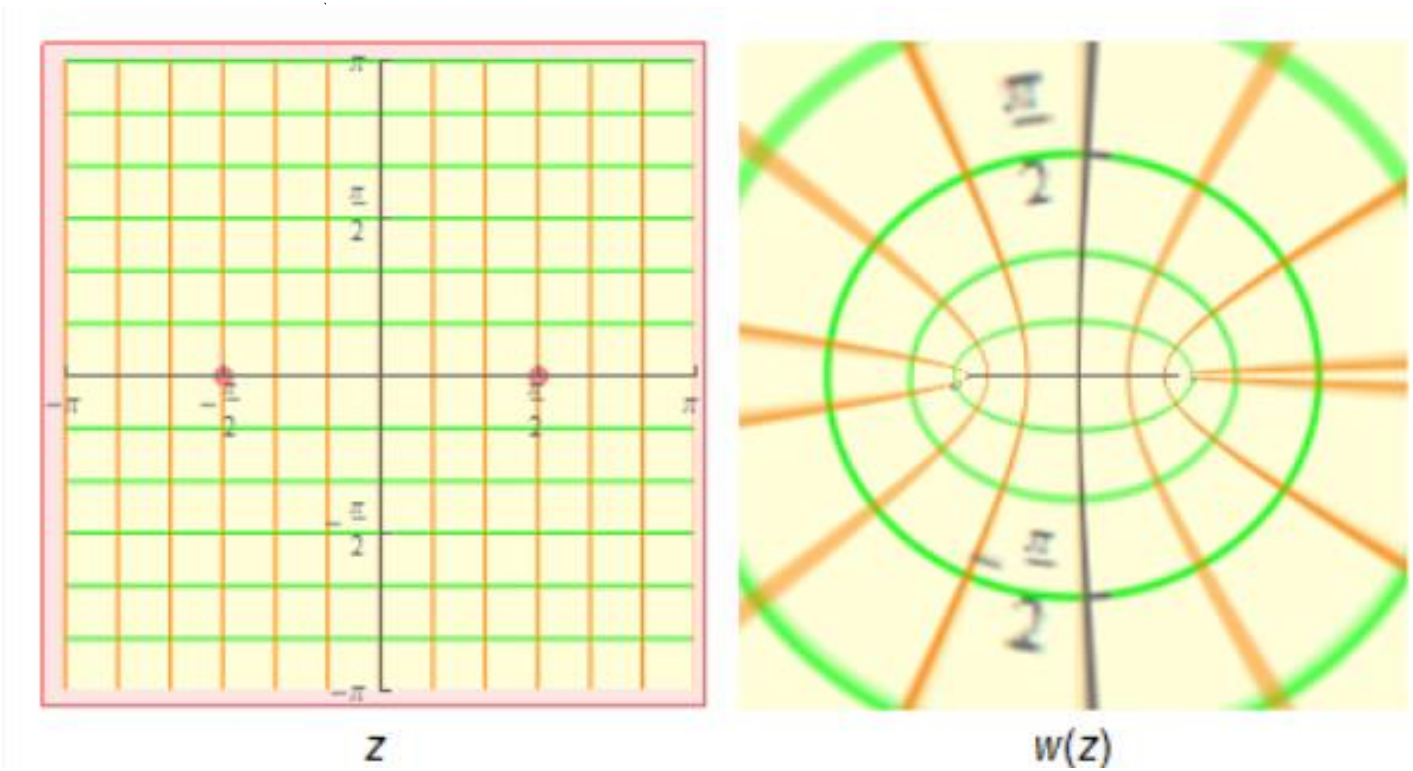
$$y = b, \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}.$$

are ellipses.



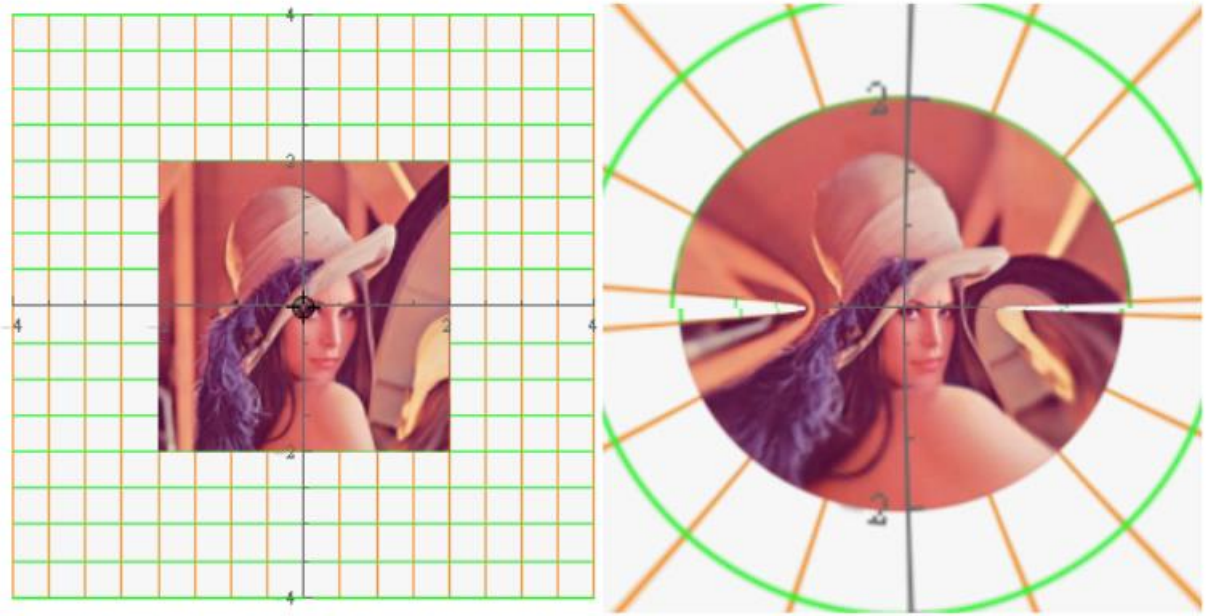


The mapping $w = \sin z$



Source: Wolfram demonstration

The mapping $w = \sin z$



Source: Wolfram demonstration

The mapping $w = cos\ z$

The mapping

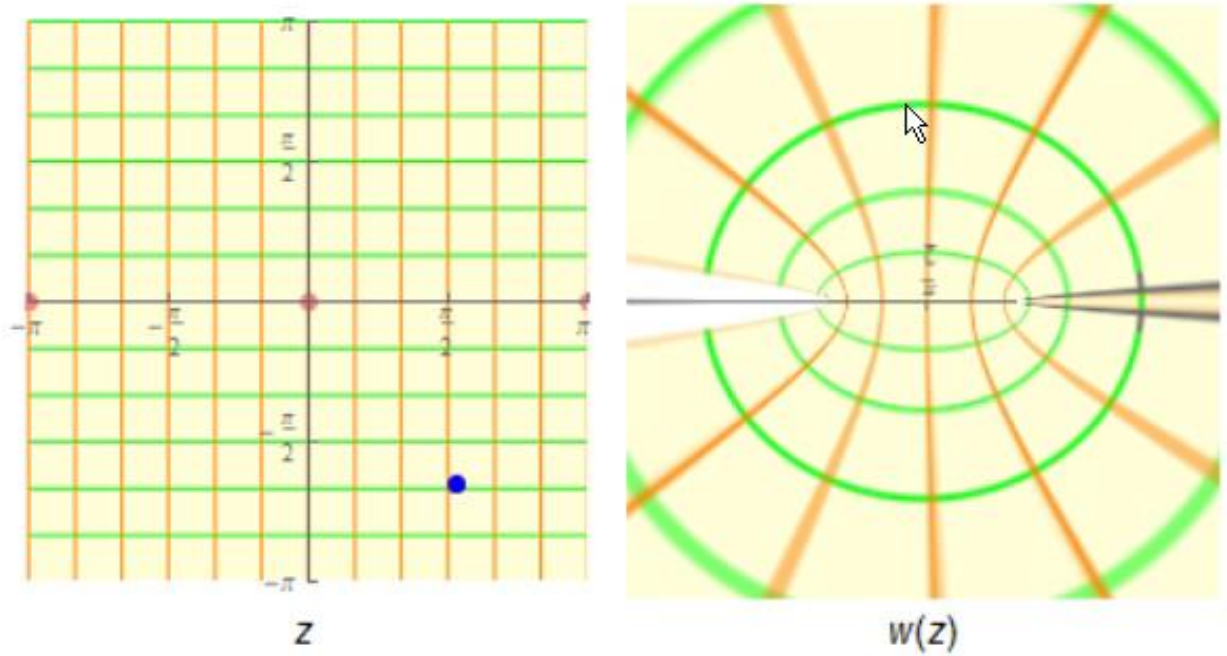
$w = \cos z$

Can be dealt in a similar way.

The mapping $w = \cos z$

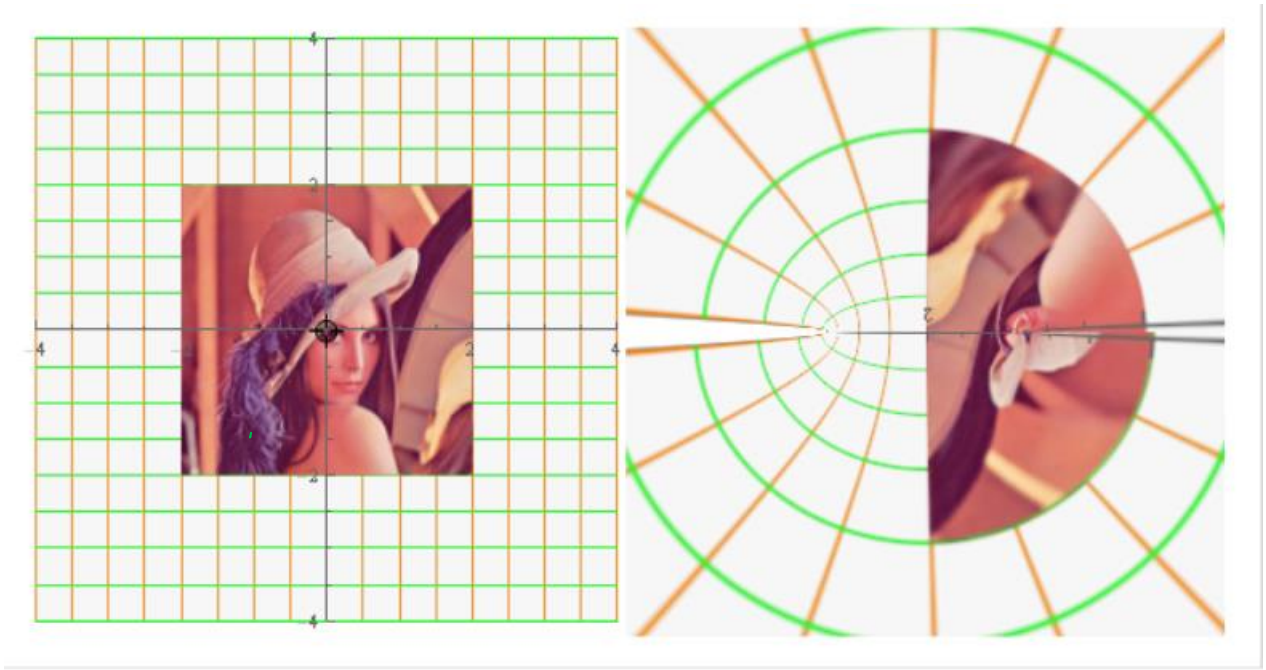
z-plane

w-plane



Source: Wolfram demonstration

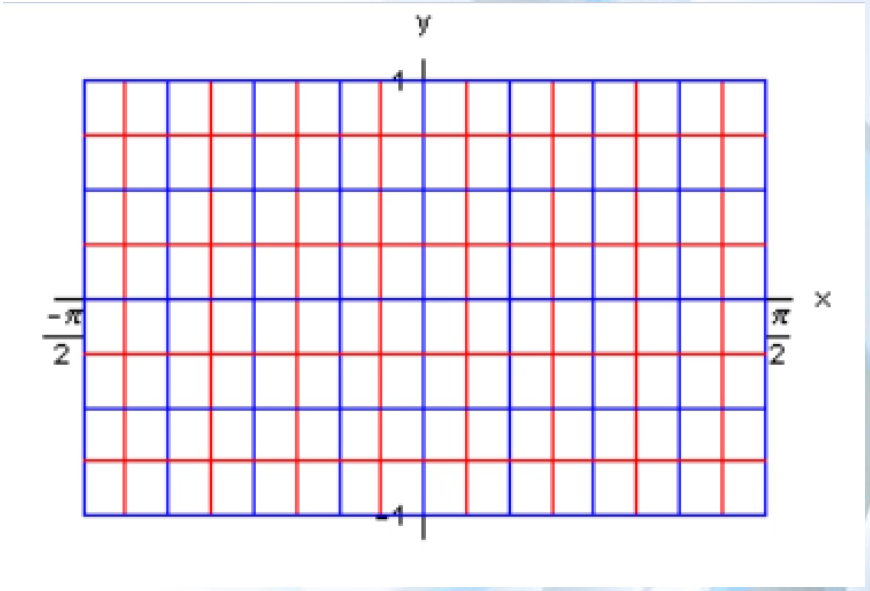
The mapping $w = cos\ z$



Source: Wolfram demonstration

Trigonometric Functions

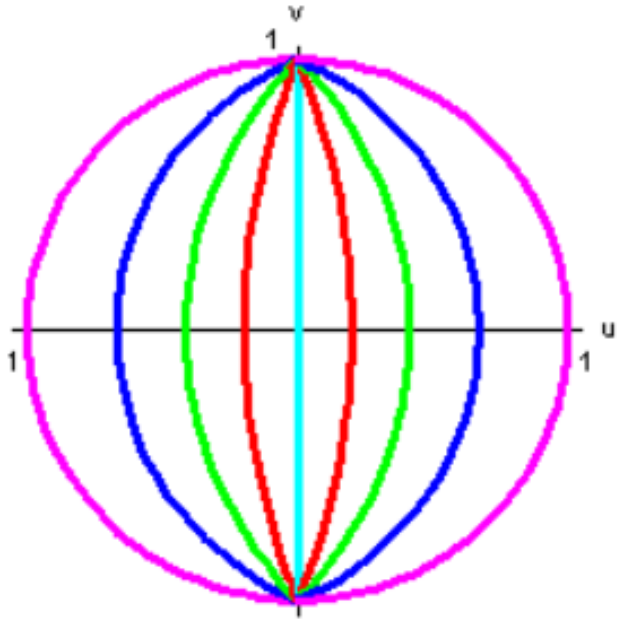
Trigonometric functions
Part III



Source: Complex Analysis by Mathews

Trigonometric Functions

Trigonometric functions
Part IV



Source: Complex Analysis by Mathews

The mapping $w = tan\ z$

Mapping

$w = \tan z$

The mapping $w = \tan z$

To understand the mapping $w = \tan z$, we study images of line segments from z-plane into w-plane.

The mapping $w = \tan z$

To understand the mapping $w = \tan z$, we study images of line segments from z-plane into w-plane. We consider line segments parallel to y-axis with x-coordinates given by

$$x = -\frac{\pi}{4} + \frac{k\pi}{16}, k = 0,1, \dots, 8$$

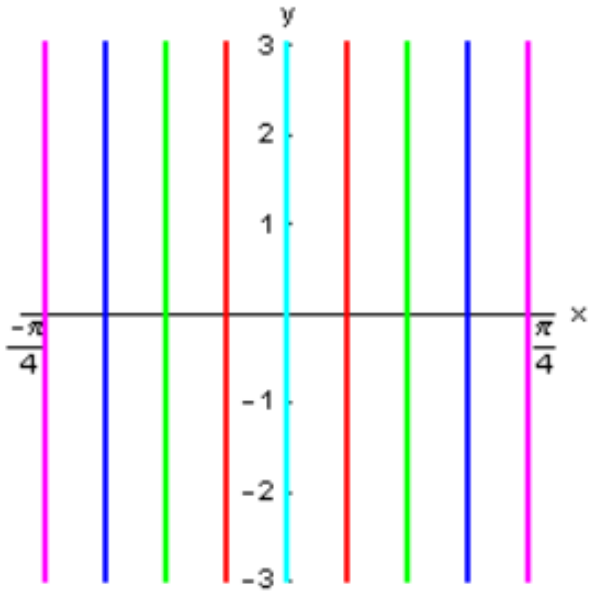
and y vary continuously in the interval $-3 \leq y \leq 3$.

The mapping $w = tan\ z$

The line segments are parallel to y-axis so the x-coordinate is constant on each line segment. So line segments are given by the equation $x = a$, where

$$-\frac{\pi}{4} \leq a \leq \frac{\pi}{4}$$

and y vary between $-3 \leq y \leq 3$.



Source: Complex Analysis by mathews

The mapping $w = tan\ z$

Let $z = x + iy$ be coordinates in the z-plane; let

$$w = \tan z = u + iv$$

be the coordinates in the w-plane.

The mapping $w = tan\ z$

We know that

$$\tan z = \tan(x + iy) = \frac{\sin(x + iy)}{\cos(x + iy)}$$

The mapping $w = tan\ z$

We know that

$$\tan z = \tan(x + iy) = \frac{\sin(x + iy)}{\cos(x + iy)}$$

$$= \frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}$$

The mapping $w = tan\ z$

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$$\tan z = \tan(x + iy) = \frac{\sin(x + iy)}{\cos(x + iy)}$$

$$= \frac{\sin x \cosh y + i \cos x \sinh y}{\cos x \cosh y - i \sin x \sinh y}$$

Simplifying, we get

$$\tan z = \frac{\cos x \sin x + i \cosh y \sinh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

The mapping $w = tan\ z$

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$$\tan z = \tan(x + iy) = \frac{\sin(x + iy)}{\cos(x + iy)}$$

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Simplifying, we get

$$\tan z = \frac{\cos x \sin x + i \cosh y \sinh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

Hence

$$u = \frac{\cos x \sin x}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y},$$

$$v = \frac{\cosh y \sinh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

The mapping $w = tan\ z$

So

$$u = \frac{\cos x \sin x}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y},$$

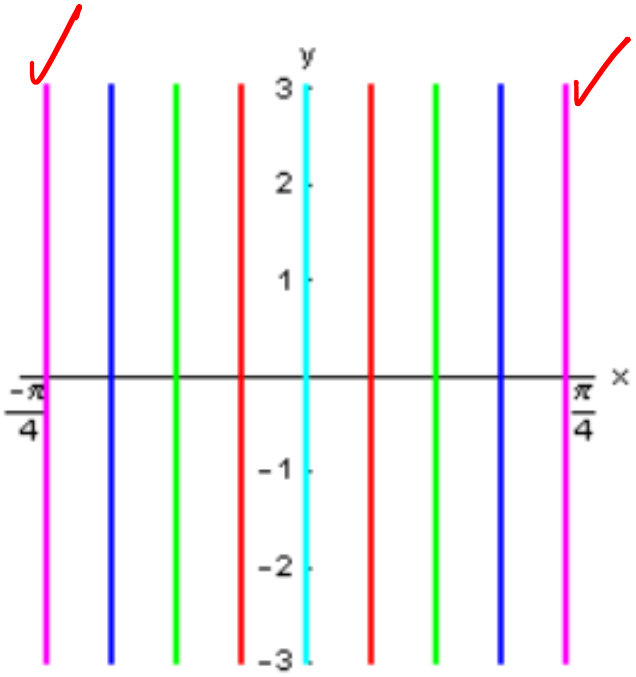
$$v = \frac{\cosh y \sinh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y}$$

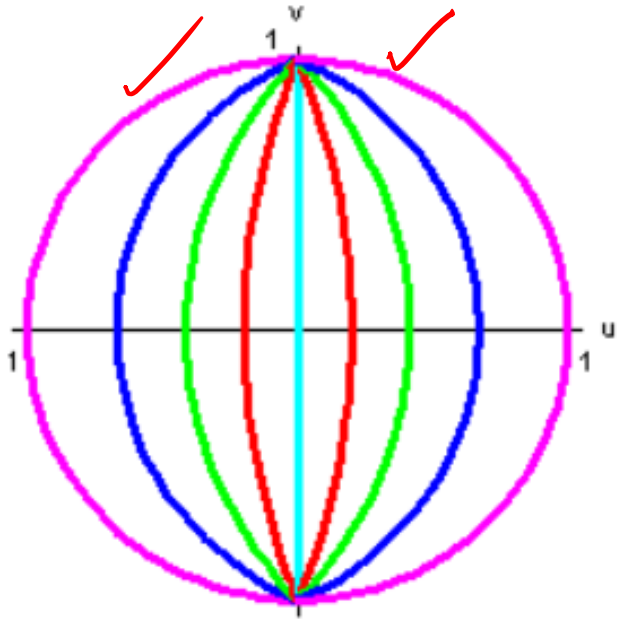
The image of the equation $x = a$, where

$$-\frac{\pi}{4} \leq a \leq \frac{\pi}{4}$$

and y vary between $-3 \leq y \leq 3$ is a circular arc.

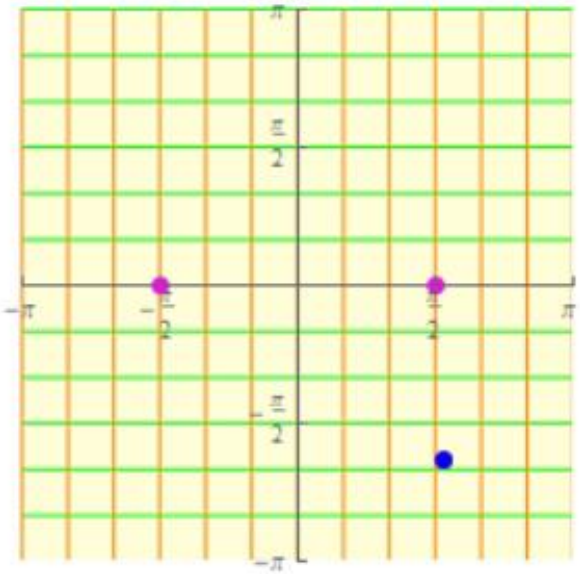
The mapping $w = tan\ z$



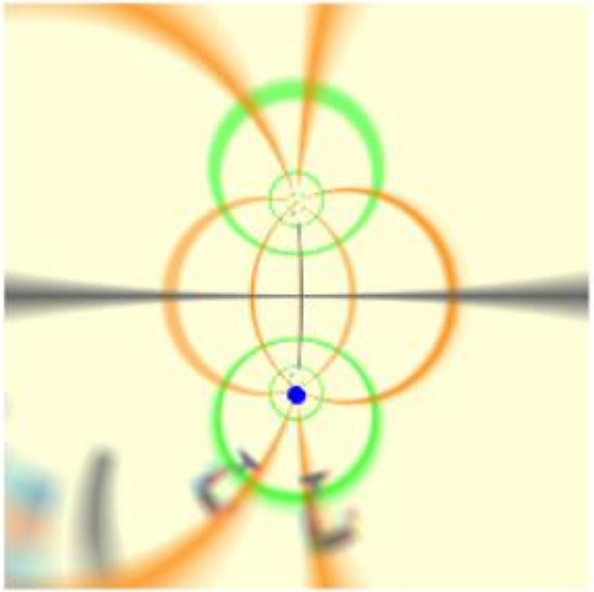


Source: Complex Analysis by mathews

The mapping $w = tan\ z$



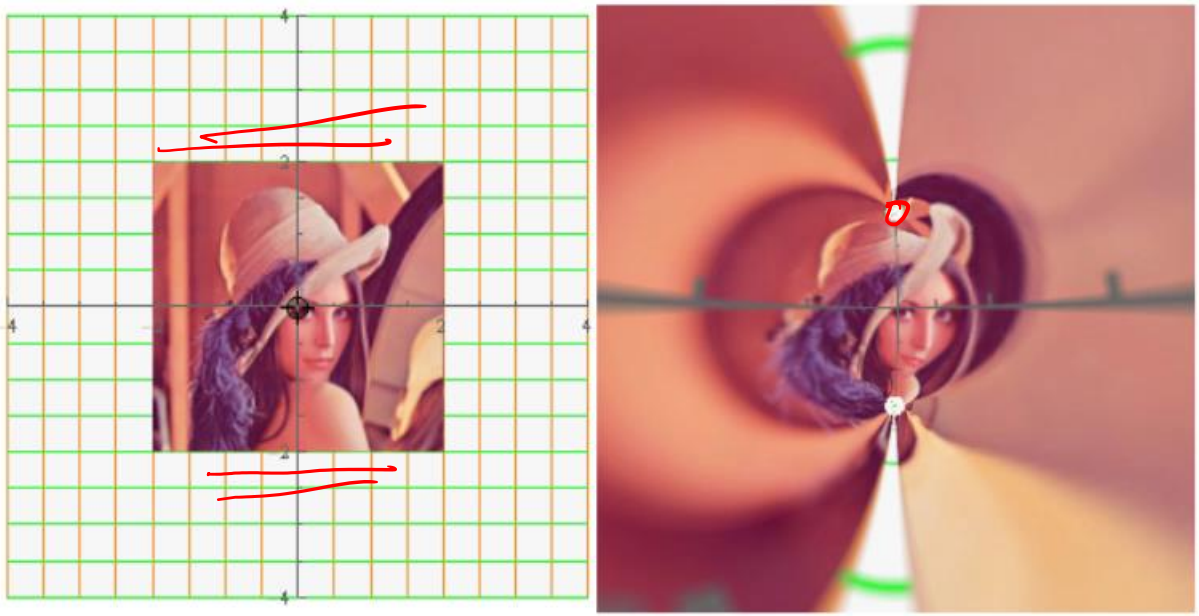
z



w(z)

Source: Wolfram demonstration

The mapping $w = tan\ z$



Source: Wolfram demonstration

The mapping $w = \cos z$

Other mappings of trigonometric functions can be understood in the similar way.

Trigonometric Functions

Trigonometric functions
Part IV



Source: Wolfram demonstration

Trigonometric Functions

Trigonometric functions
Part V

Derivatives of Trigonometric Functions

Derivatives of
trigonometric functions

Derivatives of Trigonometric Functions

Theorem: $\sin(z)$ and $\cos(z)$ are entire functions with

$$\frac{d}{dz}\sin(z) = \cos(z) \quad \text{and} \quad \frac{d}{dz}\cos(z) = -\sin(z)$$

Proof:

Derivatives of Trigonometric Functions

Proof: $(\frac{d}{dz}\sin(z) = \cos(z))$

$$\frac{d}{\mathrm{dz}}\sin(z) = \frac{d}{\mathrm{dz}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

Derivatives of Trigonometric Functions

Proof: $(\frac{d}{dz}\sin(z) = \cos(z))$

$$\frac{d}{\mathrm{dz}}\sin(z) = \frac{d}{\mathrm{dz}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}\left(\frac{d}{\mathrm{dz}}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\right)$$

Derivatives of Trigonometric Functions

Proof: $(\frac{d}{dz}\sin(z) = \cos(z))$

$$\frac{d}{\mathrm{dz}}\sin(z) = \frac{d}{\mathrm{dz}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}\left(\frac{d}{\mathrm{dz}}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n(2n+1)}{(2n+1)!}z^{2n}\right)$$

Derivatives of Trigonometric Functions

Proof: $(\frac{d}{dz}\sin(z) = \cos(z))$

$$\frac{d}{\mathrm{dz}}\sin(z) = \frac{d}{\mathrm{dz}}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty}\left(\frac{d}{\mathrm{dz}}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n(2n+1)}{(2n+1)!}z^{2n}\right)$$

$$= \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{(2n)!}z^{2n}\right) = \cos(z)$$

Derivatives of Trigonometric Functions

Theorem: $\sin(z)$ and $\cos(z)$ are entire functions with

$$\frac{d}{dz}\sin(z) = \cos(z) \quad \text{and} \quad \frac{d}{dz}\cos(z) = -\sin(z)$$

Proof: Similar proof can be done for

$$\frac{d}{dz}\cos(z) = \sin(z) \ .$$

Derivatives of Trigonometric Functions

Using the following derivatives

$$\frac{d}{dz}\sin(z) = \cos(z) \quad \text{and} \quad \frac{d}{dz}\cos(z) = -\sin(z)$$

We can prove the following:

$$\frac{d}{\mathrm{dz}}\tan z = (\sec z)^2,$$

$$\frac{d}{\mathrm{dz}}\cot z = -(\csc z)^2,$$

$$\frac{d}{\mathrm{dz}}\sec z = \sec z \tan z\,,$$

$$\frac{d}{\mathrm{dz}}\csc z = -\mathrm{cotz}\,\csc z\,.$$

Derivatives of Trigonometric Functions

Proof of the fact

$$\frac{d}{\mathrm{dz}}\tan z = (\mathrm{sec}z)^2$$

is given by.

$$\frac{d}{\mathrm{dz}}\tan z = \frac{d}{\mathrm{dz}}\frac{\sin z}{\cos z}$$

Derivatives of Trigonometric Functions

Proof of the fact

$$\frac{d}{\mathrm{dz}}\tan z = (\mathrm{sec}z)^2$$

is given by.

$$\frac{d}{\mathrm{dz}}\tan z = \frac{d}{\mathrm{dz}}\frac{\sin z}{\cos z}$$

$$= \frac{\cos z \frac{d}{\mathrm{dz}}\sin z - \sin z \frac{d}{\mathrm{dz}}\cos z}{(\cos z)^2}$$

Derivatives of Trigonometric Functions

Proof of the fact

$$\frac{d}{\mathrm{dz}}\tan z = (\mathrm{sec}z)^2$$

is given by.

$$\frac{d}{\mathrm{dz}}\tan z = \frac{d}{\mathrm{dz}}\frac{\sin z}{\cos z}$$

$$= \frac{\cos z \frac{d}{\mathrm{dz}}\sin z - \sin z \frac{d}{\mathrm{dz}}\cos z}{(\cos z)^2}$$

$$= \frac{(\cos z)^2 + (\sin z)^2}{(\cos z)^2}$$

Derivatives of Trigonometric Functions

Proof of the fact

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$$\frac{d}{\mathrm{dz}}\tan z = \frac{d}{\mathrm{dz}}\frac{\sin z}{\cos z}$$

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$$= \frac{(\cos z)^2 + (\sin z)^2}{(\cos z)^2}$$

$$= \frac{1}{(\cos z)^2}$$

Derivatives of Trigonometric Functions

Proof of the fact

$$\frac{d}{\mathrm{dz}}\tan z = (\mathrm{sec}z)^2$$

is given by.

$$\begin{aligned}
\frac{d}{\mathrm{dz}}\tan z &= \frac{d}{\mathrm{dz}}\frac{\sin z}{\cos z} \\
&= \frac{\cos z \frac{d}{\mathrm{dz}}\sin z - \sin z \frac{d}{\mathrm{dz}}\cos z}{(\cos z)^2} \\
&= \frac{(\cos z)^2 + (\sin z)^2}{(\cos z)^2} \\
&= \frac{1}{(\cos z)^2} \\
&= (\sec z)^2
\end{aligned}$$

Derivatives of Trigonometric Functions

The following can be proved in a similar way:

$$\frac{d}{\mathrm{dz}}\cot z = -(\csc z)^2,$$

$$\frac{d}{\mathrm{dz}}\sec z = \sec z \tan z\,,$$

$$\frac{d}{\mathrm{dz}}\csc z = -\mathrm{cotz}\,\csc z\,.$$

Trigonometric Functions

Trigonometric functions
Part V

Hyperbolic Functions

Hyperbolic functions

Hyperbolic Functions

How to defined?

$\sinh z$

$\cosh z$

for complex variable z

Hyperbolic Functions

We know for real variable x

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

Hyperbolic Functions

So we define

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

for complex variable z

Hyperbolic Functions

Other complex hyperbolic functions can be defined using definition of $\sinh z$ and $\cosh z$

Hyperbolic Functions

$$\tanh z = \frac{\sinh z}{\cosh z}$$

$$= \frac{(e^z - e^{-z})}{(e^z + e^{-z})}$$

Hyperbolic Functions

$$\coth z = \frac{\cosh z}{\sinh z}$$

$$= \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}}$$

Hyperbolic Functions

$$\operatorname{sech} z = \frac{1}{\cosh z}$$

$$= \frac{2}{e^{z} + e^{-z}}$$

Hyperbolic Functions

$$\operatorname{csch} z = \frac{1}{\sinh z}$$

$$= \frac{2}{e^{z} - e^{-z}}$$

Hyperbolic Functions

Derivatives of
hyperbolic functions

Hyperbolic Functions

The derivatives of hyperbolic functions are:

$$\frac{d}{dz}\cosh z = \sinh z, \quad \text{and} \quad \frac{d}{dz}\sinh z = \cosh z,$$

$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z, \quad \text{and} \quad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z,$$

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z, \quad \text{and} \quad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z.$$

Hyperbolic Functions

The derivatives of hyperbolic functions are:

$$\frac{d}{dz}\cosh z = \sinh z, \quad \text{and} \quad \frac{d}{dz}\sinh z = \cosh z,$$

$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z, \quad \text{and} \quad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z,$$

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z, \quad \text{and} \quad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z.$$

Proof:

$$\frac{d}{dz}\cosh z = \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right) = \frac{e^z - e^{-z}}{2} = \sinh z$$

Hyperbolic Functions

Two important identities

$$\cosh z = \cosh(x+iy) = \cos y \cosh x + i \sinh x \sin y$$
$$\sinh z = \sinh(x+iy) = \sinh x \cos y + i \cosh x \sin y$$

Proof:

$$\cosh(z) = \cosh(x+iy) = \frac{e^{x+iy} + e^{-x-iy}}{2}$$

$$= \frac{e^x e^{iy} + e^{-x} e^{-iy}}{2} = \frac{1}{2}\left[e^x(\cos y + i\sin y) + e^{-x}(\cos y - i\sin y)\right]$$

$$= \frac{1}{2}(e^x + e^{-x})\cos y + \frac{i}{2}\sin y\,(e^x - e^{-x})$$

$$\boxed{\cosh(x+iy) = \cosh x \cos y + i \sinh x \sin y}$$

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

Hyperbolic Functions

Two important identities

$$\cosh z = \cosh(x + iy) = \cos y \cosh x + i \sinh x \sin y$$
$$\sinh z = \sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$$

Proof: Similarly the following can be proved:

$$\sinh z = \sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$$

Hyperbolic Functions

The following can be proved using definitions.

$$\cosh(iz) = \cos(z)$$
$$\sinh(iz) = i\sin(z)$$
$$\sin(iz) = i\sinh(z)$$
$$\cos(iz) = \cosh(z)$$

$$\cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos(z)$$

$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = i\sin(z)$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

iz

Hyperbolic Functions

One of the important identities is:

$$\cosh^2 z - \sinh^2 z = 1.$$

Proof:

$$\cosh(iz) = \cos(z) \Rightarrow \cosh(z) = \cos(iz) \quad (1)$$

$z \to iz \Rightarrow \cosh(i^2 z) = \cos(iz) \Rightarrow \cosh(z) = \cos(iz)$

$$\sinh(iz) = i\sin(z) \Rightarrow \sinh(z) = -i\sin(iz) \quad (2)$$

$z \to iz \quad \sin(i^2 z) = i\sin(iz)$

$\Rightarrow \sin(-z) = i\sin(iz) \Rightarrow -\sin(z) = i\sin(iz)$

$\cosh^2 z - \sinh^2 z = \cos^2(iz) - (-i\sin(iz))^2 = \cos^2(iz) + \sin^2(iz)$

$= 1$

Hyperbolic Functions

Another important identity is following:

$$\cosh(z_1 \pm z_2) = \cosh(z_1)\cosh(z_2) \pm \sinh(z_1)\sinh(z_2)$$

Proof: We know that $\cosh(z) = \cos(iz)$

Hyperbolic Functions

The following important identity can be proved similarly:

$$\sin h(z_1 \pm z_2) = \sinh(z_1)\cosh(z_2) \pm \cosh(z_1)\sinh(z_2)$$

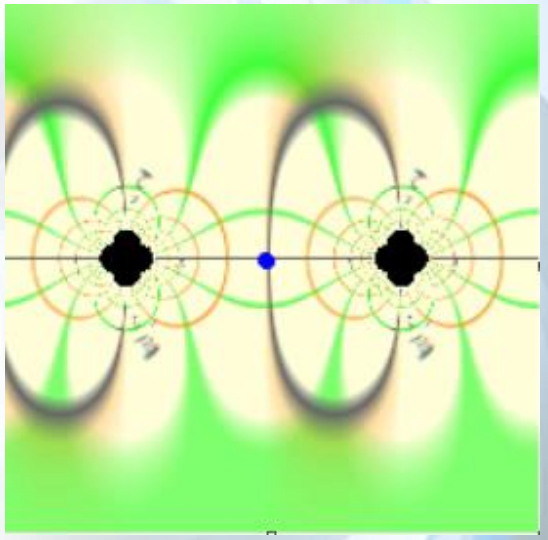
Proof: Similarly using $\sin(iz) = i\sinh(z)$ we prove above.

