Final Term Start Math 405:

Metric Spaces:

Real Valued Function:

let $f: A \rightarrow R$ be a function. Clearly domain of f is A, in other words f is defined on A. Since co-domain of f is R, we say that f is real valued function.

Metric:

let X be a non-empty set and R be a set of real numbers.

let $d: X \times X \rightarrow R$ be a function.

Then "d" is called "metric" on X, if "d" satisfies each of the following four conditions;

M1) $d(x_1, x_2) \geq 0 \quad \forall \; x_1, x_2 \in X$

M2) $d(x_1, x_2) = 0 \iff x_1 = x_2 \quad \forall \; x_1, x_2 \in X$

M3) $d(x_1, x_2) = d(x_2, x_1) \; \forall \; x_1, x_2 \in X$ (Symmetric Property).

M4) $d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3) \; \forall \; x_1, x_2, x_3 \in X$

(Triangular Inequality)

If "d" is a metric on X then the pair (X, d) is metric space.

Note: The non-negative real number $d(x_1, x_2)$ is called distance between Points x_1 and x_2 in the metric "d".

Usual Metric on R:

Let $d : R \times R \to R$ be a metric on R given by

$d(x_1, x_2) = |x_1 - x_2|$

Then "d" is called a usual metric on R and (R, d) is called usual metric space.

Usual Metric on R^2:

Let $d : R^2 \times R^2 \longrightarrow R$ be a metric on R^2 given by

$d(P_1, P_2) = d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

Then "d" is called a usual metric on R^2 and (R^2, d) is usual metric space.

Usual Metric on R^3:

let $d : R^3 \times R^3 \longrightarrow R$ be a metric on R^3 given by

$d[(x_1, y_1, z_1), (x_2, y_2, z_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$

Then "d" is called a usual metric on R^3 and (R^3, d) is usual metric space.

Note: When we say that R is a metric space without giving a metric on R then it is assumed that metric on R is used metric. Similarly we take the case of R^2 and R^3.

Example: let X be the set of all towns marked on a plane geographically map and let $d(x_1, x_2)$ be the length of the shortest rout from town x_1 to x_2.
Show that "d" is a metric on X.

Solution: Here function $d: X \times X \to R$ is defined as

$d(x_1, x_2) =$ Length of shortest route from town x_1 to x_2.

M1) Since (Length of shortest route from town x_1 to x_2) ≥ 0

$\therefore \quad d(x_1, x_2) \geq 0$

M2) Let $d(x_1, x_2) = 0 \Rightarrow$ Length of shortest route from town x_1 to $x_2 = 0$

$\Rightarrow \quad x_1 = x_2$

Let $x_1 = x_2 \Rightarrow$ Length of shortest from town x_1 to $x_2 = 0$

$\Rightarrow \quad d(x_1, x_2) = 0.$

M3) Since $d(x_1, x_2) =$ Length of shortest route from town x_1 to x_2
$=$ Length of shortest route from town x_2 to x_1.
$= d(x_2, x_1).$

M4) Let $x_1, x_2, x_3 \in X$.

Then x_1, x_2, x_3 are non-collinear or collinear.
If x_1, x_2, x_3 are non-collinear, then they form a triangle and we know that sum of length of two sides of a triangle is

always greater than the third side.

$\therefore d(x_1, x_2) + d(x_2, x_3) > d(x_1, x_3)$ — (i)

let x_1, x_2, x_3 are collinear.

Then $d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3)$ — (ii)

From (i) and (ii), we get

$d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$.

Hence "d" is a metric on X.

Example: let $X = R$ be the set of all real number: and let $d: R \times R \to R$ be defined by $d(x_1, x_2) = |x_1 - x_2|$ denotes the absolute value of the number $x_1 - x_2$. Show that (R, d) is a metric space.

Solution: Here function $d: R \times R \to R$ is defined as

$$d(x_1, x_2) = |x_1 - x_2|$$

M_1) Since $|x_1 - x_2| \geq 0$

$$\therefore d(x_1, x_2) \geq 0$$

M_2) Let $d(x_1, x_2) = 0 \Rightarrow |x_1 - x_2| = 0$

$\Rightarrow x_1 - x_2 = 0$

$\Rightarrow x_1 = x_2$

Let $x_1 = x_2 \Rightarrow x_1 - x_2 = 0$

$\Rightarrow |x_1 - x_2| = 0$

$\Rightarrow d(x_1, x_2) = 0$

Thus $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$

M3) Since $d(x_1, x_2) = |x_1 - x_2|$

$$= |-(x_2 - x_1)|$$
$$= |x_2 - x_1|$$
$$= d(x_2, x_1).$$

M4) Since $d(x_1, x_2) = |x_1 - x_2|$

$$d(x_2, x_3) = |x_2 - x_3|$$
$$d(x_1, x_3) = |x_1 - x_3|$$
$$= |x_1 - x_2 + x_2 - x_3|$$
$$\leq |x_1 - x_2| + |x_2 - x_3|$$
$$= d(x_1, x_2) + d(x_2, x_3)$$

Thus, (R, d) is a metric space.

Example:

Let $X = R^2$ be a set of all ordered pairs (x, y); $x, y \in R$. Let $P_1(x_1, y_1), P_2(x_2, y_2) \in R^2$. Show that the non-negative real valued function "d" defined by $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$ is a metric on R^2.

Solution:

Here function $d: R^2 \times R^2 \longrightarrow R$ is defined as

$$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$$

M1) Since $|x_1 - x_2| + |y_1 - y_2| \geq 0$

$$\therefore \quad d(P_1, P_2) \geq 0$$

M_2) Let $d(P_1,P_2)=0 \Rightarrow |x_1-x_2|+|y_1-y_1|=0$

$\Rightarrow |x_1-x_2|=0 \;,\; |y_1-y_2|=0$

$\Rightarrow x_1-x_2=0 \;,\; y_1-y_2=0$

$\Rightarrow x_1=x_2 \;,\; y_1=y_2$

$\Rightarrow (x_1,y_1)=(x_2,y_2)$

$\Rightarrow P_1=P_2$

Let $P_1=P_2 \Rightarrow (x_1,y_1)=(x_2,y_2)$

$\Rightarrow x_1=x_2 \;,\; y_1=y_2$

$\Rightarrow x_1-x_2=0 \;,\; y_1-y_2=0$

$\Rightarrow |x_1-x_2|=0 \;,\; |y_1-y_2|=0$

$\Rightarrow |x_1-x_2|+|y_1-y_2|=0$

$\Rightarrow$ Thus $d(P_1,P_2)=0 \Leftrightarrow P_1=P_2$

M_3) Since $d(P_1,P_2)=|x_1-x_2|+|y_1-y_2|$

$\Rightarrow |-(x_2-x_1)|+|-(y_2-y_1)|$

$\Rightarrow |x_2-x_1|+|y_2-y_1|$

$= d(P_2,P_1)$

M_4) Since $d(P_1,P_2)=|x_1-x_2|+|y_1-y_2|$

$d(P_2,P_3)=|x_2-x_3|+|y_2-y_3|$

$d(P_1,P_3)=|x_1-x_3|+|y_1-y_3|$

Since $d(P_1,P_3)=|x_1-x_3|+|y_1-y_3|$

$=|x_1-x_2+x_2-x_3|+|y_1-y_2+y_2-y_3|$

$\leq |x_1-x_2|+|x_2-x_3|+|y_1-y_2|+|y_2-y_3|$

$=|x_1-x_2|+|y_1-y_2|+|x_2-x_3|+|y_2-y_3|$

$=d(P_1,P_2)+d(P_2,P_3)$ Hence "d" is metric on R^2.

