Mth622

Vectors and Classical Mechanics.
(Important Topics Final Term Notes)

Module # 97:-

Conservation Force Field

Path Independence;

Path Independence is defined as that the work done by an object remains same between Initial point and the Final destination of the particle, it does'nt Matter which path is taken by the object.

=> The Symbolic Statement of path Independence is;

$$W = \int_{P_1}^{P_2} F \cdot d\vec{r} = V(P_1) - V(P_2)$$

∴ work done by the object depend only on the end points of the curve P_1 & P_2.

=> **Conservative Force Field;**

A Force Field is said to conservative if the total work done by the Particle

Moving along a curve is Independent of the path taken by the particle and depend upon the end points of the curve only"?

=> **Necessary and Sufficient Conditions for a Conservative Field:-**

(i) A Force Field is Conservative iff there exist a Continously differentable Scalar Field 'V' such that $\vec{F} = -\nabla V$

or $\text{Curl}\,\vec{F} = \nabla \times \vec{F} = 0$

(ii) A Continuously differentiable Force Field F is conservative iff any closed non-Intersecting Curve C (simple close curve)

$$W = \oint_c F \cdot d\bar{r} = 0$$

=> The Total work done in Moving a particle around any closed path is Zero.

=> **Examples;**

- Gravitational Force is an Example of Conservative Force.

=> Elastic Force etc

Module # 98:-
Example of Conservative Field.

Problem;

Prove that the Force Field

$\vec{F} = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2 z^3)\hat{j} + (6z^3 - 3x^2 y z^2)\hat{k}$

is Conservative.

Solution;

Force Field is Conservative

iff Curl $\vec{F}$ is zero;

$\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 - 2xyz^3 & 3 + 2xy - x^2 z^3 & 6z^3 - 3x^2 y z^2 \end{vmatrix}$$

$= (-3x^2 z^2 + 3x^2 z^2)\hat{i} - (-6xyz^2 + 6xyz^2)\hat{j} + (2y - 2xz^2 - 2y + 2xz^2)\hat{k}$

$= 0$

$\nabla \times \vec{F} = 0$

F is Conservative.

(Module # 100)

Non- Conservative Force Fields.

Forces that Cannot be expressed in the term of a ~~particle~~ potential Energy Function are Called Non- Conservative Forces.

=> If there is no Scalar Function V Such that $F = -\nabla V$ or if $\nabla \times \vec{F} \neq 0$, then F is Called Non- Conservative.

=> Friction is Non- Conservative.

=> Impulse (time dependent Force) is also Non- Conservative.

=> The work done in overcoming Friction is always Negative.

=> The Process in which work is Converted into Internal energy (due to Friction) are irreversible.

Module # 101

(Example of Non-Conservative Field)

Example:-

Show that the Force Field given by;

$$\vec{F} = x^2yz\hat{i} - xyz^2\hat{j} + 2xz\hat{k}$$

is non-conservative.

Solution;-

$$F = x^2yz\hat{i} - xyz^2\hat{j} + 2xz\hat{k}$$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2yz & -xyz^2 & 2xz \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2xz) - \frac{\partial}{\partial z}(-xyz^2)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial z}x^2yz - \frac{\partial}{\partial}x(2xz)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(-xyz^2) - \frac{\partial}{\partial y}(x^2yz)\right]$$

$$= (2xyz)\hat{i} - (x^2y - 2z)\hat{j} + (yz^2 + x^2z)\hat{k}$$

$$\nabla \times F \neq 0$$

$\vec{F}$ is Non-Conservative.

(Module # 102)

Introduction to Simple Harmonic Motion & oscillator

SHM is a particular type of oscillation and periodic Motion in which Restoring Force of an object is directly proportional to the displacement of the object acting in opposite direction of displacement.

Restoring Force is given by;

$$F_R = -Kx$$

K is the Constant of proportionality often called the spring constant or Modulus of elasticity.