Hyperbolic Functions

Hyperbolic functions

Inverse Trigonometric Functions

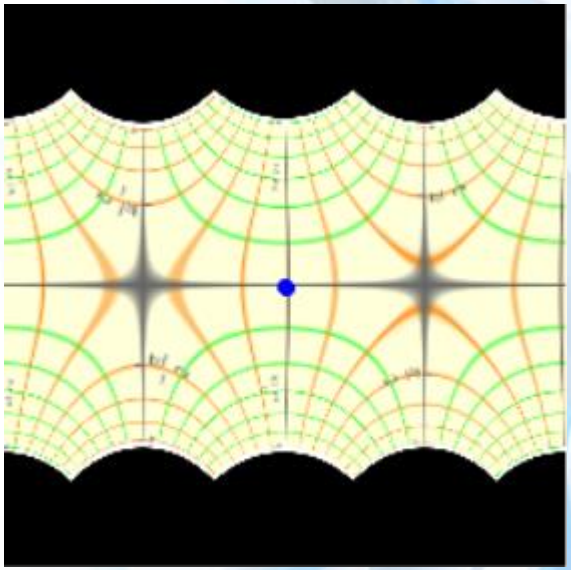
Inverse Trigonometric Functions



Source: Wolfram demonstration

Inverse Trigonometric Functions

$$arcsin(z)$$



Source: Wolfram demonstration

Inverse Trigonometric Functions

The periodic nature of $\sin(z)$ makes it a multiple-to-one function

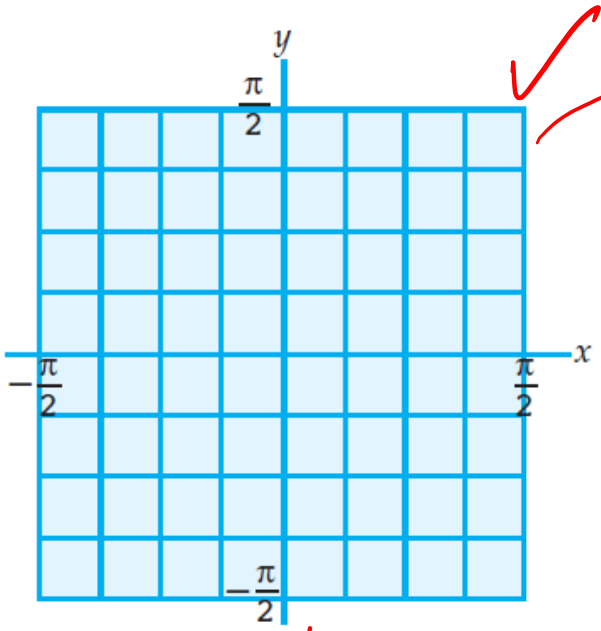
Inverse Trigonometric Functions

Riemann surface of $\sin(z)$

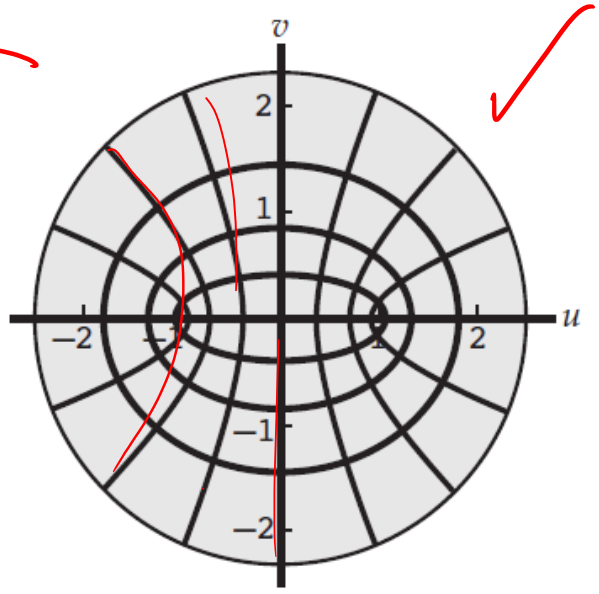
Inverse Trigonometric Functions

Step1: The function $\sin(z)$ is one-to-one on

$$A = \left\{(x, y): -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$$



Sin(z)

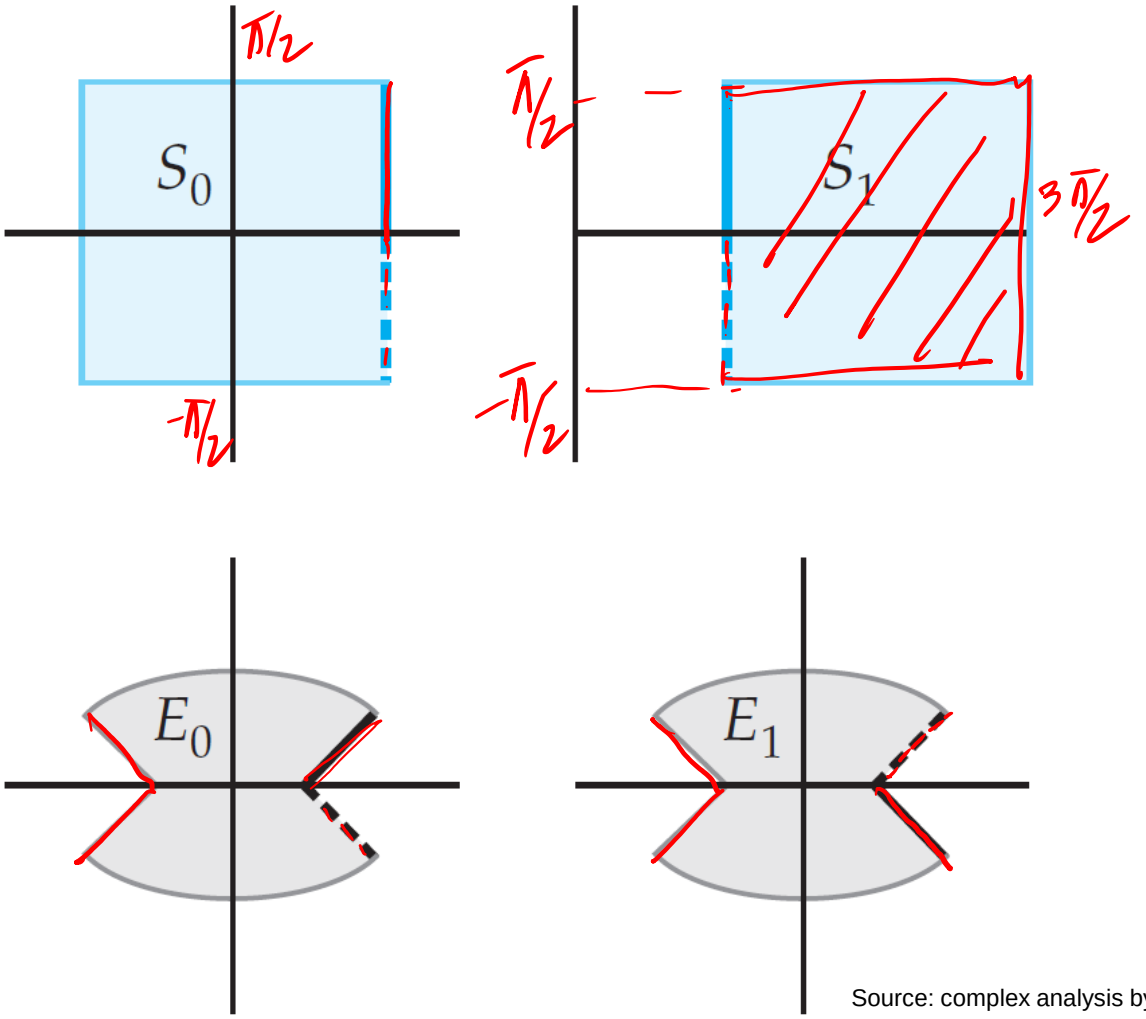


Z-plane

Source: complex analysis by Zill

Inverse Trigonometric Functions

Step 2:



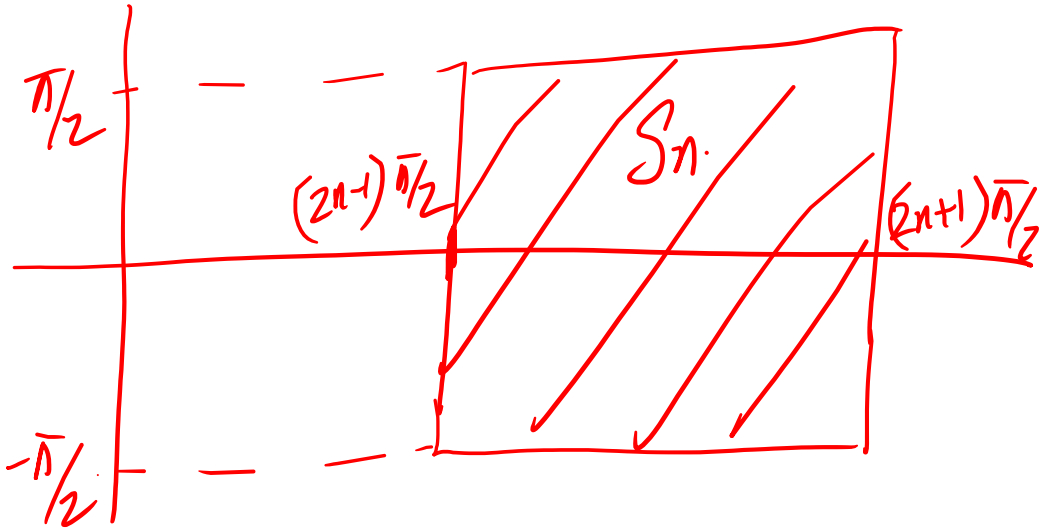
Source: complex analysis by Zill

Inverse Trigonometric Functions

Step 3: In general for an integer n an elliptical image of the following square

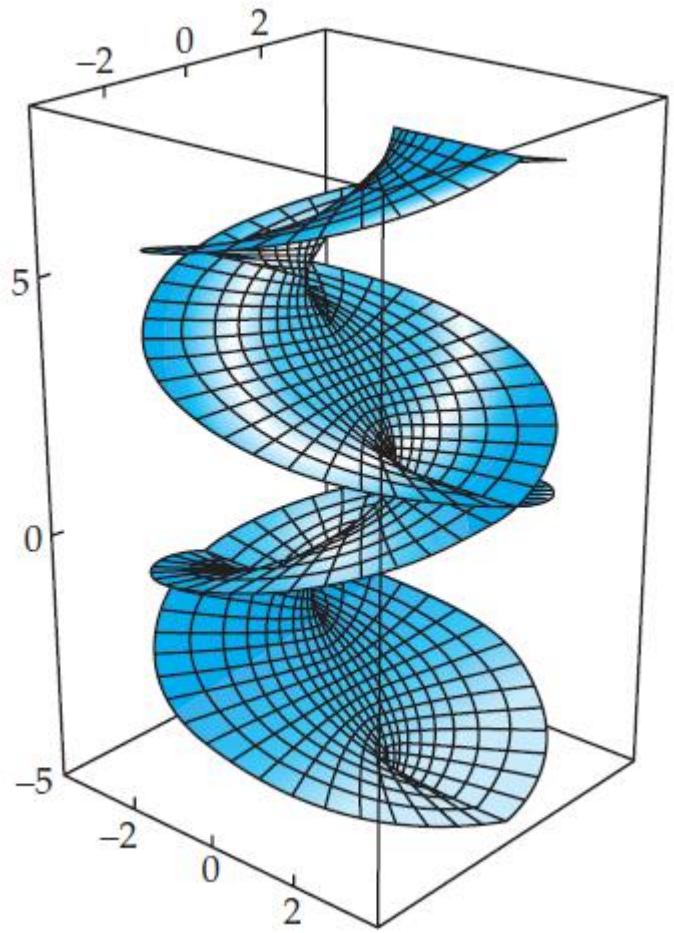
$$S_n = \left\{(x, y): (2n-1)\frac{\pi}{2} \leq x \leq (2n+1)\frac{\pi}{2}, \frac{\pi}{2} \leq y \leq \frac{\pi}{2}\right\}$$

is constructed. Each region E_n is cut open, as E_0 and E_1 were, and E_n is glued to E_{n+1} along their boundaries in a manner analogous to that used for E_0 and E_1.



Inverse Trigonometric Functions

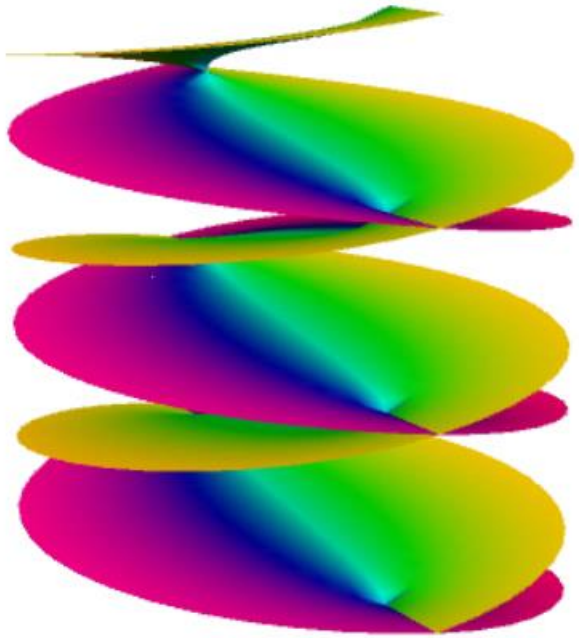
Gluing all pieces we get the following Riemann surface.



Source: complex analysis by Zill

Inverse Trigonometric Functions

Gluing all pieces we get the following Riemann surface.



Inverse Trigonometric Functions

Hence the inverse of $\sin(z)$,
denoted by $arcsin(z)$,
is a multivalued
function

Inverse Trigonometric Functions

$$arcsin(z)$$
$$= -i\log\left(iz + (1 - z^2)^{1/2}\right)$$



Inverse Trigonometric Functions

Show that

$$arcsin(z) = -i\log\left(iz + (1 - z^2)^{1/2}\right)$$ ✓

Proof:

$w = \arcsin(z) \Rightarrow z = \sin(w)$

$\Rightarrow z = \dfrac{e^{iw} - e^{-iw}}{2i}$, Find w in terms of z.

$\Rightarrow 2iz = e^{iw} - \dfrac{1}{e^{iw}} \Rightarrow 2iz e^{iw} = (e^{iw})^2 - 1$

$\Rightarrow (e^{iw})^2 - 2iz e^{iw} - 1 = 0$ → Quadratic in e^{iw}

$\Rightarrow e^{iw} = \dfrac{2iz \pm \sqrt{4i^2z^2 + 4}}{2}$

Inverse Trigonometric Functions

The function is a multivalued function

$$arcsin(z) = -i\log\left(iz + (1 - z^2)^{1/2}\right)$$

Inverse Trigonometric Functions

When principal values of $\log(z)$ and $z^{\frac{1}{2}}$ are used we denote it as

$$Arcssin(z) = -i\mathrm{Log}\left(iz + (1 - z^2)^{1/2}\right)$$

Inverse Trigonometric Functions

The function

$$Arcssin(z) = -i\mathrm{Log}\left(iz + (1 - z^2)^{1/2}\right)$$

maps upper half plane

$$\{z: Im\,(z) > 0\}$$

onto a portion of upper half plane that lies in the vertical strip

$$\left\{w: -\frac{\pi}{2} < Re(w) < \frac{\pi}{2}\right\}$$

Source: complex analysis by mathews

Inverse Trigonometric Functions

The function

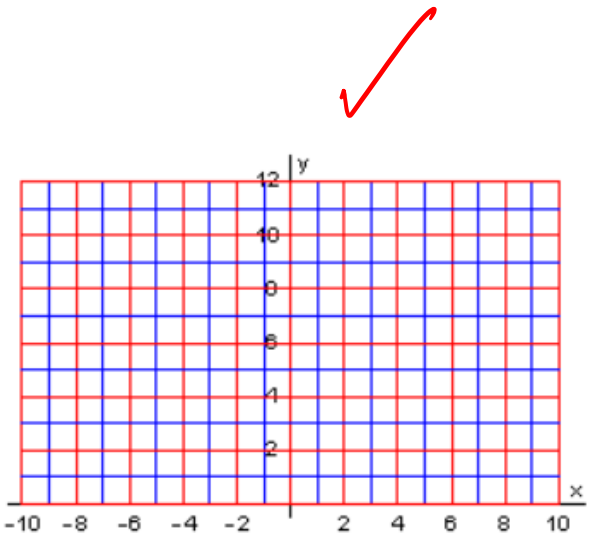
$$Arcssin(z) = -i\text{Log}\left(iz + (1 - z^2)^{1/2}\right)$$

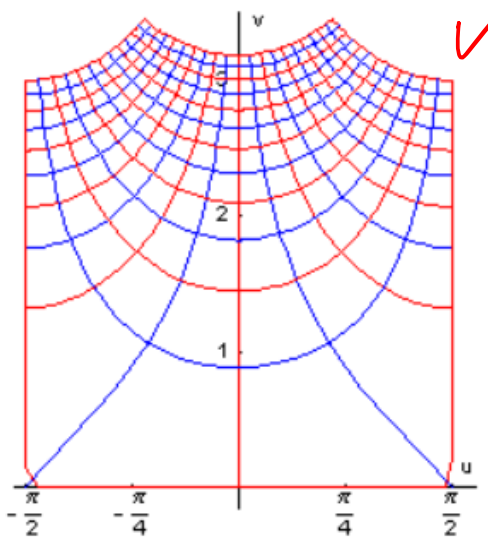
maps upper half plane

$$\{z: Im\,(z) > 0\}$$

onto a portion of upper half plane that lies in the vertical strip

$$\left\{w: -\frac{\pi}{2} < Re(w) < \frac{\pi}{2}\right\}$$

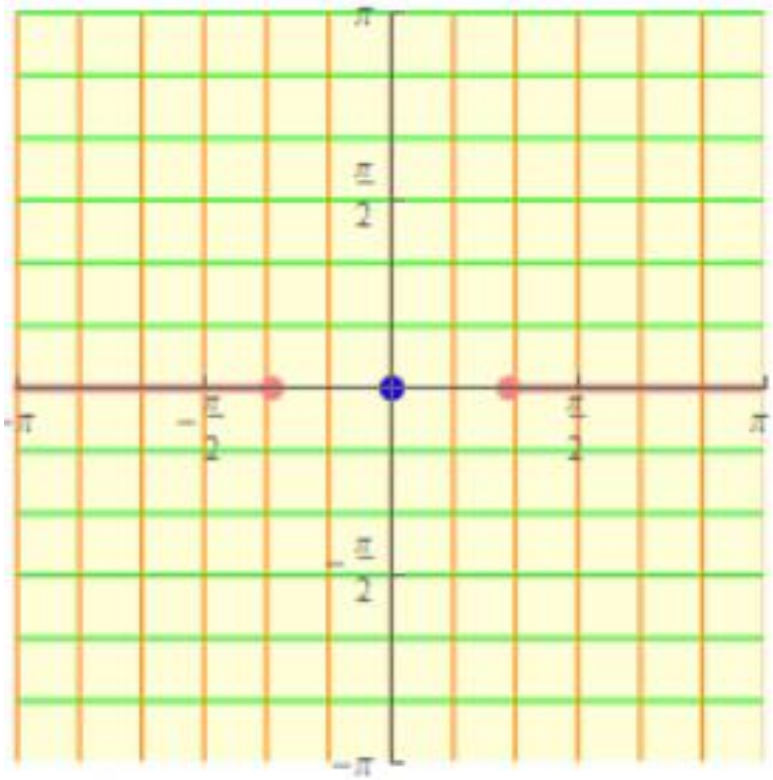


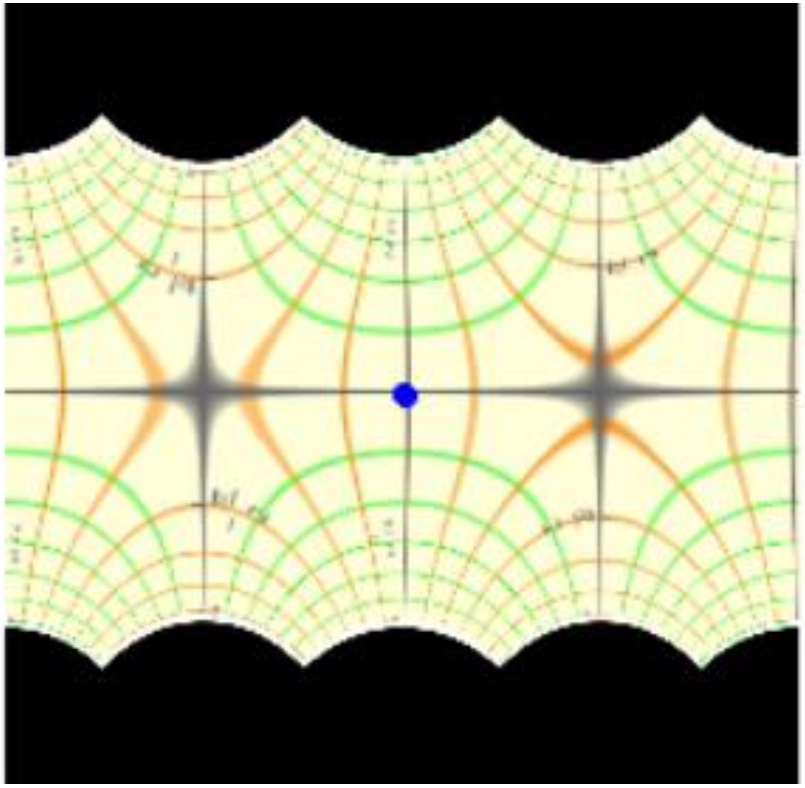


Source: complex analysis by mathews

Inverse Trigonometric Functions

$$Arcssin(z) = -i\mathrm{Log}\left(iz + (1 - z^2)^{1/2}\right)$$

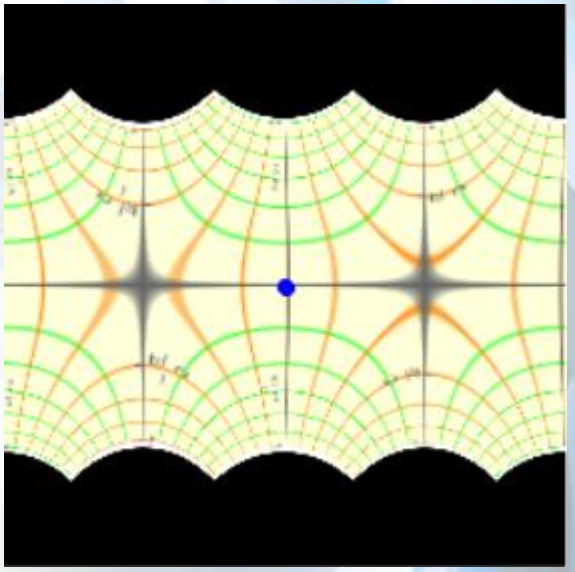




Source: wolfram demonstration

Inverse Trigonometric Functions

$\arccos z$



Source: Wolfram demonstration

Inverse Trigonometric Functions

$$\arccos z = -i\log\left(z + i(1 - z^2)^{1/2}\right)$$

Inverse Trigonometric Functions

Fact:

$$\arccos z = -i\log\left(z + i(1 - z^2)^{1/2}\right)$$

Proof:

$$\arccos z = \frac{\pi}{2} - \arcsin z = \frac{\pi}{2} + i\log\left(iz + (1 - z^2)^{1/2}\right)$$

Inverse Trigonometric Functions

Fact:

$$\arccos z = -i\log\left(z + i(1-z^2)^{1/2}\right)$$

Proof:

$$\arccos z = \frac{\pi}{2} - \arcsin z = \frac{\pi}{2} + i\log\left(iz + (1-z^2)^{1/2}\right)$$

$$= \frac{\pi}{2} - i\log\left(-iz + (1-z^2)^{1/2}\right)$$

Inverse Trigonometric Functions

Fact:

$$\arccos z = -i\log\left(z + i(1 - z^2)^{1/2}\right)$$

Proof:

$$\arccos z = \frac{\pi}{2} - \arcsin z = \frac{\pi}{2} + i\log\left(iz + (1 - z^2)^{1/2}\right)$$

$$= \frac{\pi}{2} - i\log\left(-iz + (1 - z^2)^{1/2}\right)$$

$$= -i\frac{\pi i}{2} - i\log\left(-iz + (1 - z^2)^{1/2}\right)$$

Inverse Trigonometric Functions

Fact:

$$\arccos z = -i\log\left(z + i(1-z^2)^{1/2}\right)$$

Proof:

$$\arccos z = \frac{\pi}{2} - \arcsin z = \frac{\pi}{2} + i\log\left(iz + (1-z^2)^{1/2}\right)$$

$$= \frac{\pi}{2} - i\log\left(-iz + (1-z^2)^{1/2}\right)$$

$$= -i\frac{\pi i}{2} - i\log\left(-iz + (1-z^2)^{1/2}\right)$$

$$= -i\log(i) - i\log\left(-iz + (1-z^2)^{1/2}\right)$$

$\log(i) = \ln|i| + i\arg(i)$

$= +i\frac{\pi}{2}$

Inverse Trigonometric Functions

Fact:

$$\arccos z = -i\log\left(z + i(1-z^2)^{1/2}\right)$$

Proof:

$$\arccos z = \frac{\pi}{2} - \arcsin z = \frac{\pi}{2} + i\log\left(iz + (1-z^2)^{1/2}\right)$$

$$= \frac{\pi}{2} - i\log\left(-iz + (1-z^2)^{1/2}\right)$$

$$= -i\frac{\pi i}{2} - i\log\left(-iz + (1-z^2)^{1/2}\right)$$

$$= -i\log(i) - i\log\left(-iz + (1-z^2)^{1/2}\right)$$

$$= -i\left(\log(i) + \log\left(-iz + (1-z^2)^{1/2}\right)\right)$$

Inverse Trigonometric Functions

Fact:

$$\arccos z = -i\log\left(z + i(1-z^2)^{1/2}\right)$$

Proof:

$$\arccos z = \frac{\pi}{2} - \arcsin z = \frac{\pi}{2} + i\log\left(iz + (1-z^2)^{1/2}\right)$$

$$= \frac{\pi}{2} - i\log\left(-iz + (1-z^2)^{1/2}\right)$$

$$= -i\frac{\pi i}{2} - i\log\left(-iz + (1-z^2)^{1/2}\right)$$

$$= -i\log(i) - i\log\left(-iz + (1-z^2)^{1/2}\right)$$

$$= -i\left(\log(i) + \log\left(-iz + (1-z^2)^{1/2}\right)\right)$$

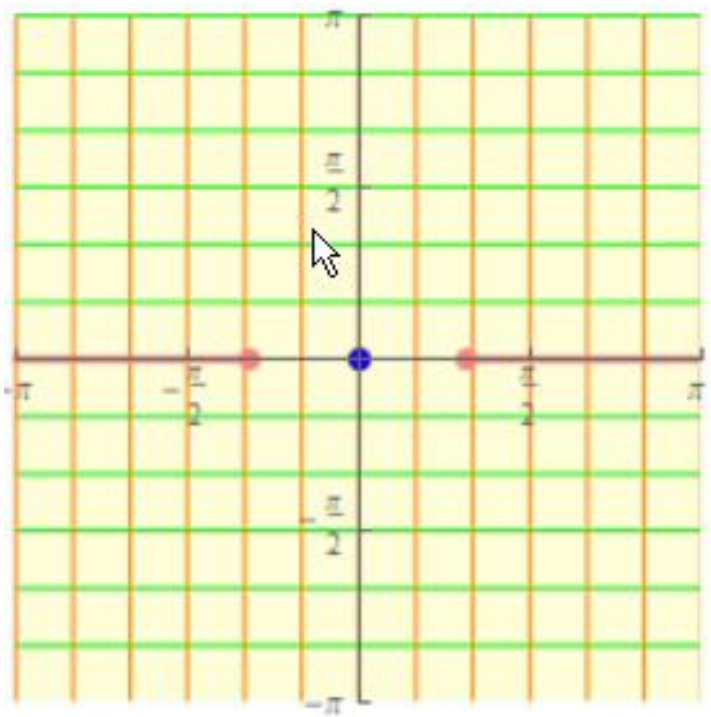
$$= -\log\left(i(-i)z + i(1-z^2)^{1/2}\right) = \boxed{-\log\left(z + i(1-z^2)^{1/2}\right)}$$

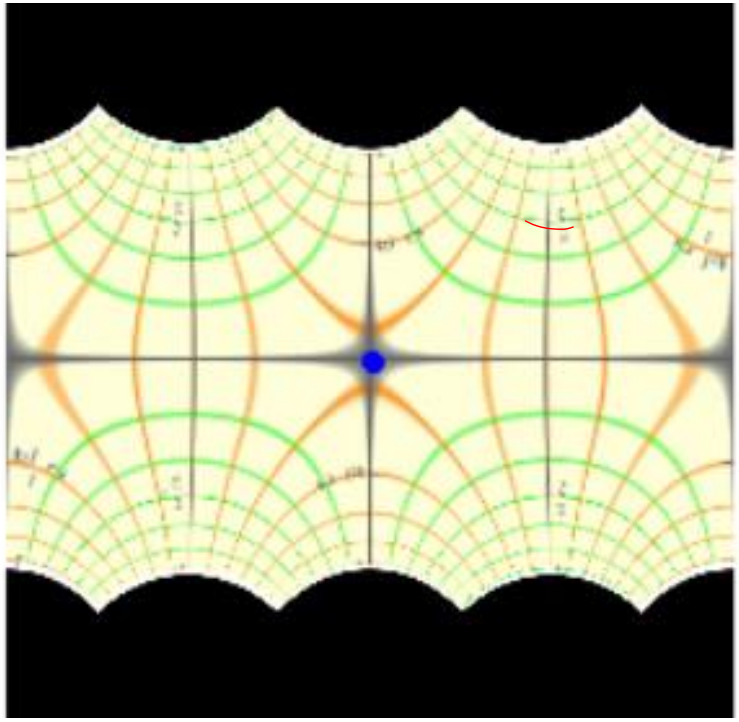
$\log(ab) = \log(a) + \log(b)$

Inverse Trigonometric Functions

Similarly we can sketch principal branch of the following function

$$\operatorname{Arccos} z = -iLog\left(z + i(1 - z^2)^{1/2}\right)$$

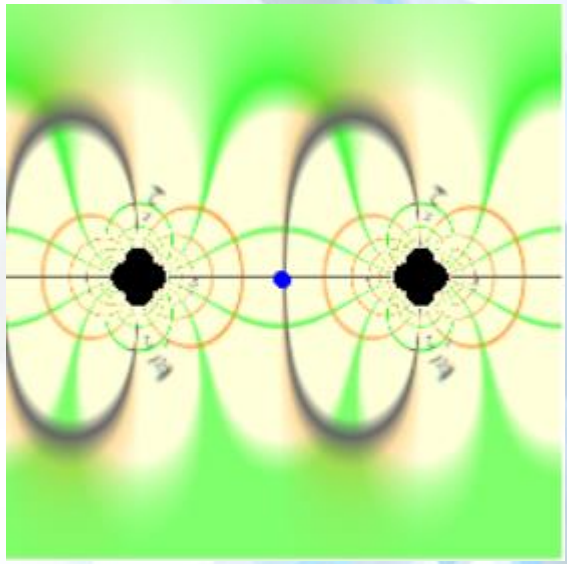




Source: Wolfram demonstration

Inverse Trigonometric Functions

$$arctan(z)$$



Source: Wolfram demonstration

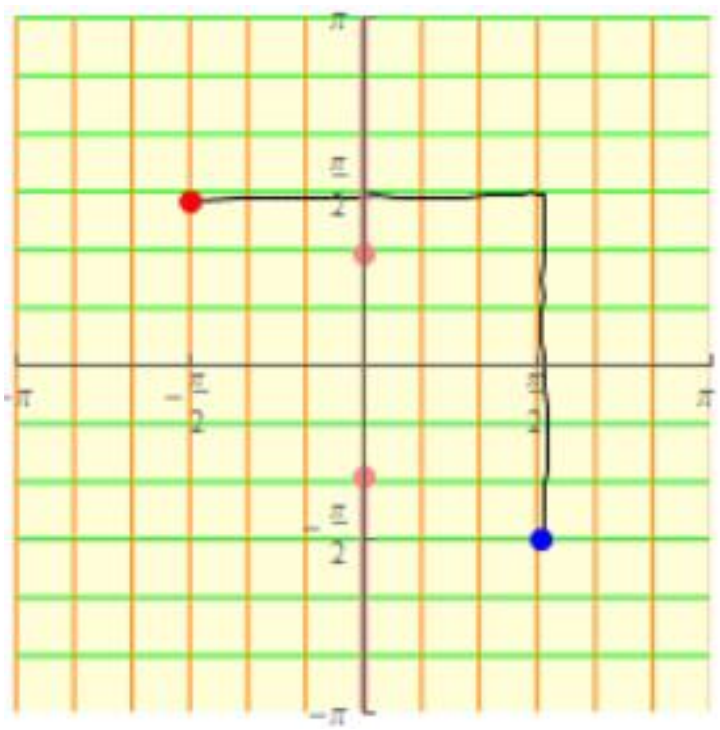
Inverse Trigonometric Functions

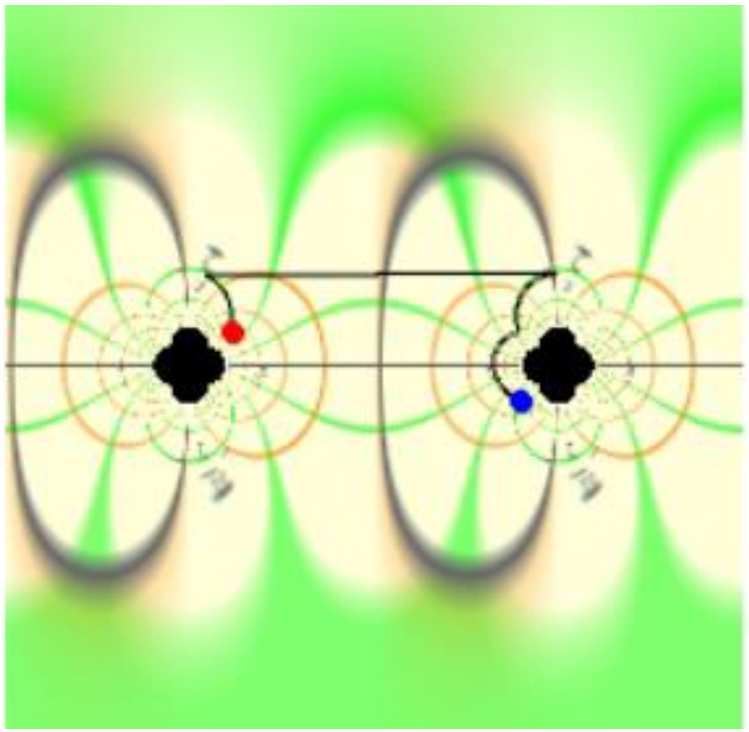
$$\arctan z = \frac{i}{2}\log\left(\frac{i+z}{i-z}\right)$$

Inverse Trigonometric Functions

we can sketch principal branch of the following function

$$\operatorname{Arctan} z = \frac{i}{2} \operatorname{Log}\left(\frac{i+z}{i-z}\right)$$

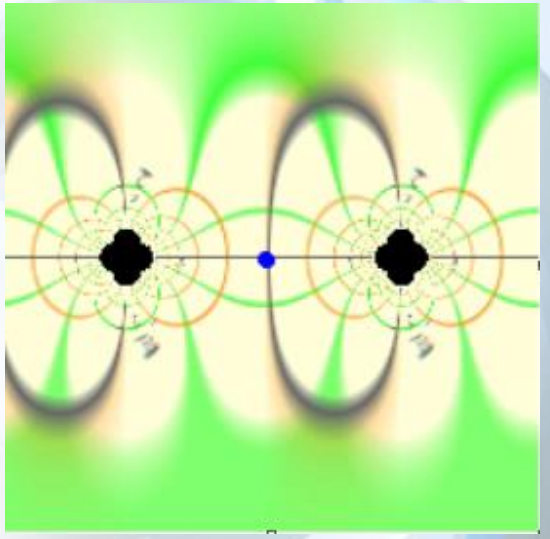




Source: Wolfram demonstration

Inverse Trigonometric Functions

Inverse Trigonometric Functions



Inverse Trigonometric Functions

Derivatives of inverse trigonometric functions

Inverse Trigonometric Functions

$arcsin(z)$
is a multivalued function

Inverse Trigonometric Functions

But we can talk about the derivative of the principal branch of $\arcsin(z)$

Inverse Trigonometric Functions

$$\frac{d}{\mathrm{dz}} \arcsin z = \frac{1}{\sqrt{1 - z^2}}$$

Inverse Trigonometric Functions

Fact:

$$\frac{d}{\mathrm{dz}}\arcsin z = \frac{1}{\sqrt{1-z^2}}$$

Proof:

$w = \arcsin z$

$\Rightarrow z = \sin w$

$$\frac{d}{dz}z = \frac{d}{\mathrm{dz}}\sin w$$

Inverse Trigonometric Functions

Fact:

$$\frac{d}{\mathrm{dz}}\arcsin z = \frac{1}{\sqrt{1-z^2}}$$

Proof:

$$\frac{d}{dz}z = \frac{d}{\mathrm{dz}}\sin w$$

$$1 = \frac{d}{\mathrm{dw}}\sin w\frac{\mathrm{dw}}{\mathrm{dz}}$$

Inverse Trigonometric Functions

Fact:

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Proof:

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$$1 = \frac{d}{\mathrm{dw}}\sin w\frac{\mathrm{dw}}{\mathrm{dz}}$$

$$1 = \cos w\frac{\mathrm{dw}}{\mathrm{dz}}$$

Inverse Trigonometric Functions

Fact:

$$\frac{d}{\mathrm{dz}}\arcsin z = \frac{1}{\sqrt{1-z^2}}$$

Proof:

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$$1 = \frac{d}{\mathrm{dw}}\sin w\frac{\mathrm{dw}}{\mathrm{dz}}$$

$$1 = \cos w\ \frac{\mathrm{dw}}{\mathrm{dz}}$$

$$\frac{\mathrm{dw}}{\mathrm{dz}} = \frac{1}{\cos w}$$

Inverse Trigonometric Functions

Fact:

$$\frac{d}{\mathrm{dz}}\arcsin z = \frac{1}{\sqrt{1-z^2}}$$

Proof:

$w = \arcsin z$

$\Rightarrow z = \sin w$

$$\frac{d}{dz}z = \frac{d}{\mathrm{dz}}\sin w$$

$$1 = \frac{d}{\mathrm{dw}}\sin w \frac{\mathrm{dw}}{\mathrm{dz}}$$

$$1 = \cos w \frac{\mathrm{dw}}{\mathrm{dz}}$$

$$\frac{\mathrm{dw}}{\mathrm{dz}} = \frac{1}{\cos w}$$

$$\frac{dw}{dz} = \frac{1}{\sqrt{1-\sin w^2}} = \frac{1}{\sqrt{1-z^2}}$$

$\cos w = \sqrt{\cos^2 w} = \sqrt{1-\sin^2 w}$

Inverse Trigonometric Functions

$$\frac{d}{\mathrm{dz}}\arccos z = \frac{-1}{\sqrt{1-z^2}}$$

Inverse Trigonometric Functions

$$\frac{d}{\mathrm{dz}} \arctan z = \frac{1}{1+z^2}$$

Inverse Trigonometric Functions

Derivatives of inverse trigonometric functions

Inverse hyperbolic Functions

Inverse hyperbolic Functions

Inverse hyperbolic Functions

$arcsinh(z)$

$\operatorname{arccosh}(z)$

$\operatorname{arctanh}(z)$

Inverse hyperbolic Functions

$$\operatorname{arcsinh} z = \log\left[z + (z^2 + 1)^{1/2}\right]$$

Inverse hyperbolic Functions

$$\operatorname{arccosh} z = \log\left[z + (z^2 - 1)^{1/2}\right]$$

Inverse hyperbolic Functions

$$\operatorname{arctanh} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

Inverse hyperbolic Functions

Fact

$$w = \operatorname{arcsinh} z = \log\left[z + (z^2+1)^{1/2}\right]$$

Proof:

$w = \operatorname{arcsinh} z \Rightarrow z = \sinh(w)$

$\Rightarrow z = \dfrac{e^w - e^{-w}}{2}$, Find w in terms of z

$\Rightarrow 2z = e^w - \dfrac{1}{e^w} \Rightarrow 2ze^w = (e^w)^2 - 1$

$\Rightarrow (e^w)^2 - 2ze^w - 1 = 0$ Quadratic equation in e^w

$e^w = \dfrac{2z \pm \sqrt{4z^2+4}}{2} \Rightarrow e^w = z + (z^2+1)^{1/2}$

Inverse hyperbolic Functions

Similarly we can prove the following

$$\operatorname{arccosh} z = \log\left[z + (z^2 - 1)^{1/2}\right]$$

$w = \operatorname{arc}\cosh z \Rightarrow z = \cosh w$

Inverse hyperbolic Functions

Similarly we can prove the following

$$\operatorname{arctanh} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

Inverse hyperbolic Functions

Derivatives of inverse hyperbolic functions

Inverse hyperbolic Functions

Derivatives of inverse hyperbolic functions are:

$$\frac{d}{\mathrm{dz}}\operatorname{arcsinh} z = \frac{1}{\sqrt{1+z^2}},$$

$$\frac{d}{\mathrm{dz}}\operatorname{arccosh} z = \frac{1}{\sqrt{z^2-1}},$$

$$\frac{d}{\mathrm{dz}}\operatorname{arctanh} z = \frac{1}{1-z^2}.$$

Inverse hyperbolic Functions

$$\frac{d}{dz}\operatorname{arcsinh} z = \frac{1}{\sqrt{1+z^2}},$$

Proof:

$w = \operatorname{arcsinh} z$, $\frac{dw}{dz} = ?$

$\Rightarrow z = \sinh w$

$\cosh^2 w - \sinh^2 w = 1$

$\Rightarrow \frac{dz}{dz} = \frac{d}{dz}\sinh w$

$\Rightarrow 1 = \cosh w \cdot \frac{dw}{dz}$

$\Rightarrow \frac{dw}{dz} = \frac{1}{\cosh w} \Rightarrow \frac{dw}{dz} = \frac{1}{\sqrt{1+\sinh^2 w}} = \frac{1}{\sqrt{1+z^2}}$

Inverse hyperbolic Functions

Inverse hyperbolic Functions

Complex Integration

Complex Integration

$$\int$$

Complex Integration

Integration



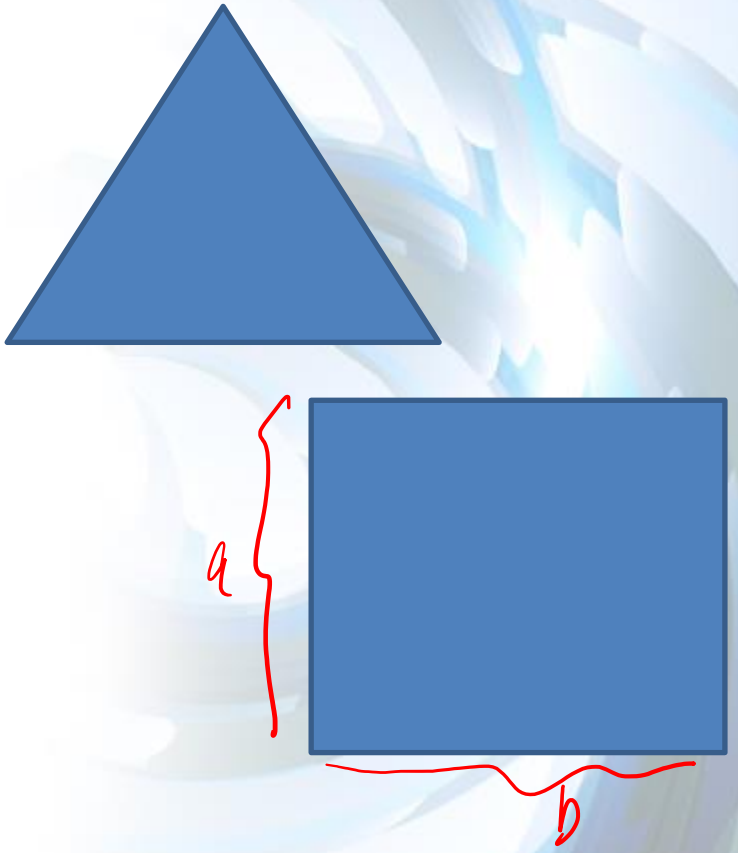
Source: https://educationaltechnology.net