Example: Let $X = R^2$ be a set of all ordered pairs (x, y); $x, y \in R$. Let $P_1(x_1, y_1), P_2(x_2, y_2) \in R^2$. Show that the non-negative real valued function "d" defined by $d(P_1, P_2) = \max(|x_1 - x_2|, |y_1 - y_2|)$ is a metric on R^2.

Solution:-

Here function $d: R^2 \times R^2 \to R$ is defined as

$d(P_1, P_2) = \max(|x_1 - x_2| + |y_1 - y_2|)$

M₁) Since $\max(|x_1 - x_2| + |y_1 - y_2|) \geq 0$

$(\because |x_1 - x_2| \geq 0 \;\&\; |y_1 - y_2| \geq 0)$

$\therefore d(P_1, P_2) \geq 0.$

M₂) Let $d(P_1, P_2) = 0 \Rightarrow \max(|x_1 - x_2|, |y_1 - y_2|) = 0$

$\Rightarrow |x_1 - x_2| = 0 \;,\; |y_1 - y_2| = 0$

$\Rightarrow x_1 - x_2 = 0 \;,\; y_1 - y_2 = 0$

$\Rightarrow x_1 = x_2 \;,\; y_1 = y_2$

$\Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow P_1 = P_2$

Let $P_1 = P_2 \Rightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow x_1 = x_2 \;,\; y_1 = y_2$

$\Rightarrow x_1 - x_2 = 0 \;,\; y_1 - y_2 = 0$

$\Rightarrow |x_1 - x_2| = 0 \;,\; |y_1 - y_2| = 0$

$\Rightarrow \max(|x_1 - x_2|, |y_1 - y_2|) = 0$

$\Rightarrow d(P_1, P_2) = 0$

Thus $d(P_1, P_2) = 0 \iff P_1 = P_2$

M3) Since $d(P_1, P_2) = \max(|x_1 - x_2|, |y_1 - y_2|)$

$$= \max(|-(x_2 - x_1)|, |-(y_2 - y_1)|)$$
$$= \max(|x_2 - x_1|, |y_2 - y_1|)$$
$$= d(P_2, P_1).$$

M4) Since

$d(P_1, P_2) = \max(|x_1 - x_2|, |y_1 - y_2|) = |x_1 - x_2|$ (say)

$d(P_2, P_3) = \max(|x_2 - x_3|, |y_2 - y_3|) = |x_2 - x_3|$ (say)

$d(P_1, P_3) = \max(|x_1 - x_3|, |y_1 - y_3|) = |x_1 - x_3|$ (say)

Now $d(P_1, P_3) = |x_1 - x_3|$

$$= |x_1 - x_2 + x_2 - x_3|$$
$$\leq |x_1 - x_2| + |x_2 - x_3|$$
$$= d(P_1, P_2) + d(P_2, P_3)$$

Hence "d" is metric on R.

Example: let $X = R^2$ be a set of all ordered pairs (x, y); $x, y \in R$. let $P_1(x_1, y_1), P_2(x_2, y_2) \in R^2$. Show that the non-negative real valued fun. "d" defined by $d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$ is a metric on R^2.

Solution: Here fun. $d : R^2 \times R^2 \to R$ is defined as

$$d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}}$$

M1) Since $[(x_1 - x_2)^2 + (y_1 - y_2)^2]^{\frac{1}{2}} \geq 0$

$$\therefore d(P_1, P_2) \geq 0$$

M2) Let $d(P_1, P_2) = 0$.

$\Rightarrow \left[(x_1-x_2)^2+(y_1-y_2)^2\right]^{\frac{1}{2}}=0$

$\Rightarrow (x_1-x_2)^2+(y_1-y_2)^2=0$

$\Rightarrow (x_1-x_2)^2=0,\ (y_1-y_2)^2=0$

$\Rightarrow x_1-x_2=0,\ y_1-y_2=0$

$\Rightarrow x_1=x_2,\ y_1=y_2$

$\Rightarrow (x_1,y_1)=(x_2,y_2).$

$\Rightarrow P_1=P_2$

Let $P_1=P_2 \Rightarrow (x_1,y_1)=(x_2,y_2)$

$\Rightarrow x_1=x_2,\ y_1=y_2$

$\Rightarrow x_1-x_2=0,\ y_1-y_2=0$

$\Rightarrow (x_1-x_2)^2=0,\ (y_1-y_2)^2=0$

$\Rightarrow (x_1-x_2)^2+(y_1-y_2)^2=0$

$\Rightarrow \left[(x_1-x_2)^2+(y_1-y_2)^2\right]^{1/2}=0$

$\Rightarrow d(P_1,P_2)=0$

M3) Since $d(P_1,P_2)=\left[(x_1-x_2)^2+(y_1-y_2)^2\right]^{\frac{1}{2}}$

$=\left[\{-(x_2-x_1)\}^2+\{-(y_2-y_1)\}^2\right]^{1/2}$

$=\left[(x_2-x_1)^2+(y_2-y_1)^2\right]^{1/2}$

$=d(P_2,P_1)$

M4) Let $P_1(x_1,y_1), P_2(x_2,y_2), P_3(x_3,y_3)\in R^2$ then P_1,P_2,P_3 are collinear or non-collinear.

If P_1,P_2,P_3 are collinear, then

$d(P_1,P_2)+d(P_2,P_3)=d(P_1,P_3)$ —— (1)

If P_1, P_2, P_3 are non-collinear, then they form a triangle and we know that, sum of length of two sides of a triangle is always greater than the third side.

$$\therefore \quad d(P_1, P_2) + d(P_2, P_3) > d(P_1, P_3) \quad (2)$$

From (1) & (2) we get,

$$d(P_1, P_2) + d(P_2, P_3) \geq d(P_1, P_3)$$

Hence "d" is metric on R^2.

Example: let $X = R^3$ be a set of all ordered Pairs (x, y); $x, y \in R$. Let $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2) \in R^3$. Show that the non-negative real valued function "d" defined $d(P_1, P_2) = \left[(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\right]^{\frac{1}{2}}$ is a metric on R^3.

Solution:

Here function $d: R^3 \times R^3 \longrightarrow R$ is defined as

$$d(P_1, P_2) = \left[(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\right]^{\frac{1}{2}}$$

M1) Since $\left[(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\right]^{\frac{1}{2}} \geq 0$

$$\therefore \quad d(P_1, P_2) \geq 0.$$

M2) Let $d(P_1, P_2) = 0$

$$\Rightarrow \left[(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2\right]^{\frac{1}{2}} = 0$$

$$\Rightarrow (x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 = 0$$

$$\Rightarrow (x_1-x_2)^2 = 0, \ (y_1-y_2)^2 = 0, \ (z_1-z_2)^2 = 0$$

$\Rightarrow x_1 - x_2 = 0, \; y_1 - y_2 = 0, \; z_1 - z_2 = 0$

$\Rightarrow x_1 = x_2, \; y_1 = y_2, \; z_1 = z_2$

$\Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2).$

$\Rightarrow P_1 = P_2.$

Let $P_1 = P_2 \Rightarrow (x_1, y_1, z_1) = (x_2, y_2, z_2).$

$\Rightarrow x_1 = x_2, \; y_1 = y_2, \; z_1 = z_2$

$\Rightarrow x_1 - x_2 = 0, \; y_1 - y_2 = 0, \; z_1 - z_2 = 0$

$\Rightarrow (x_1 - x_2)^2 = 0, \; (y_1 - y_2)^2 = 0, \; (z_1 - z_2)^2 = 0$

$\Rightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = 0$

$\Rightarrow [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}} = 0$

$\Rightarrow d(P_1, P_2) = 0.$

M3) Since

$d(P_1, P_2) = [(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2]^{\frac{1}{2}}$

$= [\{-(x_2 - x_1)\}^2 + \{-(y_2 - y_1)\}^2 + \{-(z_2 - z_1)\}^2]^{\frac{1}{2}}$

$= [(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]^{\frac{1}{2}}$

$= d(P_2, P_1)$

M4) Let $P_1(x_1, y_1, z_1), P_2(x_2, y_2, z_2), P_3(x_3, y_3, z_3) \in R^3$ then P_1, P_2, P_3 are collinear or non-collinear.

