DATA for FINAL TERM ---- MTH634 ---- Module# 086 to 183

Module : 86-89

A map f: $(X,T_X) \longrightarrow (Y,T_Y)$ is called a closed map iff the image of each closed subset of X is closed in Y.

Projection map may or may not be closed.

Is a continuous map an open map. (May or may not be Open)

Is an open map a continuous map. (May or may not be Continuous)

Restriction of an open map may or may not be an open map.

Module : 90-100

Homeomorphism: A map f: X ⟶ Y is called a Homeomorphism between two topological space (X,Tx) and (Y,Ty) iff 1. " f " is bijective. 2. " f " is continuous 3. F^{-1} : Y ⟶ X or "f" is open map

Homeomorphic Space: Two topological space are called (X,Tx) and (Y,Ty) are called homeomorphic iff there exist a homeomorphism f: X ⟶ Y we also say such spaces are topologically equivalent spaces i.e . x ≅ y

Equivalence Relation: A binary relation ~ on a set X is said to be equivalence relation iff ~ satisfied the following three properties for all a,b,c ∈ X. (1) Reflexive: i.e $a \sim a$ (2) Symmetric: i.e $a \sim b \ iff \ b \sim a$ (3) Transitive: $if \ a \sim b \ and \ b \sim c \ then \ a \sim c$

Homomorphism of topological space is an equivalence relation.

A property "p" of a topological space (X,T) is called topological property if every space homeomorphic to X has property "p" i.e. A property that is unchanged under homeomorphism.

Topological property is also called topological invariant.

Some topological properties are:

*_Cardinality of set X. *_Cardinality of T. *_Connectedness of space X. *_Being discrete.

Some non-topological properties are:

*_ Length *_ Area

A metric is a distance function.

A function that defines a distance between each pair of elements of a set.

Metric: A metric on a non empty set X is a real valued function d defines on $X \times X$ i.e. $d: X \times X \longrightarrow R$

Metric Space: A set X with a metric "d" defined on it is called a metric space.

Usual metric on ℝ is $d(x,y) = \sqrt{(x-y)^2}$

Usual metric on $\mathbb{R}^2$: Consider $\mathbb{R}^2$ and "d" is defined as $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$ where $x = (x_1, x_2),\ y = (y_1, y_2) \in \mathbb{R}^2$

Usual metric on $\mathbb{R}^n$: Consider $\mathbb{R}^n$ and "d" is defined as $d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 \ldots\ldots (x_n-y_n)^2}$ where $x = (x_1, x_2, \ldots\ldots\ldots\ldots x_n),\ y = (y_1, y_{2\ldots\ldots\ldots\ldots} y_n) \in \mathbb{R}^n$

Module : 101-107

Consider X with a metric d. For any $x \in X$ and any real number x>0, the set of all points $y \in X$ whose distance from x is less than r is called open ball of radius r and centered at x i.e. B(x,y)={ $y \in X | d(x,y) < r$}

Prepare Page # 33-34-35

Metric Topology: Let X be a non empty set with metric d. The topology T on X generated by the set of all open balls in X with respect to d is called metric topology.

A topology T on X induced by metric d.

Consider X with a metric d, this d induces a topology “metric topology”. (x,d) is called **metric space.**

The topology generated by the set of all singletons is discrete topology.

Module : 108-110

Metrizable Space: A topological space (X,T) is called Metrizable space if there exists a metric d on X that induces topology T.

Not all spaces are Metrizable.

Let X be an infinite set. Consider cofinite topology on X then it is not Metrizable space.

$\mathbb{R}$ with lower limit topology is not Metrizable space.

Module : 111-112

A topological space (X,T) is called a **First Countable Space** if for each $p \in X$ there exists a countable local base at p.

Every subspace of first countable space is first countable.

Discrete space is first countable space.

Every finite set with any topology is first countable.

$\mathbb{R}$ with usual topology is first countable i.e $(\mathbb{R}, Tu)$ is first countable.

Every metric space is first countable.

Module : 113-114

A topological space (X,T) is called **Second Countable Space** if there exist a countable a countable basis for T.

Every subspace of a second countable space is second countable.

Let X be a countable set with discrete topology then (X, T_{dis}) is second countable.

Any finite set with any topology is second countable.

$\mathbb{R}$ with usual topology is second countable i.e $(\mathbb{R}, Tu)$ is second countable.