Complex Integration

Real integrals

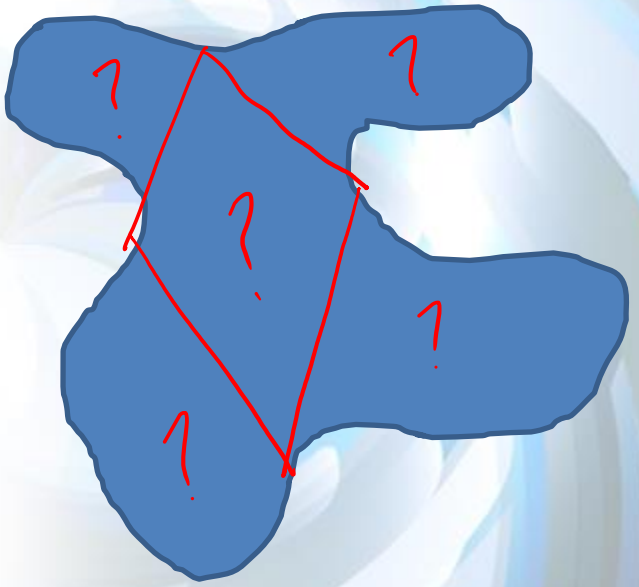
Complex Integration

Area Problem



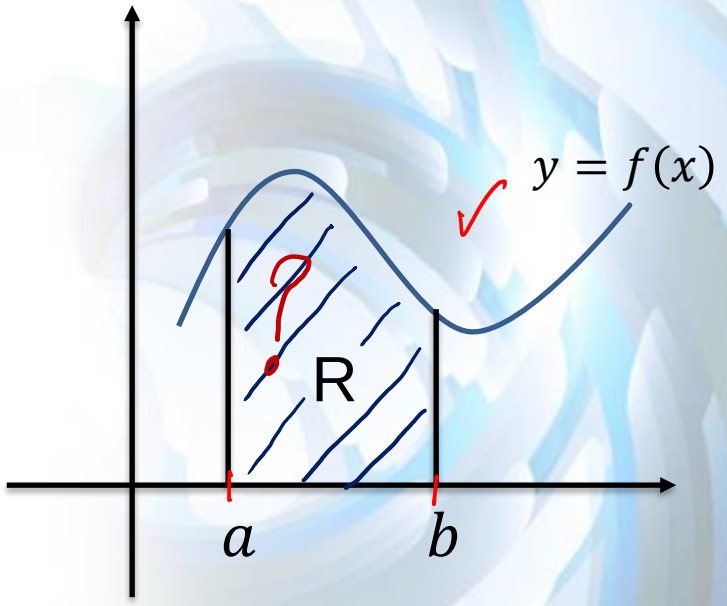
Complex Integration

Area Problem



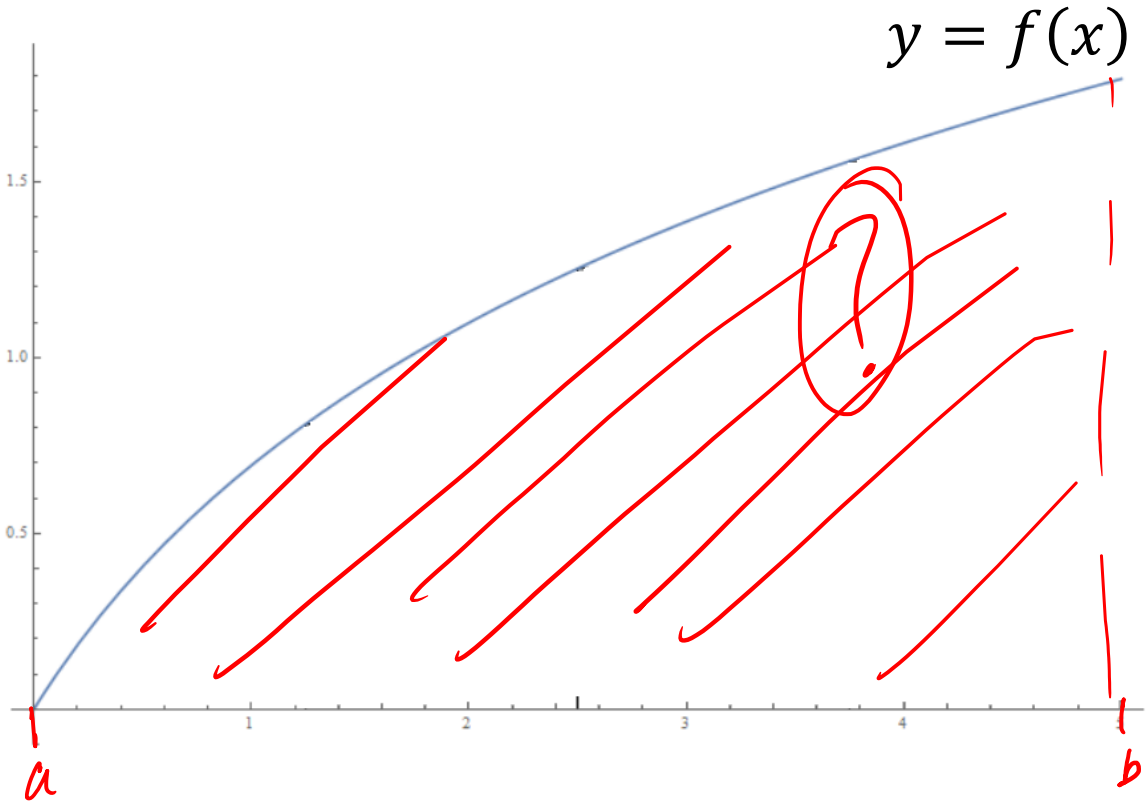
Complex Integration

Area Problem



Complex Integration

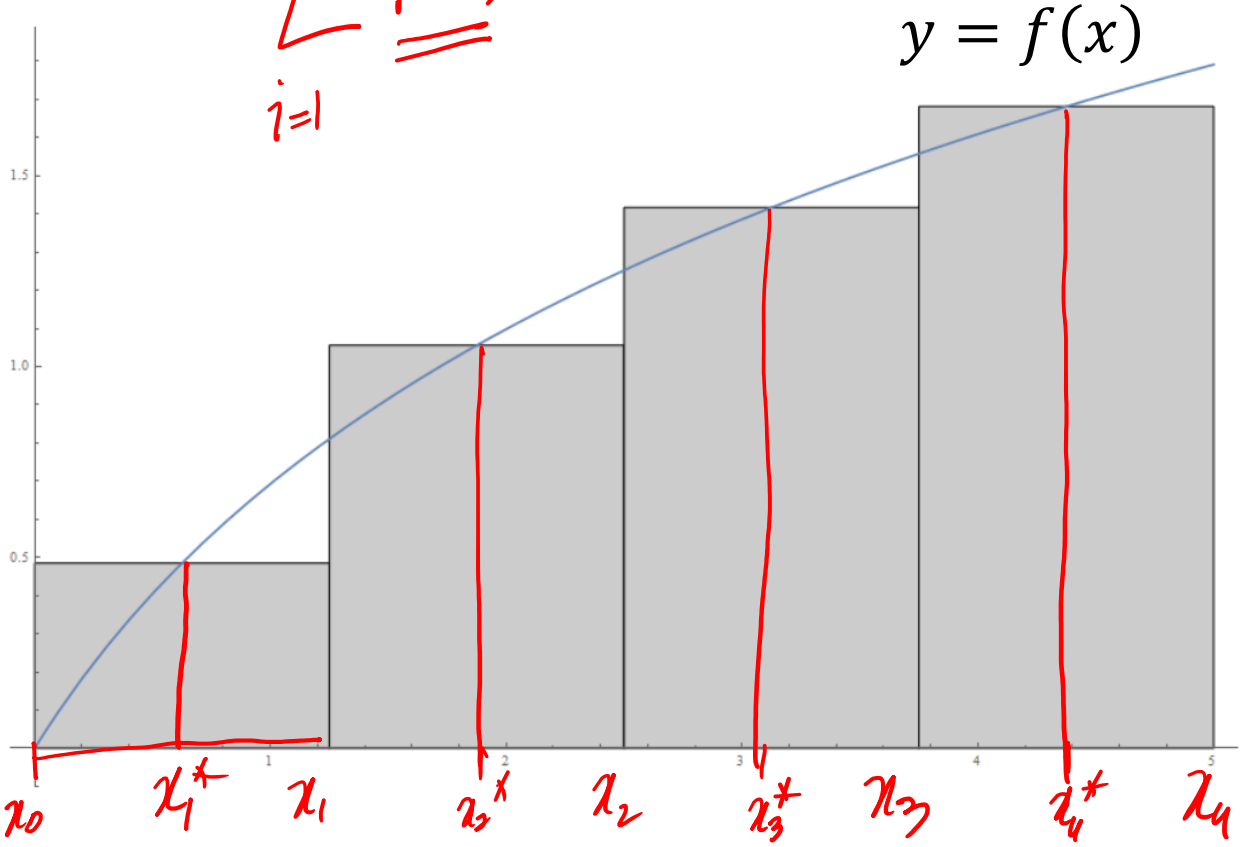
Area Problem: Given a function $y = f(x)$ that is continuous and non-negative on the interval $[a, b]$. Find the area between the graph of $f(x)$ and the interval $[a, b]$ on x-axis.



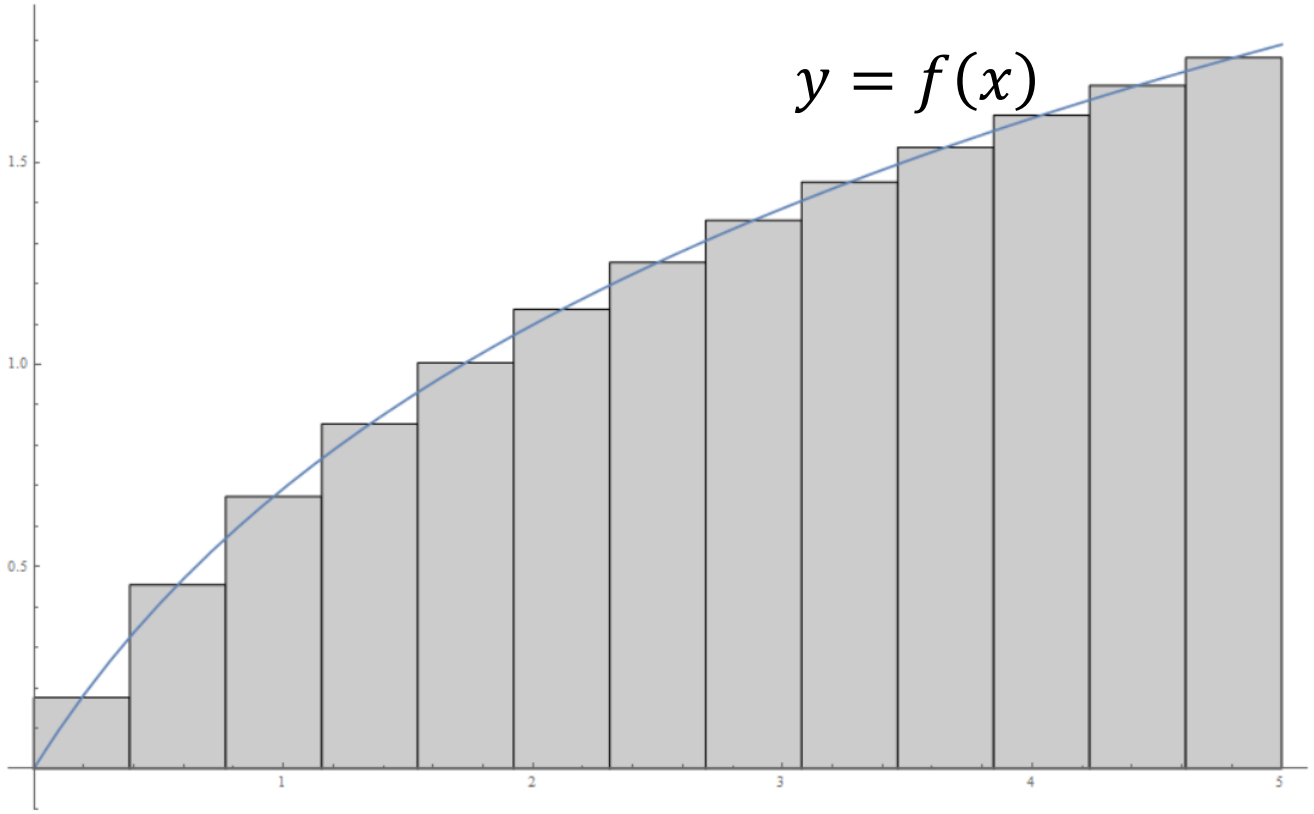
Complex Integration

$$\Delta x = x_{i+1} - x_i$$

$$\sum_{i=1}^{4} f(x_i^*)\,\Delta x$$



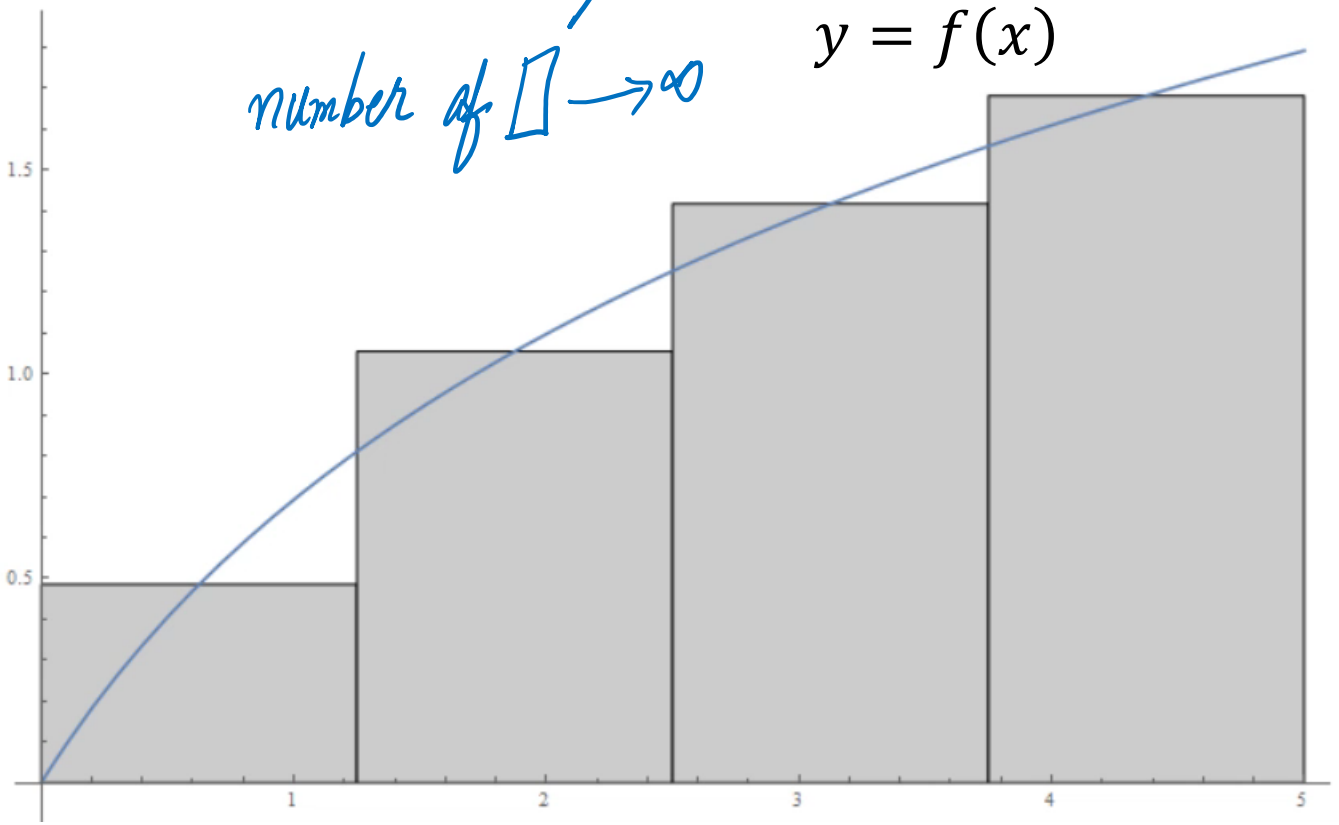
Complex Integration



Complex Integration

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

number of $\square \to \infty$



Complex Integration

Area Problem: Given a function $y = f(x)$ that is continuous and non-negative on the interval $[a, b]$. Find the area between the graph of $f(x)$ and the interval $[a, b]$ on x-axis.

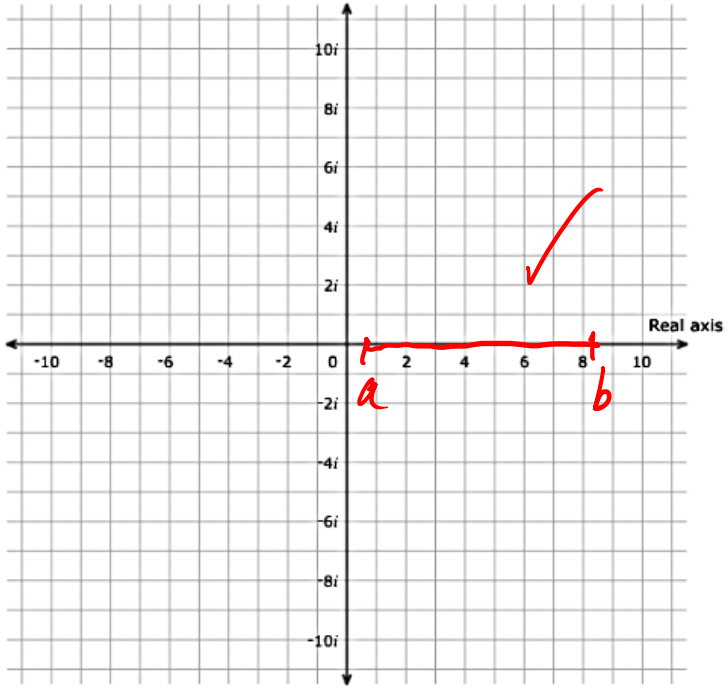
Answer :

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

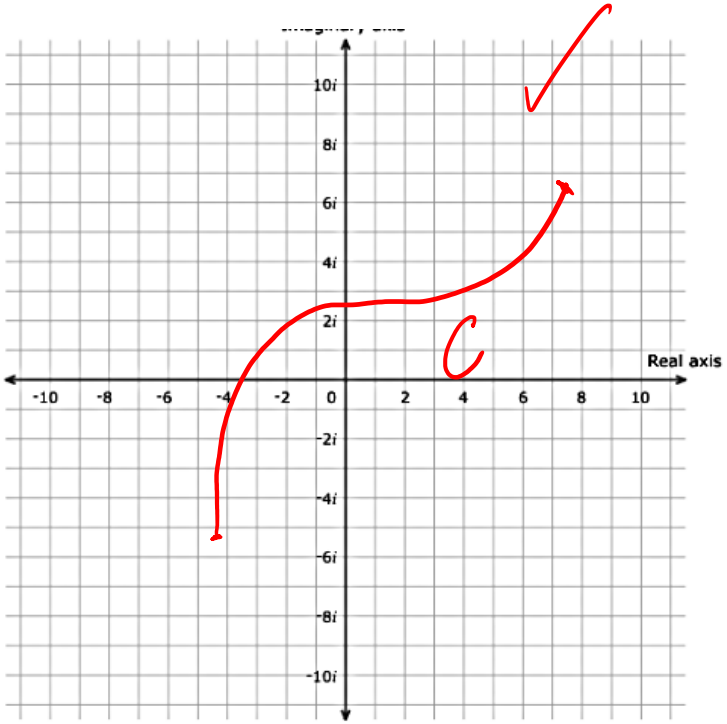
Complex Integration

A planar curve is a two dimensional analogue of the an interval.

Real integrals



Complex integrals

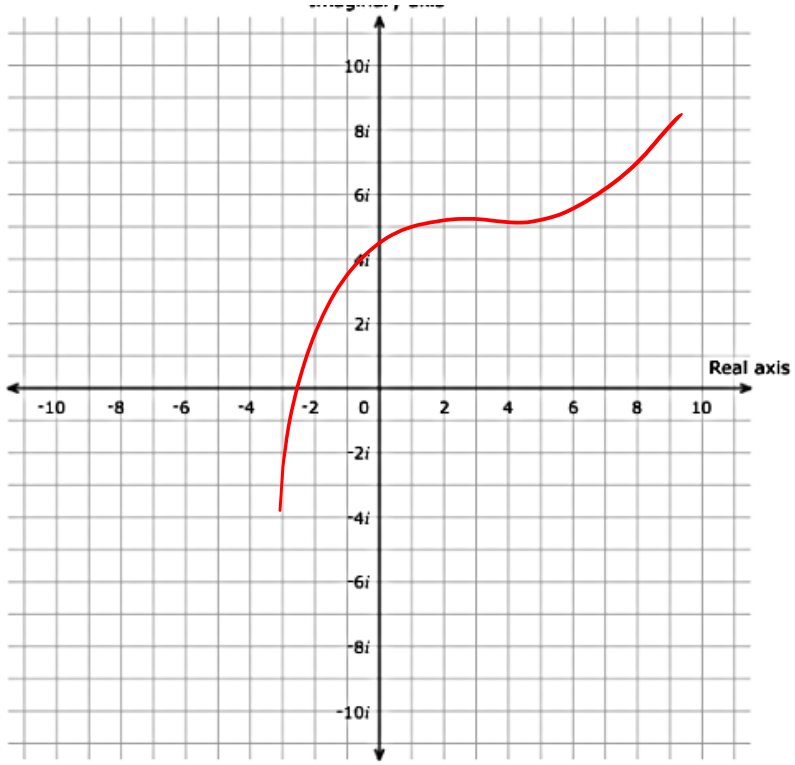


Complex Integration

Aim Define and Evaluate integrals of the form

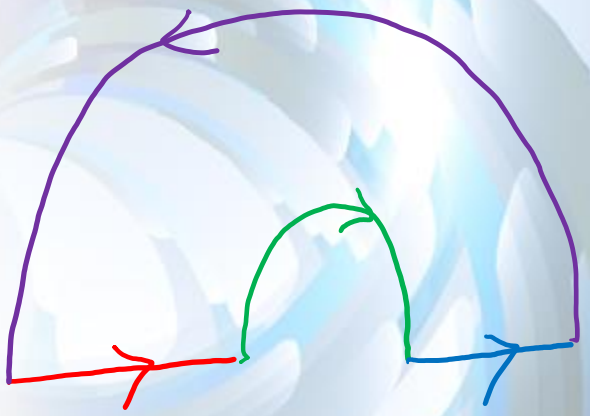
$$\int_C f(z)\,dz\ ,$$

where $f(z)$ is a complex valued function, and C is a contour in the plane.



Complex Integration

Contours



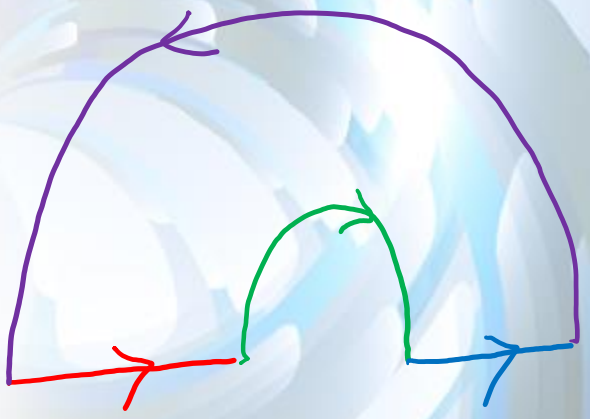
Complex Integration

Complex Integration

$$\int$$

Contours

Contours



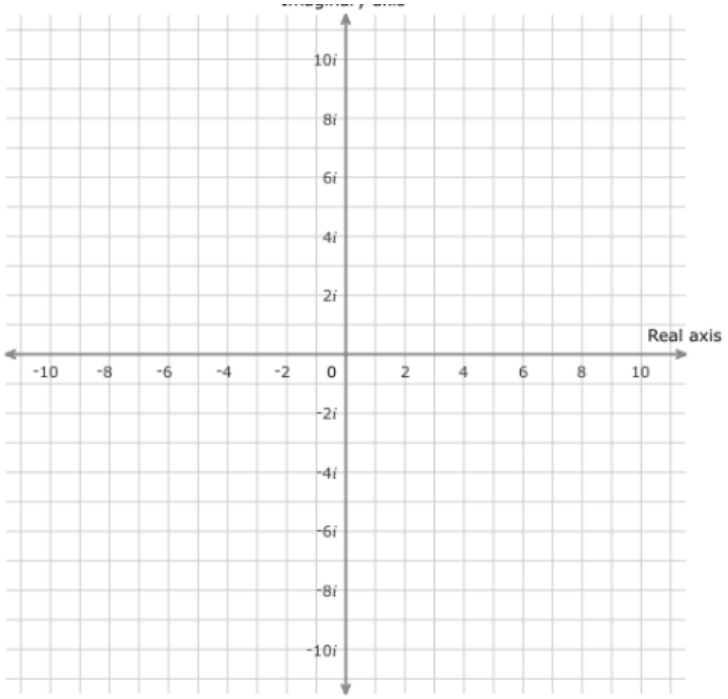
Contours

Aim Define and Evaluate integrals of the form

$$\int_C f(z)\,dz\ ,$$

where $f(z)$ is a complex valued function, and C is a contour in the plane.

?



Contours

Contours

Contours

What is a
curve (in plane)?

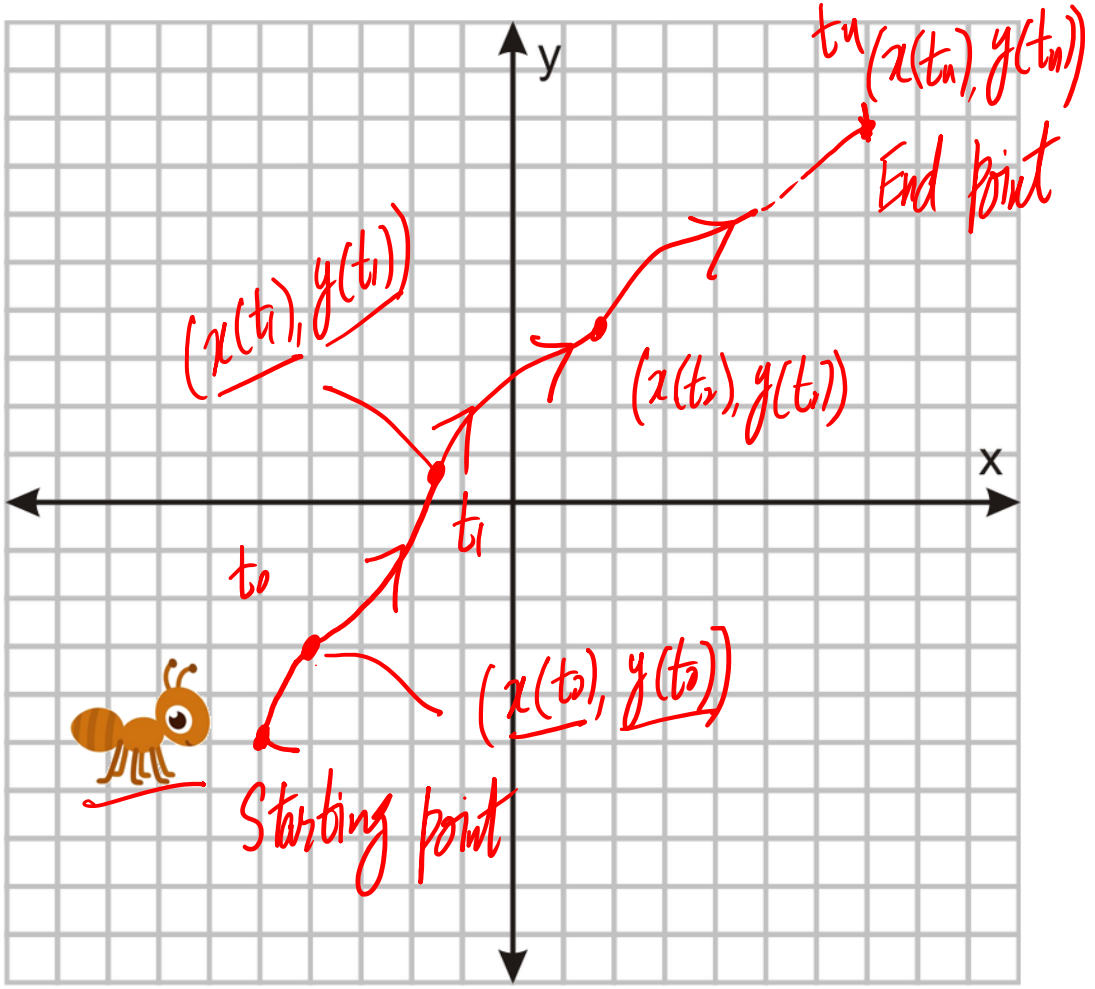
Contours

Consider motion of an ant in plane.

t

$C : (x(t), y(t))$

$0 \le t \le t_n$.



Contours

A curve in the plane is defined as

$$C: z(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

where t is a parameter.

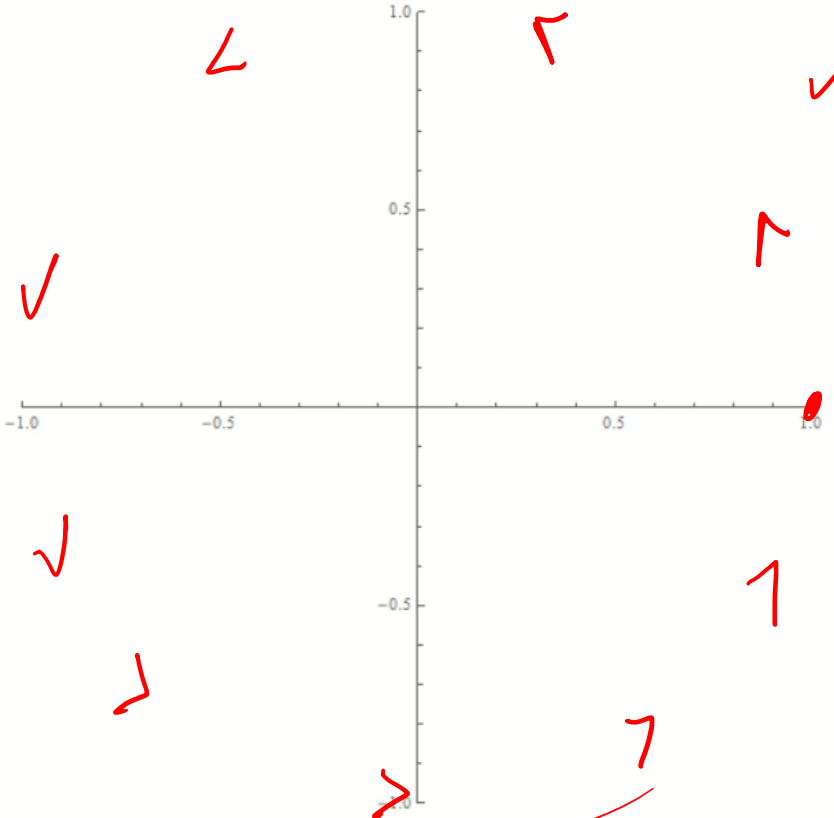
$(x(t), y(t))$

Contours

Example: the parametric representation

$$z(t) = \cos(t) + i\sin(t) \text{ for } 0 \leq t \leq 2\pi$$

represents a circle in the complex plane (observe orientation).



Contours

Example: the parametric representation

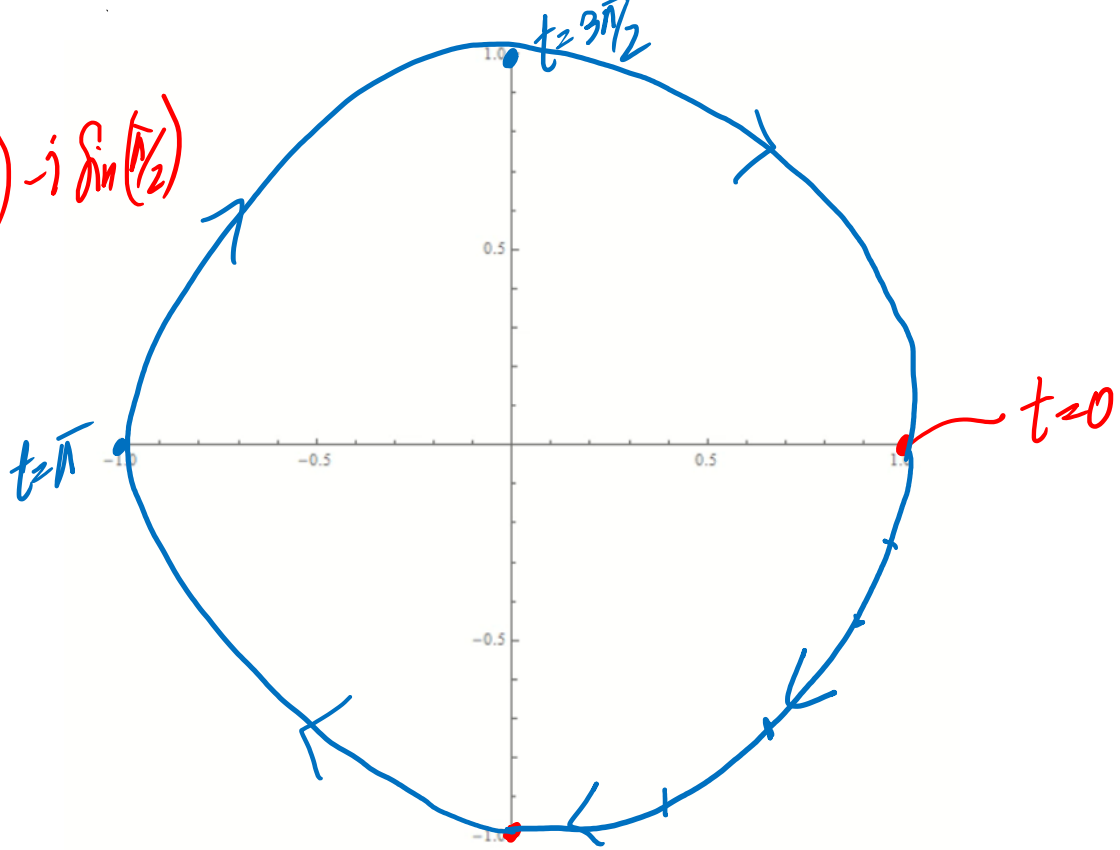
$$z(t) = \cos(t) - i\sin(t) \text{ for } 0 \le t \le 2\pi$$

represents a circle in the complex plane (observe orientation).

$t = \pi/2$

$z(\pi/2) = \cos(\pi/2) - i\sin(\pi/2)$

$= -i$

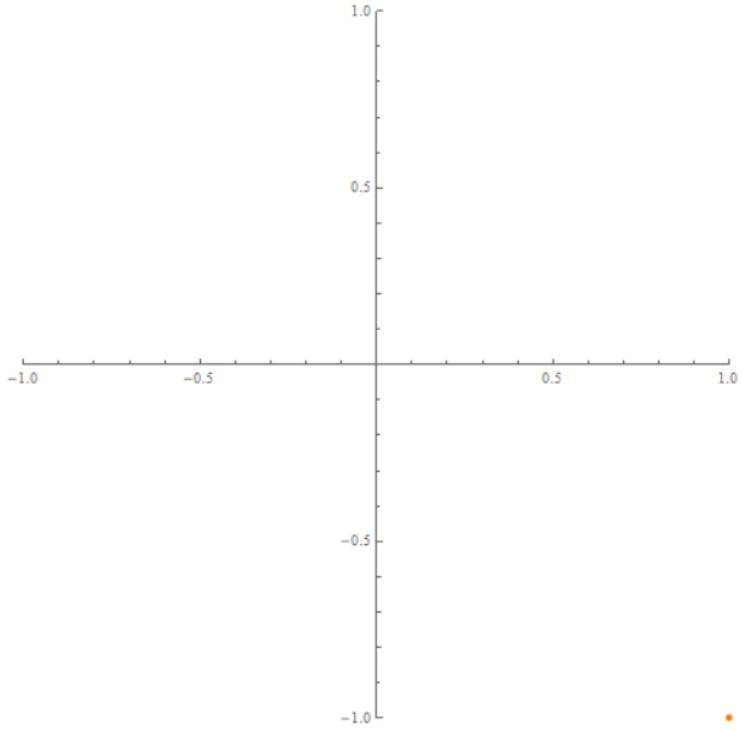


Contours

Example: the parametric representation

$$z(t) = t^2 + it \text{ for } -1 \leq t \leq 1$$

represents a parabola in the complex plane (observe orientation).



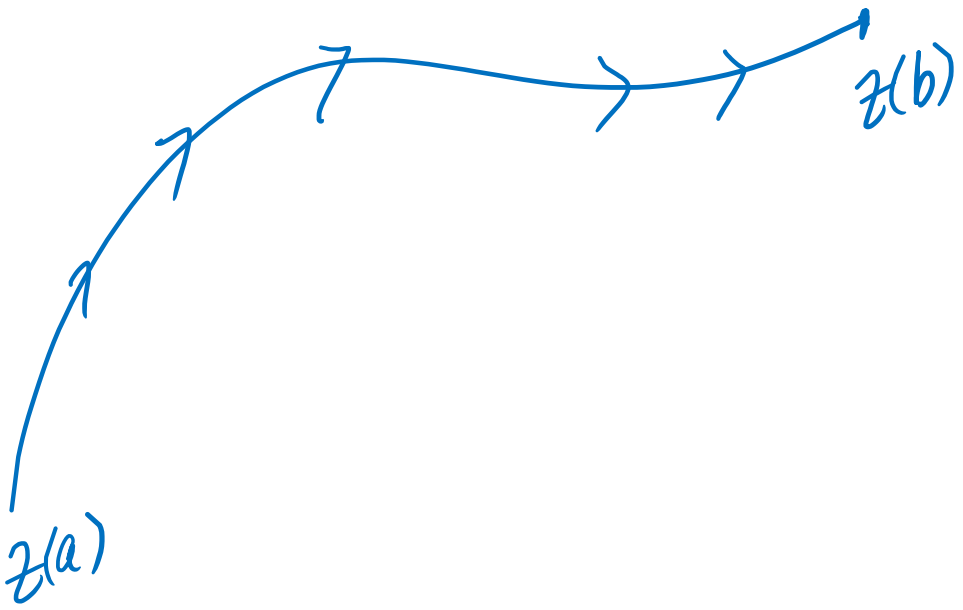
Contours

A curve in the plane is defined as

$$C: z(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

where t is a parameter.

There is orientation implicitly given in the definition.



Contours

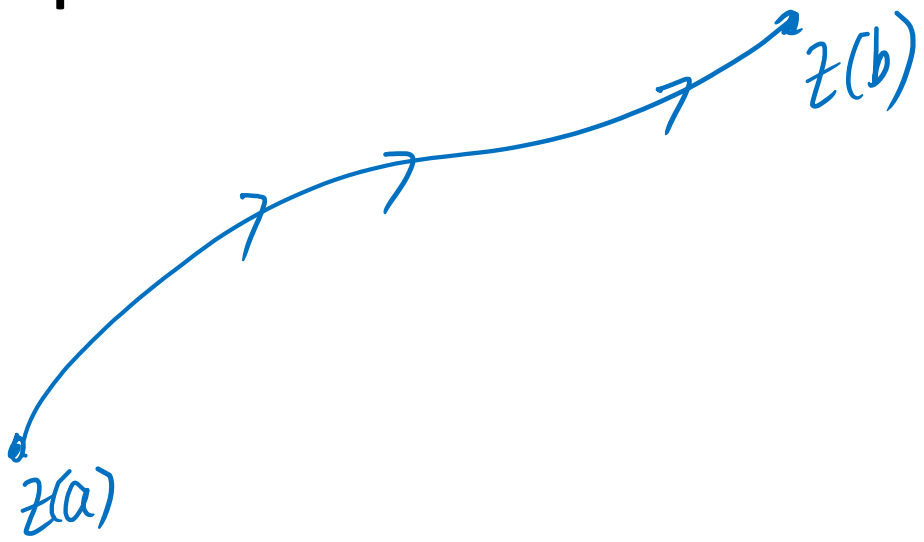
A curve in the plane is defined as

$$C: z(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

where t is a parameter.

$z(a)$ is called the **initial**.

$z(b)$ is called **terminal point**.



Contours

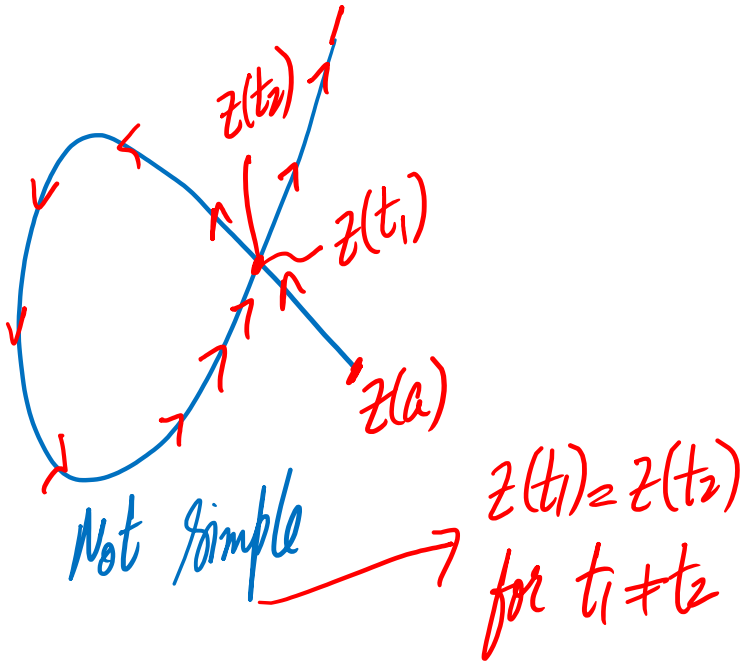
Simple curve: A curve

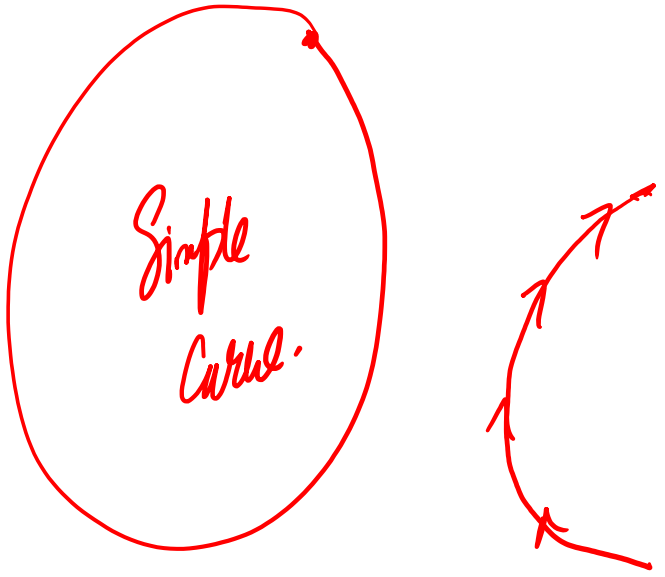
$$C: z(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

is simple if it does not cross itself, that is,

$$z(t_1) \neq z(t_2), \text{ whenever } t_1 \neq t_2$$

except possibly when $t_1 = a$ and $t_2 = b$.