If P_1, P_2, P_3 are collinear, then

$d(P_1, P_2) + d(P_2, P_3) = d(P_1, P_3)$ — (1)

If P_1, P_2, P_3 are non-collinear, then they form a triangle and we know that, sum of length of two sides of a triangle is always greater than the third side.

$\therefore \quad d(P_1, P_2) + d(P_2, P_3) > d(P_1, P_3)$ —— (2)

From (1) & (2)

$$d(P_1, P_2) + d(P_2, P_3) \geq d(P_1, P_3)$$

Hence "d" is metric on R^3.

Imp

Example :

Show that every non-empty set can be given a metric and hence can be converted into metric space.

Solution:

let 'X' be any non-empty set.

let $d_0 : X \times X \rightarrow R$ be defined by

$$d_0(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

We shall prove that d_0 is a metric on X.

M1) Here $d_0(x_1, x_2) \geq 0$ $(\because d_0(x_1, x_2) = 0 \text{ or } d_0(x_1, x_2) = 1)$

M2) Let $d_0(x_1, x_2) = 0 \Rightarrow x_1 = x_2$ (By definition)

Let $x_1 = x_2 \Rightarrow d_0(x_1, x_2) = 0$ (By def.)

Thus $d_0(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$

M3) i) Let $d_0(x_1, x_2) = 0 \Rightarrow x_1 = x_2$ (By def.)

$\Rightarrow x_2 = x_1$

$\Rightarrow d_0(x_2, x_1) = 0$

ii) Let $d_0(x_1, x_2) = 1 \Rightarrow x_1 \neq x_2$

$$\Rightarrow d_0(x_2, x_1) = 1$$

Hence in both cases $d_0(x_1, x_2) = d_0(x_2, x_1)$

M_4) Let $x_1, x_2, x_3 \in X$.

i) Let $x_1 = x_2 = x_3$ then $d_0(x_1, x_2) = 0$

$$\& \; d_0(x_2, x_3) = 0$$

$$\text{also } d_0(x_1, x_3) = 0$$

$$\therefore d(x_1, x_2) + d(x_2, x_3) = d(x_1, x_3).$$

ii) Let $x_1 \neq x_2 \neq x_3$ then $d_0(x_1, x_2) = 1$

$$\& \; d_0(x_2, x_3) = 1$$

$$\text{also } d_0(x_1, x_3) = 1$$

$$\therefore d(x_1, x_2) + d(x_2, x_3) > d(x_1, x_3)$$

$$1 + 1 > 1$$

Similar type of verification in all remaining cases leads us to the conclusion that $d_0(x_1, x_2) + d_0(x_2, x_3) \geq d_0(x_1, x_3)$

Hence (X, d_0) is a metric space.

Note: Let X be any non-empty set. Let $d_0 : X \times X \to \mathbb{R}$ be defined by

$$d_0(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

Then d_0 is called discrete metric on X.

Question: Let C be the set of all complex numbers and let $d: C \times C \to R$ be defined by $d(z_1, z_2) = |z_1 - z_2|$. d is a metric on C.

Solution:

Here function $d: C \times C \to R$ is defined as

$$d(z_1, z_2) = |z_1 - z_2|$$

M1) Since $|z_1 - z_2| \geq 0$

$$\therefore \quad d(z_1, z_2) \geq 0$$

M2) Let $d(z_1, z_2) = 0 \Rightarrow |z_1 - z_2| = 0$

$\Rightarrow z_1 - z_2 = 0$

$\Rightarrow z_1 = z_2$

Let $z_1 = z_2 \Rightarrow z_1 - z_2 = 0$

$\Rightarrow |z_1 - z_2| = 0$

$\Rightarrow d(z_1, z_2) = 0$

Thus $d(z_1, z_2) = 0 \iff z_1 = z_2$

M3) Since

$d(z_1, z_2) = |z_1 - z_2|$

$= |-(z_2 - z_1)|$

$= |z_2 - z_1|$

$= d(z_2, z_1)$

M4) Since $d(z_1, z_2) = |z_1 - z_2|$

$$d(z_2, z_3) = |z_2 - z_3|$$

$$d(z_1, z_3) = |z_1 - z_3|$$

Now $d(z_1, z_3) = |z_1 - z_3|$

$= |z_1 - z_2 + z_2 - z_3|$

$\leq |z_1 - z_2| + |z_2 - z_3|$

$= d(z_1, z_2) + d(z_2, z_3)$

Thus (C, d) is a metric space.

Question:

Let d be a metric on X and $d': X \to X \to R$ be given by $d'(x_1, x_2) = \min(1, d(x_1, x_2))$.
Is d' a metric on X?

Solution:

Here function $d': X \times X \to R$ be given by

$d'(x_1, x_2) = \min(1, d(x_1, x_2))$

M1) Since $\min(1, d(x_1, x_2)) \geq 0$

$\therefore d'(x_1, x_2) \geq 0$

M2) Let $d'(x_1, x_2) = 0$

$\Rightarrow \min(1, d(x_1, x_2)) = 0$

$\Rightarrow d(x_1, x_2) = 0 \quad \because 1 \neq 0$

$\Rightarrow x_1 = x_2 \quad \because d$ is metric on X.

Let $x_1 = x_2 \Rightarrow d(x_1, x_2) = 0 \quad \because d$ is metric on X.

$\Rightarrow \min(1, d(x_1, x_2)) = 0$

$\Rightarrow d'(x_1, x_2) = 0$

Thus $d'(x_1, x_2) = 0 \iff x_1 = x_2$

M3) Since $d'(x_1, x_2) = \min(1, d(x_1, x_2))$

$= \min(1, d(x_2, x_1)) \quad \because d$ is metric on X.

$= d'(x_2, x_1)$.

M4) $d'(x_1, x_2) = \min(1, d(x_1, x_2)) = d(x_1, x_2)$ (Say)

$d'(x_2, x_3) = \min(1, d(x_2, x_3)) = d(x_2, x_3)$ (Say)

$d'(x_1, x_3) = \min(1, d(x_1, x_3)) = d(x_1, x_3)$ (Say)

Since d is a metric on X.

$\therefore\ d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3)$

$\Rightarrow d'(x_1, x_2) + d'(x_2, x_3) \geq d'(x_1, x_3)$

$\therefore\ d'$ is a metric on X.

Question:

Let (X_1, d_1) and (X_2, d_2) be two metric space. Define $d'[(x_1, x_2), (y_1, y_2)] = \sum_{i=1}^{2} d_i(x_i, y_i)$. Is d' a metric on $X_1 \times X_2$.

Solution:

Here function $d': X_1 \times X_2 \to R$ is defined as

$$d'[(x_1, x_2), (y_1, y_2)] = \sum_{i=1}^{2} d_i(x_i, y_i)$$

$$= d_1(x_1, y_1) + d_2(x_2, y_2)$$

M1) Since $d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0$

$\because\ d_1(x_1, y_1) \geq 0,\ d_2(x_2, y_2) \geq 0$

$\because\ d_1, d_2$ are metrics on X_1 and X_2 respectively.