⇒ By Newton Second law we have SHM;

$$F = ma = m\frac{d^2x}{dt^2} = -Kx$$

OR

$$m\ddot{x} + Kx = 0$$

This vibrating system is called a simple Harmonic oscillator or linear Harmonic oscillator.

Example;

(i) Mass on a Spring;

$$T = 2\pi\sqrt{\frac{m}{k}}$$

This equation expresses that the time period of oscillation is Independent of amplitude as well as the ~~ass~~ acceleration.

(ii) Simple pendulum;

The time period of a mass m attached to a pendulum of length l with gravitational acceleration g is given by;

$$T = 2\pi\sqrt{\frac{l}{g}}$$

(Module # 103)

Amplitude, Time period, Frequency and Energy of S. H. M.

(Amplitude):-

The Maximum distance Covered by the oscillating body in one oscillation or length of a wave Measured From its Mean position.

The amplitude of a pendulum is one-half the distance that the Mass Covered in Moving from one terminal to the other.

The vibrating Sources generates waves, whose amplitude is proportional to the amplitude of the vibtrating Source.

Time period;

Time period is Minimum time Required by a oscillation System to Complete its one cycle of oscillation of the specific System.
(Measured In Seconds)

Frequency:-

The Frequency (f) of an oscillatory system is the Number of oscillations pass through a specific point in one seconds.

Measure in Hertz (Hz).

$$F = \frac{1}{T}$$

⇒ Energy of SHM:-

if T is the kinetic Energy, V is potential Energy and $E = T+V$ the Total Energy of simple ~~SHM~~ Harmonic oscillator,

$$K.E = T = \frac{1}{2} mv^2$$

and $$V = \frac{1}{2} kx^2$$

Total Energy of SHM;

$$E = \frac{1}{2} mv^2 + \frac{1}{2} kx^2$$

Module # 105

Example of Energy of S.H.M.

Q;- Find the total Energy of the force $\vec{F} = -3x\hat{i}$ acting on a simple Harmonic oscillator, wher $\hat{i}$ represents the direction.

⇒ **Solution;**

Total Energy of SHM;

$$E = T + V$$

By Newton 2nd law,

Calculate K.E Energy:

$$F = ma \Rightarrow F = m\frac{dv}{dt}$$

$$F = -3x$$

$$m\frac{dv}{dt} = -3x$$

$$\frac{dv}{dt} = -\frac{3}{m}x$$

Integrating w.r.t 'z'

$$V = -\frac{3}{2m}x^2 + C$$

Assume $V = 0$ Initially, so

$$C = 0$$

we get $\Rightarrow$ $V = \frac{-3}{2m}x^2$

$$\cancel{\boxed{E = T + V}}$$

$$T = \frac{1}{2}mv^2$$

$$= \frac{1}{2}m\left(\frac{-3}{2m}x^2\right)^2$$

$$T = \frac{1}{2}\cancel{m}\frac{9}{4m^{\cancel{2}}}x^4 \Rightarrow \frac{9}{8m}x^4$$

Calculate P.E:

Potential Energy is given by V where $F = -\nabla V$

$$F = -3x\hat{i} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$$

$$\frac{\partial V}{\partial x} = 3x, \quad \frac{\partial V}{\partial y} = 0, \quad \frac{\partial V}{\partial z} = 0$$

After Integration;

$$V = \frac{3}{2}x^2 + C_1$$

Assume ⇒

$V = 0$, So we get $C_1 = 0$

$$P.E = V = \frac{3}{2}kx^2$$

Here; Total Energy;

$$E = \frac{9}{8m}x^4 + \frac{3}{2}kx^2$$

(Module # 06)

Introduction to damped Harmonic oscillator.

The Force acting on a Harmonic osciallator are called daming Forces which tend to decrease the amplitude of the Successive oscillations or simply Force apposing the Motion.

⇒ Damping force is proportional to the velocity.

Module # 108:-

(Euler's Theorem)

Statement:-

"A Rotation of a Rigid Body about a Fixed point of the body is equivalent to a Rotation about a line which passes through the point."