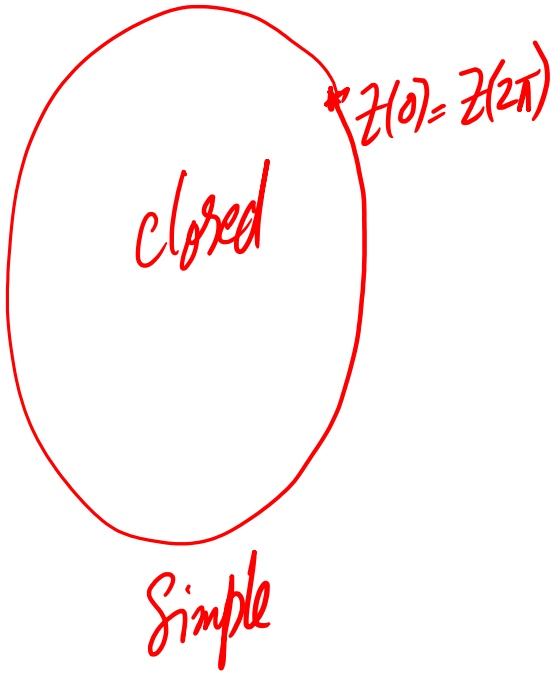
Contours

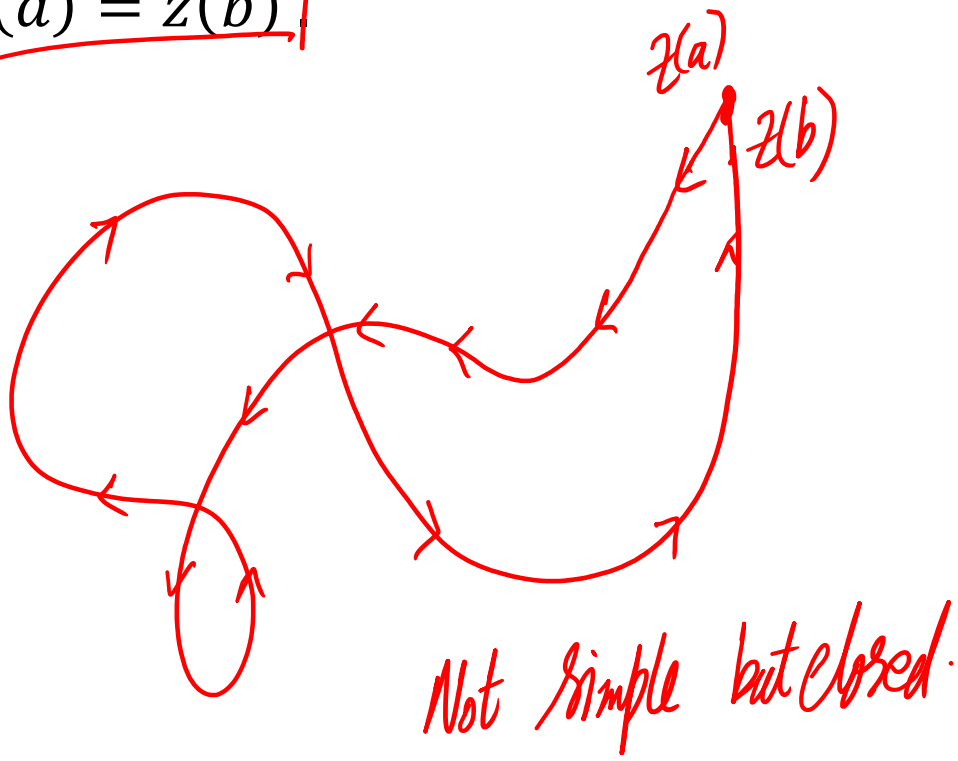
Closed curve: A curve

$$C: z(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

is closed if

$$z(a) = z(b)$$





Contours

Simple closed curve or Jordan curve: A curve

$$C: z(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

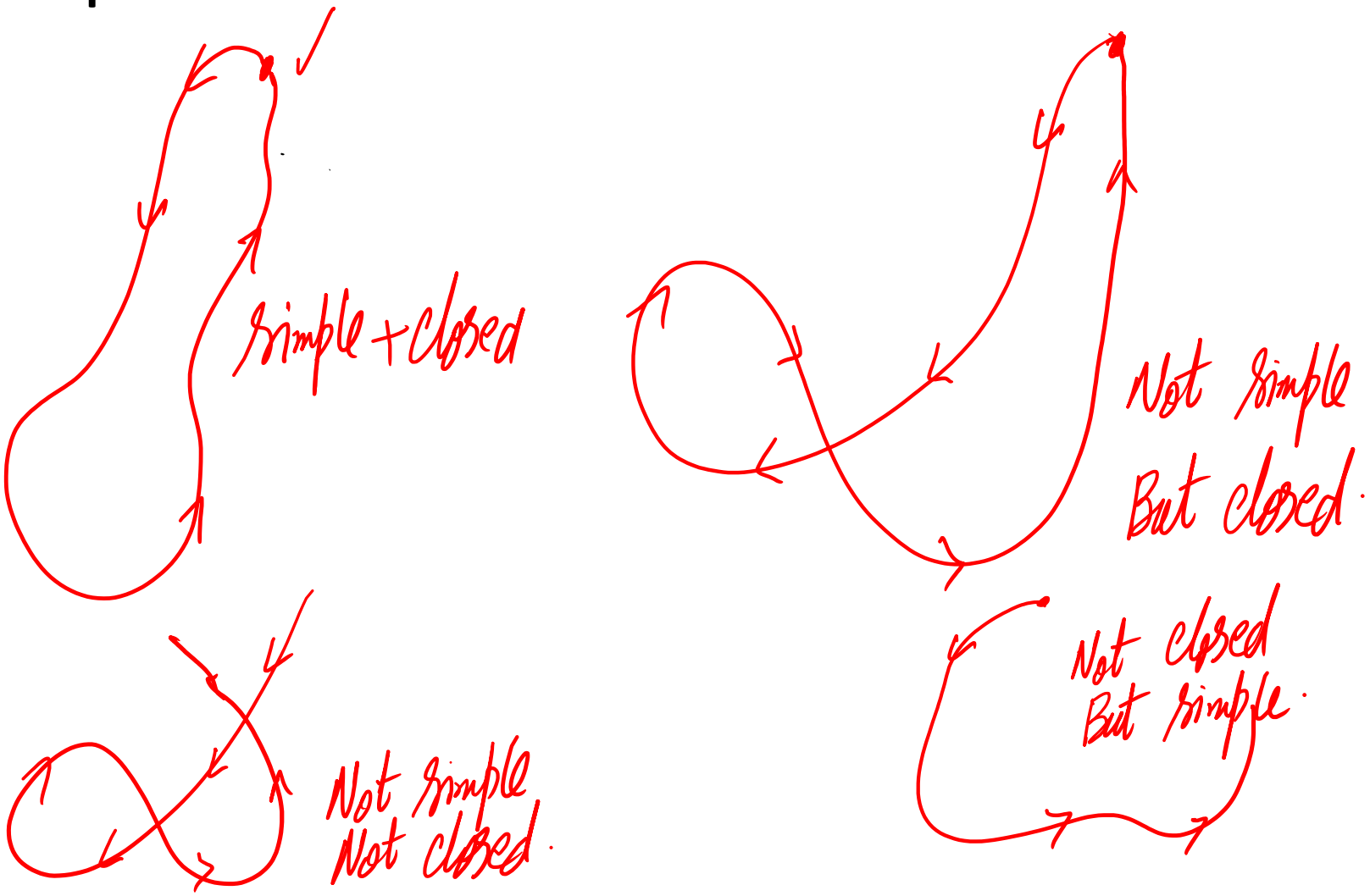
is simple if it does not cross itself, that is,

$$z(t_1) \neq z(t_2), \text{ whenever } t_1 \neq t_2$$

and $z(a) = z(b)$.

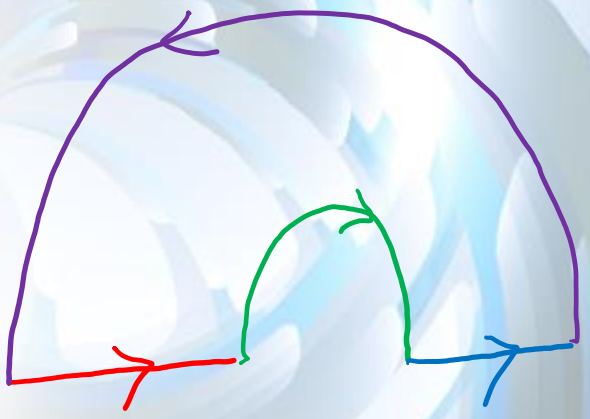
Contours

Examples



Contours

Contours



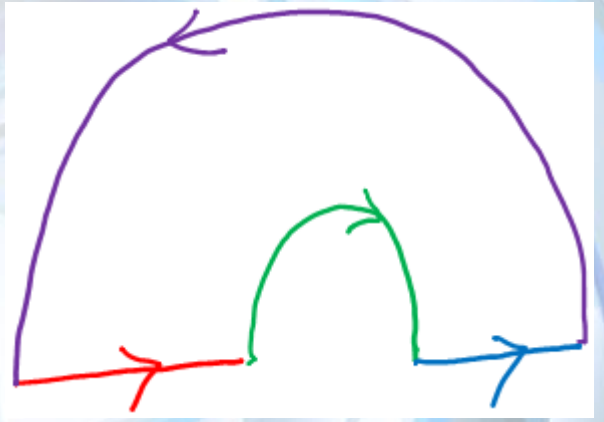
Complex Integration

Complex Integration

$$\int$$

Complex Integration

Contours

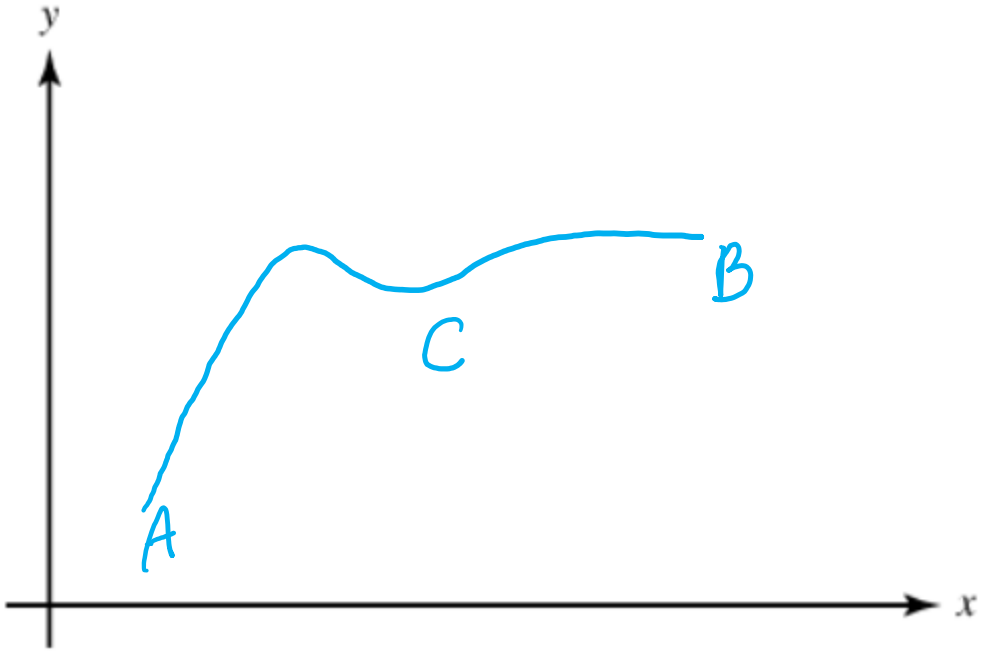


Complex Integration

Aim Define and Evaluate integrals of the form

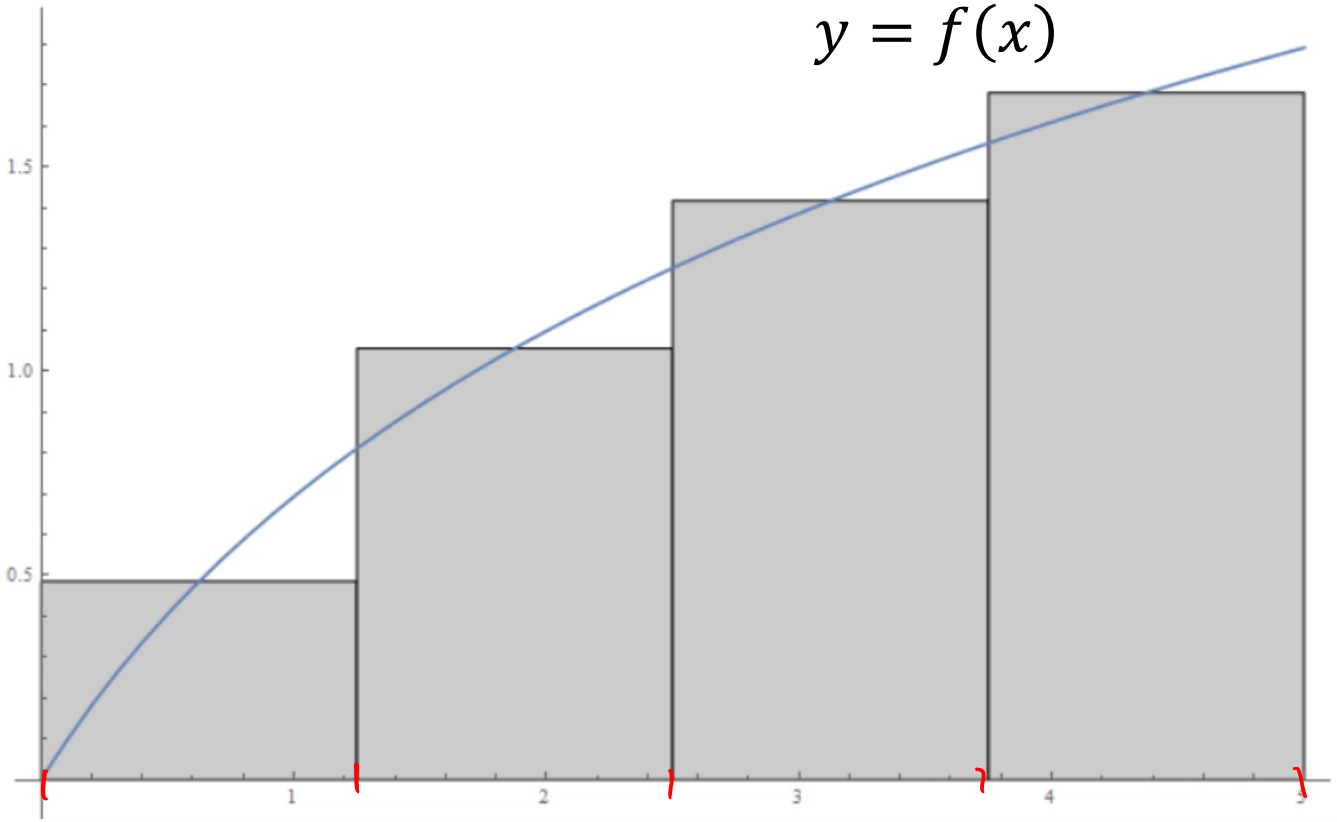
$$\int_C f(z)\,dz\ ,$$

where $f(z)$ is a complex valued function, and C is a contour in the plane from point A to point B.



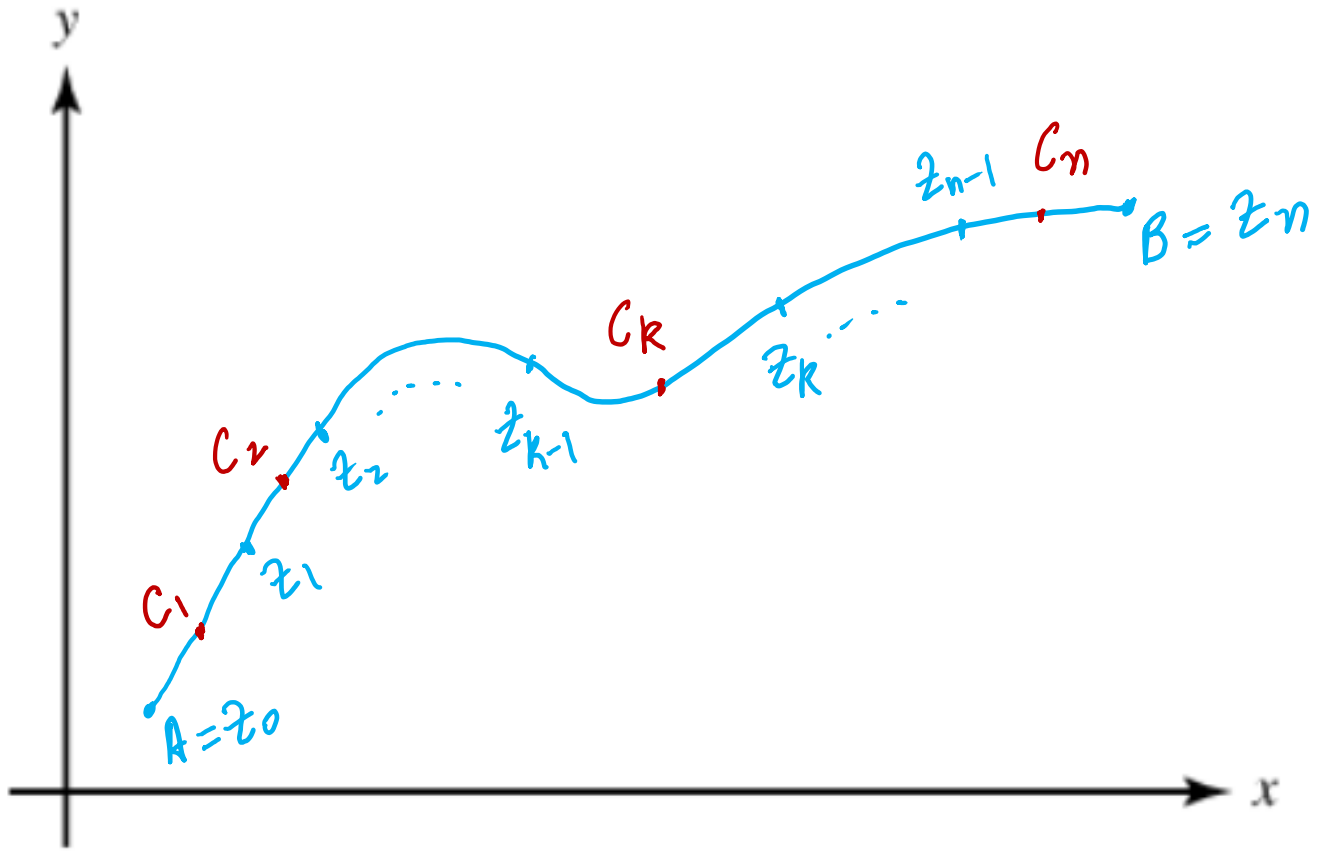
Complex Integration

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$



Complex Integration

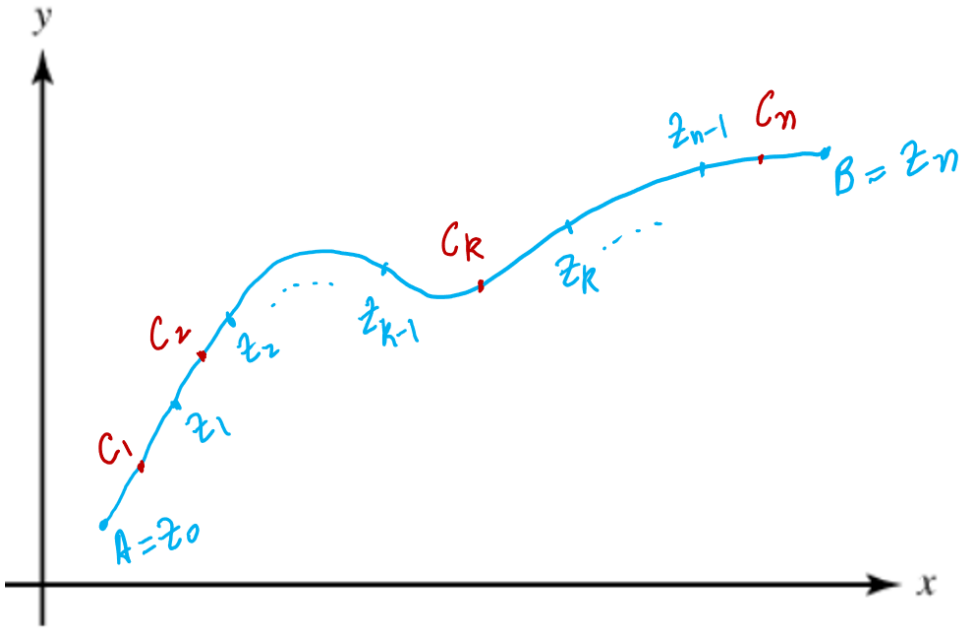
Divide C



Complex Integration

Hence we a partition

$$P_n = \{z_0 = A, z_1, z_2, \dots, z_n = B\}$$



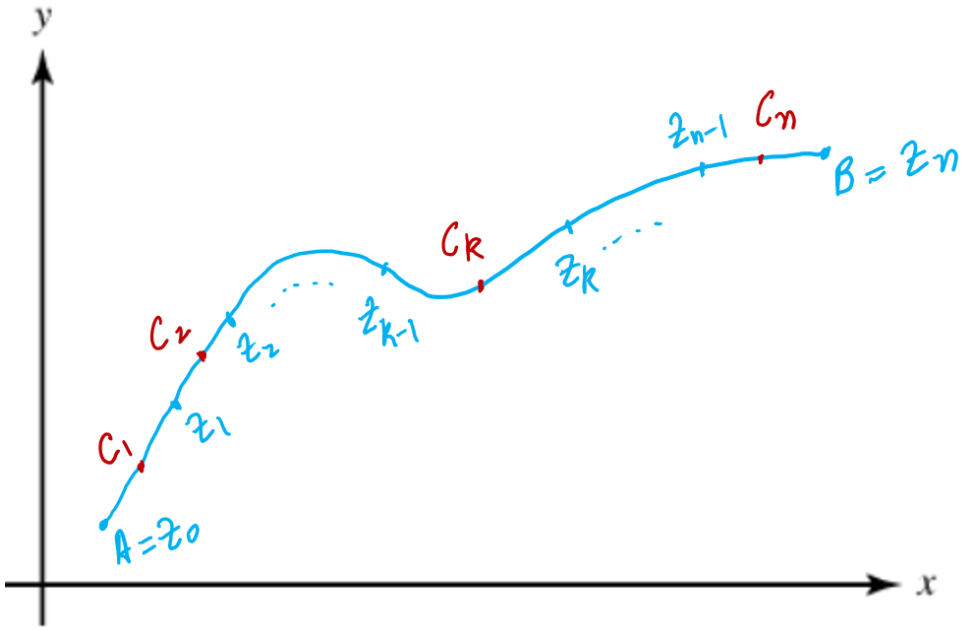
Complex Integration

Hence we a partition

$$P_n = \{z_0 = A, z_1, z_2, \dots, z_n = B\}$$

with differences

$$\Delta z_k = z_k - z_{k-1}\,, for\ k = 1,2, \dots, n$$



Complex Integration

Hence we a partition

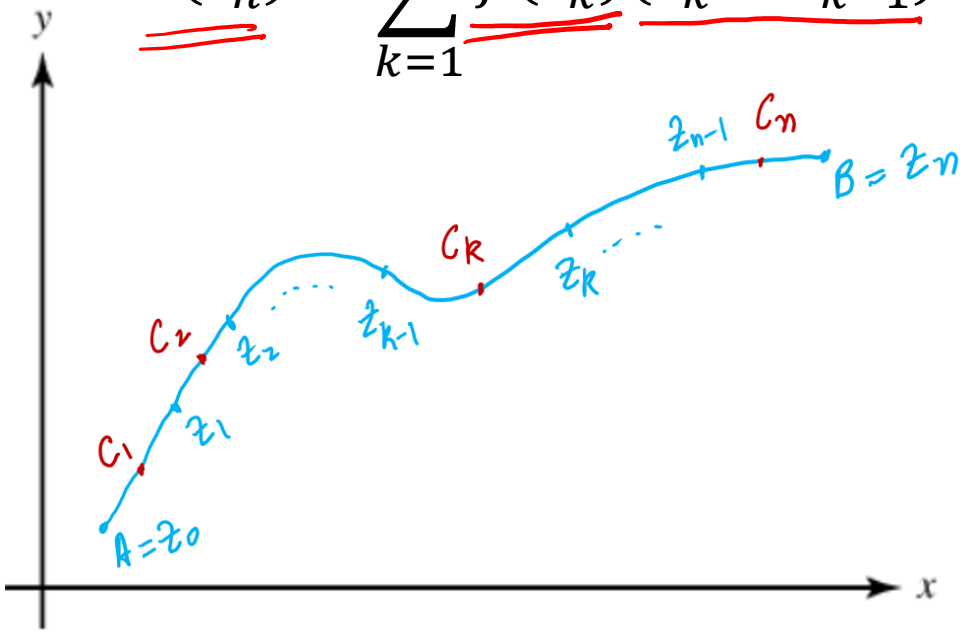
$$P_n = \{z_0 = A, z_1, z_2, \dots, z_n = B\}$$

with differences

$$\Delta z_k = z_k - z_{k-1}\,, for\ k = 1,2, \dots, n$$

and the Riemann sum corresponding to partition P_n is

$$S(P_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\Delta z_k$$

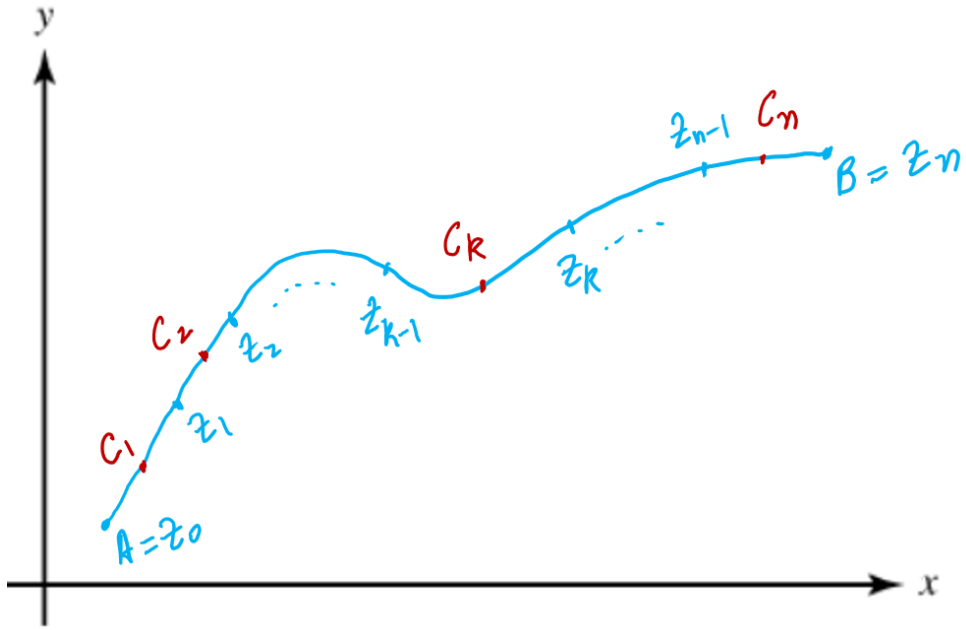


Complex Integration

$$S(P_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\Delta z_k$$

$S(P_n)$ depends on n so it forms a sequence $\{S(P_n)\}_{n=1}^{\infty}$.

$S(P_1), S(P_2), S(P_3), \cdots S(P_n), \cdots$



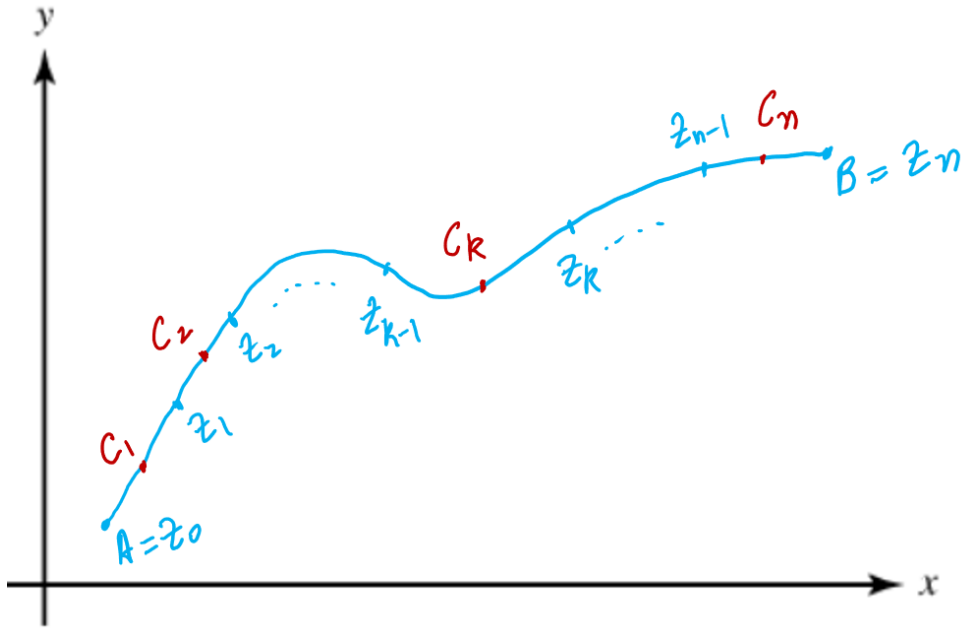
Complex Integration

$$S(P_n) = \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(c_k)\Delta z_k$$

$S(P_n)$ depends on n so it forms a sequence $\{S(P_n)\}_{n=1}^{\infty}$.

Assume the sequence converges with limit L, that is,

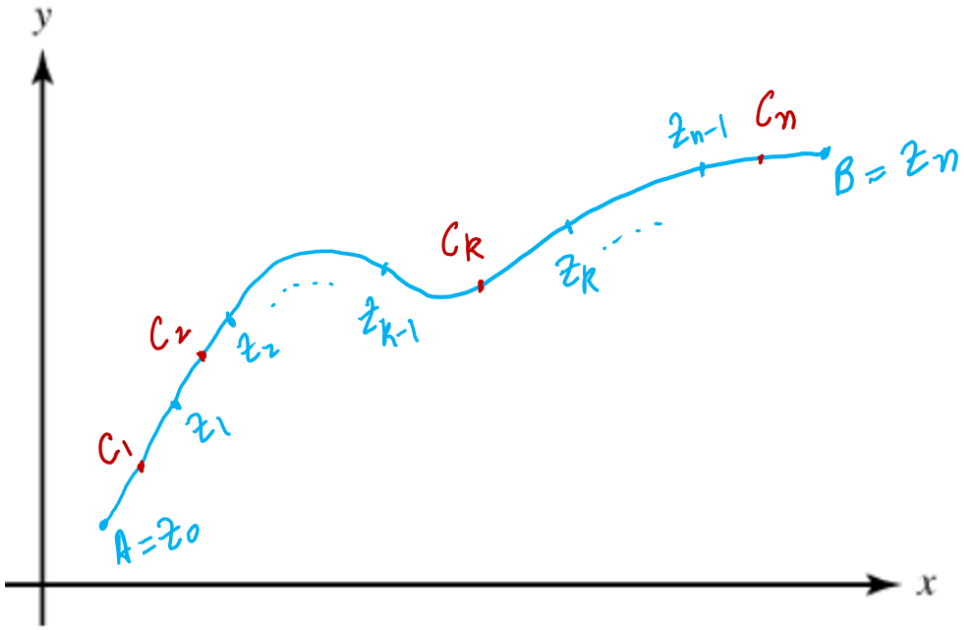
$$L = \lim_{n\to\infty} S(P_n)$$



Complex Integration

The value L depends on the partition chosen

$$L = \lim_{n \to \infty} S(P_n)$$



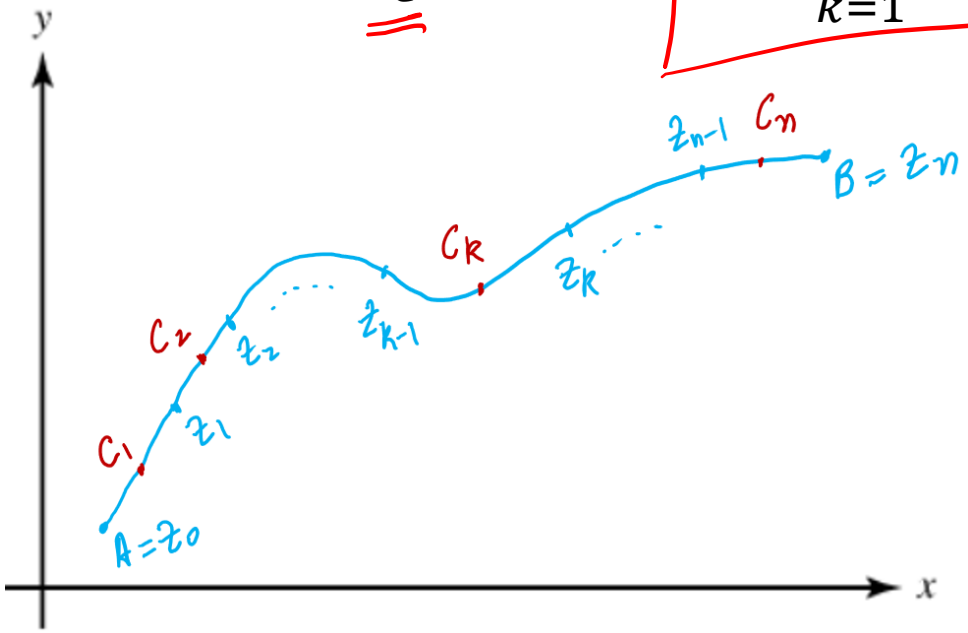
Complex Integration

The value L depends on the partition chosen

$$L = \lim_{n\to\infty} S(P_n)$$

If we get the same limit L for every partition of the contour then we say that

$$\int_C f(z)dt = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta z_k$$

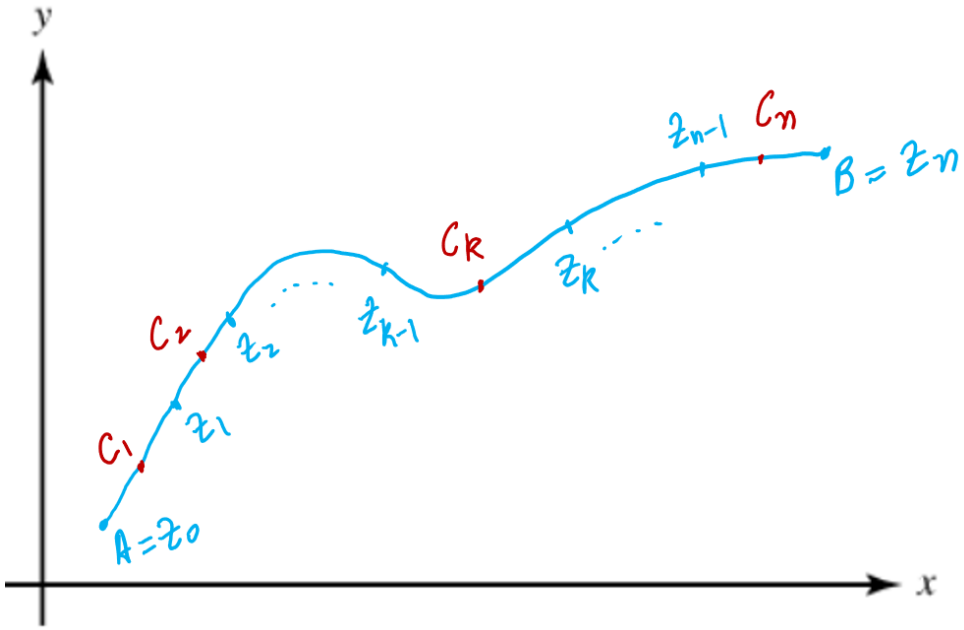


Complex Integration

Definition (Complex Integral)

Let $f(z)$ be a complex valued function and C be a contour then

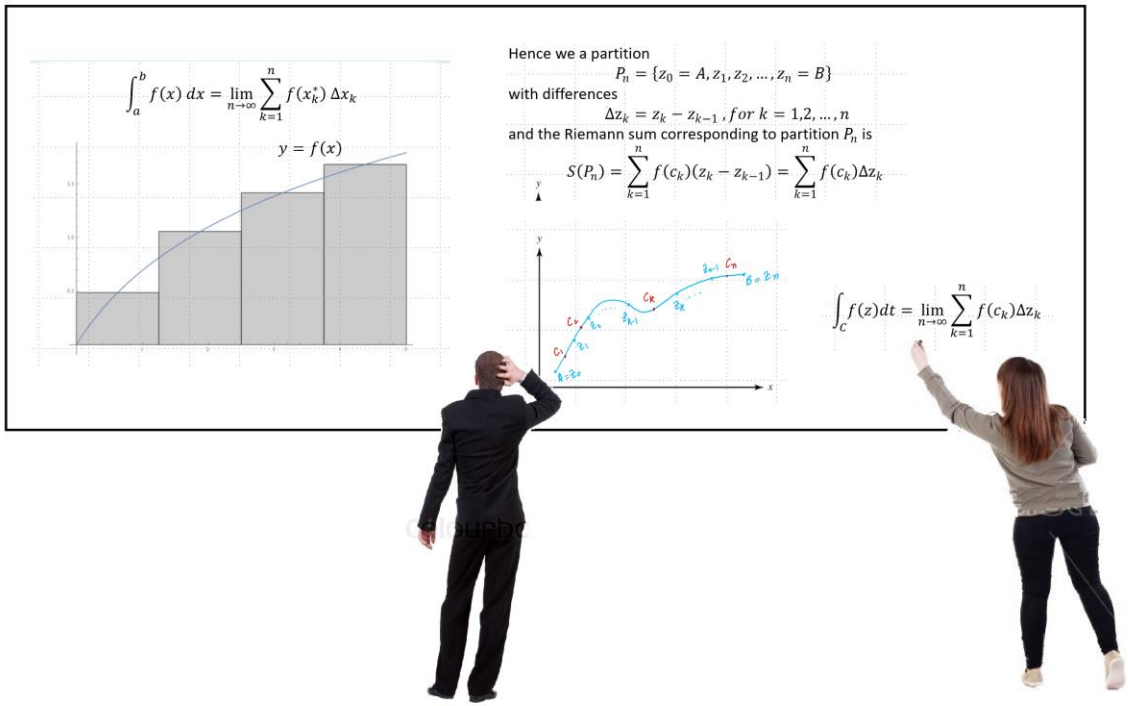
$$\int_C f(z)dt = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta z_k$$