$\therefore\ d'((x_1, x_2), (y_1, y_2)) \geq 0$

M2) Let $d'((x_1, x_2), (y_1, y_2)) = 0$

$\Rightarrow d_1(x_1, y_1) + d_2(x_2, y_2) = 0$

$\Rightarrow d_1(x_1, y_1) = 0,\ d_2(x_2, y_2) = 0$

$\Rightarrow x_1 = y_1,\ x_2 = y_2$

$\because\ d_1, d_2$ are metrices on $X_1 \times X_2$.

$\Rightarrow (x_1, x_2) = (x_2, y_2)$.

Let $(x_1, x_2) = (y_1, y_2)$

$\Rightarrow x_1 = y_1, \quad x_2 = y_2$

$\Rightarrow d_1(x_1, y_1) = 0, \quad d_2(x_2, y_2) = 0$

($\because d_1, d_2$ are metrics on X_1 and X_2 resp.)

$\Rightarrow d_1(x_1, y_1) + d_2(x_2, y_2) = 0$

$\Rightarrow d'((x_1, x_2), (y_1, y_2)) = 0$

Thus $d'((x_1, x_2), (y_1, y_2)) = 0 \Leftrightarrow (x_1, x_2) = (y_1, y_2)$

M3) Since $d'((x_1, x_2), (y_1, y_2))$

$= d_1(x_1, y_1) + d_2(x_2, y_2)$

$= d_1(y_1, x_1) + d_2(y_2, x_2)$

$= d'((y_1, y_2), (x_2, x_1))$

M4) Since $d'((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2)$

$d'((y_1, y_2), (z_1, z_2)) = d_1(y_1, z_1) + d_2(y_2, z_2)$

$d'((x_1, x_2), (z_1, z_2)) = d_1(x_1, z_1) + d_2(x_2, z_2)$

Now $d'((x_1, x_2), (y_1, y_2)) + d'((y_1, y_2), (z_1, z_2))$

$= d_1(x_1, y_1) + d_2(x_2, y_2) + d_1(y_1, z_1) + d(y_2, z_2)$

$= d_1(x_1, y_1) + d_1(y_1, z_1) + d_2(x_2, y_2) + d_2(y_2, z_2)$

$\geq d_1(x_1, z_1) + d_2(x_2, z_2)$

$\because d_1, d_2$ are metric on X_1 & X_2 resp.

$\therefore d_1(x_1, y_1) + d_1(y_1, z_1) \geq d_1(x_1, z_1)$

$+ d_2(x_2, y_2) + d_2(y_2, z_2) \geq d_2(x_2, z_2)$

$= d'((x_1, x_2), (z_1, z_2))$

$\therefore d'$ is a metric on $X_1 \times X_2$.

Question: Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Let $d''[(x_1, x_2), (y_1, y_2)] = \max(d_1(x_1, y_1), d_2(x_2, y_2))$.

Is d'' a metric on $X_1 \times X_2$.

Solution:

Here function $d'' : X_1 \times X_2 \to R$ is defined as

$d''[(x_1, x_2), (y_1, y_2)] = \max(d_1(x_1, y_1), d_2(x_2, y_2))$

M1) Since $\max(d_1(x_1, y_1) + d_2(x_2, y_2) \geq 0)$

$\because d_1(x_1, y_1) \geq 0, \quad d_2(x_2, y_2) \geq 0$

$\because d_1, d_2$ are metrics on X_1 and X_2 resp.

$\therefore \quad d''((x_1, x_2), (y_1, y_2)) \geq 0$

M2) Let $d''((x_1, x_2), (y_1, y_2)) = 0$

$\Rightarrow \max(d_1(x_1, y_1), d_2(x_2, y_2)) = 0$

$\Rightarrow d_1(x_1, y_1) = 0 \quad , \quad d_2(x_2, y_2) = 0$

$\Rightarrow \quad x_1 = y_1 \quad , \quad x_2 = y_2$

($\because d_1, d_2$ are metrics on X_1, X_2 resp.)

$\Rightarrow \quad (x_1, x_2) = (y_1, y_2)$

Let $(x_1, x_2) = (y_1, y_2) \Rightarrow x_1 = y_1 \ , \ x_2 = y_2$

$\Rightarrow d_1(x_1, y_1) = 0 \ , \ d_2(x_2, y_2) = 0$

($\because d_1, d_2$ are metrics on X_1, X_2 resp.)

$\Rightarrow \max(d_1(x_1, y_1), d_2(x_2, y_2)) = 0$

$\Rightarrow \quad d''((x_1, x_2), (y_1, y_2)) = 0$

$d''((x_1, x_2), (y_1, y_2)) = 0 \Leftrightarrow (x_1, x_2) = (y_1, y_2)$

M3) Since $d''[(x_1,x_2),(y_1,y_2)] = \max(d_1(x_1,y_1), d_2(x_2,y_2))$

$= \max(d_1(y_1,x_1), d_2(y_2,x_2))$

$(\because d_1, d_2$ are metrics on X_1, X_2 resp.$)$

$= d''((y_1,y_2),(x_1,x_2))$

M4) Let $d''[(x_1,x_2),(y_1,y_2)] = \max(d_1(x_1,y_1), d_2(x_2,y_2)) = d_1(x_1,y_1)$ (say)

$d''[(y_1,y_2),(z_1,z_2)] = \max(d_1(y_1,z_1), d_2(y_2,z_2)) = d_1(y_1,z_1)$ (say)

$d''[(x_1,x_2),(z_1,z_2)] = \max(d_1(x_1,z_1), d_2(x_2,z_2)) = d_1(x_1,z_1)$ (say)

Since d_1 is a metric on X_1.

$\therefore d_1(x_1,y_1) + d_1(y_1,z_1) \geq d_1(x_1,z_1)$

$\Rightarrow d''[(x_1,x_2),(y_1,y_2)] + d''[(y_1,y_2),(z_1,z_2)] \geq d''[(x_1,x_2),(z_1,z_2)]$

(We get the same result in the remaining cases.)

$\therefore d''$ is a metric on $X_1 \times X_2$.

Question:

Let (X,d) be a metric space and let $d' : X \times X \to R$ be given by

$$d'(x_1,x_2) = \frac{d(x_1,x_2)}{1+d(x_1,x_2)}.$$

Prove that d' is metric on X.

Solution:

Here function $d' : X \times X \to R$ be defined by

$$d'(x_1, x_2) = \frac{d(x_1, x_2)}{1+d(x_1, x_2)}$$

M1) Since $\frac{d(x_1, x_2)}{1+d(x_1, x_2)} \geq 0$

$\because$ $d(x_1, x_2) \geq 0$

$\because$ d is a metric on X.

$\therefore$ $d'(x_1, x_2) \geq 0$

M2) Let $d'(x_1, x_2) = 0 \Rightarrow \frac{d(x_1, x_2)}{1+d(x_1, x_2)} = 0$

$$\Rightarrow d(x_1, x_2) = 0$$

$\Rightarrow x_1 = x_2$ ($\because$ d is a metric on X)

Let $x_1 = x_2 \Rightarrow d(x_1, x_2) = 0$ ($\because$ d is a metric on X.)

$$\Rightarrow \frac{d(x_1, x_2)}{1+d(x_1, x_2)} = 0$$

$$\Rightarrow d'(x_1, x_2) = 0$$

Thus $d'(x_1, x_2) = 0 \iff x_1 = x_2$ d is metric on X.