Module : 115-116

Second accountability implies first accountability.

Let a topological space (X,T) is second countable space then it is also first countable space.

First accountability does not imply second accountability.

Consider $\mathbb{R}$ with usual topology then it is both first countable and second countable.

Consider $\mathbb{R}$ with lower limit topology then it is first countable but not second countable.

A first countable space may or may not be second countable.

Module : 117-119

Cover: A cover of a set X is a collection of sets U={Ui | $i \in$ I} where I is index set such that $x \subset U_{i \in \mathrm{I}} U_i$

Cover: Let (X,T) be topological space. A cover of a X is a collection of subsets of X i.e. $U = \{U_i | \; U_i \subset X \; i \in I\}$ where I is index set such that $X = U_{i \in \mathrm{I}} U_i$

Open Cover: Let (X,T) be topological space. An open cover of a X is a collection of subsets of X i.e. $U = \{U_i | \; U_i \in T \; i \in I\}$ where I is index set such that $X = U_{i \in \mathrm{I}} U_i$

Module : 120-121

Lindelof Space: A topological space (X,T) is called a Lindelof space iff every open cover of X has a countable sub cover.

Let X be a finite set with any topology then it is Lindelof.

A set with indiscrete topology then it is Lindelof.

X= $\mathbb{R}$ with usual topology is Lindelof.

A subspace of Lindelof space needs not to be Lindelof.

Every countable space is second Lindelof. (ALSO PREPARE ITS PROOF FROM P#69-70)

Module : 122-125

A topological space (X,T) is said to be separable if there exist a countable dense subset A of X.

Every second countable space is separable.

$\mathbb{R}$ with lower limit topology is separable but not second countable.

Let X be a finite set then X with any topology is separable.

Consider a set X with indiscrete topology the X is separable.

$\mathbb{R}$ with usual topology is separable.

$\mathbb{R}$ with discrete topology is not separable.

A discrete space is separable iff it is countable. **[ALSO PROOF FROM PAGE # 74]**

X is separable iff X is countable.

(X,T_{cof}) is separable. **[ALSO PROOF FROM PAGE # 74-75]**

In general subspace of separable is not separable.

Module : 126

A metric space is not a second countable and not separable in general.

An infinite set X with trivial metric is not second countable and not separable.

A separable metric space is second countable. **[ALSO PROOF FROM PAGE # 78]**

Module : 127-143

There are 7 separation axioms. T_0-Spaces, T_1-Spaces, T_2-Spaces (Hausdorff), T_3-Spaces, Normal Spaces,T_4-Spaces,

T_0-Space: A topological space (X,T) is said to be **T_0-Space** iff for each $x, y \in X$ there exist an open set $U_x \subset X$ containing X such that $y \notin U_x$ OR There exist an open set $V_y \subset X$ containing y such that $x \notin V_y$.

T_0 Property is a topological property.

T_1-Space: A topological space (X,T) is said to be **T_1-Space** iff for each $x, y \in X$ such that $x \neq y$ there exist open sets U_x, U_y *of* X containing x,y respectively such that $y \notin U_x$ *and* $x \notin U_y$.

T_1-Propery is a topology property.

Every discrete space is T_1 space.

Every metric space is T_1 space.

Every cofinite space is T_1 space.

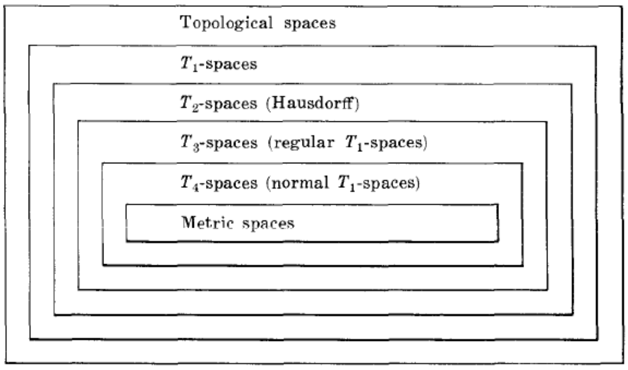
T_1 property is much stronger property than T_0 property.