"The line Referred to is called the Instantaneous axis of Rotation. Rotations can be Considered as Finite or Infinitesimal. Finite Rotations Cannot be Represented by the vectors since the Commutative law Fails. However, Infintesimal Rotations Can be Represented by vectors.

(Module # 109)

(Chasle's' Theorem)

Statement;

Chasle's Theorem States that the Most General Rigid Body displacement can be produced by a translation along a line (Called its screw axis) followed by a Rotation about that line.

=> A Rigid Body has six degree of Freedom.

=> By Euler's theorem, three of these are associated with pure Rotation.

=> The Remaining three must be associated with translation.

(Module # 110)

Kinematics of a System of Particles (Space, time & Mass)

"Kinematics is the Branch of Machanics deals with the Moving objects without Reference to the Forces which Cause the Motion".

"Kinematics are the Features of Motion of Concerned with System of particles

Features of Rigid Body Motion;

=> Displacement ★ position ★ velocity

★ linear velocity & Angular velocity

★ Motion of Rigid Body (Translation & Rotations)

* **Space:-**

This is closely Related to the concept of point, position, direction and Displacement.

* **Time:-**

Measurement of time is achieved by use of clock. units of time ~~is~~ are seconds, Hours years etc.

* **Matter;**

Physical objects are compared of "Small bits of Matter" such as atoms and Molecules.

"A Measure of the "Quantity of Matter" Associated with a particle is called its Mass.

units of Mass are grams, kilograms, etc.

(Module # 111)

Concept of Rectilinear Motion of Particles, uniform Rectilinear Motion, uniformly Accelerated Rectilinear Motion

When a Moving particle Remains on a single ~~Motion~~ stright line, the Motion is said to be Rectilinear.

General Equation of Motion;

$$F = ma = m\ddot{x}$$

=> Rectilinear Motion for a particle;

Rectilinear Motion of a body is defined by Considering the two point of a body Covered the same distance in the parallel direction.

=> Rectilinear Motion Can be of two type:-

(i) uniform Rectilinear Motion.

(ii) Non-uniform Rectilinear Motion.

=> Uniform Rectilinear Motion :-

uniform Rectilinear Motion is a type of Motion in which the body Moves with uniform velocity or Zero acceleration.

=> Non-uniform Rectilinear Motion is such type of Motion with variable velocity or Non-zero acceleration.

⇒ uniformly Accelerated Rectilinear Motion;

uniformly accelerated Rectilinear Motion is a Special case of Non-uniform Rectilinear Motion along a line is that which Arises when an object is Subjected to Constant acceleration.

This kind of Motion is Called uniformly accelerated Motion.

uniformly accelerated Motion in a type of Motion in which the velocity of an object changes by an equal amount in Every equal Interval of time.

($F = mg$)

(Module #112)

Concept of Curvilinear Motion of particles.

The Motion of a particle Moving in a curved path is Called Curvilinear Motion

Tangential and Normal unit vectors are usually denoted

$\bar{e}_t$ and $\bar{e}_n$ Respectively.

⇒ velocity of Curvilinear Motion;

if the tangential and Normal unit vectors are $\vec{e}_t$ and $\vec{e}_n$ Respectively, then the velocity will be:

$$\vec{V} = \frac{ds}{dt}\vec{e}_t$$

OR $\quad v = vT$

⇒ Acceleration of Curvilinear Motion;

if the tangential and Normal unit vectors are $\vec{e}_t$ and $\vec{e}_n$ Respectively, then the acceleration will be:

$$\vec{a} = \frac{d^2s}{dt^2}\vec{e}_t + \frac{(ds/dt)^2}{\rho}e_n$$

OR $\quad a = T\frac{dv}{dt} + \frac{v^2}{r}N$

(Module # 113)

Example Related to Curvilinear Coordinate.

⇒ Cylindrical Coordinate system is orthogonal.

Example;

prove

$$\frac{de_\rho}{dt} = \varphi \dot{e}_\varphi$$

and $$\frac{de_\varphi}{dt} = -\dot{\varphi} e_\rho$$

where dots denote differentiation w.r.t time t.