Complex Integration

Problem: The expression is too complicated to be calculated

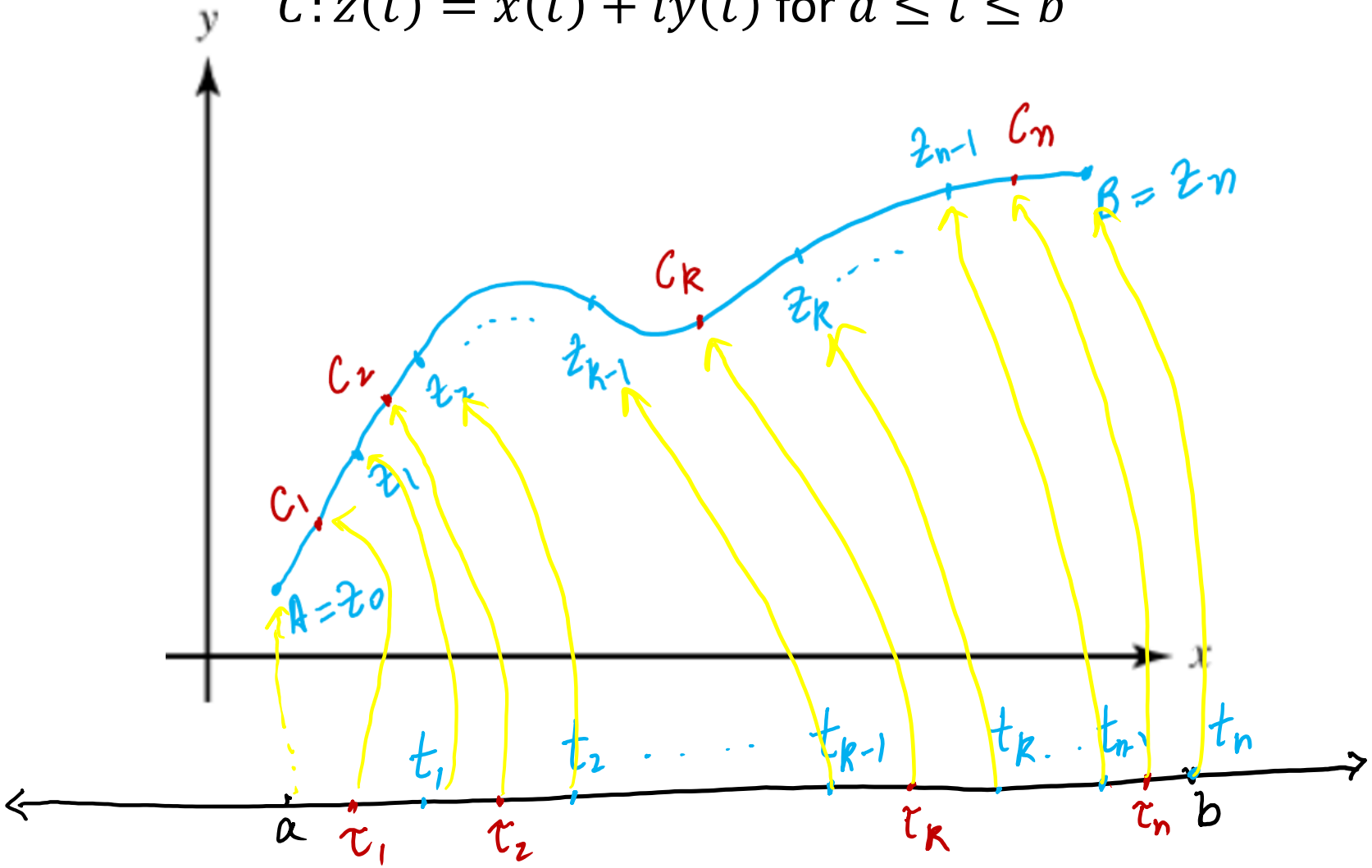
$$\int_C f(z)dt = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta z_k$$



Complex Integration

The contour C is defined as

$$C: z(t) = x(t) + iy(t) \text{ for } a \le t \le b$$



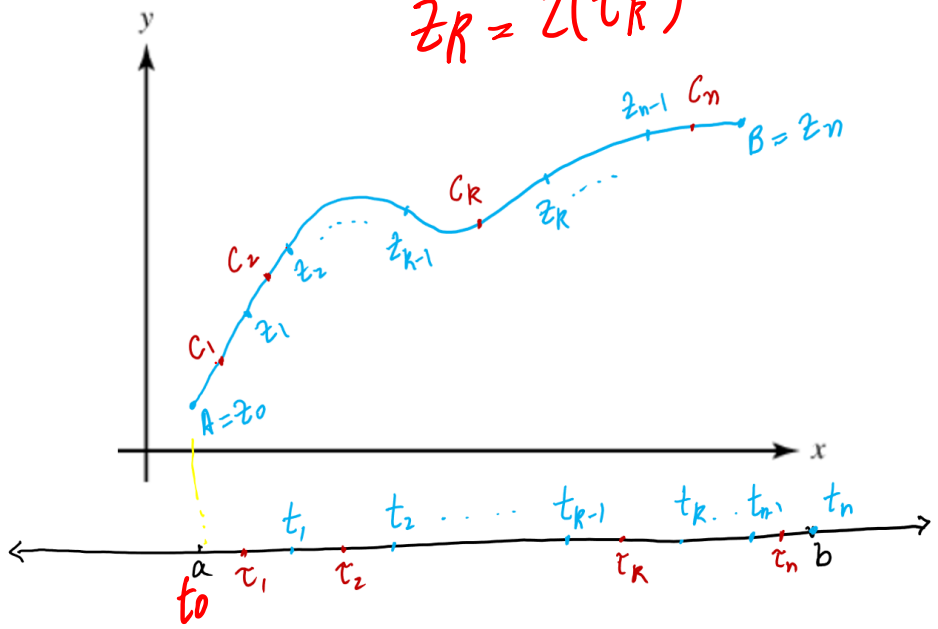
Complex Integration

$$\int_C f(z)\,dz = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta z_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)(z_k - z_{k-1})$$

$c_k = z(\tau_k)$

$z_k = z(t_k)$

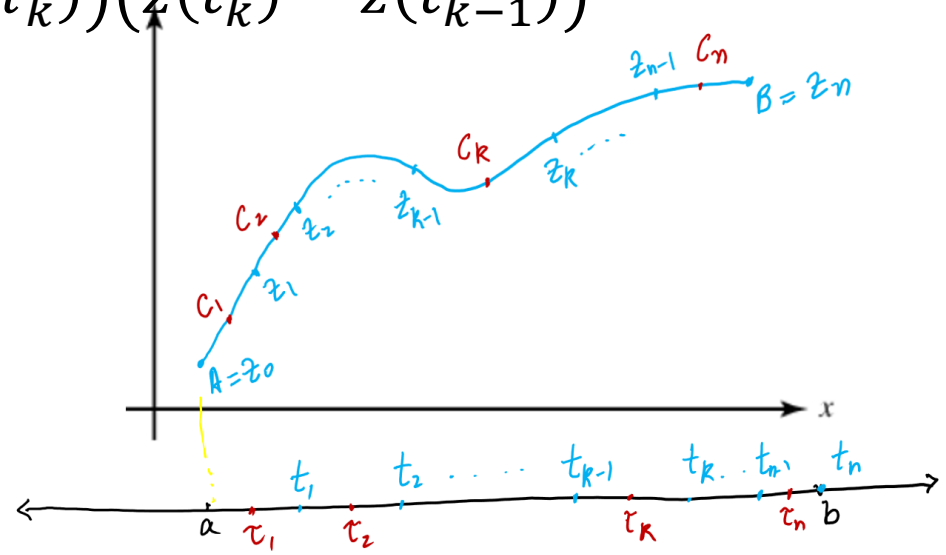


Complex Integration

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta z_k$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)(z_k - z_{k-1})$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f\big(z(\tau_k)\big)\big(z(t_k) - z(t_{k-1})\big)$$



Complex Integration

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta z_k$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)(z_k - z_{k-1})$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f\big(z(\tau_k)\big)\big(z(t_k) - z(t_{k-1})\big)$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f\big(z(\tau_k)\big)\left(\frac{z(t_k) - z(t_{k-1})}{t_k - t_{k-1}}\right)(t_k - t_{k-1})$$

$\lim_{t_{k-1}\to t_k} \dfrac{z(t_k)-z(t_{k-1})}{t_k - t_{k-1}} = z'(t)$

Complex Integration

$$\lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)\Delta z_k$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f(c_k)(z_k - z_{k-1})$$

$$= \lim_{n\to\infty}\sum_{k=1}^{n} f\big(z(\tau_k)\big)\big(z(t_k) - z(t_{k-1})\big)$$

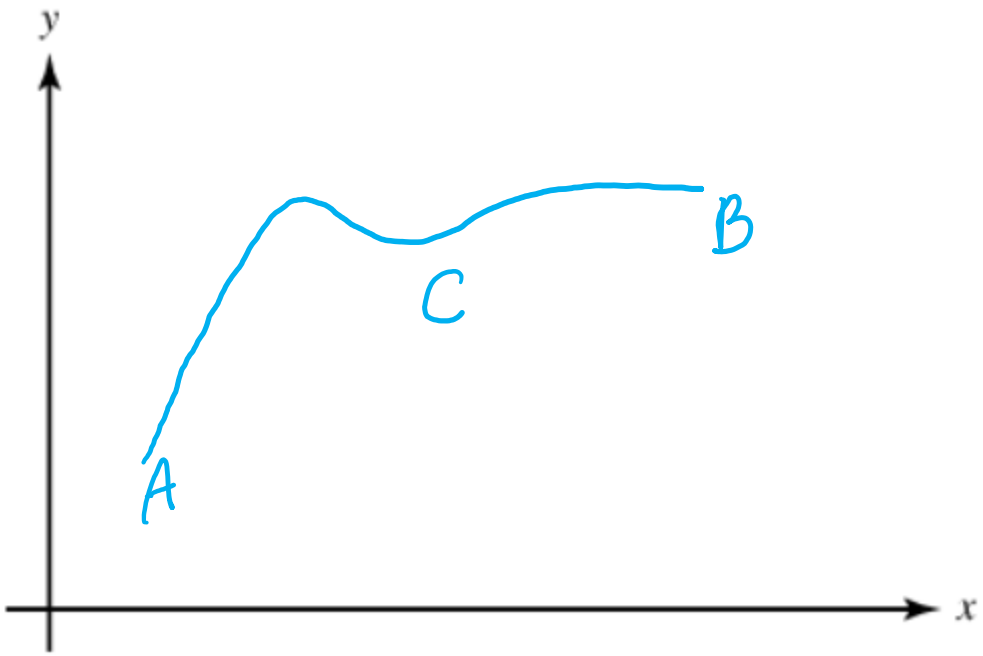
$$= \lim_{n\to\infty}\sum_{k=1}^{n} f\big(z(\tau_k)\big)\left(\frac{z(t_k) - z(t_{k-1})}{t_k - t_{k-1}}\right)(t_k - t_{k-1})$$

$$= \int_a^b f\big(z(t)\big)\, z'(t)\, dt$$

Complex Integration

Theorem: Suppose that $f(z)$ is a continuous complex-valued function defined on a set containing the contour C. Let $z(t)$ be any parametrization of C for $a \le t \le b$ then

$$\int_C f(z)\,dz = \int_a^b f\big(z(t)\big)\,z'(t)\,dt$$



Complex Integration

Theorem: Suppose that $f(z)$ is a continuous complex-valued function defined on a set containing the contour C. Let $z(t)$ be any parametrization of C for $a \leq t \leq b$ then

$$\int_C f(z) = \int_a^b f\big(z(t)\big)\, z'(t)\, dt$$

.

Complex Integration

Theorem: Suppose that $f(z)$ is a continuous complex-valued function defined on a set containing the contour C. Let $z(t)$ be any parametrization of C for $a \leq t \leq b$ then

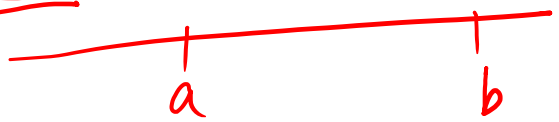
$$\int_C f(z) = \int_a^b f\big(z(t)\big)\, z'(t)\, dt$$

Note: we are allowed to choose any parametrization of the contour as we want.

Complex Integration

Theorem: Suppose that $f(z)$ is a continuous complex-valued function defined on a set containing the contour C. Let $z(t)$ be any parametrization of C for $a \leq t \leq b$ then

$$\int_C f(z) = \int_a^b f\big(z(t)\big)\, z'(t)\, dt$$



Note: we are allowed to choose any parametrization of the contour as we want.

The above integral involves integrating complex functions over real intervals.

Complex Integration

Next Task

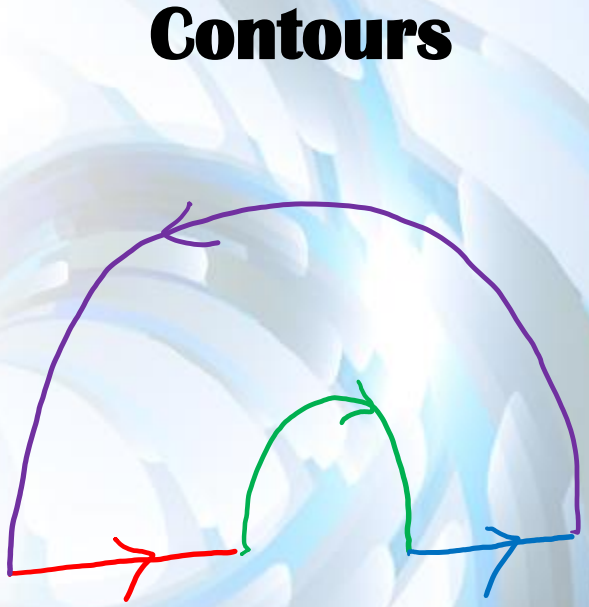
Understand integration of complex functions on real intervals

Complex Integration

Complex Integration

$$\int$$

Contours



Contours

Module # 87

Contours

Contours

A curve in the plane is defined as

$$C: z(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

where t is a parameter.

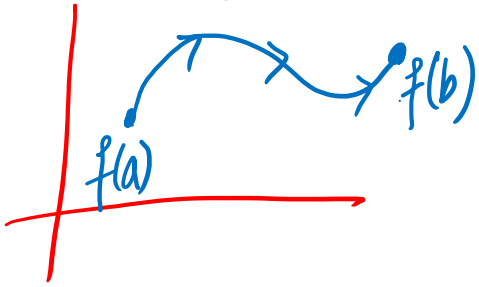
real-valued functions

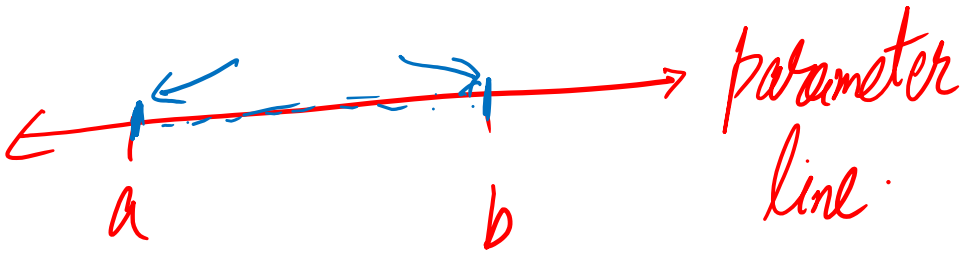
Contours

Differentiability

The complex-valued function $f(t) = u(t) + iv(t)$ is said to be differentiable $[a, b]$ if both $x(t)$ and $y(t)$ both are differentiable on $[a, b]$.

Here we require one-sided differentiability at the end points $x = a$ and $x = b$.





Contours

Example: Check differentiability of the curve C, where

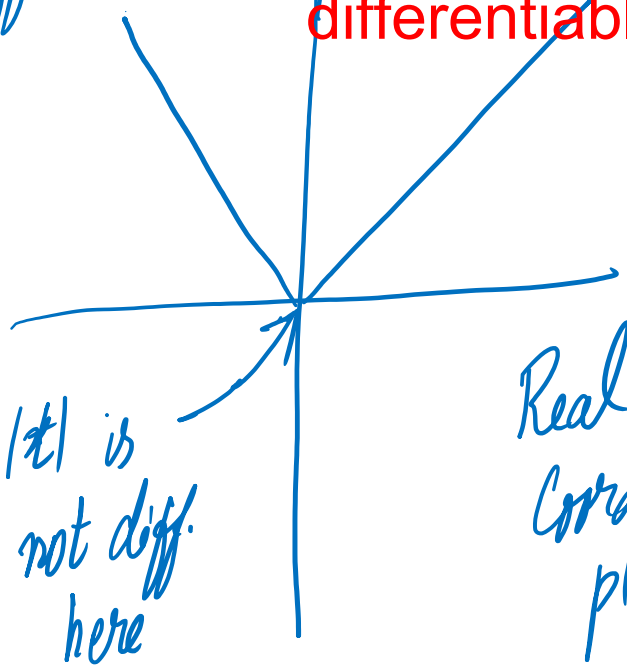
$$C: z(t) = |t| + i\,t^2 \text{ for } -1 \le t \le 1$$

$x(t)$ $y(t)$

$y(t) = t^2$ is always differentiable.

y(t) is a polynomial so all polymonials are differentiable

$x(t) = |t|$



$|t|$ is not diff. here

Real Coordinate plane

$\Rightarrow z(t)$ is not differentiable in $[-1, 1]$

Contours

Example: Check differentiability of the curve C, where

$$C: z(t) = \cos(t) + i\sin(t) \text{ for } 0 \le t \le 2\pi$$

$x(t) = \cos t$ is always differentiable $[0, 2\pi]$

$y(t) = \sin(t)$ " " $[0, 2\pi]$

Contours

Smooth curve: A curve

$$C: z(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

is said to be smooth curve if $z'(t)$ is continuous and nonzero on the interval. $[a,b]$

agar derivative hold kary to continous ho ga

$\rightarrow z'(t)$ is Cont.

$\rightarrow z'(t) \neq 0, \quad \forall\ t \in [a,b]$

Contours

Example: Check whether C is a smooth curve, where C is given by

$$C: z(t) = t^2 + it^3 \text{ for } -1 \le t \le 1.$$

first property the derivative exist

① ✓ $z'(t) = 2t + i(3t^2) \Rightarrow z'(t)$ is cont.

2nd property f'(t) not= 0

polynomials.

② ✗ $z'(t) \neq 0, \forall\, t \in [-1,1]$

$z'(0) = 2(0) + i\,3(0)^2 = 0$

$\Rightarrow z(t)$ is not smooth

Contours

Contour: A curve C is said to be a contour if it is constructed by joining finitely many smooth curves end to end.

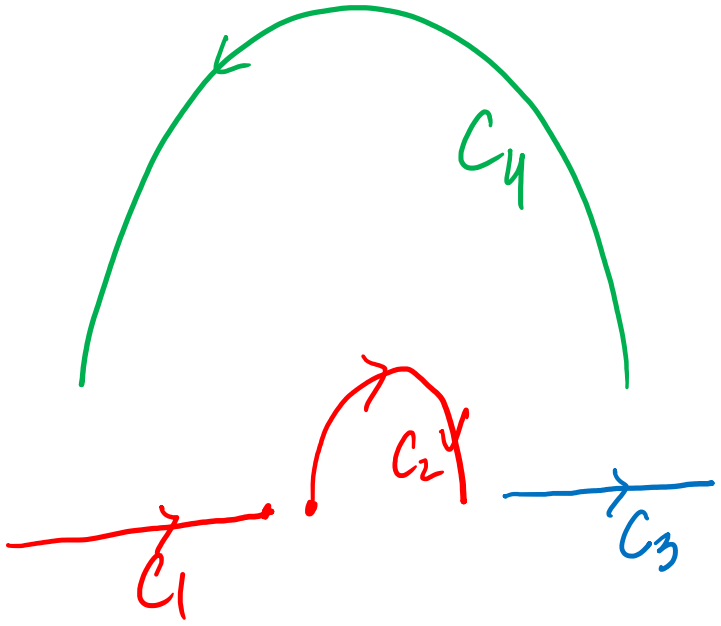
Contours

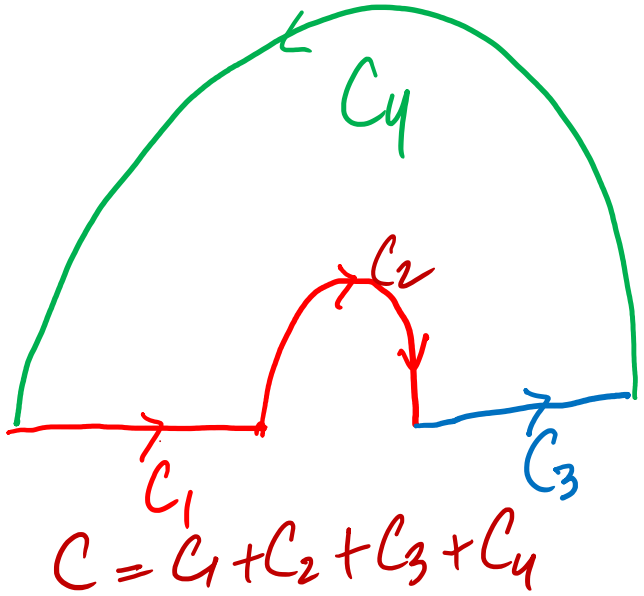
MCQ

Contour: A curve C is said to be a contour if it is constructed by joining finitely many smooth curves end to end.

We express a contour obtained by joining sequence of curves $C_1, C_2, \ldots, C_n$ in the following way

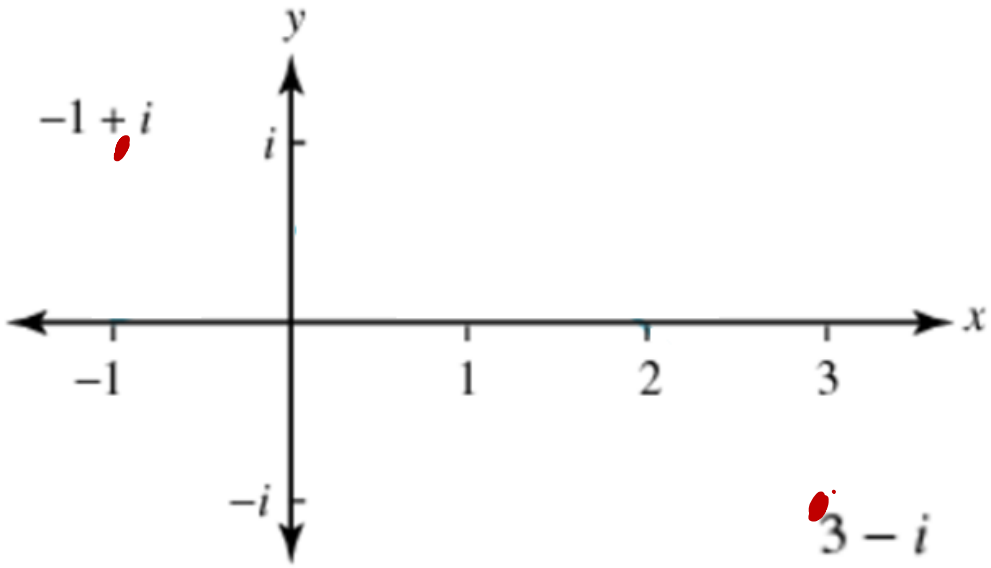
$$C = C_1 + C_2 + \cdots + C_n$$





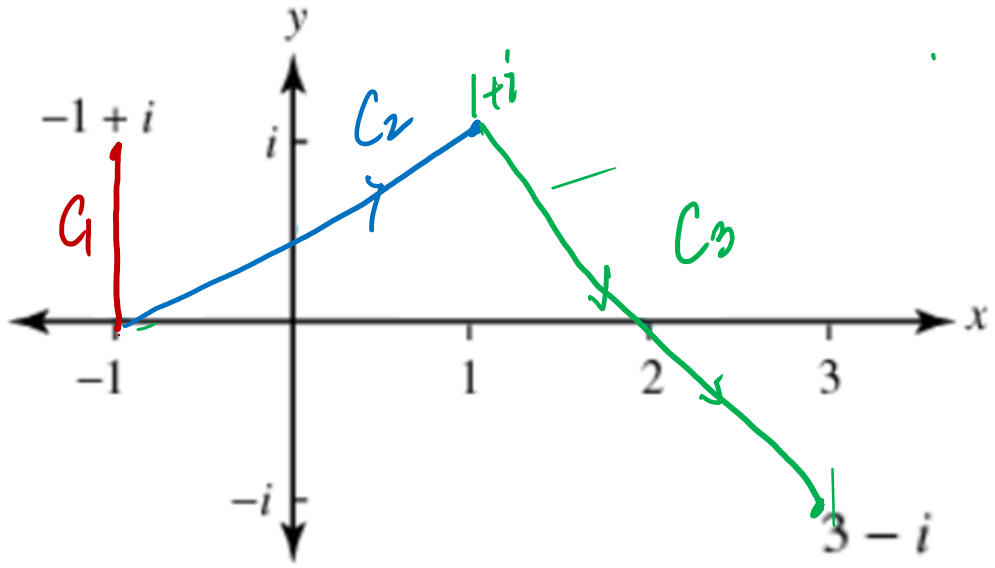
Contours

Example: Find parametrization of the a polygonal path C from $-1+i$ to $3-i$.



Contours

Example: Find parametrization of the a polygonal path C from $-1+i$ to $3-i$.

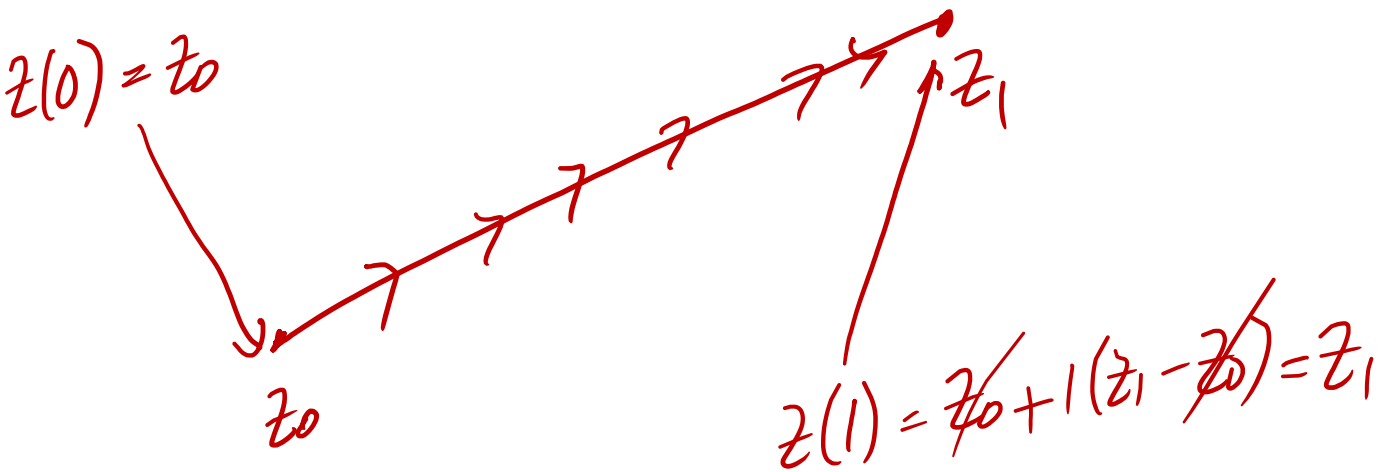


Let C_1 be line segment from $-1+i$ to -1
Let C_2 be line segment from -1 to $1+i$
Let C_3 be line segment from $1+i$ to $3-i$

Contours

Parametrization of line segment from z_0 to z_1 is given by

$$z(t) = z_0 + t(z_1 - z_0), \qquad 0 \le t \le 1$$



Contours

Parametrization of line segment from z_0 to z_1 is given by

$$z_0 + t(z_1 - z_0), \qquad 0 \le t \le 1$$

Equation of C_1, line segment from $-1 + i$ to -1

$z_0 = -1 + i, \quad z_1 = -1$

$$z_1(t) = (-1+i) + t(-1 - (-1+i))$$

$$= (-1+i) + t(-i) = (-1+i) - it$$

$$= \underbrace{-1}_{x(t)} + i\underbrace{(1-t)}_{y(t)}$$

Contours

Parametrization of line segment from z_0 to z_1 is given by

$$z_0 + t(z_1 - z_0), \qquad 0 \le t \le 1$$

Equation of C_1, line segment from $-1 + i$ to -1

$$C_1 : z_1(t) = -1 + i(1 - t), \qquad for\ 0 \le t \le 1$$

Equation of C_2, line segment from -1 to $1 + i$

$z_0 = -1, \quad z_1 = 1+i$

perametric equ. = Z(t)=Zo+t(Z1-Zo)
Z2(t)=-1+t(1+i-(-1))
=-1+t(2+i)
=-1+2t+it
=(-1+2t)+it 0<=t<=1

Contours

Parametrization of line segment from z_0 to z_1 is given by

$$z_0 + t(z_1 - z_0), \qquad 0 \le t \le 1$$

Equation of C_1, line segment from $-1 + i$ to -1

$$C_1 : z_1(t) = -1 + i(1 - t), \qquad for\ 0 \le t \le 1$$

Equation of C_2, line segment from -1 to $1 + i$

$$C_2 : z_2(t) = (-1 + 2t) + i\,t, \qquad for\ 0 \le t \le 1$$

Equation of C_3, line segment from $1 + i$ to $3 - i$

$z_0 = 1 + i, \quad z_1 = 3 - i$

Contours

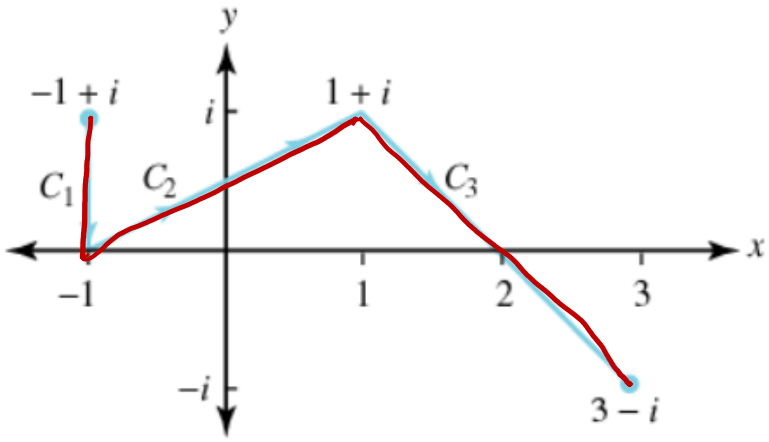
Example: Find parametrization of the a polygonal path C from $-1+i$ to $3-i$.

Solution: The polygonal path is the contour C, given by

$$C = C_1 + C_2 + C_3$$

where

$$\begin{cases} C_1 : z_1(t) = -1 + i(1-t), & for\ 0 \le t \le 1 \\ C_2 : z_2(t) = (-1+2t) + i\,t, & for\ 0 \le t \le 1 \\ C_3 : z_3(t) = (1+2t) + i(1-2t), & for\ 0 \le t \le 1 \end{cases}$$



Source: Complex analysis by Mathews

Contours

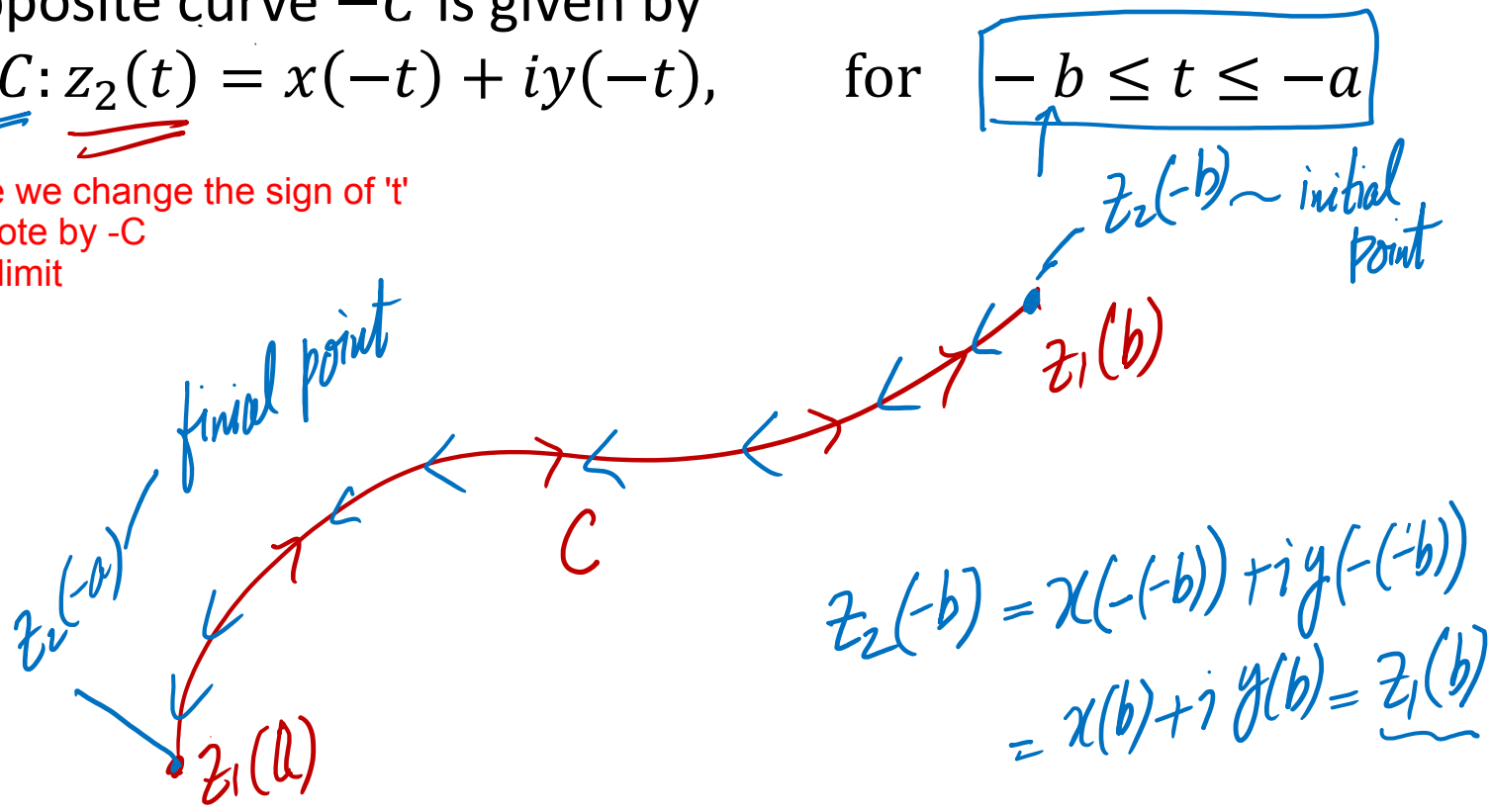
Opposite curve: Consider a curve C

$$C: z_1(t) = x(t) + iy(t), \quad \text{for} \quad a \le t \le b$$

The opposite curve $-C$ is given by

$$-C: z_2(t) = x(-t) + iy(-t), \quad \text{for} \quad -b \le t \le -a$$

in opposite curve we change the sign of 't'
and curve C denote by -C
also change the limit



$$z_2(-b) = x(-(-b)) + iy(-(-b))$$

$$= x(b) + iy(b) = z_1(b)$$

Contours

Opposite curve: Consider a curve C

$$C: z_1(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

The opposite curve $-C$ is given by

$$-C: z_2(t) = x(-t) + iy(-t), \qquad \text{for} \quad -b \le t \le -a$$

Geometrically $-C$ traces out the same set of points in the plane as C, but in the reverse order.

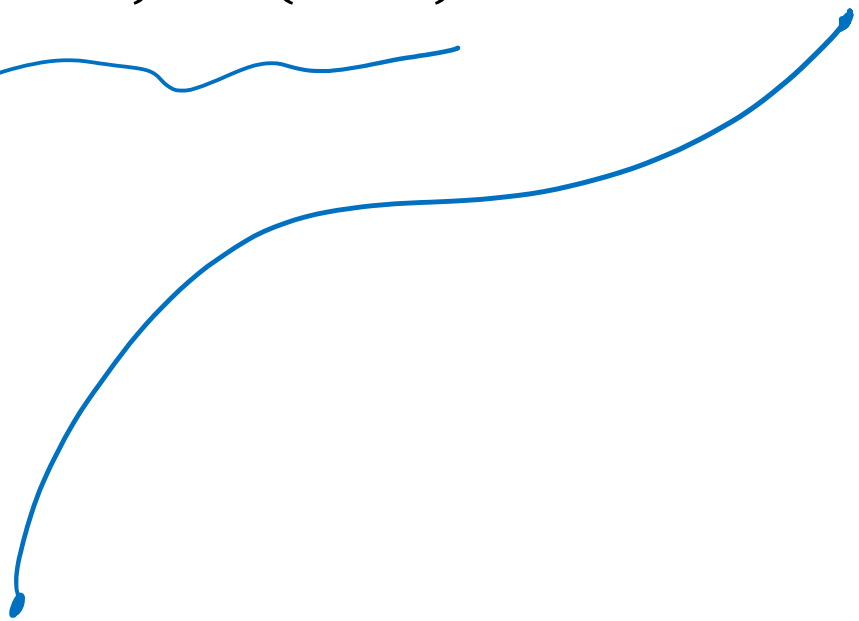
Contours

Arc length

$$C: z(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

We can find arc length of C from $z(a)$ to $z(b)$ in the following way:

$$L(C) = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$



Contours

Since C is smooth so $z'(t)$, given by

$$z'(t) = x'(t) + iy'(t) \text{ for } \quad a \leq t \leq b$$

is continuous. Hence the following function is continuous

$$\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}$$

and hence it is integrable on $[a, b]$.

Contours

Example: Find arc length of circle

$$z(t) = \cos(t) + i\sin(t), for\ 0 \le t \le 2\pi$$

$$\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\ dt$$

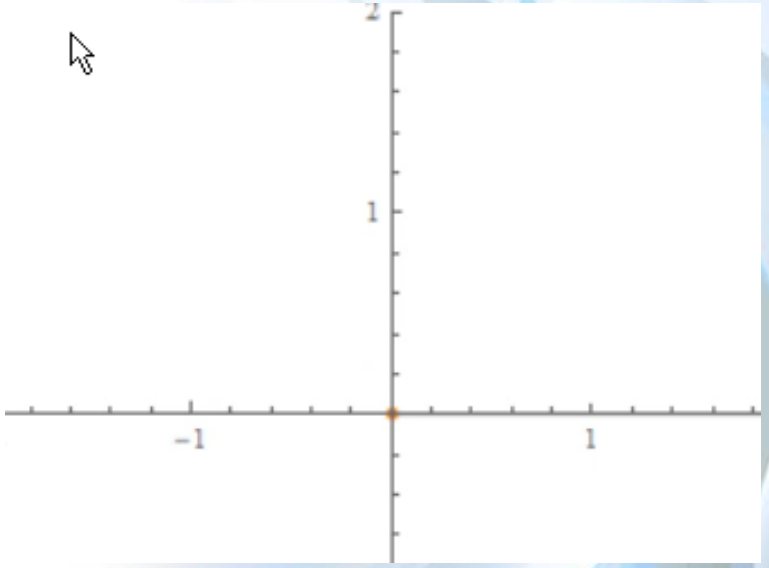
the above term =1 from the trignometric formula

$$= \int_0^{2\pi} dt = [t]_0^{2\pi} = 2\pi$$

Contours

Module # 89

Contours



Source: wolfram demonstration

Complex Integration

Complex Integrals over real intervals
Part II

$$\int_a^b f(t)\,dt$$

Complex Integration

Properties of

$$\int_a^b \boldsymbol{f(t)\,dt}$$

where

$$f(t) = u(t) + iv(t)$$

Complex Integration

Recall

Definition (Integral of $f(t)$)

Let $f(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real-valued functions of the real variable t for $a \leq t \leq b$. Then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Complex Integration

Definition (Integral of $f(t)$)

Let $f(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real-valued functions of the real variable t for $a \le t \le b$. Then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Hence

$$Re\left[\int_a^b f(t)\,dt\right] = \int_a^b Re[f(t)]\,dt$$

Complex Integration

Definition (Integral of $f(t)$)

Let $f(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real-valued functions of the real variable t for $a \leq t \leq b$. Then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Hence

$$Re\left[\int_a^b f(t)\,dt\right] = \int_a^b Re[f(t)]\,dt$$

Similarly

$$Im\left[\int_a^b f(t)\,dt\right] = \int_a^b Im[f(t)]\,dt$$

Complex Integration

For

$$f(t) = u(t) + iv(t)$$

and

$$g(t) = p(t) + iq(t)$$

we have

$$\int_a^b \big(f(t) + g(t)\big)\, dt$$

$$= \int_a^b f(t)\, dt + \int_a^b g(t)\, dt$$

Complex Integration

For

$$f(t) = u(t) + iv(t)$$

and a partition of the interval $[a, b]$ given as

$$a \le t \le d, d \le t \le b$$

we have



$$\int_a^b f(t)\,dt$$

$$= \int_a^d f(t)\,dt + \int_d^b f(t)\,dt$$

Complex Integration

Fundamental theorem of calculus (FTC) for Complex functions

Let $f(t) = u(t) + iv(t)$

and

$$F'(t) = f(t)$$

such that $f(t)$ and $F(t)$ are continuous on $[a, b]$, then

$$\int_a^b f(t)dt = F(t)\Big|_{t=a}^{t=b}$$

$$= F(b) - F(a)$$

Complex Integration

A consequence of FTC for complex functions

For a differentiable function

$$f(t) = u(t) + iv(t)$$

and $a \le t \le b$, we have

$$\int_a^b f'(t)dt = f(b) - f(a)$$

Complex Integration

The following is a useful bridge for evaluating real integrals as well

$$\int_a^b f(t)\,dt$$

$$= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Complex Integration

Example: Evaluate the following

$$\int_0^{\pi/2} \exp(t + it)\, dt$$

Complex Integration

Example: Evaluate the following

$$\int_0^{\pi/2} \exp(t+it)\,dt$$

In the last module we evaluated the above integral by evaluating two real integrals, that is,

$$\int_0^{\pi/2} e^{t+it}dt$$

$$= \int_0^{\pi/2} e^t e^{it}dt$$

$$= \int_0^{\pi/2} e^t(\cos t + i\sin t)dt$$

$$= \int_0^{\pi/2} e^t \cos t\,dt + i\int_0^{\pi/2} e^t \sin t\,dt$$

Complex Integration

Example: Evaluate the following

$$\int_0^{\pi/2} \exp(t + it)\, dt$$

The above integral can also be calculated in much simple way using properties of complex integrals over real intervals.

$$F(t) = \frac{1}{1+i} e^{t(1+i)}, \quad F'(t) = \exp(t+it)$$

$$\int_0^{\pi/2} \exp(t+it)\,dt = \left[\frac{1}{1+i} e^{t(1+i)}\right]_0^{\pi/2} = \frac{1}{1+i}\left(ie^{\pi/2} - 1\right)$$

$$= \frac{1}{2}(1-i)\left(ie^{\pi/2} - 1\right) = \frac{1}{2}\left(e^{\pi/2} - 1\right) + i\left(e^{\pi/2} + 1\right)$$

Complex Integration

Example: Evaluate the following

$$\int_0^{\pi/2} \exp(t + it)\, dt$$

$$= \int_0^{\pi/2} e^t \cos t\, dt + i \int_0^{\pi/2} e^t \sin t\, dt$$

$$\longrightarrow = \frac{1}{2}\left(e^{\pi/2} - 1\right) + \frac{i}{2}\left(e^{\pi/2} + 1\right)$$

$$\Rightarrow \int_0^{\pi/2} e^t \cos t\, dt = \frac{1}{2}\left(e^{\pi/2} - 1\right), \quad \int_0^{\pi/2} e^t \sin t\, dt = \frac{1}{2}\left(e^{\pi/2} + 1\right)$$

Complex Integration

Although there are many similarities between real and complex integrals over a real interval. But there are some important differences as well.