A topological space (X,T) is T_1 iff every singleton subset $\{x\} \subset X$ is closed.**[ALSO PROOF FROM PAGE # 90]**

Every subspace of a T_1 space is also a T_1 space. **[ALSO PROOF FROM PAGE # 92]**
Every finite subset of T_1 space is closed.
Let A be a finite subset of a T_1 space X then $\acute{A}$ is empty.
T_2-Space: A topological space (X,T) is said to be **T_2-Space** iff for every distinct pair of element $x, y \in X$ there exist open sets $U_x, U_y\ of\ X$ containing x,y respectively such that $U_x \cap U_y = \phi$.
T_2-space is also called Hausdorff discovered by Felix Hausdorff (1868-1942)
T_2 property is a topological property.
Every discrete space is Hausdorff.
A nonempty set $X \neq \{x\}$ with indiscrete topology is not Hausdorff.
$\mathbb{R}$ with lower limit topology is Hausdorff .
$\mathbb{R}$ with topology generated $\{(a, \infty) \| a \in \mathbb{R}\}$ is not Hausdorff.
An infinite set with cofinite topology is not Hausdorff space.
The property of being a Hausdorff space is inherently.
Let (X,T) is a Hausdorff space and $A \subset X$ then (A,T_A) is Hausdorff. **[ALSO PROOF FROM PAGE # 102]**
An infinite set X with cofinite topology is not Metrizable.
Every metric space is Hausdorff. **[ALSO PROOF FROM PAGE # 104]**
T_2 property is a stronger property than T_1 property.
Every T_2 space is T_1 space but every T_1 space is not T_2 space.
An infinite set X with cofinite topology is T_1 space but not T_2 space.
Let X be a Hausdorff space then every convergent sequence of points has a unique limit point.
Module : 144
A topological space (X,T) is said to be regular iff for every closed subset F of X and for every $x \in F^c$ $\exists$ open subsets U_F and U_x containing F and x such that $U_F \cap U_x = \phi$
A topological space (X,T) is said to be "T_3 space" iff it is either Regular+T_0 or Regular+T_1 or Regular+T_2.
Every T_3 space will be regular but every regular space will not T_3 space.
Property of being regular is a topological property.
Every subspace of regular space is a regular space.
Every subspace of a T1 space is also a T1 space i.e. T1 space is heredity.
T_3 property is a topological property.
Every subspace of a T_3 space is a T_3 space.
Module : 145
Discrete space is a regular space.
Indiscrete space is a regular space.
A discrete topology space has a complete power set.
Every subset of discrete space is also closed as well as open so open collection is also close collection.
$\mathbb{R}$ with usual property is regular and this space is also T_3 space.
An infinite set with co finite topology is not regular so it is not T_3.

K-Topology on real line topology T_K.....($\mathbb{R}, Tk$) is not a regular space.
Module : 146
A topological space is said to be normal iff for every pair of disjoint closed subset $F_1, F_2 \subset X$, there exist open subset U_{F1} and U_{F2} containing F_1 and F_2 respectively such that $U_{F1} \cap U_{F2} = \phi$
A topological space (X,T) is said to be T_4 space iff it is either Normal +T_1 or Normal+T_2
Property of being normal is a topology property.
Subspace of normal space needs not to be normal space.
T_4 space is a topology property.
A topological space (X,T) is said to be complete normal space iff every subspace of X with subspace topology is normal.
Module : 147
Consider: x={a,b,c,d} T={ ϕ ,{d},{b,d},{c,d},{b,c,d},X} Claim: (X,T) is normal space but (X,T) is not T_4.
Consider: y={b,c,d} T={ ϕ ,{d},{b,d},{c,d},X} Claim: (y,T_4) is not a normal space.
Consider: x={a,b,c,d} T={ ϕ ,{d},{b,d},{c,d},{b,c,d},X} Claim: (X,T_4) is not completely normal space.
($\mathbb{R}$,T_4) is normal space.
$\mathbb{R}$ with usual topology is a T_4 space.
Module : 148
Metric spaces are T_4 space.
Every metric space is a Hausdorff space.
Every metric space is normal.
All the metric spaces are normal spaces.
Every separable metric space is second countable.
Module : 149
T_i properties: Topological space gives T0-spaces gives T1-spaces gives T2-spaces (Hausdorff) gives T3-spaces (Regular T1 spaces) gives T4-spaces (Normal T1-spaces) gives Metric Spaces.

diagram illustrates the relationship between the spaces discussed in this chapter.