Solution;

we have;

$$e_\rho = \cos\varphi\, i + \sin\varphi\, j$$

$$e_\varphi = -\sin\varphi\, i + \cos\varphi\, j$$

Then;

$$\frac{de_\rho}{dt} = \frac{d}{dt}(\cos\varphi\, i + \sin\varphi\, j)$$

$$= (-\sin\varphi\, \dot{\varphi}\, i + \cos\varphi\, \dot{\varphi}\, j)$$

$$= \dot{\varphi}(-\sin\varphi\, i + \cos\varphi\, j)$$

$$= \dot{\varphi}\, e_\varphi$$

$$\frac{de_\varphi}{dt} = \frac{d}{dt}(-\sin\varphi\, i + \cos\varphi\, j)$$

$$= -\cos\varphi\, \dot{\varphi}\, i - \sin\varphi\, \dot{\varphi}\, j$$

$$= -\varphi(\cos\varphi\, i + \sin\varphi\, j) = -\varphi\, e_\rho$$

(Module # 114)

Introduction to projectile, Motion of a projectile.

=> **Introduction to projectile;-**

if a ball is thrown from one person to another or an object is dropped from a Moving plane, then their path of traveling /Motion is often Called projectile.

=> if Air Resistance is Negligible a projetile can be Considered as a Freely Falling body of the Eq of Motion will be,

$$m\frac{d^2r}{dt^2} = -mg$$

$$\frac{d^2r}{dt^2} = -g$$

Time of Flight $= T = \frac{2v_1}{g}\left(1-\frac{kv_1}{3g}\right)$

(Module # 115)

Conservation of Energy for a System of particles.

Statement;

"The law of Conservation of Energy describes that the Net Energy of an isolated System Remains Conserved. Energy Can neither be Created Nor destroyed; Rather it transforms from one form to another."

(Theorem Statement)

(Principle of Conservation of Energy)

In case of Conservation force Field, the total Energy is a Constant.

If T is for Kinetic Energy and V for potential Energy, then the total Energy E is

$$E = T + V = \text{Constant.}$$

Explanation;

Work done = Change in K.E =

$$W = T_2 - T_1$$

and also

Work done = Change in P.E

$$W = V_1 - V_2$$

$$T_2 - T_1 = V_1 - V_2$$

Results a Corresponding vector Change in its linear Momentum along the Same direction.

=> SI unit of Impluse is the Newton - second (NS)

The time Integral of force F given by;

$$\int_{t_1}^{t_2} F\,dt$$

is Called Impulse of the force F.

Theorem Statement:-

The Implulse is equal to the Change in Momentum, or in Symbols;

$$\int_{t_1}^{t_2} F\,dt = Mv_2 - Mv_1 = P_2 - P_1$$

(Module # 118)

Example of Impulse

Examples:-

What is the Magnitude of the Impulse developed by a mass of 200 gm which Changes its velocity From $5i - 3j + 7k$ m/sec?

Solution;-

$$m = 200 gm$$

$$V_1 = 5i - 3j + 7k$$

$$V_2 = 2i + 3j + k$$

$$\text{Impulse} = P_2 - P_1$$

$$= mV_2 - mV_1 \Rightarrow m(V_2 - V_1)$$

Putting values;

$$\text{Impulse} = 200(2i + 3j + k - (5i - 3j + 7k))$$

$$= 200(-3i + 6j - 6k)$$

Magnitude of Impulse;

$$= 200\sqrt{9 + 36 + 36}$$

$$= 200\sqrt{81}$$

$$= 200(9) = 1800 \text{ mgm/sec}$$

$$= 1.8 \text{ N Sec.}$$

(Module No 119)

(Torque)

Torque is defined as the Turning effect of a body. It is trend of an acting force due to which the Rotational Motion of a body changes.

"It is also called twist and Rotational force on an object"

Mathematically;

$$\tau = \vec{r} \times \vec{F}$$

The Magnitude of torque depends upon the applied force.