M3) Since $d'(x_1, x_2) = \frac{d(x_1, x_2)}{1+d(x_1, x_2)}$

$$= \frac{d(x_2, x_1)}{1+d(x_2, x_1)}$$

$$= d'(x_2, x_1)$$

M4) Since $d'(x_1, x_2) = \dfrac{d(x_1, x_2)}{1+d(x_1, x_2)}$

$$d'(x_2, x_3) = \frac{d(x_2, x_3)}{1+d(x_2, x_3)}$$

$$d'(x_1, x_3) = \frac{d(x_1, x_3)}{1+d(x_1, x_3)}$$

Now $d'(x_1, x_2) + d'(x_2, x_3) = \dfrac{d(x_1, x_2)}{1+d(x_1, x_2)} + \dfrac{d(x_2, x_3)}{1+d(x_2, x_3)}$

$$\geq \frac{d(x_1, x_2)}{1+d(x_1, x_2)+d(x_2, x_3)} + \frac{d(x_2, x_3)}{1+d(x_1, x_2)+d(x_2, x_3)}$$

$$= \frac{d(x_1, x_2) + d(x_2, x_3)}{1+d(x_1, x_2)+d(x_2, x_3)}$$

$\therefore d'(x_1, x_2) + d'(x_2, x_3) \geq \dfrac{d(x_1, x_3)}{1+d(x_1, x_3)}$

$= d'(x_1, x_3)$

$(\because d(x_1, x_2) + d(x_2, x_3) \geq d(x_1, x_3))$

$\because$ d is a metric on X.

$\therefore$ d' is a metric on X.

Question: let $X = R$ and $d(x_1, x_2) = |x_1| + |x_2|$.
Show that d is not a metric on R.

Solution:

Let $d(x_1, x_2) = 0 \Rightarrow |x_1| + |x_2| = 0$

$\Rightarrow |x_1| = 0 \quad , \quad |x_2| = 0$

$\Rightarrow x_1 = 0 \quad , \quad x_2 = 0$

$\Rightarrow x_1 = x_2$

Let $x_1 = x_2 \Rightarrow |x_1| = |x_2|$

$\Rightarrow |x_1| + |x_2| = |x_2| + |x_2|$ (Adding $|x_2|$ both sides).

$\Rightarrow d(x_1, x_2) = 2|x_2|$

$\Rightarrow d(x_1, x_2) = 0$ if $|x_2| = 0$

i.e. $d(x_1, x_2)$ is not always zero.

$\because$ d is not a metric on X.

Question:

Let $X = R$ and $d(x_1, x_2) = \max(|x_1|, |x_2|)$.
Show that d is not a metric on R.

Solution:

let $d(x_1, x_2) = 0 \Rightarrow \max(|x_1|, |x_2|) = 0$

Show that d is not a metric on R.

$\Rightarrow |x_1| = 0 \quad , \quad |x_2| = 0$

$\Rightarrow x_1 = 0 \quad , \quad x_2 = 0$

$\Rightarrow x_1 = x_2$

Let $x_1 = x_2 \Rightarrow |x_1| = |x_2|$

$\Rightarrow \max(|x_1|, |x_2|) = |x_2|$

$\Rightarrow d(x_1, x_2) = 0$ if $|x_2| = 0$

i.e. $d(x_1, x_2)$ is not always zero.

Thus d is not a metric on X.

Question:

Let (X, d) be a metric space and let $d'': X \times X \to R$ be given by

$$d''(x_1, x_2) = \frac{1 - d(x_1, x_2)}{1 + d(x_1, x_2)}.$$

Prove that d'' is not a metric on X.

Solution:

Let $d''(x_1, x_2) = 0$.

$$\Rightarrow \frac{1 - d(x_1, x_2)}{1 + d(x_1, x_2)} = 0$$

$$\Rightarrow 1 - d(x_1, x_2) = 0$$

$$\Rightarrow d(x_1, x_2) = 1$$

$$\Rightarrow x_1 \neq x_2$$

$\because$ d is a metric on X and $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$

Thus $d''(x_1, x_2) = 0 \not\Rightarrow x_1 = x_2$

Thus d'' is not metric on X.

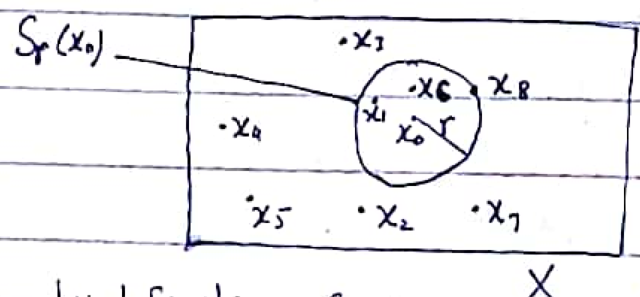
Open Sphere:

Let (X, d) be a metric space. let $x_0 \in X$ and $r > 0$, then open sphere with centre at x_0 and radius equal to r is denoted by $S_r(x_0)$ and is defined as

$$S_r(x_0) = \{x \mid x \in X,\ d(x, x_0) < r\}.$$

Note:

i) Let $X = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$ and $r > 0$; $S_r(x_0) = ?$



Then by definition of open sphere $S_r(x_0) = \{x_0, x_1, x_6\}$.

ii) $S_r(x_0) \subseteq X$

iii) $S_r(x_0) \neq \phi$

iv) Here we shall study the open spheres of the following shapes.

(a) Open interval **(b)** Open disc

(c) Open ball

The shape of an open sphere depends upon the metric space (X, d).

Example: let R be the metric space. Let $x_0 = 1$, $r = \frac{1}{2}$. Find $S_{\frac{1}{2}}(1)$.

Solution:

Here metric space is (R, d), where metric $d : R \times R \to R$ is defined as $d(x_1, x_2) = |x_1 - x_2|$

We know that

$$S_r(x_0) = \{x \mid x \in X, \ d(x, x_0) < r\}$$

Put $X = R$, $x_0 = 1$, $r = \frac{1}{2}$

$$\therefore S_{\frac{1}{2}}(1) = \{x \mid x \in R, \ d(x,1) < \tfrac{1}{2}\}$$

$$= \{x \mid x \in R, \ |x-1| < \tfrac{1}{2}\}$$

$$= \{x \mid x \in R, \ x-1 < \tfrac{1}{2}, \ x-1 > -\tfrac{1}{2}\}$$

$$= \{x \mid x \in R, \ x < 1 + \tfrac{1}{2}, \ x > 1 - \tfrac{1}{2}\}$$

$$= \{x \mid x \in R, \ \tfrac{1}{2} < x < \tfrac{3}{2}\}$$

$$= \left] \tfrac{1}{2}, \tfrac{3}{2} \right[$$

$|x-1| < \frac{1}{2}$

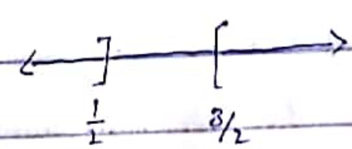
$\pm(x-1) < \frac{1}{2}$

$x-1 < \frac{1}{2}$ & $-(x-1) < \frac{1}{2}$

$x < 1+\frac{1}{2}$ & $x > \frac{1}{2}$

$x < \frac{3}{2}$ & $x > \frac{1}{2}$

Open sphere in this case is an open interval.



Note:

An open sphere in a usual metric space R is always an "open interval".

Example: Let the metric space be R^2 and let $P_0 = (a,b)$ and $r = 1$. Find $S_r(P_0)$.

Solution:

Here metric space is (R, d), where metric $d: R \times R \to R$ is defined as

$$d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

We know that

$$S_r(P_0) = \{P \mid P \in X,\ d(P, P_0) < r\}$$

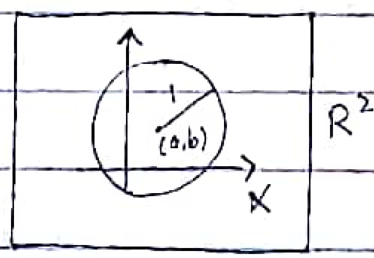
Put $X = R^2$, $P_0 = (a,b)$, $P = (x,y)$, $r = 1$

$$\therefore S_1(a,b) = \{(x,y) \mid (x,y) \in R^2,\ d((x,y),(a,b)) < 1\}$$

$$= \{(x,y) \mid (x,y) \in R^2,\ \sqrt{(x-a)^2 + (y-b)^2} < 1\}$$

$$= \{(x,y) \mid (x,y) \in R^2,\ (x-a)^2 + (y-b)^2 < 1\}$$

This is an open disc with centre at (a,b) and radius 1.