Module : 150

Urysohn's Lemma is a classical and important result of topology.

An important application of Urysohn's Lemma is the partial solution of Metrization Theorem.

Urysohn's Metrization Theorem : Every **normal** T1 space with countable basis is metrizable.

Urysohn's Metrization Theorem : Every **regular** T1 space with countable basis is metrizable.

Module : 151

Cover: Let (X,T) be a topological space, Let C={Ci} be a class of subsets of X such that

$X=\cup_{i\in I}C_i$ then C is called a cover of X.

Subcover: A subclass S of a cover C of X which is also a cover is called a subcover.

Open Cover: A cover C of X is said to be Open Cover iff $C \subset T$

Compact Space: A topological space (X,T) is said to be compact if every open cover C of X contains a finite subcover S of X.

Module : 152

A set X with indiscrete topology T is compact.

A set X with topology T containing finite number subsets of X is compact.

A finite set X with topology T is compact.

A finite set X with discrete topology T is not compact.

Module : 153

An open interval in ℝ with respect to usual topology is not compact.

ℝ with usual topology/ $(\mathbb{R}, T_u)$ is not compact.

Module : 154

A set X with co finite topology T_{cof} is compact.

Proof:

If X is finite then $T_{cof}=P(X)$

$\Longrightarrow (X,T_{cof})$ is compact

A set X with cofinite topology is compact.

Module : 155

ℝ with usual topology/ $(\mathbb{R},T_u)$ is not compact.

ℝ with usual topology/ $(\mathbb{R},T_{indis})$ is compact.

ℝ with usual topology/ $(\mathbb{R},T_{dis})$ is not compact.

$\mathbb{R}$ with usual topology/ $(\mathbb{R},T_{cof})$ is compact.

Module : 156

Every closed subspace of a compact space is compact.

If (X,T) is compact space and A be a closed subset of X then (A,T_A) will be closed subspace of X where T_A is subspace topology on A.

Module : 157

If (A,T_A) be compact subspace of a Hausdorff space (X,T) then A is closed in X.

Every compact subspace of a Hausdorff space is closed.

Module : 158

If (X,T) be a compact space and f: $(X,T)\longrightarrow(Y,T^{/})$ be continuous map then f(x) is compact.

Image of compact space under a continuous map is compact.

Module : 159

Let (X,T) be a compact space and $(Y,T^{/})$ be a Hausdorff space then a continuous map f: $(X,T)\longrightarrow(Y,T^{/})$ is a closed map.

Let (X,T) be a compact space and $(Y,T^{/})$ be a Hausdorff space then a bijective continuous map f: $(X,T)\longrightarrow(Y,T^{/})$ is a homeomorphism.

Module : 160

Consider $\mathbb{R}$ with usual topology/ $(\mathbb{R}, T_u)$, there exist no homeomorphism between an open interval of $\mathbb{R}$ and closed interval of $\mathbb{R}$.

Open interval in $\mathbb{R}$ with usual topology/ $(\mathbb{R}, T_u)$ are not compact.

The only compact subset of $\mathbb{R}$ are closed and bounded subsets [a,b] compact and (c,d) not compact.

$[0,1)\neq S^1$ since S^1 Compact and $[0,1)$ not compact.

Module : 161

Let (X,T) be a topological space and $\{(X_i,T_i)|i\in I\}$ be a finite family of compact subspace of X then $\cup_{i\in I} X_i$ is compact.

Module : 162

Let (X,T) be a compact Hausdorff space then (X,T) is normal.

For every pair of disjoint closed subsets F_1, F_2 of a compact Hausdorff space X there exist a pair of disjoint open subsets V_{F1}, V_{F2} of X containing F_1,F_2 respectively.

Module : 163

Let A and B be two subsets of a topological space (X,T) then A and B are said to be separated sets if and only if

$A\cap B = \phi \text{ and } \acute{A}\cap B = \phi \text{ and } A\cap \acute{B} = \phi$

Let A and B be two subsets of a topological space (X,T) then A and B are said to be separated sets if and only if there exists open subsets U_A and U_B of X containing A and B respectively.

$A\cap U_B = \phi \text{ and } B\cap U_A = \phi$

Let A and B be two subsets of a topological space (X,T) then A and B are said to be separated sets if and only if

$\bar{A}\cap B = \phi \text{ and } A\cap \bar{B} = \phi$

Module : 164

Consider ℝ with usual topology/(ℝ, T_u) then A=(0,1) and B=(4,9] are separated sets.