Symbolically we can write it as;

$$\tau = |r|\,|F|\sin\theta.$$

=> Torque is a vector Quantity.

=> SI unit of Torque is the Newton Meter (Nm)

=> The direction of torque Can be Approximate using Right Hand Rule.

Theorem:-

The Torque acting on a particle equals the time Rate of Change in its angular Momentum.

$$\tau = \frac{d\Omega}{dt}$$

=> where $\Omega = r \times P$ is ~~dep~~ defined angular Momentum of system.

=> $P = mv$ is linear Momentum of System.

(Module # 120)

(Example of Torque)

((Go through Handouts)

(Module # 121)

Introduction to Rigid Bodies and Elastic Bodies.

=> (Definition of Rigid Bodies)

=> When a force is applied to an object/system of particles, and if the object Maintains its overall Shape, then the object is called a **Rigid Body.**

=> Gap between two Fixed points on the Rigid Body ~~Re~~ Remains Same Regardless of external Forces exerted on it.

=> we Can Neglect the deformation of such bodies.

=> A Rigid body usually has Continuous distribution of Mass.

(Definition of Elastic Bodies)

=> When a force is applied to ~~byt~~ system of particles, it Changes the distance between Individual particles. Such System are often Called deformable or elastic bodies.

(Examples)

=> A Spring and Rubber band are some Common example of elastic bodies.

=> A wheel is a Common example of Rigid body.

(Module No. 122)

(Properties of Rigid Bodies)

(Degree of Freedom):-

"The number of Coordinates Required to specify the position of a System of one or More particles is Called the Number of Degree of Freedom of the System."

- (Translations)

A displacement of a Rigid body is a direct Change of position of its particles. Translational Motion is the displacement of all particles of the body by the same amount and the line segment Joining the Initial and the Final position of the particles represented by parallel vectors.

- (Rotations)

Circular Motion of a body about a Fixed point or axis is called Rotation.

if during a displacement the points of the Rigid Body on some line Remains Fixed and All other are displaced through the same Angle, then this ~~pt~~ displacement is Called Rotation.

A Rigid performs Rotation around an Imaginary line called a Rotation axis.

=> if the axis of Rotation passes through the Center of Mass of the Rigid body then body is said to be Spin ~~on~~ or Rotate upon itself.

=> if a body Rotate about some external Fixed point is called Revolution or orbital Motion of the Rigid Body.

=> Rotational Motion concerns only with Rigid bodies.
The Reverse Rotation of a body is also a Rotation.

(Module # 123)

Instantaneous Axis and Center of Rotation.

=> General Plane Motion;
* Translational Motion along the given Fixed plane and

- and Rotational Motion about a Suitable axis perpendicular to the plane.

=) This Fixed axis is Specifically Chosen to pass through the Center of Mass of the Rigid Body.

=) Instantaneous Axis of Rotation.

The axis about which the Rigid body Rotates is called Instantaneous axis of Rotation, where this axis is prepenolicular to the plane.

=) Instantaneous Center of Rotation.

The point where Instantaneous axis Mets the Fixed plane along which the body performs translation Motion is described as the Instantaneous center of Rotation.

(Module # 124)

(Centre of Mass and Motion of the Center of Mass)

The Center of Mass of the System of particles is defined as :,

$$r_c = \frac{\sum_i m_i r_i}{\sum_i m_i}$$

When the System is Moving the position vector will depend on time t and therefore r_c will also be a function of t,

the velocity and acceleration of Center of Mass will be given by:,

$$\dot{r}_c = v_c = \frac{\sum_i m_i \dot{r}_i}{\sum_i m_i}$$

$$\ddot{r}_c = a_c = \frac{\sum_i m_i \ddot{r}_i}{\sum_i m_i}$$

(Module # 125)

Centre of Mass & Motion of the Center of Mass.

The Moment of Inertia of a Rigid body is a property which depends upon its Mass and shape and Determine its behavior in Rotational Motion.