Note: An open sphere in a usual metric space R^2 is always an "open disc".

Example:

Let the metric space be R^2 and d_1 be the metric on R^2 defined by

$$d_1(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$$

Let $P_0 = (0,0)$ and $r = \frac{1}{\sqrt{2}}$. Find $S_r(x_0)$.

Solution:

Here metric space is (R^2, d_1), where metric $d_1 : R^2 \times R^2 \to R$ is defined

$$d_1[(x_1, y_1), (x_2, y_2)] = |x_1 - x_2| + |y_1 - y_2|$$

We know that

$$S_r(P_0) = \{P \mid P \in X,\ d(P, P_0) < r\}$$

Put $X = R^2$, $P_0 = (0,0)$, $P = (x, y)$, $r = \frac{1}{\sqrt{2}}$

$$\therefore S_{\frac{1}{\sqrt{2}}}(0,0) = \left\{(x,y) \mid (x,y) \in R^2,\ d_1((x,y),(0,0)) < \frac{1}{\sqrt{2}}\right\}$$

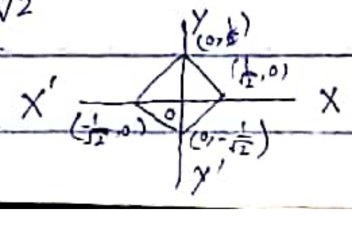
$$= \left\{(x,y) \mid (x,y) \in R^2,\ |x-0| + |y-0| < \frac{1}{\sqrt{2}}\right\}$$

$$= \left\{(x,y) \mid (x,y) \in R^2,\ |x| + |y| < \frac{1}{\sqrt{2}}\right\}$$ — and which is required open sphere.

$$= \left\{(x,y) \mid (x,y) \in R^2,\ \pm x \pm y < \frac{1}{\sqrt{2}}\right\}$$

$$= \left\{(x,y) \mid (x,y) \in R^2,\ \frac{x}{\pm\frac{1}{\sqrt{2}}} \pm \frac{y}{\pm\frac{1}{\sqrt{2}}} < 1\right\}$$

This is an open square with x-intercepts $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$ and y-intercepts $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$



Example: Let (X, d_0) be a discrete metric space. Let $x \in X$ and $r > 0$.
Find $S_r(x_0)$, when (i) $r \leq 1$ (ii) $r > 1$.

Solution:

Here metric space is (X, d_0), where
$d_0 : X \times X \to R$ is defined by

$$d_0(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

We know that

$$S_r(x_0) = \{x \mid x \in X, \underset{(1,0)}{d(x, x_0)} < r\} \quad \text{---} (1)$$

<u>When $r \leq 1$</u>

If $x \neq x_0$ then from eq. (1) we get $1 < r$ (False)
If $x = x_0$ then from eq. (1) we get $0 < r$ (True)
Thus $S_r(x_0) = \{x \mid x \in X, x = x_0\}$

When $r > 1$

If $x \neq x_0$ then from eq. (1) we get $1 < r$ (True)
If $x = x_0$ then from eq. (1) we get $0 < r$ (True)
Thus $S_r(x_0) = \{x \mid x \in X, x = x_0 \text{ or } x \neq x_0\}$

$= \{x \mid x \in X, x = x_0\} \cup \{x \mid x \in X, x \neq x_0\}$

$= \{x_0\} \cup X - \{x_0\}$

$= X$

Note: From above example we conclude that

i) An open sphere with radius less than or equal to 1 in a discrete metric space is always singleton.

ii) An open sphere with radius greater than in a discrete metric space is always the full space X.

Question: Let C be the set of all complex numbers and let $d: C \times C \to R$ be defined by $d(z_1, z_2) = |z_1 - z_2|$. Find $S_r(x_0)$ when $x_0 = 1$, $r = 0.01$

Solution:

The given metric space is (C, d), where $d: C \times C \to R$ be defined by

$$d(z_1, z_2) = |z_1 - z_2|$$

Now $S_r(x_0) = \{x \mid x \in X,\ d(x, x_0) < r\}$

Put $X = C$, $x_0 = 1$ $r = 0.01$

$\therefore S_{0.01}(1) = \{x \mid x \in C,\ d(x, 1) \leq 0.01\}$

$= \{x \mid x \in C,\ |x-1| < 0.01\}$

Since $x \in C \therefore x = a + ib$

$\Rightarrow x - 1 = a + ib - 1$

$\Rightarrow x - 1 = (a-1) + ib$

$\Rightarrow |x-1| = \sqrt{(a-1)^2 + b^2}$

$\therefore (1) \Rightarrow S_{0.01}(1) = \{(a+ib) \mid (a+ib) \in \mathbb{C}, |(a-1)+ib| < 0.01\}$

$= \{(a+ib) \mid (a+ib) \in \mathbb{C}, (a-1)^2 + (b-0)^2 < (.01)^2\}$

This is an open disc with centre at $(1, 0)$ and radius equal to 0.01.

Question: Let d be a metric on X and let $d' : X \times X \to \mathbb{R}$ be given by

$$d'(x_1, x_2) = \min(1, d(x_1, x_2)).$$

Describe $S_r(x_0)$.

Solution:

Here given metric space is (X, d'), where $d' : X \times X \to \mathbb{R}$ be given by

$$d'(x_1, x_2) = \min(1, d(x_1, x_2))$$

$$S_r(x_0) = \{x \mid x \in X, d'(x, x_0) < r\}$$

$$= \{x \mid x \in X, \min(1, d(x_1, x_2)) < r\}$$

This is the required open sphere.

Question:

Let (X, d) be a metric space and let $d' : X \times X \to R$ be given by
$d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}$. Describe $S_r(x_0)$.

Solution:

Here given metric space is (X, d'), where $d' : X \times X \to R$ be given by

$$d'(x_1, x_2) = \frac{d(x_1, x_2)}{1 + d(x_1, x_2)}$$

Now $S_r(x_0) = \{x \mid x \in X,\ d'(x, x_0) < r\}$

$$= \left\{x \mid x \in X,\ \frac{d(x_1, x_2)}{1 + d(x_1, x_2)} < r\right\}$$

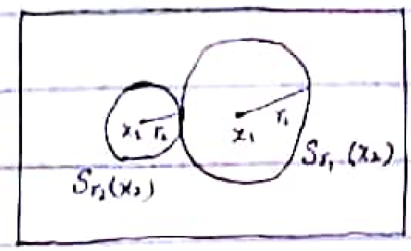
This is the required open sphere.

Theorem: Let x_1, x_2 be any two distinct points of a metric space X. Prove that there exist two open spheres $S_{r_1}(x_1)$ and $S_{r_2}(x_2)$ in X such that

$$S_{r_1}(x_1) \cap S_{r_2}(x_2) = \phi .$$

Proof:

Let $S_{r_1}(x_1)$ and $S_{r_2}(x_2)$ be two open spheres with centers x_1 and x_2 radii r_1 and r_2 resp.



Let $d(x_1, x_2) = r_1 + r_2$

We are to prove that

$$S_{r_1}(x_1) \cap S_{r_2}(x_2) = \phi$$

We shall prove it by contradiction method.