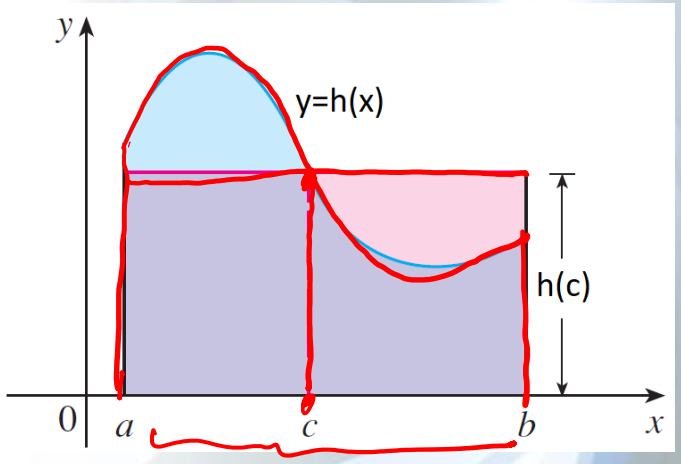
Source: https://www.1001freedownloads.com

Complex Integration

Mean value theorem for real integrals

If $h(x)$ is continuous on $[a, b]$ then there exist $c \in [a, b]$ such that

$$\int_a^b h(x)\,dx = h(c)\,(b-a)$$



Source:calculus by James Stewart

Complex Integration

Example: Mean value theorem for real integrals does not hold in case of integrals of complex functions over real intervals.

Complex Integration

Example: Mean value theorem for real integrals does not hold in case of integrals of complex functions over real intervals. Consider

$$f(t) = e^{it} \quad \text{for } 0 \leq t \leq 2\pi$$

Complex Integration

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$$f(t) = e^{it} \quad \text{for } 0 \leq t \leq 2\pi$$

Let us evaluate

$$\int_0^{2\pi} e^{it}\, dt$$

Complex Integration

Example: Mean value theorem for real integrals does not hold in case of integrals of complex functions over real intervals. Consider

$$f(t) = e^{it} \quad \text{for } 0 \le t \le 2\pi$$

Let us evaluate

$$\int_0^{2\pi} e^{it}\, dt$$

$$= \frac{e^{it}}{i}\Big]_0^{2\pi}$$

Complex Integration

Example: Mean value theorem for real integrals does not hold in case of integrals of complex functions over real intervals. Consider

$$f(t) = e^{it} \quad \text{for } 0 \le t \le 2\pi$$

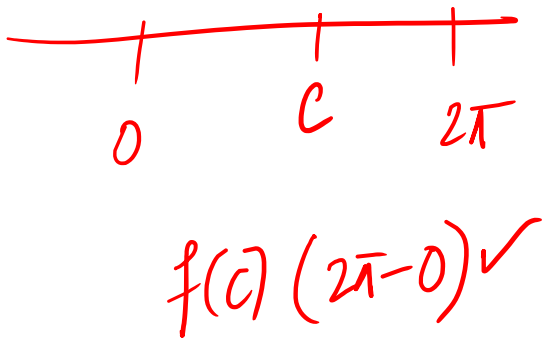
Let us evaluate

$$\int_0^{2\pi} e^{it}\, dt$$

$$= \frac{e^{it}}{i}\Big]_0^{2\pi}$$

$$= e^{i2\pi} - e^0$$

$$= 0$$



Complex Integration

Example: Mean value theorem for real integrals does not hold in case of integrals of complex functions over real intervals.

Consider

$$f(t) = e^{it} \quad \text{for } 0 \le t \le 2\pi$$

Let us evaluate

$$\int_0^{2\pi} e^{it}\, dt$$

$$= \frac{e^{it}}{i}\Big]_0^{2\pi}$$

$$= e^{i2\pi} - e^0$$

$$= 0$$

For any $c \in [0,2\pi]$ we have

$$|f(c)(2\pi - 0)| = |e^{ic}||2\pi| = 2\pi$$

Complex Integration

Complex Integrals over real intervals
Part II

$$\int_a^b f(t)\,dt$$

Complex Integration

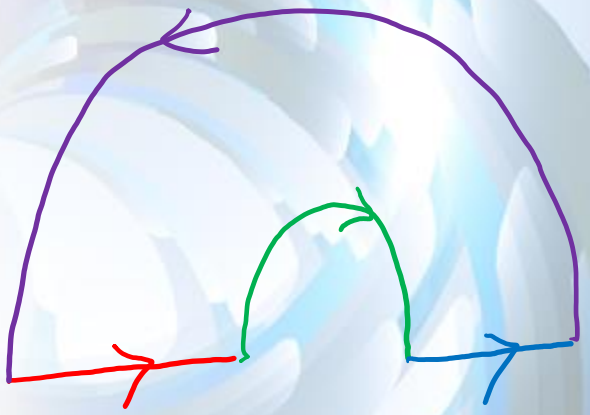
Complex Integration

$$\int$$

Complex Integration

Module # 90

Contours



Complex Integration

Let $f(z)$ is a continuous complex-valued function defined on a set containing the contour C, then

$$\int_C f(z) = \int_a^b f\big(z(t)\big)\, z'(t)\, dt$$

Where

$$C: z(t) = x(t) + iy(t)$$

for $a \leq t \leq b$

Complex Integration

Next Task

Understand integration of complex functions on real intervals

Complex Integration

Problem

How to evaluate the following

$$\int_a^b f(t)\,dt$$

where

$$f(t) = u(t) + i\,v(t)$$

for $a \le t \le b$

Complex Integration

Definition (Integral of $f(t)$)

Let $f(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real-valued functions of the real variable t for $a \leq t \leq b$. Then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Complex Integration

Example: Evaluate the following

$$\int_1^2 (t + i\,t^2)\,dt$$

Complex Integration

Example: Evaluate the following

$$\int_0^{\pi/2} \exp(t + it)\, dt$$

Complex Integration

Example: Evaluate the following

$$\int_0^{\pi/2} \exp(t + it)\, dt$$

Complex Integration

Properties of integration

Complex Integration

For

$$f(t) = u(t) + iv(t)$$

and

$$g(t) = p(t) + iq(t)$$

we have

property 1

$$\int_a^b \big(f(t) + g(t)\big)\, dt$$

$$= \int_a^b f(t)\, dt + \int_a^b g(t)\, dt$$

Complex Integration

For

$$f(t) = u(t) + iv(t)$$

and a partition of the interval $[a, b]$ given as

$$a \leq t \leq d, d \leq t \leq b$$

we have

$$\int_a^b f(t)\, dt$$

$$= \int_a^d f(t)\, dt + \int_d^b f(t)\, dt$$

Complex Integration

Fundamental theorem of calculus (FTC) for Complex functions

Let $f(t) = u(t) + iv(t)$ such that

$$F'(t) = f(t)$$

then

$$\int_a^b f(t)dt = F(t)\Big|_{t=a}^{t=b}$$

$$= F(b) - F(a)$$

Complex Integration

A consequence of FTC for complex functions

For a differentiable function

$$f(t) = u(t) + iv(t)$$

and $a \leq t \leq b$, we have

$$\int_a^b f'(t)dt = f(b) - f(a)$$

Complex Integration

The following is a useful bridge for evaluating real integrals as well

$$\int_a^b f(t)\,dt$$

$$= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Complex Integration

Example: Evaluate the following

$$\int_0^{\pi/2} \exp(t + it)\, dt$$

Complex Integration

module # 91

Complex Integration

$$\int$$

Contour Integrals

Contour Integrals
Examples
Part I

$$\int_C$$

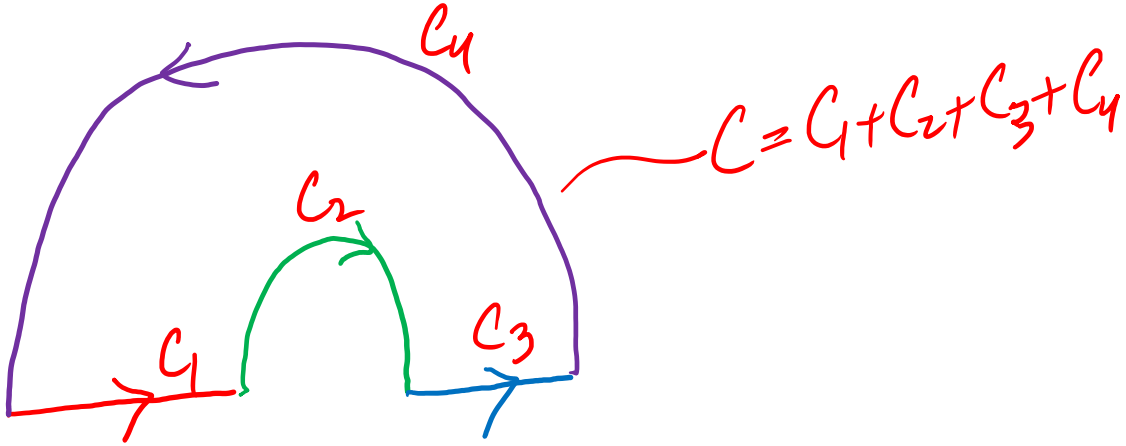
Contour Integrals

Recall:

Contour: A curve C is said to be a contour if it is constructed by joining finitely many smooth curves end to end.

We express a contour obtained by joining sequence of curves $C_1, C_2, \ldots, C_n$ in the following way

$$C = C_1 + C_2 + \cdots + C_n$$

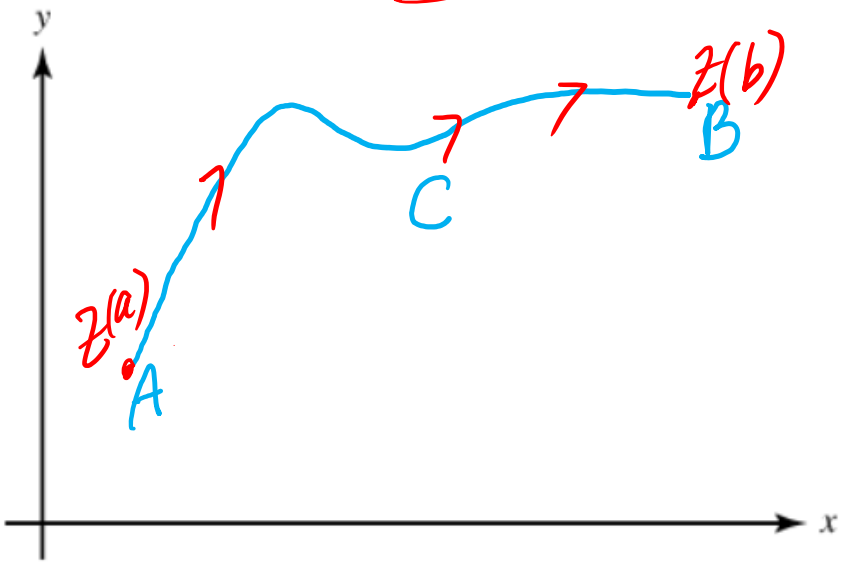


Contour Integrals

Recall

Theorem: Suppose that $f(z)$ is a continuous complex-valued function defined on a set containing the contour C. Let $z(t)$ be any parametrization of C for $a \leq t \leq b$ then

$$\int_C f(z) = \int_a^b f\big(z(t)\big)\, z'(t)\, dt$$



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Theorem: Suppose that $f(z)$ is a continuous complex-valued function defined on a set containing the contour C. Let $z(t)$ be any parametrization of C for $a \leq t \leq b$ then

$$\int_C f(z) = \int_a^b f\big(z(t)\big)\, z'(t)\, dt$$

Note: we are allowed to choose any parametrization of the contour as we want.

The above integral involves integrating complex functions over real intervals.

Contour Integrals

Recall:

Definition (Integral of $f(t)$)

Let $f(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real-valued functions of the real variable t for $a \leq t \leq b$. Then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

Contour Integrals

Example: Evaluate the integral

$$\int_C \exp(z)\, dz$$

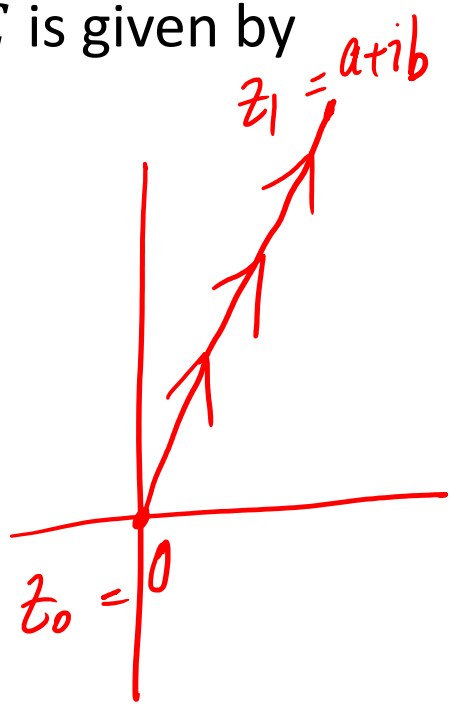
where the contour C is a line segment from $0 + i0$ to $a + ib$.

Solution: The parametric representation of C is given by

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt$$

$$z(t) = z_0 + t(z_1 - z_0), \quad 0 \le t \le 1$$

$$= 0 + t(a+ib) = t(a+ib)$$



Contour Integrals

Example: Evaluate the integral

$$\int_C \exp(z)\, dz$$

where the contour C is a line segment from $0 + i0$ to $a + ib$.

Solution: The parametric representation of C is given by

$$\mathrm{C}: \; z(t) = (a + ib)t \;\text{ for }\; 0 \le t \le 1$$

Which implies

$$z'(t) = (a + ib)$$

Contour Integrals

Example: Evaluate the integral

$$\int_C \exp(z)\, dz$$

where the contour C is a line segment from $0 + i0$ to $a + ib$.

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Which implies

$$z'(t) = (a + ib)$$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\, z'(t)\, dt$$

Hence

$$\int_C \exp(z)\; dz = \int_0^1 \exp\big(z(t)\big)\;\; z'(t)\, dt$$

Contour Integrals

Example: Evaluate the integral

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Which implies

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Hence

$$\int_C \exp(z)\; dz = \int_0^1 \exp\big(z(t)\big)\;\; z'(t)\, dt$$

$$= \int_0^1 \exp\big((a + ib)t\big)\;(a + ib)\, dt$$

Contour Integrals

Example: Evaluate the integral

$$\int_C \exp(z)\, dz$$

where the contour C is a line segment from $0 + i0$ to $a + ib$.

Solution cont.:

$$\int_C \exp(z)\ dz = \int_0^1 \exp\big((a+ib)t\big)\ (a+ib)\, dt$$

Contour Integrals

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$$\int_C \exp(z)\ dz = \int_0^1 \exp\big((a+ib)t\big)\ (a+ib)\, dt$$

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$$= (a+ib)\ \frac{\exp\big((a+ib)t\big)}{(a+ib)}\Bigg]_0^1$$

Contour Integrals

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$$= \exp(a+ib) - \exp 0$$

Contour Integrals

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$$= \exp(a+ib) - \exp 0$$

$$= \exp(a+ib) - 1$$

Contour Integrals

Contour Integrals
Examples
Part I

$$\int_C$$

Contour Integrals

Contour Integrals
Examples
Part II

$$\int_C$$

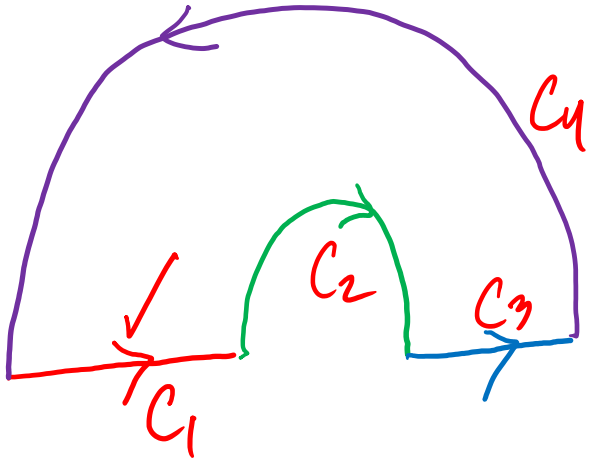
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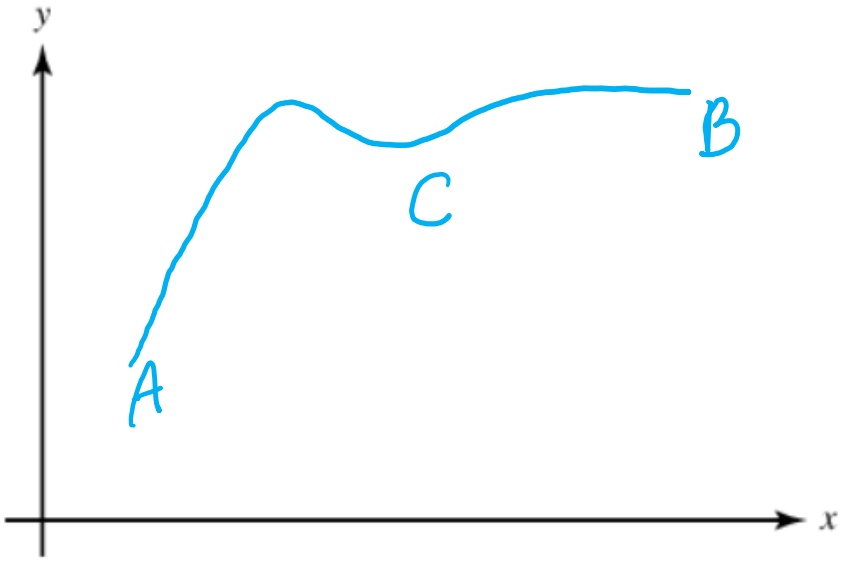


Contour Integrals

Recall

Theorem: Suppose that $f(z)$ is a continuous complex-valued function defined on a set containing the contour C. Let $z(t)$ be any parametrization of C for $a \leq t \leq b$ then

$$\int_C f(z) = \int_a^b f\big(z(t)\big)\, z'(t)\, dt$$



Contour Integrals

Recall

Theorem: Suppose that $f(z)$ is a continuous complex-valued function defined on a set containing the contour C. Let $z(t)$ be any parametrization of C for $a \leq t \leq b$ then

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Contour Integrals

Recall:

Definition (Integral of $f(t)$)

Let $f(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real-valued functions of the real variable t for $a \leq t \leq b$. Then

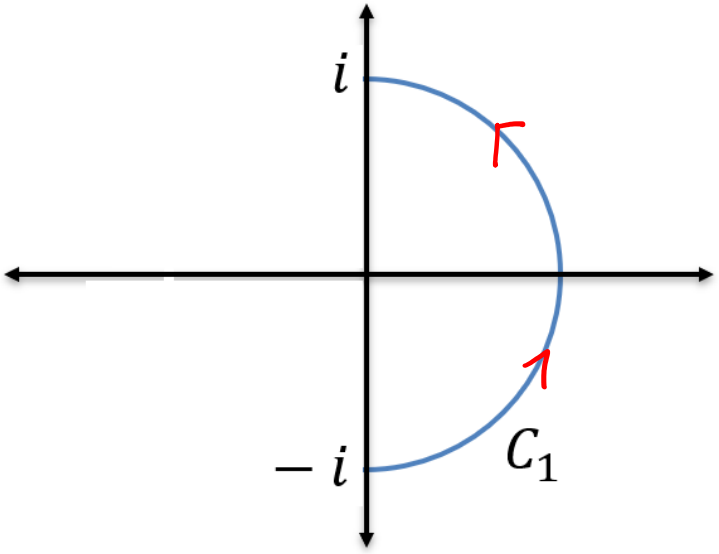
$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

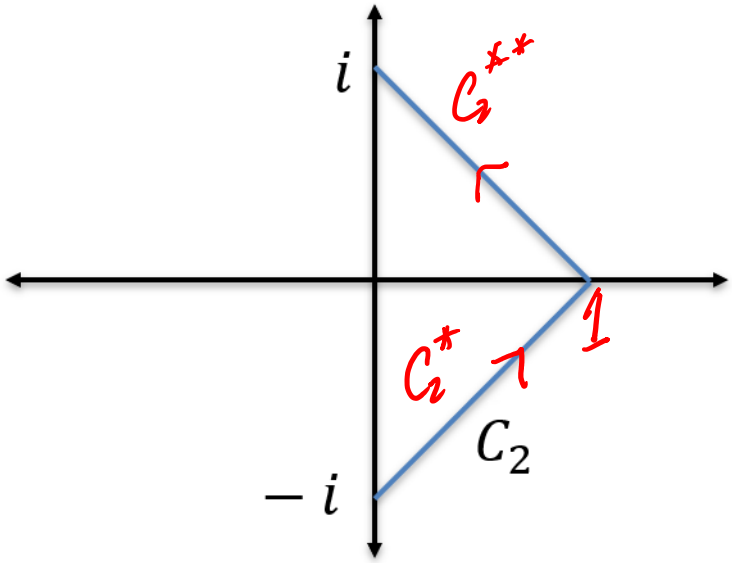
Contour Integrals

Example1: Evaluate the integrals

$$\int_{C_1} z\,dz \quad and \int_{C_2} z\,dz$$

where the contour C_1 is semicircle from $-i$ to i (counterclockwise), and C_2 is the polygonal path from $-i$ to i as shown in the figure.





Contour Integrals

Solution ($\int_{C_1} z\,dz$):

C_1: $z_1(t) = \cos t + i\sin t = e^{it}$ $\quad -\pi/2 \le t \le \pi/2$

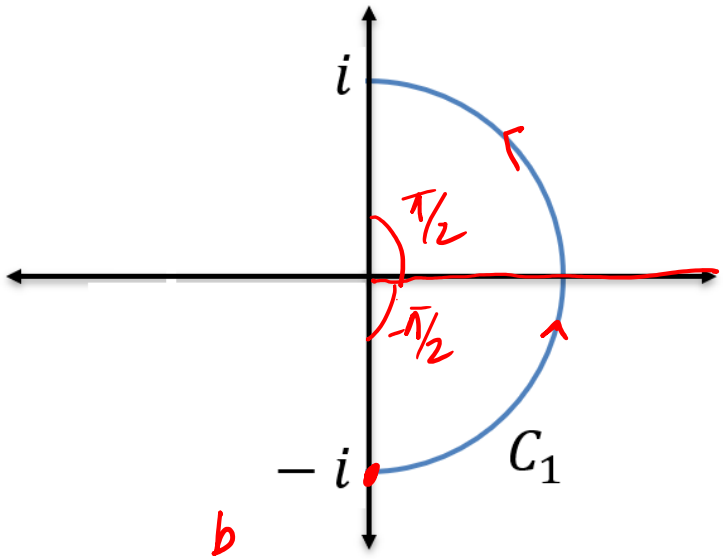
$z_1'(t) = ie^{it}$

$$\int_{C_1} z\,dz = \int_{-\pi/2}^{\pi/2} e^{it}(ie^{it})\,dt = i\int_{-\pi/2}^{\pi/2} e^{i2t}\,dt$$

$$= i\left[\frac{e^{i2t}}{2i}\right]_{-\pi/2}^{\pi/2} = \frac{1}{2}\left[e^{i2t}\right]_{-\pi/2}^{\pi/2}$$

$$= \frac{1}{2}e^{i\pi} - \frac{1}{2}e^{-i\pi}$$

$$= \frac{1}{2}(-1) - \frac{1}{2}(-1) = 0$$



$$\int_a^b f(z(t))\,z'(t)\,dt$$

$f(z) = z$

$f(z(t)) = z(t)$

Contour Integrals

Solution cont. ($\int_{C_2} z\,dz$):

C_2^*: $y_1(t) = -i + (1+i)t$, $0 \le t \le 1$

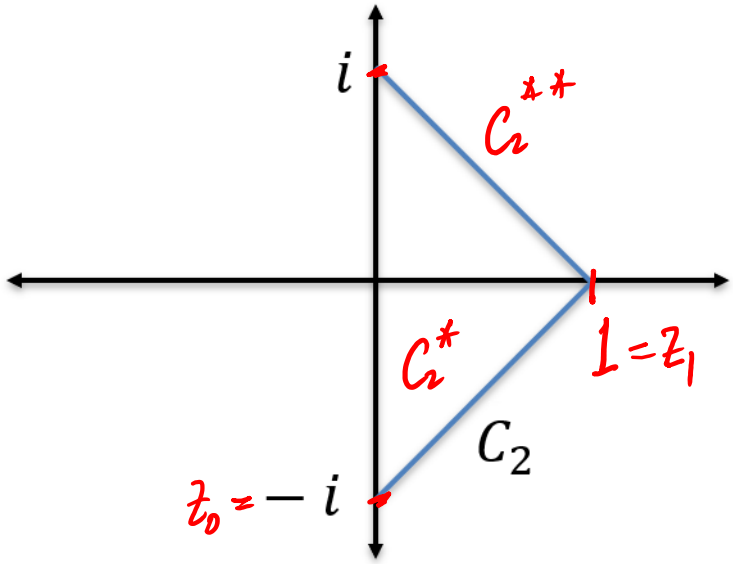
$y_1'(t) = 1+i$

$$\int_{C_2^*} z\,dz = \int_0^1 f(y_1(t))\, y_1'(t)\,dt, \quad f(z) = z$$

$$= \int_0^1 (-i + (1+i)t)(1+i)\,dt$$

$$= (1+i)\left[-it + (1+i)\frac{t^2}{2}\right]_0^1$$

$$= (1+i)\left(-i + \frac{(1+i)}{2}\right) = 1$$



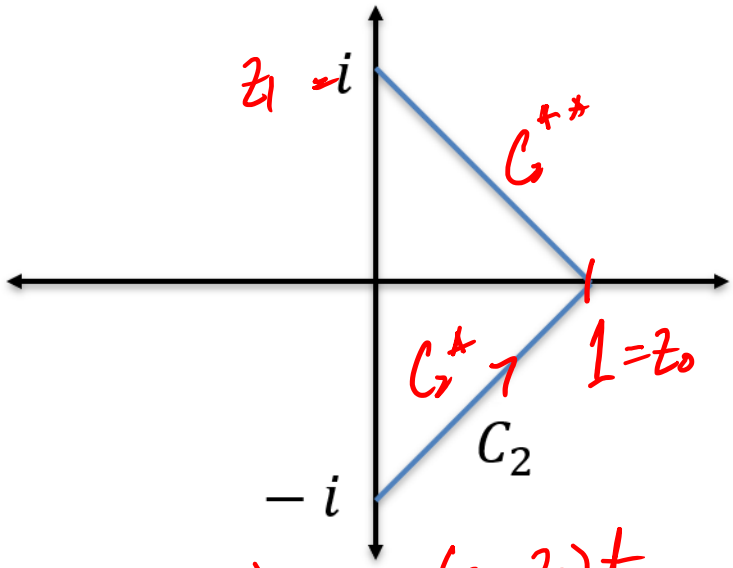
$y_1(t) = z_0 + (z_1 - z_0)t$

$0 \le t \le 1$

$y_1(t) = -i + (1+i)t$

Contour Integrals

Solution cont. ($\int_{C_2} z\,dz$):



C_1^{**}: $y_2(t) = 1 + (i-1)t$, $0 \le t \le 1$

$y_2'(t) = i - 1$

$$\int_{C_1^{**}} z\,dz = \int_0^1 (1 + (i-1)t)(i-1)\,dt$$

$$= (i-1)\int_0^1 (1 + (i-1)t)\,dt$$

$$= (i-1)\left[t + (i-1)\frac{t^2}{2}\right]_0^1 = (i-1)\left[1 + \frac{(i-1)}{2}\right]$$

$$= -1$$

$y_2(t) = z_0 + (z_1 - z_0)t$

$0 \le t \le 1$

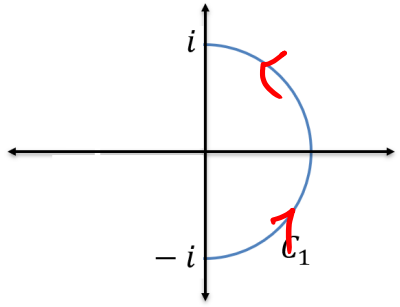
$= 1 + (i-1)t$

$$\Rightarrow \int_{C_2} z\,dz = \int_{C_1^{*}} z\,dz + \int_{C_1^{**}} z\,dz = -1 = 0$$

Contour Integrals

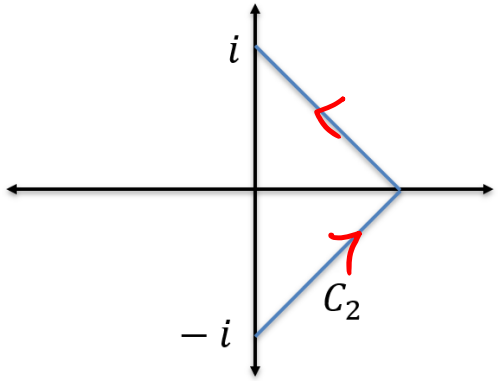
Example1: Evaluate the integrals

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where the contour C_1 is semicircle from $-i$ to i (counterclockwise), and C_2 is the polygonal path from $-i$ to i as shown in the figure.

$\int_{C_1} z\,dz = 0$, $\int_{C_2} z\,dz = 0$



Contour Integrals

When a contour integral is independent of path?

Contour Integrals

Contour Integrals
Examples
Part II

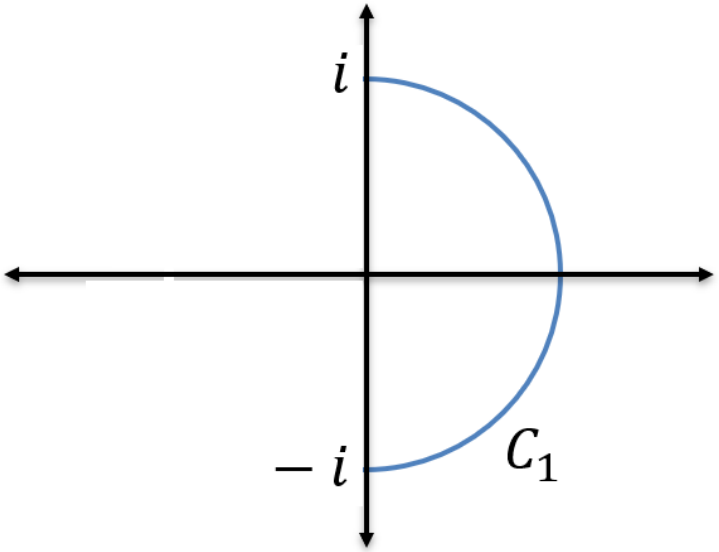
$$\int_C$$

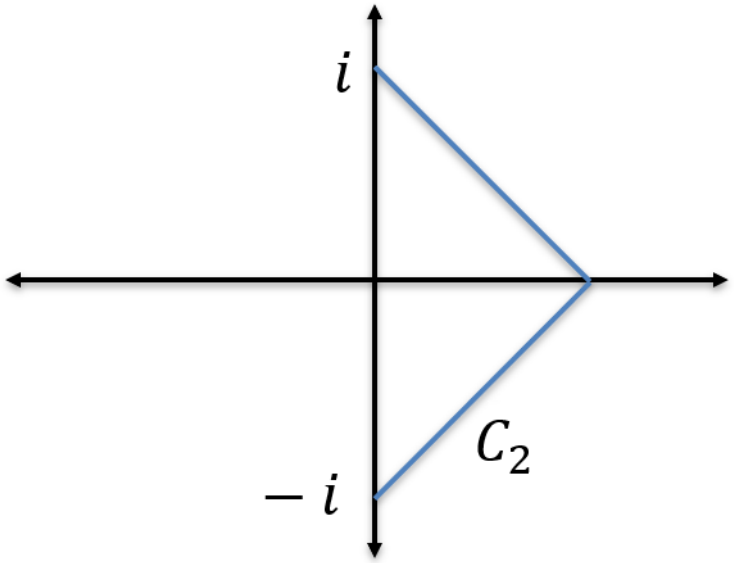
Contour Integrals

Example2: Evaluate the integrals

$$\int_{C_1} \bar{z}\, dz \quad and \int_{C_2} \bar{z}\, dz$$

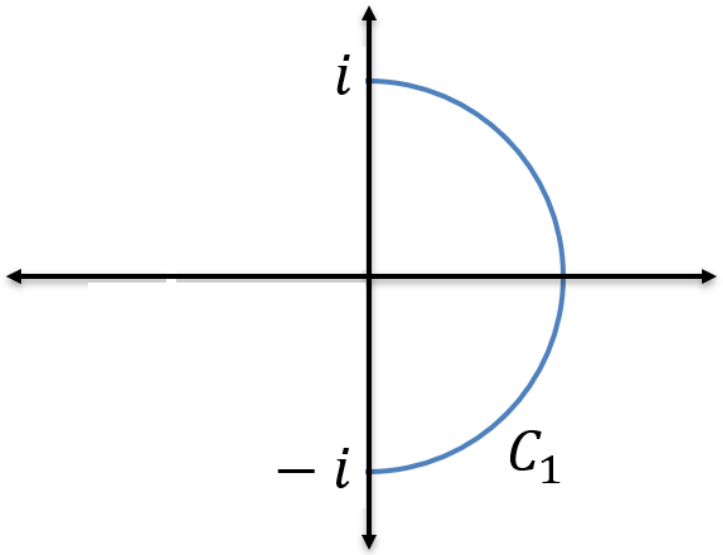
where the contour C_1 is semicircle from $-i$ to i (counterclockwise), and C_2 is the polygonal path from $-i$ to i as shown in the figure.





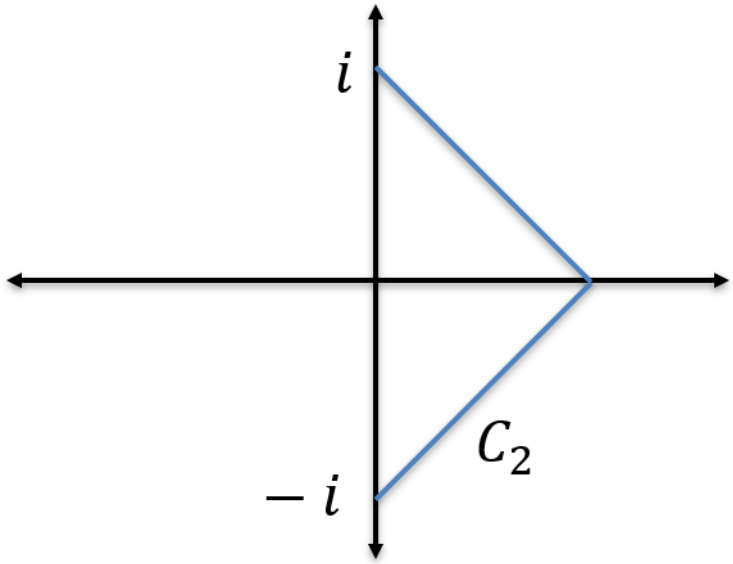
Contour Integrals

Solution ($\int_{C_1} \bar{z}\, dz$):



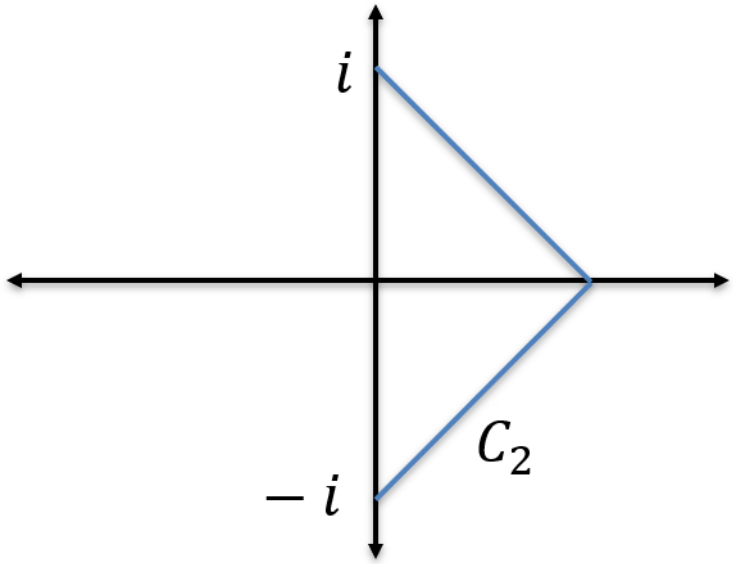
Contour Integrals

Solution cont. ($\int_{C_2} \bar{z}\, dz$):



Contour Integrals

Solution cont. ($\int_{C_2} \bar{z}\, dz$):



Contour Integrals

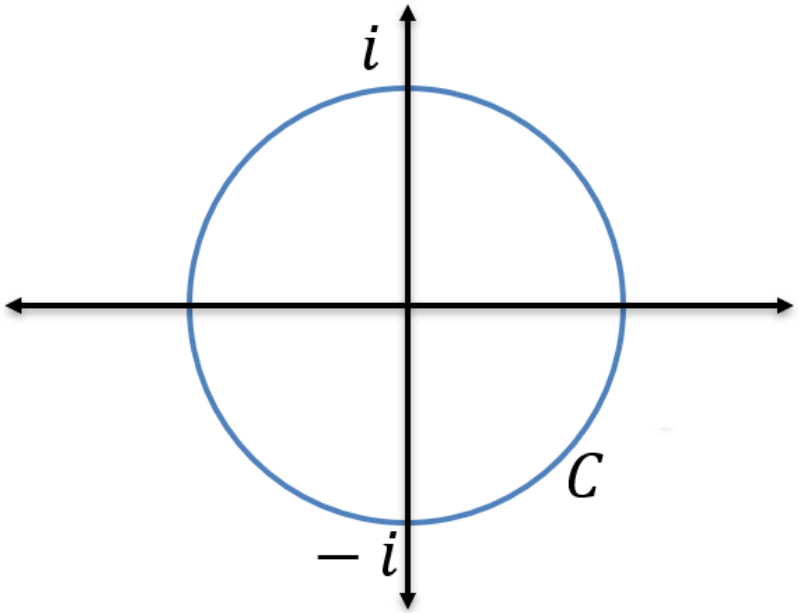
When a contour integral is independent of path?

Contour Integrals

Example3: Evaluate the integrals

$$\int_C z\,dz$$

where the contour C is circle from $-i$ to $-i$ (counterclockwise) as shown in the figure.