Definiton of Moment of Inertia;

The

The Moment of Inertia I of the particle of mass m about a line is defined by;

$$I = md^2.$$

'd' is the perpendicular distance between the particle and the line.

Moment of Inertia of a System of particles with Masses $m_1, m_2, m_3 — m_N$.

$$I = \sum_{i=1}^{N} m_i d_i^2$$

$$= m_1 d_1^2 + m_2 d_2^2 — m_n d_N^2$$

$$[I] = [M][L^2]$$

=) Moment of Inertia in Coordinate System:-

$$I_{xx} = m(y^2 + z^2)$$

$$I_{yy} = m(z^2 + x^2)$$

$$I_{zz} = m(x^2 + y^2)$$

=) Product of Inertia;

$$I_{xy} = -mxy$$

$$I_{yz} = -myz$$

$$I_{zx} = -mzx$$

(Module # 127)

Radius of Gyration

Radius of Gyration specifies the distribution of the elements of body around the axis in term of the Mass of Inertia.

$$K = \sqrt{\frac{I}{M}} \quad \text{OR} \quad K^2 = I/M$$

is called the Radius of Gyration of the System AB.

Example:-

Find the Radius of Gyration, K, of the triangular

Lamina of mass M and Momentum of Inertia $I = \frac{1}{6} Mh^2$

Solution;

Radius of Gyration is given by;

$$k^2 = \frac{I}{M}$$

$$I = \frac{1}{6Mh^2} \Rightarrow k^2 = \frac{1/6 Mh^2}{M}$$

$$\sqrt{k^2} = \sqrt{\frac{1}{6} h^2}$$

$$k = \frac{h}{6} \text{ Ans.}$$

(Module # 128)

Principal Axes for the Inertia Matrix

When a Rigid body is Rotating about a Fixed point O, the Angular velocity vector ω and the angular Momentum vector L are not in General in the Same direction. Such directions are called principal directions.

Axes along them are Referred to as ~~direction~~ principal axes of Inertia.

The Corresponding Moments of Inertia are called principal Moments of Inertia.

ORthogonality of Principal Axes:-

if the principal axes at each point of the body exist then their orthogonality can be proved by stating that axes relative to which product of Inertia are zero at ~~are~~ the principal axes.

(Module # 129)

Introduction to the Dynamics of a System of Particles.

=) Dynamics:-

Dynamics is the branch of Mechanics deals with forces & Relationship Fundamentally to the Motion but Sometime also to the equilibrium of bodies.

The Dynamics of a system of particles:-

These will be two type of forces, Internal and External.

Internal forces;

act between particles of the System; all other are external.

Internal forces between a pair of particles are equal and opposite.

If $F_{ij}^{(int)}$ denote the Internal forces on the ith particles due to the Jth Particle, then,

$$F_{ij}^{(int)} = -F_{ji}^{(int)}$$

(Module # 130)

Introduction to Center of Mass and linear Momentum.

=> Center of Mass:

The Center of the Mass of the System as the point whose position vector r_{cm} is given by;

$$r_{cm} = \frac{\sum_{i=1}^{n} m_i r_i}{m}$$

Where $m = \sum_{i=1}^{n} m_i$ is the total Mass of the system.

Also;

$$x_{cm} = \frac{\sum_{i=1}^{n} m_i x_i}{m}$$

$$Y_{cm} = \frac{\sum_{i=1}^{n} m_i y_i}{m}$$

$$Z_{cm} = \frac{\sum_{i=1}^{n} m_i z_i}{m}$$

these are the Rectangular Coordinate of the center of Mass of the System.

⇒ Linear Momentum:-

The linear momentum P of the system as the vector sum of the linear momentum of the individual particles,

$$P = \sum_i P_i = \sum_i m_i v_i$$

$$r_{cm} = \frac{\sum_{i=1}^{n} m_i r_i}{m}$$

$$\dot{r}_{cm} = V_{cm} = \frac{\sum_{i=1}^{n} m_i v_i}{m}$$

it follows that

$$P = mV_{cm}$$

the linear momentum of a system of particles is the product of the velocity of the centre of mass and the total mass of the system.