Suppose $S_{r_1}(x_1) \cap S_{r_2}(x_2) \neq \phi$

Let $x \in S_{r_1}(x_1) \cap S_{r_2}(x_2)$

$\Rightarrow x \in S_{r_1}(x_1)$ and $x \in S_{r_2}(x_2)$

$\Rightarrow d(x, x_1) < r_1$ and $d(x, x_2) < r_2$ — (1)

Since $r_1 + r_2 = d(x_1, x_2) \leq d(x_1, x) + d(x, x_2)$

$\because$ d is a metric on X.

$\Rightarrow r_1 + r_2 \leq d(x_1, x) + d(x, x_2)$

$\Rightarrow r_1 + r_2 < r_1 + r_2$ [By (1)]

It is impossible.

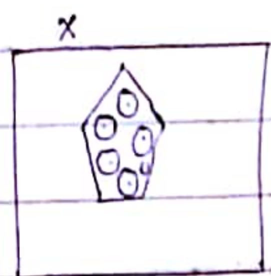
Thus our supposition $S_{r_1}(x_1) \cap S_{r_2}(x_2) \neq \phi$ *is wrong.*

Hence $S_{r_1}(x_1) \cap S_{r_2}(x_2) = \phi$

Open Set:

★ A subset U of a metric space (X, d) is called an open set if for every $x \in U$ there exists a real number $r > 0$ such that $x \in S_r(x) \subseteq U$.

★ let (X, d) be a metric space. let $U \subseteq X$. Then U is called an open set, if for each $x \in U$, $\exists\ r > 0$, such that $S_r(x) \subseteq U$.
i.e. U is called an open set, if each point of U is the centre of some open sphere, which is contained in U.



U is an open set.

U is not an open set.

Example: let R be a usual metric space (The ordinary real number line) and let $U =]0,1[$. Then show that U is open.

Solution:

Here metric space is (R, d), where $d: R \times R \to R$ is given by

$$d(x_1, x_2) = |x_1 - x_2|$$

Let $x_0 \in U$, Let $r > 0$.

Then $S_r(x_0) = \{x \mid x \in R,\ d(x, x_0) < r\}$

$$= \{x \mid x \in R,\ |x - x_0| < r\}$$

$$= \{x \mid x \in R,\ x - x_0 < r,\ x - x_0 > -r\}$$

$$= \{x \mid x \in R,\ x < x_0 + r,\ x > x_0 - r\}$$

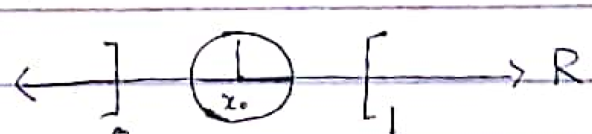
$$= \{x \mid x \in R,\ x_0 - r < x < x_0 + r\}$$

$$=]x_0 - r, x_0 + r[$$

We can find a value of r for which

$$S_r(x_0) =]x_0 - r, x_0 + r[\subseteq U =]0,1[$$

Thus $U =]0,1[$ is an open set.



Note:

In the above example if we take $x_0 = 0.99$. Let $r = 0.001$

Then $S_{0.001}(0.99) =]0.99 - 0.001, 0.99 + 0.001[=]0.989, 0.991[$
$\subseteq]0,1[$

Example: Let R^2 be a usual metric space.
(The ordinary real plane)
Let $U = \{(x, y) \mid (x, y) \in R^2, x^2 + y^2 < 1\}$.
Show that U is an open set.

Solution:

Here metric space is (R^2, d), where $d : R^2 \times R^2 \to R$ is given by
$$d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

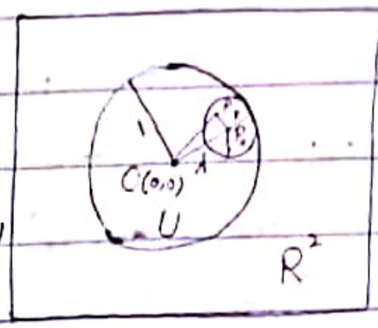
Let $P_0 \in U$

Let $d(O, P_0) = \lambda$

Let $r = 1 - \lambda$, then $r > 0$

We shall prove that $S_r(P_0) \subseteq U$

Let $P \in S_r(P_0) \Rightarrow d(P, P_0) < r$



Since (R^2, d) is a metric space,

$\therefore d(P, P_0) + d(P_0, O) \geq d(P, O)$

$\Rightarrow r + \lambda > d(P, O) \quad \because r > d(P, P_0)$

$\Rightarrow 1 - \lambda + \lambda > d(P, O)$

$\Rightarrow d(P, O) < 1$

$\Rightarrow P \in U$

Since $P \in S_r(P_0) \Rightarrow P \in U$

$\therefore S_r(P_0) \subseteq U$

Hence U is an open set.

Example: Let R be a usual metric space (The ordinary real number line) and let $U = \{x \mid x \in R,\ 0 \le x < 1\}$, then show that open set.

Solution:

Here metric space is (R, d) where $d : R \times R \to R$ is given by $d(x_1, x_2) = |x_1 - x_2|$

Here $U = \{x \mid x \in R,\ 0 \le x < 1\}$

$= [0, 1[$

Let $x_0 = 0 \in U$. Let $r > 0$

Then $S_r(0) = \{x \mid x \in R,\ d(x, 0) < r\}$

$= \{x \mid x \in R,\ |x - 0| < r\}$

$= \{x \mid x \in R,\ |x| < r\}$

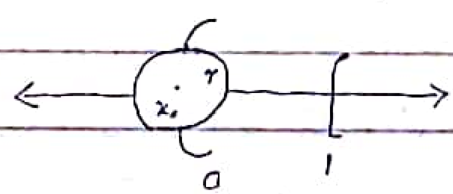
$= \{x \mid x \in R,\ x < r;\ x > -r\}$

$= \{x \mid x \in R,\ -r < x < r\}$

$= \ ]-r, +r[$

We can find a value of r for which $S_r(0) =]-r, +r[$

Thus $U = [0, 1[$ is not an open set. $\not\subset U = [0,1[$



Theorem

Let (X, d) be a metric space, then

i) Union of any collection $\{U_\alpha : \alpha \in I\}$ of open sets is open.

ii) Intersection of finite number of open set is open.

iii) The whole space X and the empty set ϕ are both open.

Proof:

i) Let $\{U_\alpha : \alpha \in I\}$ be any collection of open sets in (X, d).

We are to prove that, $\bigcup_{\alpha \in I} U_\alpha$ is an open set.

Let $x \in \bigcup_{\alpha \in I} U_\alpha$

Then $x \in U_\alpha$ for some $\alpha \in I$

Since each U_α is an open set therefore there exist $r > 0$ such that $S_r(x) \subseteq U_\alpha$ for some $\alpha \in I$

$$\Rightarrow S_r(x) \subseteq \bigcup_{\alpha \in I} U_\alpha$$

$$\Rightarrow \bigcup_{\alpha \in I} U_\alpha \text{ is an open set.}$$

ii) Let $\{U_\alpha : \alpha = 1, 2, \ldots, n\}$ be finite collection of open sets in (X, d).

We are to prove that $\bigcap_{\alpha=1}^{n} U_\alpha$ is an open set.