Consider ℝ with usual topology/(ℝ, T_u) then A=(-1,3] and B=[3,5) are not separated sets.

Consider ℝ with usual topology/(ℝ, T_u) then A=(-1,3) and B=[3,5) are not separated sets.

Module : 165

Connected Set:

A subset A of a topological space (X,T) is said to be connected if and only if there exist no pair of non empty open subset of U and V of X such that $A\cap U$ *and* $A\cap V$ are non empty disjoint sets and $A = (A\cap U)U(A\cap V)$

Disconnected Set:

A subset A of a topological space (X,T) is said to be disconnected if it is not connected i.e If there exist a pair of non empty open subset of U and V of X such that $A\cap U$ *and* $A\cap V$ are non empty disjoint sets and $A = (A\cap U)U(A\cap V)$

Connected Space (Definition I):

A topological space (X,T) is said to be connected space if it cannot be written as union of two non empty open disjoint subsets U and V of X.

Disconnected Space:

A topological space (X,T) is said to be disconnected space if it can be written as union of two non empty open disjoint subsets U and V of X.

Connected Space (Definition II):

A topological space (X,T) is said to be connected space if $X = D_1 U D_2$ such that D_1, D_2 both open and disjoint i.e. $D_1 \cap D_2 = \phi$ implies $D_1 = \phi$ *or* $D_2 = \phi$

Module : 166

Consider ℝ with usual topology/(ℝ, T_u) then A=(0,1)U(3,5] is disconnected subset of ℝ

Consider X={a,b,c,d,e} with $T = \{\phi, \{a\}, \{a,b\}, \{a,b,c\}, \{a,d,e\}, \{a,b,d,e\}, X\}$ then $A = \{c,e\}$ is disconnected subset of X.

Consider X={a,b,c,d,e} with $T = \{\phi, \{a,b\}, \{a,b,c\}, X\}$ then X is a connected space.

Consider X={a,b,c,d,e} with $T = \{\phi, \{a,b\}, \{d,e\}, \{a,b,c\}, \{a,b,d,e\}, X\}$ then X is a disconnected space.

Module : 167

Connected Space:

A topological space (X,T) is said to be connected iff $X = D_1 U D_2$ such that D_1, D_2 both open and disjoint implies $D_1 = \phi$ *or* $D_2 = \phi$

Disconnected Space:

A topological space (X,T) is said to be disconnected space iff $X = D_1 U D_2$ such that D_1, D_2 both open and disjoint implies that neither D_1 is empty nor D_2 is empty.

Module : 168-183

Let A be connected subset of a topological space (X,T) and $A \subset B \subseteq \bar{A}$ then B is connected.**[PROOF PAGE # 166]**

Image of a connected space under a continuous map is connected. **[ALSO SEE PROOF PAGE # 168]**

Prepare Page # 170 and 171
A subset of A of $(\mathbb{R}, Tu)$ is connected iff A is an interval.
Fixed Point: A point $x \in X$ is said to be a fixed point of a map $f: X \longrightarrow X$ iff $f(x) = x$
Let A and B be two connected subsets of a topological spaces (X,T) such that $x \in A \cap B$ then $C = AUB$ is connected subset of X. **[ALSO SEE PROOF Page#179-180]**
Consider a topological space (X,T), Let $a \in X$ and C be a connected subset of X containing a then $C_a = U_{a \in C} C$ is called connected component of X containing a.
The connected component Ca is the largest connected subset of X containing a.
Prepare Page#182
Let C be connected component of X then $C = \bar{C}$
A connected component is a closed subset of X.
In general a connected component C of X is not open.
A topological space (X,T) is locally connected iff it is locally connected at each of its points.
Prepare Page#186
Locally connectedness **does not imply** connectedness
Connectedness **does not imply** locally connectedness
A topological space (X,T) is said to be connected iff for every $x, y \in X$ there exist a path P from x to y in X.
Let (X,T) be a path connected topological space then (X,T) is connected.
Path connectedness **implies** connectedness.
Connectedness **does not imply** Path connectedness.
Let (X,T) be a path connected topology space and $f: (X, T) \longrightarrow (y, \acute{T})$ be a continuous map then f(x) is path connected.