Contour Integrals

Contour Integrals
Examples
Part II

$$\int_C$$

Contour Integrals

Contour Integrals
Examples
Part III

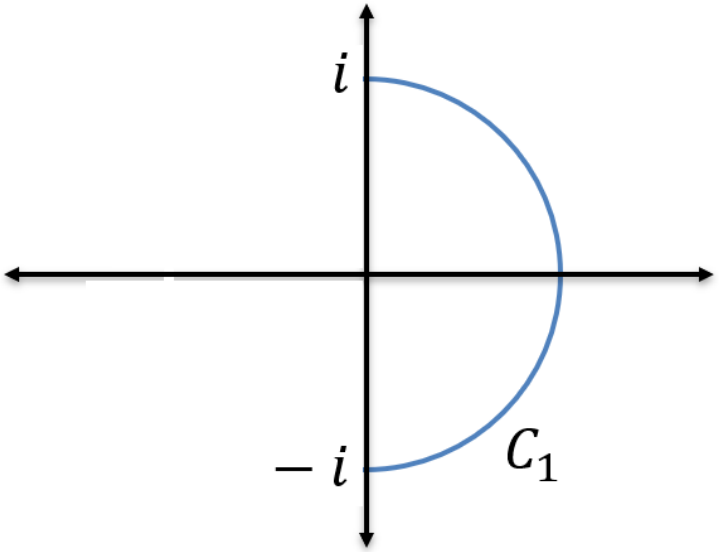
$$\int_C$$

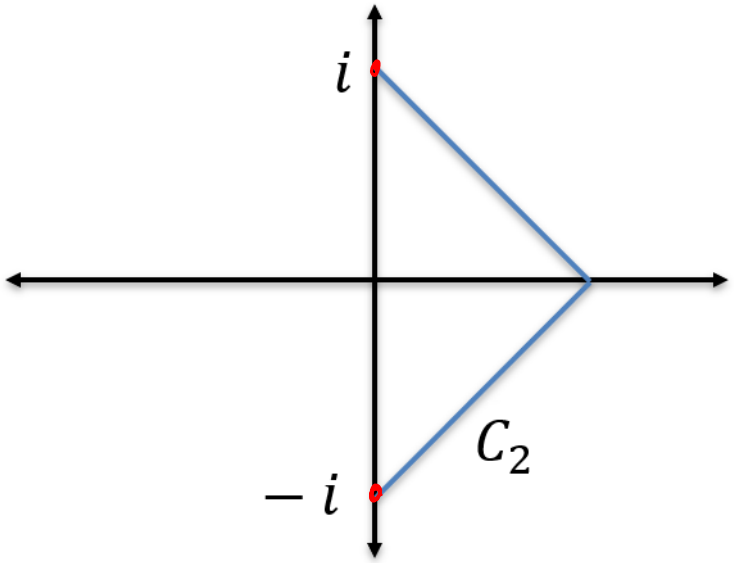
Contour Integrals

Recall: Example1: Evaluate the integrals

$$\int_{C_1} z\,dz \quad and \quad \int_{C_2} z\,dz$$

where the contour C_1 is semicircle from $-i$ to i (counterclockwise), and C_2 is the polygonal path from $-i$ to i as shown in the figure.





Contour Integrals

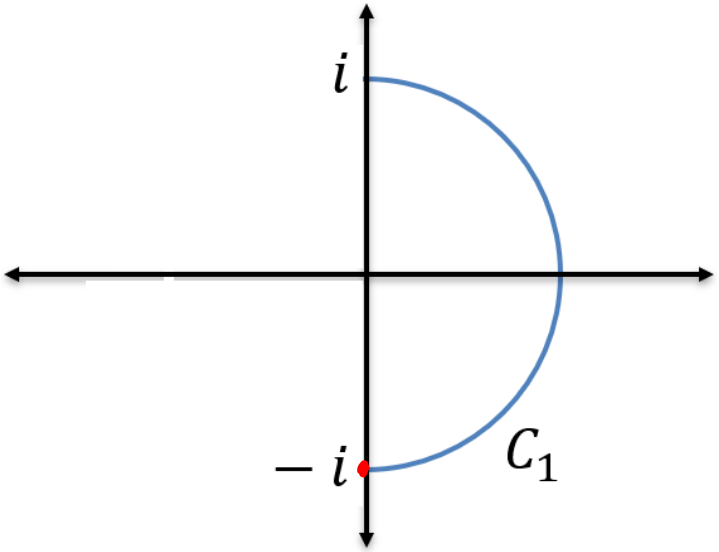
When a contour integral is independent of path?

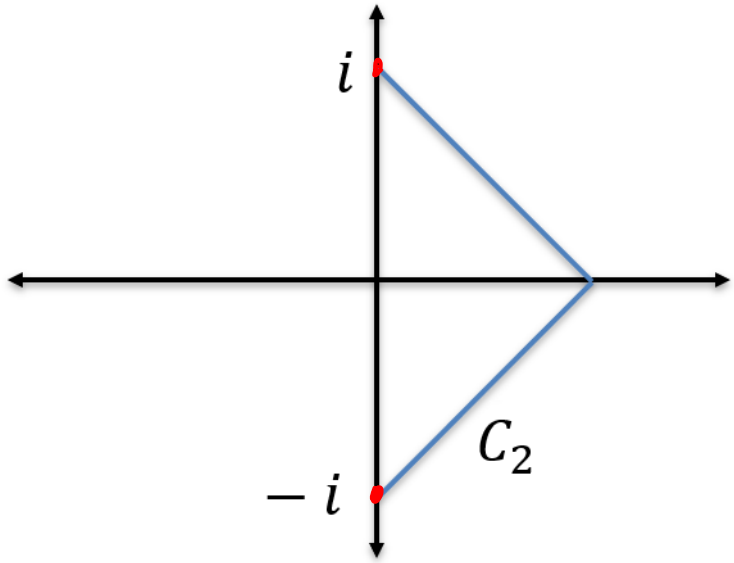
Contour Integrals

Example2: Evaluate the integrals

$$\int_{C_1} \bar{z}\, dz \quad and \int_{C_2} \bar{z}\, dz$$

where the contour C_1 is semicircle from $-i$ to i (counterclockwise), and C_2 is the polygonal path from $-i$ to i as shown in the figure.





Contour Integrals

Solution ($\int_{C_1} \bar{z}\,dz$):

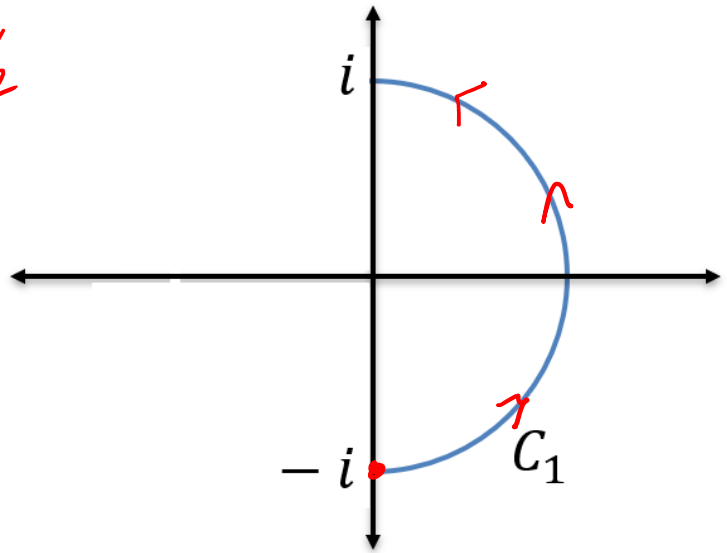
$C_1: z(t) = e^{it} = \cos t + i\sin t, \quad -\pi/2 \le t \le \pi/2$

$z'(t) = ie^{it}$

$$\int_{C_1} \bar{z}\,dz = \int_{-\pi/2}^{\pi/2} \overline{e^{it}}\,(ie^{it})\,dt$$

$$= i\int_{-\pi/2}^{\pi/2} e^{-it}e^{it}\,dt$$

$$= i\int_{-\pi/2}^{\pi/2} dt = i\left[\pi/2 - (-\pi/2)\right] = i\pi$$



$\bar{z}z = |z|^2 = 1$

$\Rightarrow \bar{z} = \frac{1}{z}$

$$\Rightarrow \int_{C_1} \bar{z}\,dz = \int_{C_1} \frac{dz}{z} = i\pi$$

Contour Integrals

Solution cont. ($\int_{C_2} \bar{z}\,dz$):

$C_2^*: \; y_1(t) = -i + (1+i)t, \quad 0 \le t \le 1$

$y_1(t) = t + i(t-1), \quad 0 \le t \le 1$

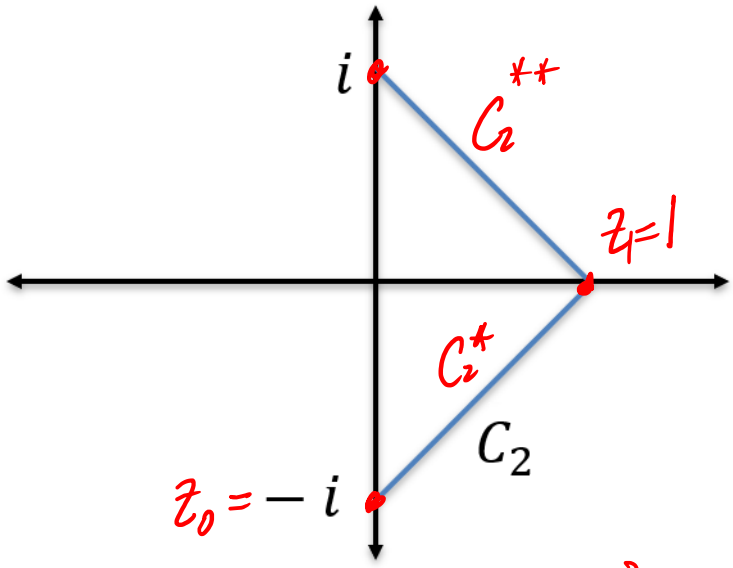
$y_1'(t) = 1 + i$

$$\int_{C_2^*} \bar{z}\,dz = \int_0^1 \overline{[t + i(t-1)]}\,(1+i)\,dt$$

$$= (1+i)\int_0^1 t - i(t-1)\,dt$$

$$= (1+i)\left[\frac{t^2}{2} - i\left(\frac{t^2}{2} - t\right)\right]_0^1 = (1+i)\left[\frac{1}{2} - i\left(\frac{1}{2} - 1\right)\right]$$

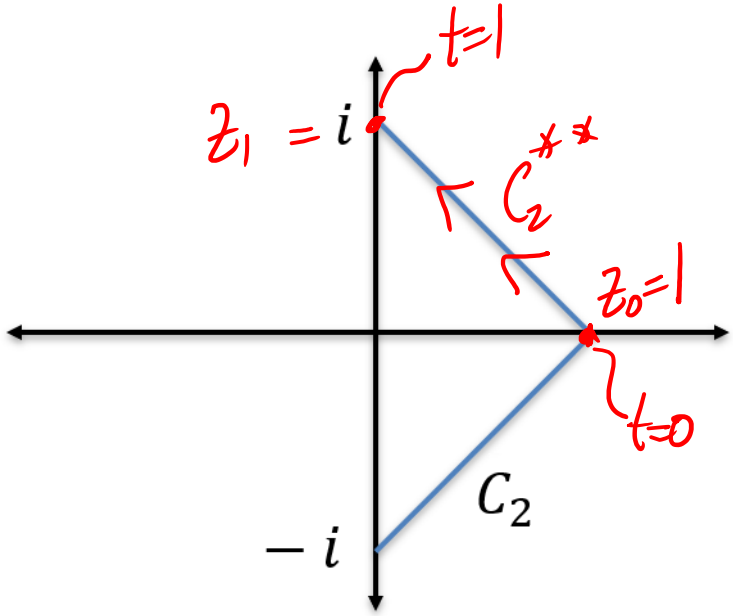
$$= i$$



$$\left\{ \begin{array}{c} z_0 \text{ —— } z_1 \\ z_0 + (z_1 - z_0)t \\ 0 \le t \le 1 \end{array} \right\}$$

Contour Integrals

Solution cont. ($\int_{C_2} \bar{z}\,dz$):



C_2^{**}: $y_2(t) = 1+(i-1)t$, $0 \le t \le 1$

$y_2(t) = (1-t)+it$, $0 \le t \le 1$

$y_2'(t) = i-1$

$$\int_{C_2^{**}} \bar{z}\,dz = \int_0^1 \overline{((1-t)+it)}\,(i-1)\,dt$$

$$= (i-1)\int_0^1 (1-t)-it\,dt$$

$$= (i-1)\left[t - \frac{t^2}{2} - i\frac{t^2}{2}\right]_0^1 = (i-1)\left[1-\frac{1}{2}-\frac{i}{2}\right] = \frac{1}{2}(-1-2i+1) = -i$$

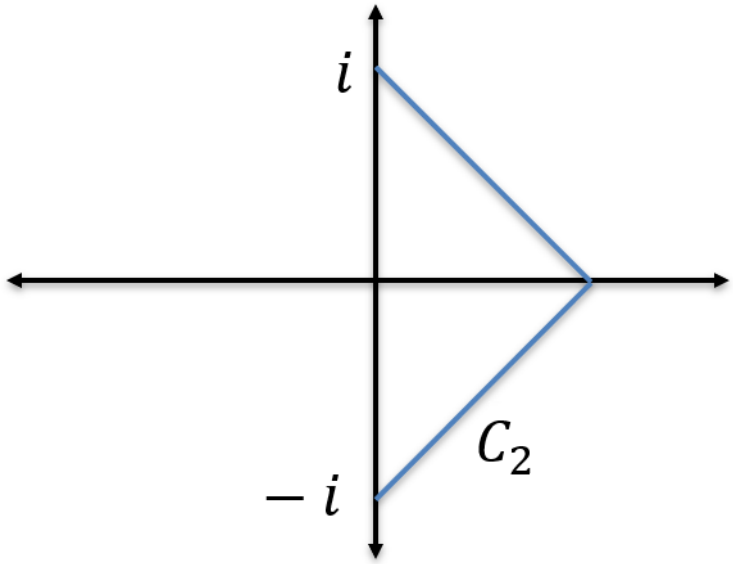
Contour Integrals

Solution cont. ($\int_{C_2} \bar{z}\,dz$):

$$\int_{C_2} \bar{z}\,dz = \int_{C_2^*} \bar{z}\,dz + \int_{C_2^{**}} \bar{z}\,dz$$

$$= i + (-i)$$

$$= 0$$



Contour Integrals

When a contour integral is independent of path?

Contour Integrals

Example3: Evaluate the integral

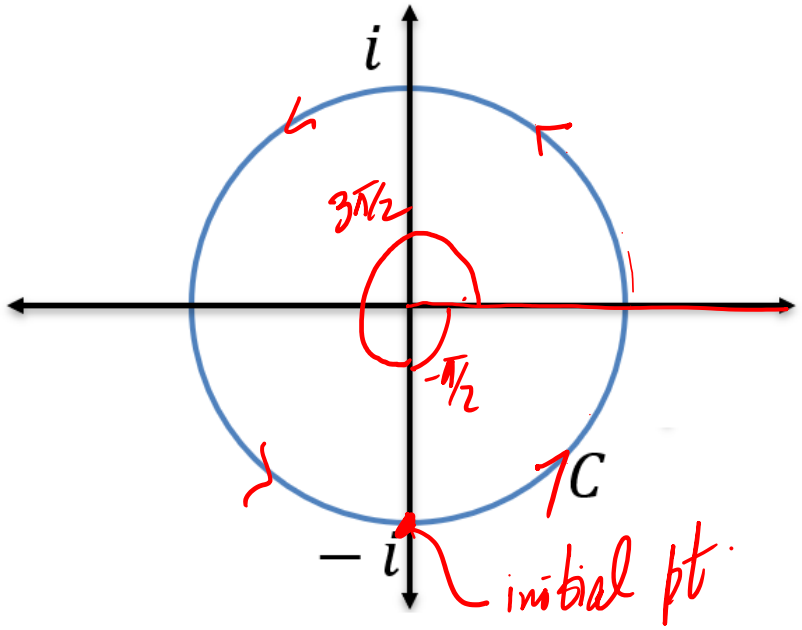
$$\int_C z\,dz$$

where the contour C is circle from $-i$ to $-i$ (counterclockwise) as shown in the figure.

$C: z(t) = e^{it}, \quad -\pi/2 \le t \le 3\pi/2$

$z'(t) = ie^{it}$

$$\int_C f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$



Contour Integrals

Example3: Evaluate the integral

$$\int_C z\,dz$$

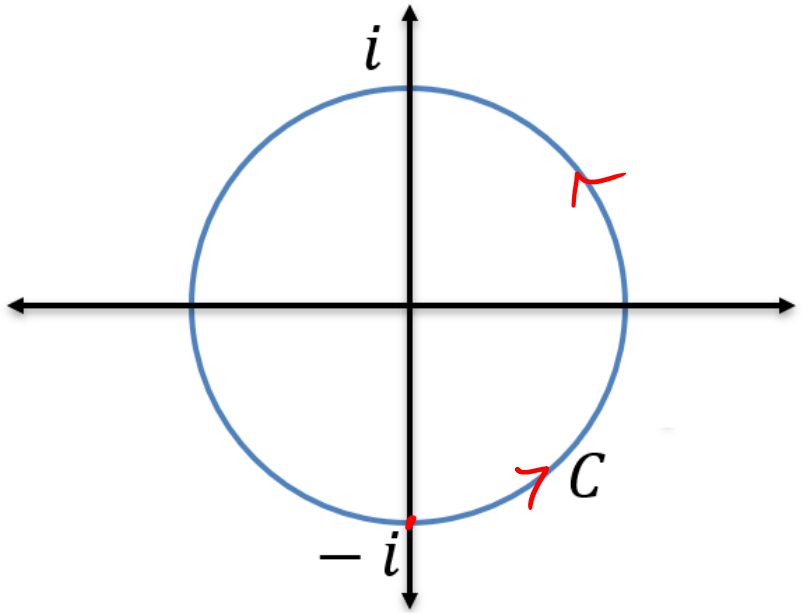
where the contour C is circle from $-i$ to $-i$ (counterclockwise) as shown in the figure.

$$\int_C z\,dz = i\int_{-\pi/2}^{3\pi/2} e^{it}\,e^{it}\,dt$$

$$= i\left[\frac{e^{i2t}}{2i}\right]_{-\pi/2}^{3\pi/2}$$

$$= i\left[e^{i3\pi} - e^{-i\pi}\right]$$

$$= i\left[-1-(-1)\right] = 0$$



Contour Integrals

When a contour integral around a closed contour is zero?

Contour Integrals

Contour Integrals
Examples
Part III

$$\int_C$$

Contour Integrals

Contour Integrals
Examples
Part IV
(Contours with a branch cut passing through them)

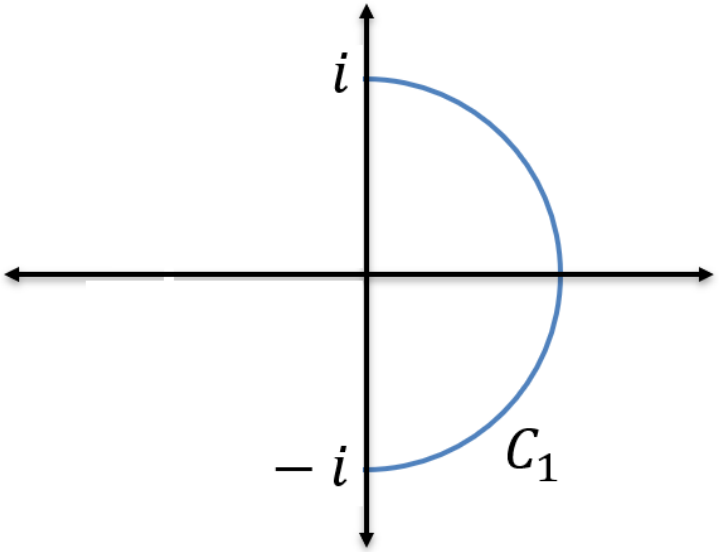
$$\int_C$$

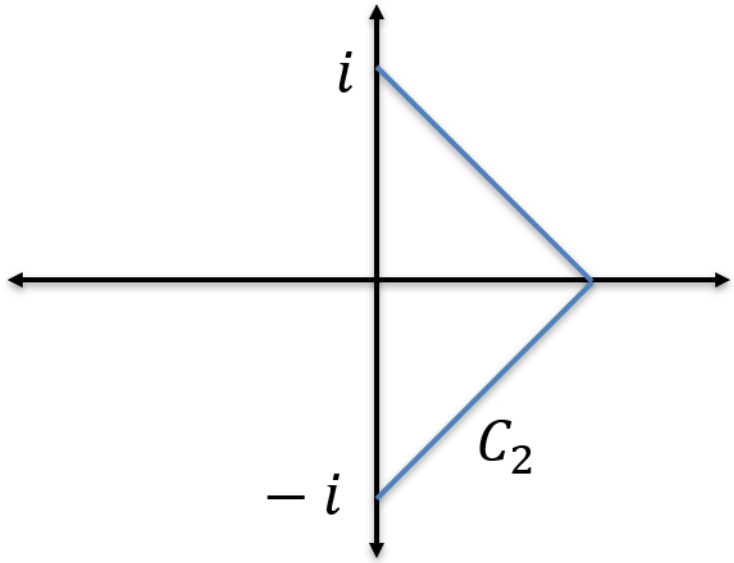
Contour Integrals

Recall: Example1: Evaluate the integrals

$$\int_{C_1} z\,dz = 0 \quad and \quad \int_{C_2} z\,dz = 0$$

where the contour C_1 is semicircle from $-i$ to i (counterclockwise), and C_2 is the polygonal path from $-i$ to i as shown in the figure.



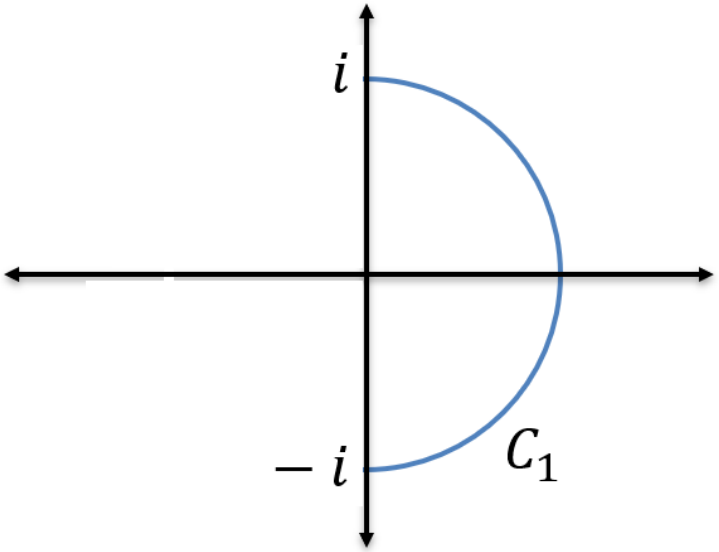


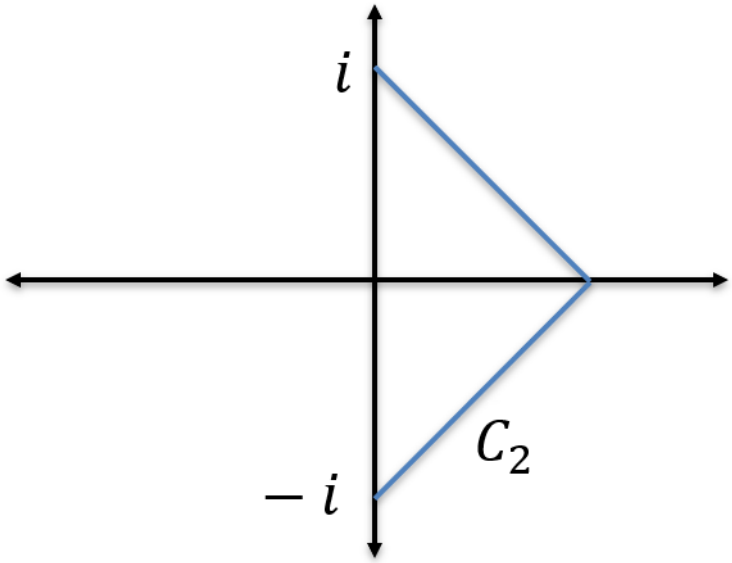
Contour Integrals

Recall: Example2: Evaluate the integrals

$$\int_{C_1} \bar{z}\,dz \quad and \quad \int_{C_2} \bar{z}\,dz$$

where the contour C_1 is semicircle from $-i$ to i (counterclockwise), and C_2 is the polygonal path from $-i$ to i as shown in the figure.



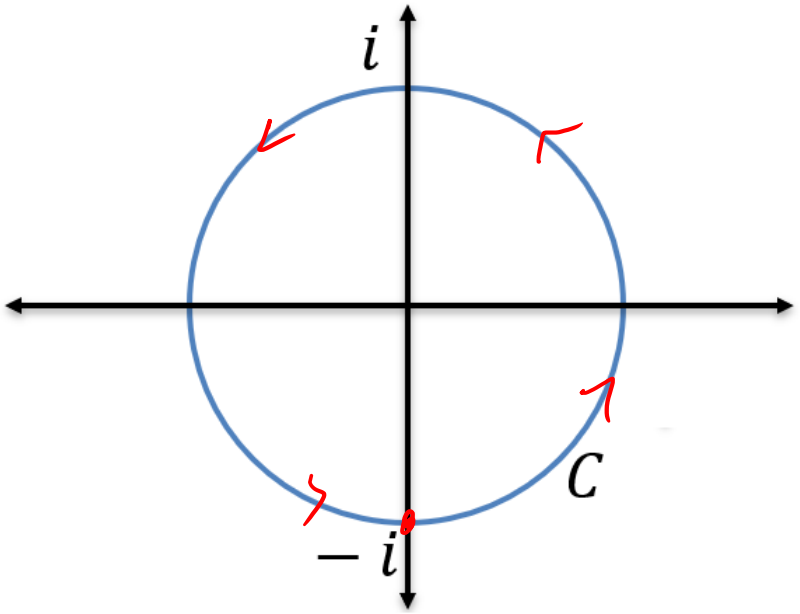


Contour Integrals

Recall: Example3: Evaluate the integrals

$$\int_C z\,dz$$

where the contour C is circle from $-i$ to $-i$ (counterclockwise) as shown in the figure.



Contour Integrals

When a contour integral is independent of path?

When a contour integral around a closed contour is zero?



Contour Integrals

How to compute contour integrals if a branch cut passes through our contour?

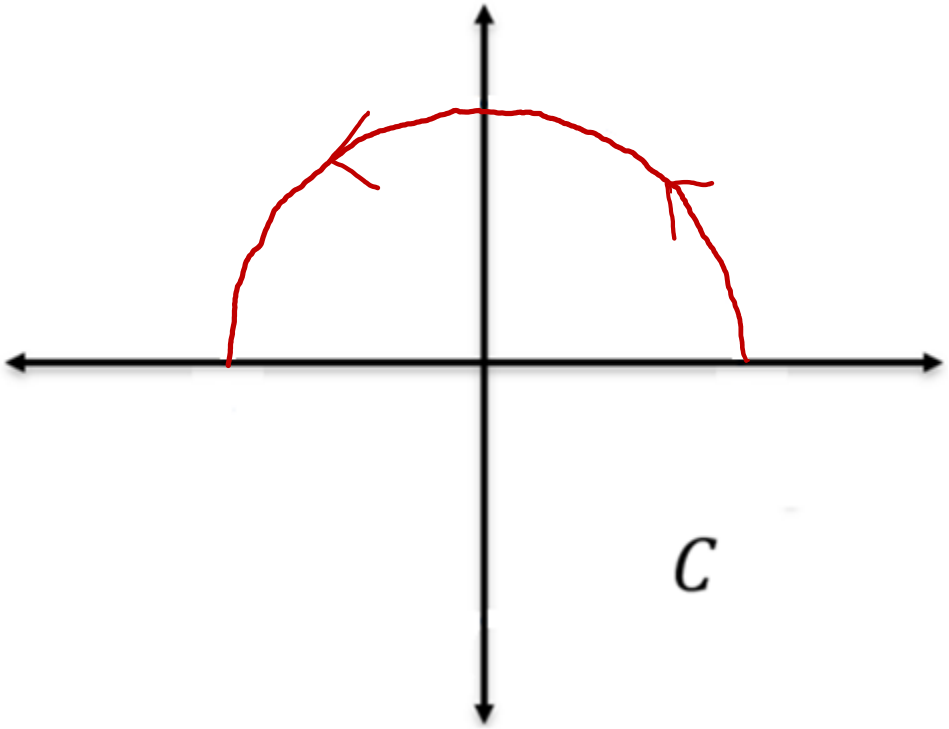


Contour Integrals

Example4: Evaluate the integral

$$\int_C z^a \, dz$$

where the contour C is a semicircle from 2 to -2 (counterclockwise) as shown in the figure.



Contour Integrals

Solution: We know that

$$z^a = \exp(a \log(z))$$

Contour Integrals

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$$z^a = \exp(a \log(z))$$

and $\log(z)$ is a multivalued function defined as

$$\log(z) = \ln|z| + i \arg(z),$$

Contour Integrals

Solution: We know that

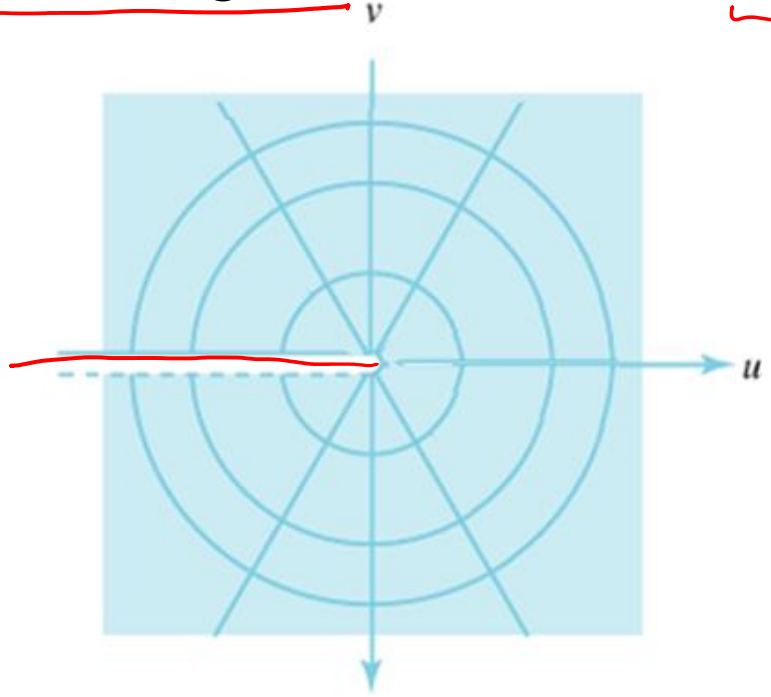
$$z^a = \exp(a \log(z))$$

and $\log(z)$ is a multivalued function defined as

$$\log(z) = \ln|z| + i \arg(z),$$

and the principal branch of $\log(z)$ is given by

$$\text{Log}\,(z) = \ln|z| + i\,Arg\,(z), \qquad where\ -\pi < Arg\,(z) \leq \pi$$



Contour Integrals

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$$z^a = \exp(a \log(z))$$

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and the principal branch of $\log(z)$ is given by

$$\mathrm{Log}\,(z) = \ln|z| + i\, Arg\,(z), \qquad where\ -\pi < Arg\,(z) \leq \pi$$

So the principal branch of z^a is given by

$$z^a = \exp(a\, \mathrm{Log}(z))$$

where

$$-\pi < Arg\,(z) \leq \pi$$

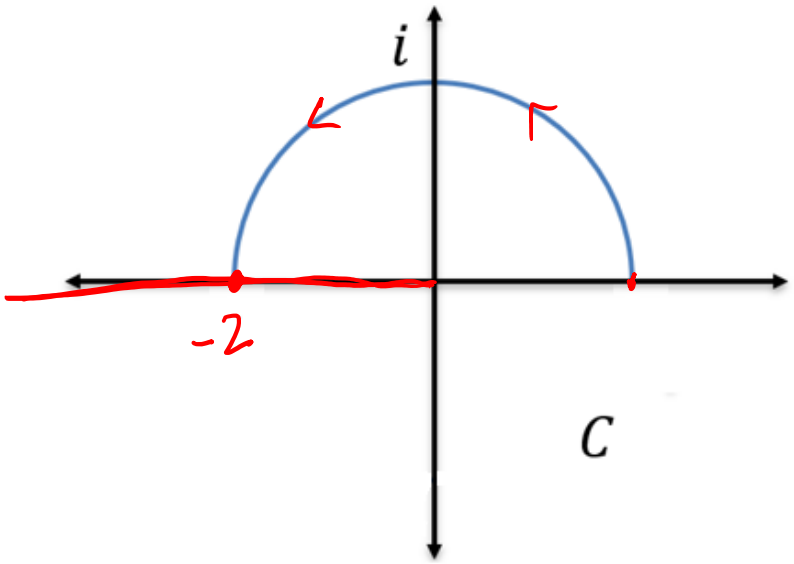
Contour Integrals

Solution (cont.): The contour is given by:

$$C: z(t) = 2e^{i\theta}, \text{ where } 0 \le \theta \le \pi$$

So the contour passes through a branch cut.

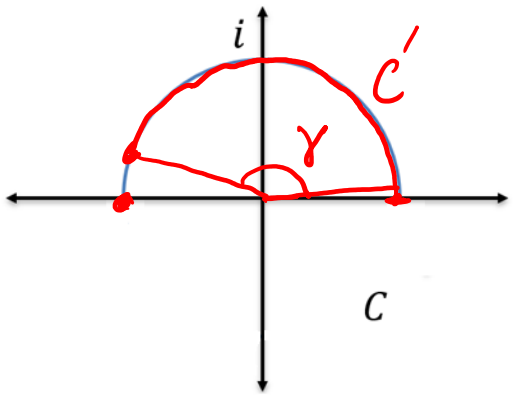
$\int f(z(t))\, z'(t)\, dt$, $f(z) = z^9$



Contour Integrals

Solution (cont.) Method 1

$C' : z(t) = 2e^{i\theta}, \quad 0 \le \theta \le \gamma, \quad 0 < \gamma < \pi$



$$\int_{C'} z^a dz = \int_0^{\gamma} (2e^{i\theta})^a (2i\, e^{i\theta})\, d\theta$$

$$= 2^{a+1} i \int_0^{\gamma} e^{i(a+1)\theta}\, d\theta$$

$$= 2^{a+1} i \left[\frac{e^{i(a+1)\theta}}{(a+1)i} \right]_0^{\gamma}$$

Aim

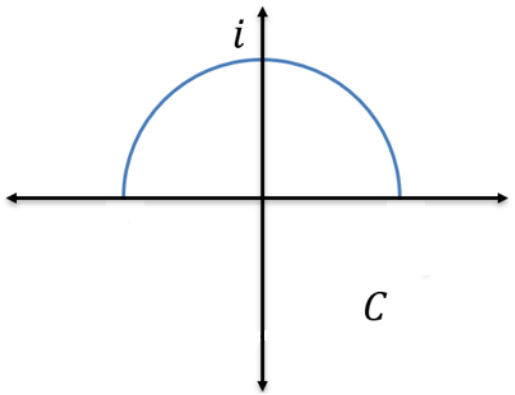
$$\lim_{\gamma \to \pi} \int_{C'} z^a dz = \int_C z^a dz$$

Contour Integrals

Solution (cont.) Method 1

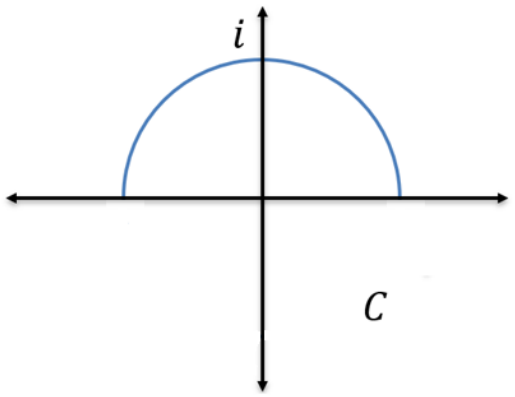
$$= \frac{2^{a+1} \cdot i}{(a+1)i} \left[e^{i(a+1)\gamma} - e^{0} \right]$$

$$\int_{C'} z^a \, dz = \frac{2^{a+1}}{a+1} \left[e^{i(a+1)\gamma} - 1 \right]$$



Contour Integrals

Solution (cont.) Method 1



$$\int_C z^a dz = \lim_{r \to \pi} \frac{r^{a+1}}{a+1}\left[e^{i(a+1)\theta} - 1 \right]$$

$$= \frac{2^{a+1}}{a+1}\left[e^{i(a+1)\pi} - 1 \right]$$

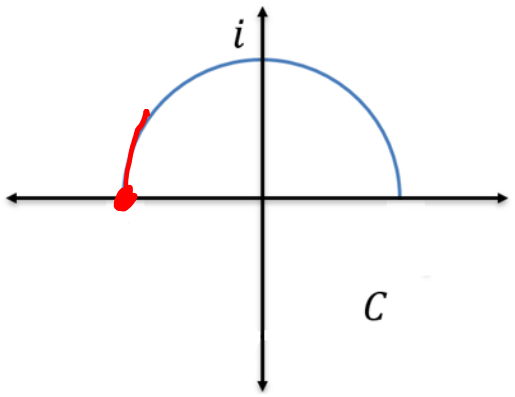
If a is integer

$$\int_C z^a dz = \frac{2^{a+1}}{a+1}\left[\cos(a+1)\pi + \sin(a+1)\pi - 1 \right]$$

$$= \frac{2^{a+1}}{a+1}\left[(-1)^{a+1} - 1 \right]$$ A.