(Module # 131)

Law of Conservation of Momentum for Multiple particles.

(Just Read Statment of Theorem)

In the absence of the external force, the momentum of the System of particles will be Conserved.

Module # 132

Examples of Conservation of Momentum.

(Just go through the Examples From Handouts)

(Module # 133)

Angular Momentum :-

Angular Momentum of a particle of Mass m, position vector r and linear Momentum p is defined as $r \times P$. it is also called Moment of Momentum.

The Total Angular Momentum L is given by:

$$L = \sum_i r_i \times p_i = \sum_i r_i \times (m v_i)$$

To find the Relation ~~bew~~ between L and $\vec{\omega}$:

$$= \sum_i m_i [(r_i \cdot r_i)\vec{\omega} - (r_i \cdot \vec{\omega}) r_i]$$

Here $(r_i \cdot r_i)\vec{\omega}$ and $(r_i \cdot \vec{\omega}) r_i$ are Not parallel to each other.

The definition of Moment of Inertia and product of Inertia be defined as:

$$I_{xx} = \sum_i m_i (y_i^2 + z_i^2)$$

$$I_{yy} = \sum_i m_i (z_i^2 + x_i^2)$$

$$I_{zz} = \sum_i m_i (x_i^2 + y_i^2)$$

and:

$$I_{xy} = -\sum_i m_i x_i y_i$$

$$I_{yz} = -\sum_i m_i y_i z_i$$

$$I_{zx} = -\sum_i m_i x_i z_i$$

using these definition and Noting that $I_{xy} = I_{yx}$ we can write:

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}$$
$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$
$$L_z = I_{zx}\omega_x + I_{yz}\omega_y + I_{zz}\omega_z$$

Module # 135

Law of Conservation of Angular Momentum.

Angular Momentum:

The Angular Momentum of a single particle is defined as the cross product of linear Momentum and position vector of concerned particle.

$$L = r \times mv.$$

Angular Momentum of System of particles.

The Angular Momentum L of a system of particles is defined accordingly, as the vector sum of the individual angular Momentum, Namely:

$$L = \sum_{i=1}^{n} r_i \times m_i v_i$$

Law of Conservation of Angular Momentum:-

The time rate of change of Angular Momentum in the absence of some external forces is zero.

$$\frac{dL}{dt} = 0 \Rightarrow L = \text{Constant}$$

$\Rightarrow$ The time Rate of change of the Angular Momentum of a system is equal to the total Moment of all the External Forces acting on the system.

$$\frac{dL}{dt} = N$$

if a system is Isolated, then $N = 0$, and the angular Momentum Remains Constant in both Magnitude and direction

$$L = \sum_{i=1}^{n} r_i \times m_i v_i = \text{constant vector}$$

(Module # 136)

Example Related to Angular Momentum.

Skip or Check from Handouts.

(Module # 137)

Kinetic Energy of a System about principal Axes.

$$K.E = \frac{1}{2} m v^2$$

K. E of a Rigid Body:

For a Rigid body experiencing planar (two-dimentional) Motion, the kinetic Energy is given by the following General scalar equation:

$$T = \frac{1}{2} m v_c^2 + \frac{1}{2} I_c \omega^2$$

K. E of a Rigid Body w.r.t Origin:

if the Rigid Body is Rotating about a Fixed point O that is Attached to ground we Can express the K.E As;

$$T = \frac{1}{2} I_o \omega^2$$

K.E of a Rigid Body about Principal Axes;

if the Rigid Body has a Fixed point O that is Attached to ground, we can give an alternate Scalar equation for the K.E of the Rigid Body;

$$T = \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2$$

Module # 138;

Moment of Inertia of a Rigid Body about a Given line.

Let M be the Mass of the System and $\hat{e}$ a unit vector along the line l. Then

$\hat{e} = \lambda \hat{i} + \mu \hat{j} + v\hat{k}$, where (λ, μ, v) are direction Cosines of the line.