Contour Integrals

Solution (cont.) Method 2



$\int_C z^a dz, \quad C: z(t) = 2e^{i\theta}, \quad 0 \le \theta \le \pi$

$$\int_C f(z)dz = \int_a^b f(z(t))\,z'(t)\,dt$$

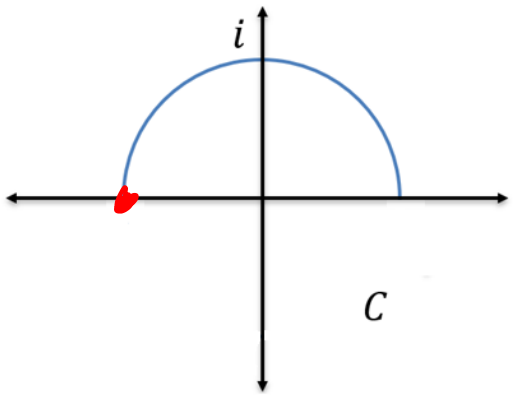
Now we assign value to $f(z(t))z'(t)$ at $t=\pi$ such that $f(z(t))z'(t)$ is piecewise Cont. on $[0,\pi]$

$$f(z(t))z'(t) = (2e^{i\theta})^a (2ie^{i\theta}) = 2^{a+1} i e^{i(a+1)\theta}$$ ✓

$$\lim_{\theta \to \pi^-} 2^{a+1} i e^{i(a+1)\theta} = 2^{a+1} i e^{i(a+1)\pi} = f(z(\pi))z'(\pi)$$

Contour Integrals

Solution (cont.) Method 2

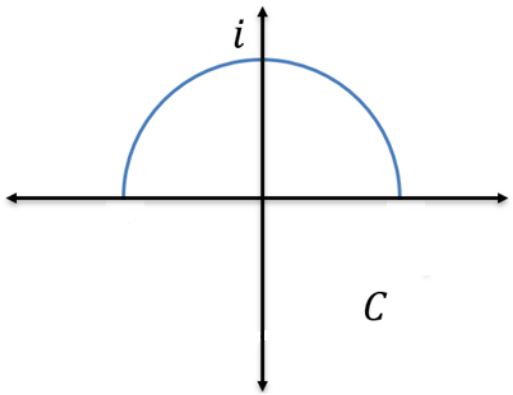


$f(z(t))z'(t)$ is left-hand cont. at $\theta = \pi$

$\Rightarrow \int_0^{\pi} f(z(t))z'(t)\,dt$ can be evaluated

Contour Integrals

Solution (cont.) Method 2



$$\int_C z^a dz = \int_0^{\pi} (e^{i\theta})^{a+1} \, i e^{i\theta} d\theta$$

$$= i^{a+1} \int_0^{\pi} e^{i(a+1)\theta} d\theta$$

$$= i^{a+1} \left[\frac{e^{i(a+1)\theta}}{i(a+1)} \right]_0^{\pi}$$

$$= \frac{i^{a+1}}{(a+1)i} \left[e^{i(a+1)\pi} - e^{0} \right] = \frac{i^{a+1}}{a+1} \left[e^{i(a+1)\pi} - 1 \right]$$

Contour Integrals

Similar procedure can be applied to compute the following contour integral

$$\int_C z^a \, dz$$

where

$$C: z(t) = 2e^{i\theta} \text{ for } -\pi \leq \theta \leq \pi \, .$$

Same But different

Contour Integrals

Contour Integrals
Examples
Part IV
(Contours with a branch cut passing through them)

$$\int_C$$

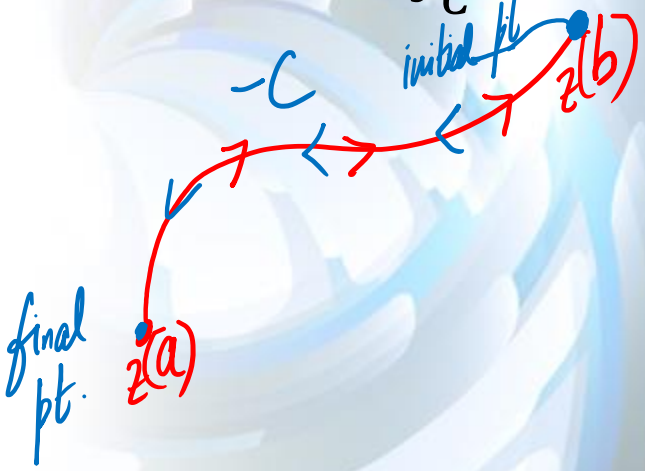
Properties of Contour Integrals

Properties of contour integrals

$$\int_C$$

Properties of Contour Integrals

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$$



Properties of Contour Integrals

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$$

Properties of Contour Integrals

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$$

Consider a curve C

$$C: z_1(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

Properties of Contour Integrals

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$$

Consider a curve C

$$C: z_1(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

The opposite curve $-C$ is given by

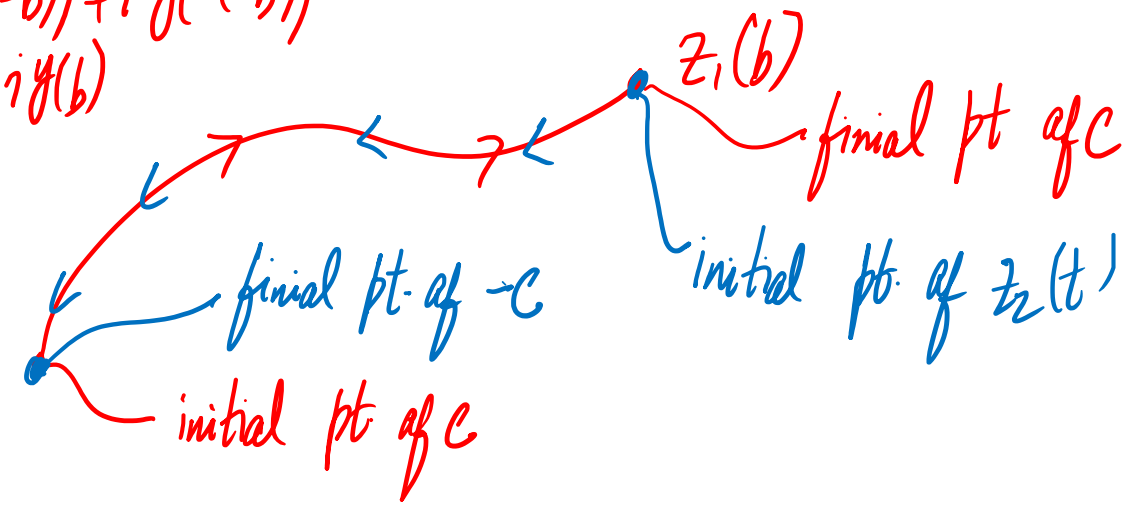
$$-C: z_2(t) = x(-t) + iy(-t), \qquad \text{for} \quad -b \le t \le -a$$

$z_2(-b) = x(-(-b)) + i\,y(-(-b))$

$= x(b) + iy(b)$

$= z_1(b)$

$z_2(-a) = z_1(a)$



Properties of Contour Integrals

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$$

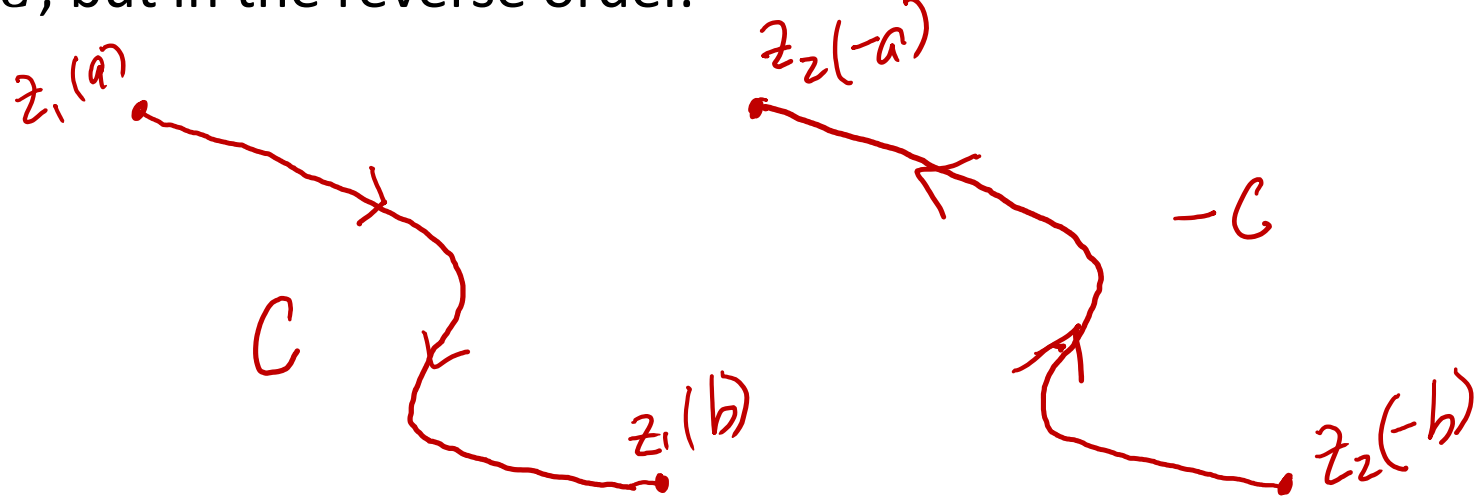
Consider a curve C

$$C: z_1(t) = x(t) + iy(t), \qquad \text{for} \quad a \le t \le b$$

The opposite curve $-C$ is given by

$$-C: z_2(t) = x(-t) + iy(-t), \qquad \text{for} \quad -b \le t \le -a$$

Geometrically $-C$ traces out the same set of points in the plane as C, but in the reverse order.



Properties of Contour Integrals

Proof($\int_C f(z)\,dz = -\int_{-C} f(z)\,dz$)

Here

$$C: z_1(t) = x(t) + iy(t), \quad \text{for} \quad a \le t \le b$$
$$-C: z_2(t) = x(-t) + iy(-t), \quad \text{for} \quad -b \le t \le -a$$

Hence

$$z_2(-t) = z_1(t)$$

$$z_2(-t) = x(-(-t)) + i\,y(-(-t))$$
$$= x(t) + i\,y(t)$$
$$= z_1(t)$$

Properties of Contour Integrals

Proof cont. ($\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$)

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f\big(z_2(t)\big)\frac{d}{dt}\big(z_2(t)\big)dt$$

Properties of Contour Integrals

Proof cont. ($\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$)

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f\big(z_2(t)\big)\frac{d}{dt}\big(z_2(t)\big)dt$$

Take $t = -s$

$$\int_{-C} f(z)\,dz = \int_b^a f\big(z_2(-s)\big)\left[\frac{d}{d(-s)}\big(z_2(-s)\big)\right](-ds)$$

$t = -b \Rightarrow t = -s$

$-b = -s$

$\Rightarrow \boxed{s = b}$

$t = -s$

$dt = -ds$

Properties of Contour Integrals

Proof cont. ($\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$)

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f\big(z_2(t)\big)\frac{d}{dt}\big(z_2(t)\big)dt$$

Take $t = -s$

$$\int_{-C} f(z)\,dz = \int_b^a f\big(z_2(-s)\big)\left[\frac{d}{d(-s)}\big(z_2(-s)\big)\right](-ds)$$

$$= -\int_b^a f\big(z_1(s)\big)\left[\frac{d}{d(-s)}\big(z_1(s)\big)\right]ds$$

Properties of Contour Integrals

Proof cont. ($\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$)

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f\big(z_2(t)\big)\frac{d}{dt}\big(z_2(t)\big)dt$$

Take $t = -s$

$$\int_{-C} f(z)\,dz = \int_b^a f\big(z_2(-s)\big)\left[\frac{d}{d(-s)}\big(z_2(-s)\big)\right](-ds)$$

$$= -\int_b^a f\big(z_1(s)\big)\left[\frac{d}{d(-s)}\big(z_1(s)\big)\right]ds$$

$$= \int_a^b f\big(z_1(s)\big)\left[\frac{d}{ds}\big(z_1(s)\big)\,\frac{d\,s}{d(-s)}\right]ds$$

Properties of Contour Integrals

Proof cont. ($\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$)

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f\big(z_2(t)\big)\frac{d}{dt}\big(z_2(t)\big)dt$$

Take $t = -s$

$$\int_{-C} f(z)\,dz = \int_b^a f\big(z_2(-s)\big)\left[\frac{d}{d(-s)}\big(z_2(-s)\big)\right](-ds)$$

$$= -\int_b^a f\big(z_1(s)\big)\left[\frac{d}{d(-s)}\big(z_1(s)\big)\right]ds$$

$$= \int_a^b f\big(z_1(s)\big)\left[\frac{d}{ds}\big(z_1(s)\big)\,\frac{d\,s}{d(-s)}\right]ds$$

$$= \int_a^b f\big(z_1(s)\big)\left[\frac{d}{ds}\big(z_1(s)\big)\;1\Big/\left(\frac{d\,(-s)}{d(s)}\right)\right]ds$$

Properties of Contour Integrals

Proof cont. ($\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$)

$$\int_{-C} f(z)\,dz = \int_{-b}^{-a} f\big(z_2(t)\big)\frac{d}{dt}\big(z_2(t)\big)dt$$

Take $t = -s$

$$\int_{-C} f(z)\,dz = \int_b^a f\big(z_2(-s)\big)\left[\frac{d}{d(-s)}\big(z_2(-s)\big)\right](-ds)$$

$$= -\int_b^a f\big(z_1(s)\big)\left[\frac{d}{d(-s)}\big(z_1(s)\big)\right]ds$$

$$= \int_a^b f\big(z_1(s)\big)\left[\frac{d}{ds}\big(z_1(s)\big)\,\frac{d\,s}{d(-s)}\right]ds$$

$$= \int_a^b f\big(z_1(s)\big)\left[\frac{d}{ds}\big(z_1(s)\big)\;1/\left(\frac{d\,(-s)}{d(s)}\right)\right]ds$$

$$= -\int_a^b f\big(z_1(s)\big)\,z_1'(s)\,ds = -\int_C f(z)\,dz$$

Properties of Contour Integrals

$$\int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$$ ✓

Example:

$\int_C f(z)\,dz$, $C: z(\theta) = e^{i\theta}$, $0 \le \theta \le 2\pi$

$\int_{-C} f(z)\,dz$, $-C: z(-\theta) = e^{-i\theta}$, $-2\pi \le \theta \le 0$

$$\Rightarrow \int_{-C} f(z)\,dz = -\int_{C} f(z)\,dz$$ ✓

Properties of Contour Integrals

$$\int_C \big(f(z)+g(z)\big)dz = \int_C f(z)dz + \int_C g(z)dz$$

Example: $f(z)=z^2$, $g(z)=\bar{z}^2$

C is simple closed contour in plane.

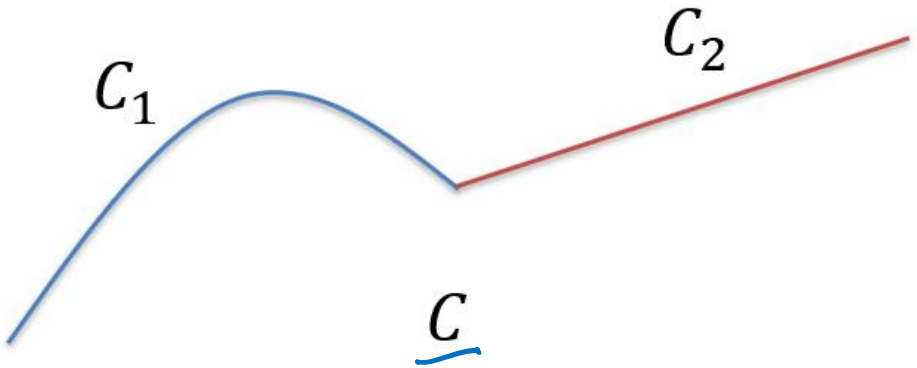
$$\int_C z^2+\bar{z}^2 = \int_C z^2\,dz + \int_C \bar{z}^2\,dz$$

Properties of Contour Integrals

$$\int_C (a+ib)\, f(z)dz = (a+ib)\int_C f(z)dz$$

Properties of Contour Integrals

What happens if a contour consists of many contours (called legs)



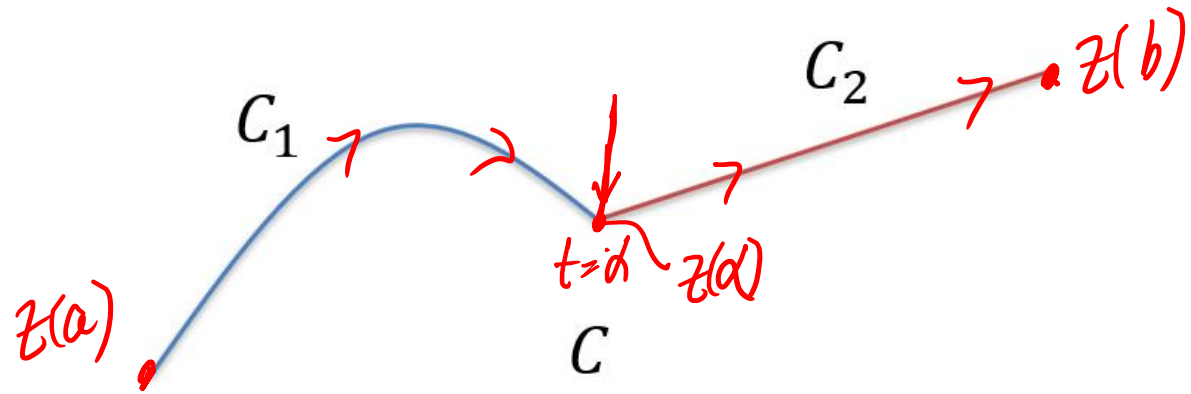
Properties of Contour Integrals

What happens if a contour consists of many contours (called legs)

Consider a contour

$$C: z(t), \qquad a \le t \le b$$

such that C has two legs, as shown in the figure



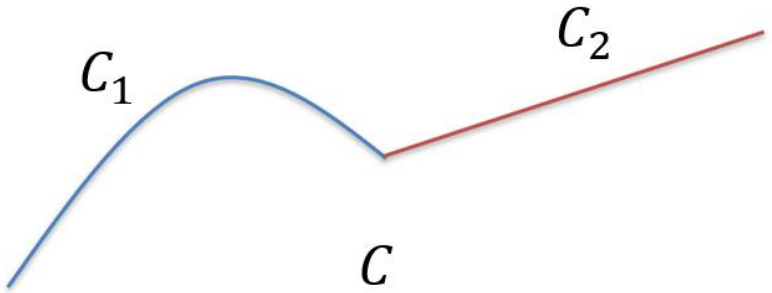
There exist a value α, such that $a \le \alpha \le b$ and

$$C_1: z(t), \qquad a \le t \le \alpha$$
$$C_2: z(t), \qquad \alpha \le t \le b$$

Properties of Contour Integrals

$$\int_{C_1+C_2} f(z)dz$$

$$= \int_C f(z)dz$$

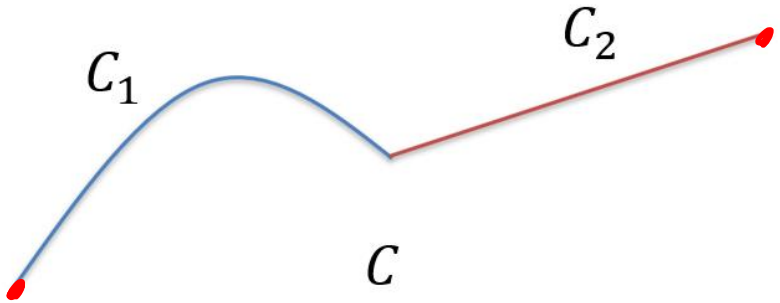


Properties of Contour Integrals

$$\int_{C_1+C_2} f(z)dz$$

$$= \int_{C} f(z)dz$$

$$= \int_{a}^{b} f\big(z(t)\big)\, z'(t)dz$$



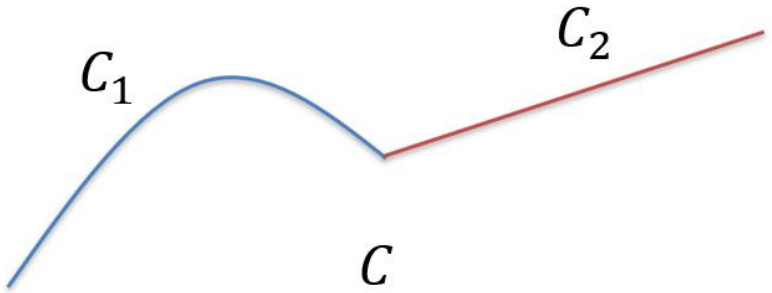
Properties of Contour Integrals

$$\int_{C_1+C_2} f(z)dz$$

$$= \int_{C} f(z)dz$$

$$= \int_{a}^{b} f\big(z(t)\big)\, z'(t)dz$$

$$= \int_{a}^{\alpha} f\big(z(t)\big)\, z'(t)dz + \int_{\alpha}^{b} f\big(z(t)\big)\, z'(t)dz$$



Properties of Contour Integrals

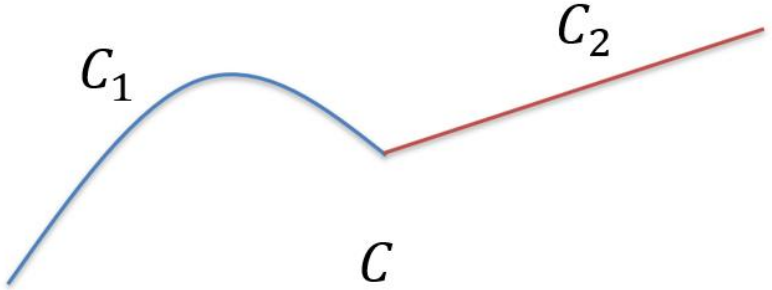
$$\int_{C_1+C_2} f(z)dz$$

$$= \int_{C} f(z)dz$$

$$= \int_{a}^{b} f\big(z(t)\big)\, z'(t)dz$$

$$= \int_{a}^{\alpha} f\big(z(t)\big)\, z'(t)dz + \int_{\alpha}^{b} f\big(z(t)\big)\, z'(t)dz$$

$$= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$



Properties of Contour Integrals

Hence we have

$$\int_{C_1+C_2} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$



Properties of Contour Integrals

Effect of
reparameterization
of a contour
on contour integral

Properties of Contour Integrals

Effect of reparameterization of a contour on contour integral

Let C be a contour given by

$$C: z_1(t) = x_1(t) + iy_1(t), \qquad a \le t \le b$$

Properties of Contour Integrals

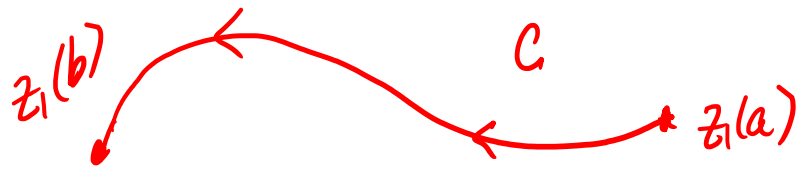
Effect of reparameterization of a contour on contour integral

Let C be a contour given by

$$C: z_1(t) = x_1(t) + iy_1(t), \qquad a \le t \le b$$

A reparameterization of C, given by

$$C: z_2(s) = x_2(s) + iy_2(s), \qquad c \le s \le d$$





Properties of Contour Integrals

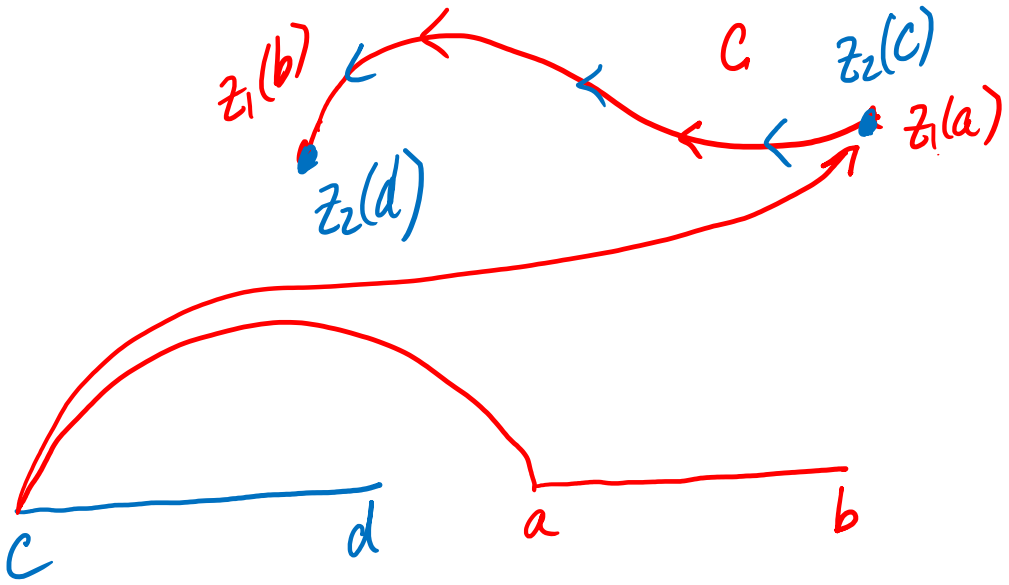
Effect of reparameterization of a contour on contour integral

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A reparameterization of C, given by

$$C: z_2(s) = x_2(s) + iy_2(s), \qquad c \le s \le d$$



Properties of Contour Integrals

Effect of reparameterization of a contour on contour integral

Let C be a contour given by

$$C: z_1(t) = x_1(t) + iy_1(t), \qquad a \le t \le b$$

A reparameterization of C, given by

$$C: z_2(s) = x_2(s) + iy_2(s), \qquad c \le s \le d$$

There exist a differentiable function t $= h(s)$, such that

$$h(a) = c, \qquad h(b) = d \quad \text{and} \quad h'(s) > 0$$

and

$$z_1(t) = z_1\big(h(s)\big) = z_2(s)$$

Properties of Contour Integrals

Effect of reparameterization of a contour on contour integral

Let C be a contour given by

$$C: z_1(t) = x_1(t) + iy_1(t), \qquad a \le t \le b$$

A reparameterization of C, given by

$$C: z_2(s) = x_2(s) + iy_2(s), \qquad c \le s \le d$$

There exist a differentiable function t $= h(s)$, such that

$$h(a) = c, \qquad h(b) = d \quad \text{and} \quad h'(s) > 0$$

and

$$z_1(t) = z_1\big(h(s)\big) = z_2(s)$$

Then

$$\int_a^b f\big(z_1(t)\big)z_1'(t)dt = \int_c^d f\big(z_2(s)\big)z_2'(s)\,ds$$

Properties of Contour Integrals

Properties of contour integrals

$$\int_C$$

Inequalities Involving Contour Integrals

Inequalities involving moduli of contour integrals

$$\left| \int_C f(z)\, dz \right|$$

$$\int_C \frac{\cos z}{z^2+1} \sin z \tan z \, dz$$

Inequalities Involving Contour Integrals

Absolute value inequality

For a piecewise continuous complex-valued function of real variable t

$$f(t) = u(t) + i\,v(t)$$
$$a \le t \le b$$

We have

$$\left| \int_a^b f(t)dt \right| \le \int_a^b |f(t)|dt$$

Inequalities Involving Contour Integrals

Theorem (Absolute value inequality)

For a piecewise continuous complex-valued function of real variable t

$$f(t) = u(t) + i\,v(t), \qquad a \le t \le b$$

We have

$$\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt$$

Proof:

Inequalities Involving Contour Integrals

Proof (Absolute value inequality)

If $\int_a^b f(t)\,dt = 0$ ✓ trivial

Assume $\int_a^b f(t)\,dt \neq 0$

$$\int_a^b f(t)\,dt = r_0 e^{i\theta_0} \quad \text{—— (*)}$$

$$\Rightarrow r_0 = e^{-i\theta_0}\int_a^b f(t)\,dt = \int_a^b e^{-i\theta_0} f(t)\,dt \quad \text{— (I)}$$

$$r_0 = \left| r_0 e^{i\theta_0} \right| = \left| \int_a^b f(t)\,dt \right| \quad \text{— (II)}$$

Inequalities Involving Contour Integrals

Proof (Absolute value inequality)

$$I_0 = \int_a^b e^{-i\theta} f(t)\,dt \quad \checkmark$$

$$\Rightarrow \quad I_0 = \mathrm{Re}(I_0) = \mathrm{Re}\left[\int_a^b e^{-i\theta_0} f(t)\,dt\right] = \int_a^b \mathrm{Re}\left[e^{-i\theta_0} f(t)\right]dt \qquad (**)$$

$$\mathrm{Re}\left[e^{-i\theta_0} f(t)\right] \le \left|e^{-i\theta_0} f(t)\right| = |f(t)| \quad \checkmark$$

$$I_0 = \int_a^b \mathrm{Re}\left[e^{-i\theta_0} f(t)\right]dt \le \int_a^b |f(t)|\,dt$$

Since $I_0 = \left|\int_a^b f(t)\,dt\right| \Rightarrow \left|\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt\right|$

$h_1(t) \le h_2(t)$ $\forall\, t \in [a,b]$

$\int_a^b h_1(t)\,dt \le \int_a^b h_2(t)\,dt$

Inequalities Involving Contour Integrals

Theorem (ML inequality)

If $f(z)$ is continuous on contour C, then

$$\left| \int_C f(z)\, dz \right| \leq ML$$

Where L is length of the contour and M is upper bound for the modulus $|f(z)|$ on C, that is,

$$|f(z)| \leq M, \quad for\ all\ z \in C\ .$$

Proof:



Inequalities Involving Contour Integrals

Proof ($\left|\int_C f(z)\,dz\right| \le ML$) ✓

$$\left|\int_C f(z)\,dz\right| = \left|\int_a^b f(z(t))\,z'(t)\,dt\right|$$

$$\le \int_a^b \left|f(z(t))\,z'(t)\right|dt,$$

$$\le \int_a^b M\,|z'(t)|\,dt.$$

$$= M\int_a^b |z'(t)|\,dt = ML$$

L

$$\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$$

$|f(z)| \le M \quad \forall\, z \in C$

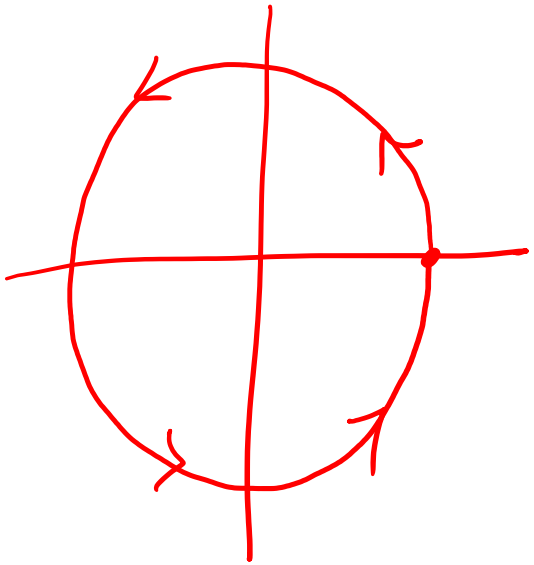
Inequalities Involving Contour Integrals

Example ($\left|\int_C f(z)\,dz\right| \le ML$): Find upper bound of

$$\int_C \bar{z}\,dz$$

where $C: z(t) = e^{it},\ 0 \le t \le 2\pi.$

$\left|\int_C \bar{z}\,dz\right| = ?$, $L = 2\pi$



$|f(z)| \le M,\ \forall\ z \in C$

$f(z) = \bar{z}$, $z = e^{it}$

$f(e^{it}) = \overline{e^{it}} = \cos t - i\sin t$, $|f(e^{it})| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$\Rightarrow M = 1$ $\Rightarrow$ $\left|\int_C f(z)\,dz\right| \le 1(2\pi) = 2\pi$

Inequalities Involving Contour Integrals

Example: C: arc of the circle $|z|=2$, from $z=2$ to $z=2i$

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \le \frac{6\pi}{7}$$

$$|z+4| \le |z|+4 = 6$$

$$|z^3-1| \ge |\,|z|^3-1| = 7$$

$$\Rightarrow \left| \frac{z+4}{z^3-1} \right| = \frac{|z+4|}{|z^3-1|} \le \frac{6}{7}$$

Hints

Inequalities Involving Contour Integrals

Inequalities involving moduli of Contour integrals

$$\left| \int_C f(z)\, dz \right|$$

LECTURE 39
MODULE 97

THE CAUCHY GOURSAT THEOREM

PART 1

From this module, we start to discuss most powerful and beautiful result of complex analysis. It is Cauchy Goursat theorem. It is a contribution of two French mathematicians – Augustin Louis Cauchy and Edouard Goursat.

It is a kind of series of two results. First result was given by Cauchy, known as Cauchy Theorem in 1825, then Goursat generalized it in 1883. So the most generalized version of these results is the Cauchy Goursat Theorem.

This result actually links the two branches of mathematics – Topology and Complex Analysis. When we will discuss it, we will see some topological terms involved in it. These topological terms will define the domain in which we will work.

If the domain is simply connected, then it is a basic requirement of this theorem.

If the domain is simply connected, then the contour integral around the simple closed curve is 0. That is more or less a statement of Cauchy –Goursat Theorem.

This theorem has many important consequences.

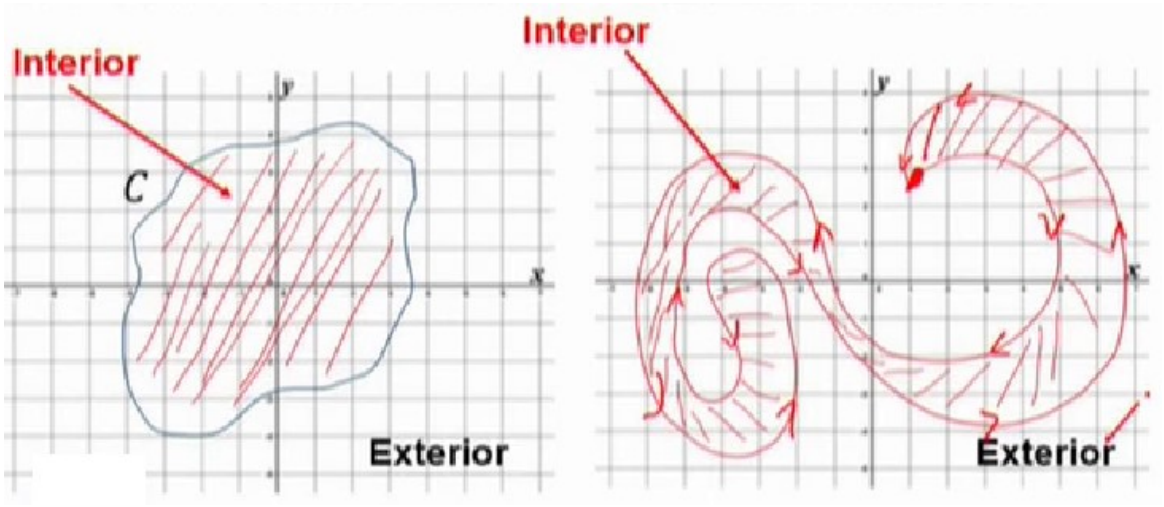
For example, it helps us in solving the contour integral if the contour is extremely complicated. Similarly, it gives us the conditions under which an analytic function has an antiderivative. So it will help us prove that an analytic function has antiderivative.

Let us recall Topology.

INTERIOR AND EXTERIOR OF A CURVE C

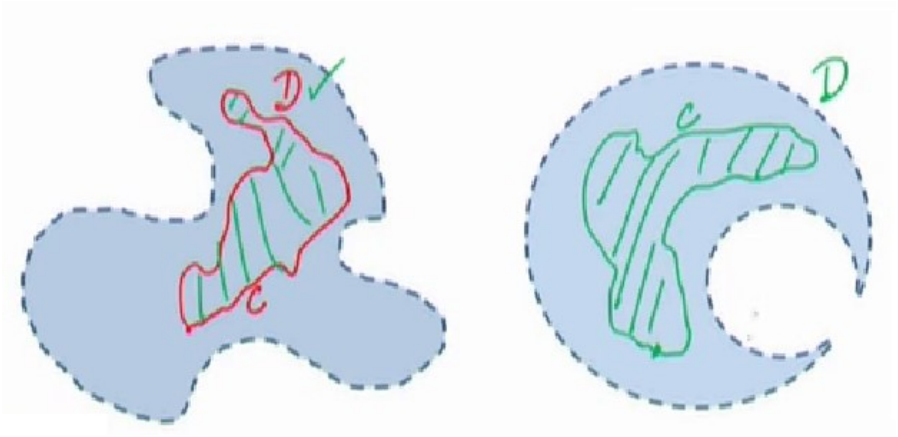
(JORDAN CURVE THEOREM)

Each simple closed curve C divides the plane into two domains (connected open sets). One is bounded and is called interior of C. The other is unbounded and is called the exterior of C.

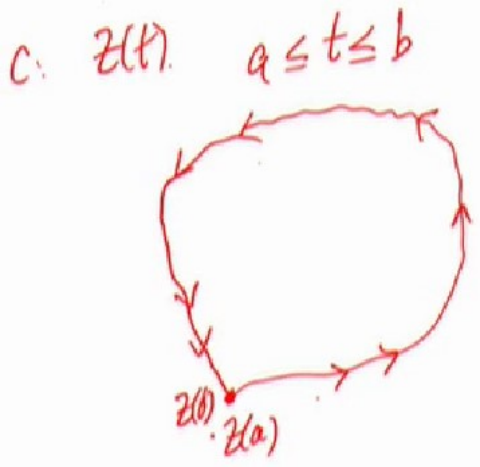


SIMPLY CONNECTED DOMAIN

A domain *D* (connected open set) is called simply connected if interior of every simple closed contour C is contained in *D*.

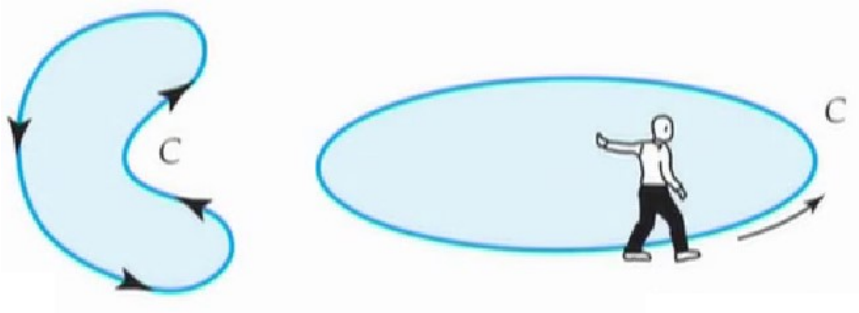


A domain D (connected open set) that is not simply connected is called multiply connected or doubly connected domain.



Negative direction of a contour C is the direction opposite to the positive direction.

In case of simple closed contour positive orientation, roughly speaking, corresponds to counter-clockwise direction. The negative direction corresponds to clockwise direction.



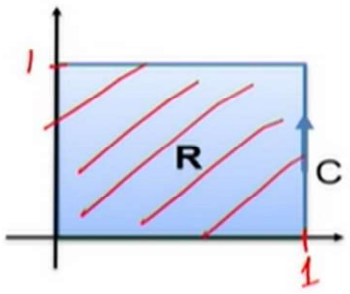
If C is positively oriented, then - C is negatively oriented.

Now we see one of the important theorem from the multivariable Calculus, which we need to prove Cauchy-Goursat Theorem; it is Green Theorem.

GREEN THEOREM

Let C be a simple closed contour with positive orientation and let R be the domain that forms the interior of C. If P and Q are continuous and have continuous partial derivatives P_x, P_y, Q_x, Q_y at all points on C and R, then

$$\int_C P(x,y)dx + Q(x,y)dy = \iint_R \big(Q_x(x,y) - P_y(x,y)\big)dx\,dy$$



In order to evaluate this line integral, we will convert it into surface integral.

$$\int_C P\,dx + Q\,dy = \int_C y\,dx + x\,dy$$

$$\therefore \quad P = y, \quad Q = x$$

$$\int_C P\,dx + Q\,dy = \int_0^1 \int_0^1 \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\ dy$$

$$= \int_0^1 \int_0^1 \left(\frac{\partial x}{\partial x} - \frac{\partial y}{\partial y} \right) dx\ dy$$

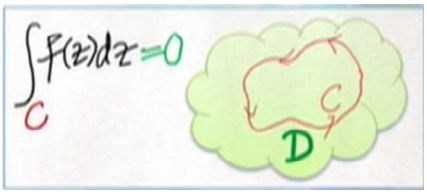
$$= \int_0^1 \int_0^1 (1-1)\ dx\ dy$$

$$= \int_0^1 \int_0^1 0\ dx\ dy = 0$$

CAUCHY's THEOREM

Let *f* be analytic in a simply connected domain *D* and *f '* is continuous in *D*. If *C* is a simple closed contour that lies in *D*, then

$$\int_C f(z)\ dz = 0$$



This is Cauchy's theorem introduced by Cauchy in 1825. In 1883, Goursat generalized this theorem. Goursat removed the condition of "*f '* is continuous in *D*".

We will prove Cauchy Theorem in the next module.

LECTURE 39
MODULE 98

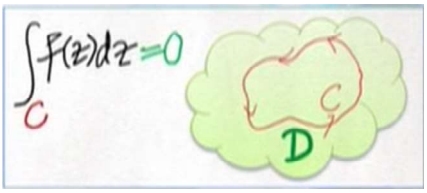
THE CAUCHY GOURSAT THEOREM

PART II

CAUCHY's THEOREM

Let f be analytic in a simply connected domain D and f' is continuous in D. If C is a simple closed contour that lies in D, then

$$\int_C f(z)\, dz = 0$$



PROOF

$f(z) = u(z) + i\, v(z)$

The curve $C : z(t) = x(t) + i\, y(t), \quad a \le t \le b$ is a positively oriented curve.

$$\int_C f(z)\, dz = \int_a^b f(z(t))\, z'(t)\, dt$$

$$= \int_a^b \left[u(z(t)) + i\, v(z(t))\right] \left(x'(t) + i\, y'(t)\right) dt$$

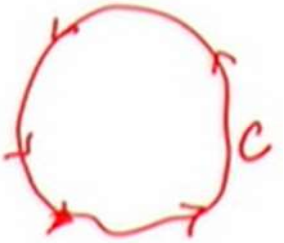
$$= \int_a^b \left(u\, x' - v\, y'\right)\, dt + i \int_a^b \left(u\, y' + v\, x'\right) dt$$

$$= \int_a^b \left(u\, x'\right)\, dt - v\, y'\, dt + i \int_a^b u\, y'\, dt + v\, x'\, dt$$

Since $x = x(t) \Rightarrow dx = x'\, dt$

Since $y = y(t) \Rightarrow dy = y'\, dt$

$$\int_C f(z)\,dz = \int_a^b u\,dx - v\,dy + i\int_a^b u\,dy + v\,dx \quad \text{____(1)}$$



By Green's Theorem,

Since $\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dx\,dy$

So, $$\int_C u\,dx - v\,dy = \iint_R \left(\frac{\partial(-v)}{\partial x} - \frac{\partial u}{\partial y}\right)dx\,dy$$
$$= \iint_R \left(-v_x - u_y\right)dx\,dy \qquad (2)$$

Similarly, $$\int_C u\,dy + v\,dx = \int_C v\,dx + u\,dy$$
$$= \iint_R \left(\frac{\partial(u)}{\partial x} - \frac{\partial v}{\partial y}\right)dx\,dy$$
$$= \iint_R \left(u_x - v_y\right)dx\,dy \qquad (3)$$

Use the results (2) and (3) in (1), we get

$$\int_C f(z)\,dz = \int_a^b u\,dx - v\,dy + i\int_a^b u\,dy + v\,dx$$
$$= \iint_R \left(-v_x - u_y\right)dx\,dy + \iint_R \left(u_x - v_y\right)dx\,dy$$

f' exists, so f satisfies CR.equations,

$$u_x = v_y,\ \ u_y = -v_x$$

$$\Rightarrow \int_C f(z)\,dz = \iint_R \left(u_y - u_y\right)dx\,dy + \iint_R \left(u_x - u_x\right)dx\,dy$$
$$= \iint_R (0)\,dx\,dy + \iint_R (0)\,dx\,dy$$
$$= 0 + 0$$
$$= 0$$

This is the case when C is positively oriented.

Now we see the case when C is negatively oriented.

When C is negatively oriented, then $-C$ is positively oriented.

$$\int_{-C} f(z)\, dz = -\int_{C} f(z)\, dz = 0$$

Hence the result is the same for the negatively oriented curve C.

EXAMPLE: Show that

$$\int_{C} \exp\left(z^2 + 1\right) dz = 0$$

Where C is a simply closed contour in complex plane.

SOLUTION:

Since $\exp(z)$ and $z^2 + 1$ are entire functions.

$$f(z) = \exp\left(z^2 + 1\right)$$
$$f'(z) = \exp\left(z^2 + 1\right)\ (2z)$$

We know that the complex plane is simply connected.

And the given function is analytic, so all the conditions of Cauchy's theorem are satisfied.

$$\Rightarrow \quad \int_{C} \exp\left(z^2 + 1\right) dz = 0$$

EXAMPLE: Show that

$$\int_{C} \left(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\right) dz = 0$$

Where C is a simply closed contour in complex plane.

SOLUTION:

Since $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$ are entire functions.

We know that the complex plane is simply connected.

And the given function is analytic, so all the conditions of Cauchy's theorem are satisfied.

$$\Rightarrow \int_C \left(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0\right) dz = 0$$