If I_i denotes the Record Moment of Inertia, then

$$I_i = \sum_i m_i d_i^2$$

(Module No 141)

Rotational kinetic Energy

K.E is the Energy produced in any body during its Motion. It is equal to the Half of the product of Mass and Square of the velocity of the Moving body.

$$K.E = T = \frac{1}{2} m v^2$$

if v is the velocity of C then the velocity v_i of the ith particle is given by;

$$v_i = v \times \omega \times r_i$$

=> The definition of Position vector center of Mass we have;

$$r_c = \frac{\Sigma_i m_i r_i}{\Sigma_i m_i}$$

=> Rotational K.E is given by;

$$T_{rot} = \frac{1}{2} I \omega^2$$

(Module # 143)

Introduction to special Moments of Inertia

=> **Solid Circular cylinder;**

The Radius of cylinder is a and Mass M about axis of cylinder,

$$I = \frac{1}{2} Ma^2$$

=> **Hollow Circular cylinder;**

The Radius of cylinder is a and Mass M about axis of cylinder;

$$I = Ma^2$$

=> **Solid Sphere;**

The Radius of sphere a and Mass M about a diameter;

$$I = \frac{2}{5} Ma^2$$

=> **Hollow Sphere;**

The Radius of sphere is a and Mass M about a diameter

$$I = Ma^2$$

=> **Rectangular plate**

$$I = \frac{1}{12} M (a^2 + b^2)$$

=> Thin Rod;

$$I = \frac{1}{12} Ma^2$$

=> Triangular Lamina;

$$I = \frac{1}{6} Mh^2$$

=> Right Circular Cone;

$$I = \frac{3}{10} Ma^2$$

(Imp Objective from
Modules 144 to)

=> The Moment of Inertia of
the Ring;

$$I_{ring} = r^2 dm$$

=> The Surface area of the ring is;

$$Area = 2\pi r dr$$

=> The Surface Area of the
annulus is;

$$\pi(R_2^2 - R_1^2)$$

(Rectangular Plate)

we Consider a Rectangular plate (lamina) of sides of Length a & b, we Consider an element of Length dx & dy.

=> Mass of slected element will be;

$$dm = \rho dx dy$$

=> Moment of Inertia about the Y-axis;

$$\rho dx dy x^2 = \rho x^2 dx dy$$

=> Total Mass of Rectangular plate is;

$$M = \rho ab$$

(M.I square plate)

Let ρ be the density of plate and the total Area of square plate is a^2. So density will be;

$$\rho = \frac{M}{a^2}$$

(M.I of a Solid Circular Cylinder - Derivation)

The Mass of one shell is dm, Height is same as h, thickness be dr and the Radius be r then the density of shell will be;

$$\rho = \frac{dm}{2\pi r dr h}$$

and Mass dm is,

$$dm = 2\pi \rho r dr h$$

(M.I of Hollow cylindrical Shell)

=> The Mass of one Ring is dm, Height is dh, thickness be dr and the Radius be R, then the Mass of one Ring will be;

$$dm = \rho 2 \pi R dR dh$$

=> The Moment of Inertia of the Ring of Radius R will be;

$$I_{ring} = dm R^2$$

OR $I_{ring} = 2\rho\pi R^3 dR dh$

(M.I of Solid Sphere)-

for a uniform Solid Sphere, due to Symmetry, we have;

$$I_{xx} = I_{yy} = I_{zz}$$

The expression for the Moment of Inertia of a representative disc of Radius x, which is;

$$I_{disk} = \frac{1}{2} x^2 dm$$

of an elementary disc of Mass dm and the Radius x.

Mass = (density)(Area of disc)

therefore;

$$dm = \rho \pi x^2$$

(M.I of the Hollow sphere)

=> The volume of the elementary Ring is;

$$dV = 2\pi x R d\theta dR$$

=> Mass = (density)(volume)

$$dm = \rho dV$$

$$dm = \rho 2 \pi x R d\cancel{Q} R d\theta.$$