Let $x \in \bigcap_{\alpha=1}^{n} U_\alpha$

$$\Rightarrow x \in U_\alpha \quad \forall\ \alpha = 1, 2, \ldots, n.$$

Since each U_α is an open set therefore there exist $r>0$

Such that $S_{r_\alpha}(x) \subseteq U_\alpha \quad \forall\, \alpha = 1,2,\dots,n.$

Let $r = \min\{r_1, r_2, r_3, \dots r_n\}$

Then $S_r(x) \subseteq S_{r_\alpha}(x) \subseteq U_\alpha \quad \forall\, \alpha = 1,2,\dots,n.$

$\Rightarrow \quad S_r(x) \subseteq U_\alpha \quad \forall\, \alpha = 1,2,\dots,n.$

$\Rightarrow \quad S_r(x) \subseteq \bigcap_{\alpha=1}^{n} U_\alpha$

$\Rightarrow \quad \bigcap_{\alpha=1}^{n} U_\alpha$ is an open set.

iii) To show that empty set ϕ is an open set, we have to show that each point in ϕ is the centre of some open sphere which is contained in ϕ. But since there is no point in ϕ, the condition is automatically satisfied.

Hence ϕ is an open set.

Since every open sphere centered at a point of X is contained in X.

$\therefore$ X is an open set.

Theorem: Every non-empty subset of a discrete metric space is open.

Proof:

Let (X, d) be a discrete metric space.

Let $U \subseteq X$ such that $U \neq \phi$

We shall prove that U is an open set.

Let $x_0 \in U$.

Let $0 < r < 1$

Then $S_r(x_0) = \{x \mid x \in X, d(x, x_0) < r\}$

$= \{x_0\}$

$\therefore$ The open sphere in a discrete metric space, whose radius is less then 1, is always singleton.

Since $S_r(x_0) = \{x_0\} \subseteq U$

$\Rightarrow$ U is an open set.

Example:

Let R be a usual metric space (The ordinary real number line) and let $U = \{0\}$, then show that U is not open.

Solution:

Here metric space is (R, d), where $d: R \times R \rightarrow R$ is given by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $U = \{0\}$

let $x_0 = 0 \in U$. let $r > 0$

Then $S_r(0) = \{x \mid x \in R,\ d(x,0) < r\}$

$= \{x \mid x \in R,\ |x - 0| < r\}$

$= \{x \mid x \in R,\ |x| < r\}$

$= \{x \mid x \in R,\ x < r,\ x > -r\}$

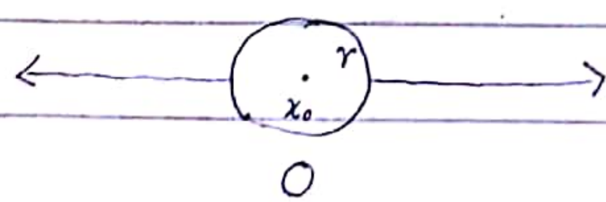
$= \{x \mid x \in R,\ -r < x < r\}$

$=]-r, +r[$

We can find a value of r for which

$S_r(0) =]-r, +r[\not\subseteq U = \{0\}$

Thus $U = \{0\}$ is not an open set.

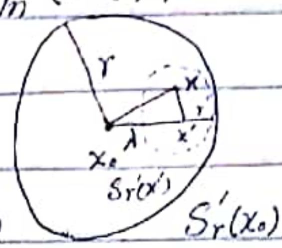


Theorem: An open sphere in a metric space (X, d) is an open set.

Proof:

Let $S_r(x_0)$ be an open sphere in (X, d).

Let $x' \in S_r(x_0) \Rightarrow d(x', x_0) < r$



Let $d(x', x_0) = \lambda$

Let $r' = r - \lambda$, then $r' > 0$

We shall prove that $S_{r'}(x') \subseteq S_r(x_0)$

Let $x \in S_{r'}(x') \Rightarrow d(x, x') < r'$

Since (X, d) is a metric space,

$\therefore \; d(x, x') + d(x', x_0) \geq d(x, x_0)$

$\Rightarrow \; r' + \lambda > d(x, x_0) \quad \because r' > d(x, x')$

$\Rightarrow \; r - \lambda + \lambda > d(x, x_0) \quad \because r' = r - \lambda$

$\Rightarrow \; d(x, x_0) < r$

$\Rightarrow \; x \in S_r(x_0)$

Since $x \in S_{r'}(x') \Rightarrow x \in S_r(x_0)$

$\therefore \; S_{r'}(x') \subseteq S_r(x_0)$

Thus $S_r(x_0)$ is an open set.

Hence open sphere in a metric space is an open set.

Theorem: A subset U of a metric space X is open if and only if U is union of open spheres.

Proof:

let (X, d) be a metric space. let $U \subseteq X$.
We have to prove that

U is an open set $\Leftrightarrow$ U is the union of open spheres.

We suppose that U is an open set. Since U is open therefore each point of U is the centre of some open sphere which is contained in U.

Thus U is the union of open spheres.

Conversely suppose that U is the union of open spheres. Thus U is the union of open sets.
($\because$ Open spheres in metric space are open sets.)

Since the union of any number of open sets in a metric space is an open set.

Thus U is an open set.

Theorem:

let X be a metric space and let $\{x_0\}$ be a singleton subset of X.

Then $X - \{x_0\}$ is open.

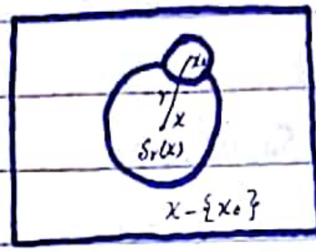
Proof:

Let $x \in X - \{x_0\}$

Let $d(x, x_0) = r \longrightarrow (1)$

We shall prove that

$S_r(x) \subseteq X - \{x_0\}$



Let $x' \in S_r(x)$

$\Rightarrow d(x', x) < r \longrightarrow (2)$

From (1) and (2) we get

$\Rightarrow d(x', x) \neq d(x, x_0)$

$\Rightarrow d(x, x') \neq d(x, x_0)$ $\left[(\because d \text{ is a metric on } X. \text{ So } d(x', x) = d(x, x') \right]$

$\Rightarrow x' \neq x_0$

$\Rightarrow x' \notin \{x_0\}$

$\Rightarrow x' \in X - \{x_0\}$

Since $x' \in S_r(x) \Rightarrow x' \in X - \{x_0\}$

$\therefore S_r(x) \subseteq X - \{x_0\}$

Since every point x of $X - \{x_0\}$ is the centre of some open sphere contained in $X - \{x_0\}$

Hence $X - \{x_0\}$ is an open set.

Question:

Can a finite subset of a metric space be open?

Solution: We know that

i) If (X, d_0) is a discrete metric space, then every subset of X is open.
Therefore a finite subset of a metric space is open.

ii) If $(\mathbb{R}, d)$ is a usual metric space then $\{0\} \subseteq \mathbb{R}$ is not open.
Therefore a finite subset $\{0\}$ of $\mathbb{R}$ is not open.

$\therefore$ Thus in general, we can say that, finite subset of a metric space may or may not open.

Metric Topology:

The topology determined by a metric is called "metric topology".

Closed Set: Let (X, d) be a metric space.
Let $F \subseteq X$.
Then F is closed $\Longleftrightarrow$ $F' = \underbrace{X - F}_{\text{complement}}$ is open.

Example: Let $X = R$ be the metric space and let $A = [a,b]$, where $a, b \in R$, & $a<b$. Show that A is closed set.

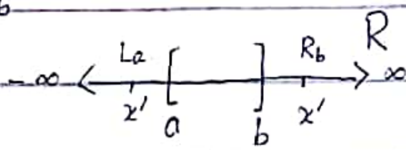
Solution:

The given metric space is (R, d), where $d : R \times R \to R$ is given by

$$d(x_1, x_2) = |x_1 - x_2|$$

Since $A = [a, b]$

$$A' = R - [a,b]$$
$$=]-\infty, a[\cup]b, \infty[$$
$$= L_a \cup R_b$$



In order to prove A is closed; we will have to prove that A' is open.

Let $x' \in A' \Rightarrow x' \in L_a \cup R_b$

$\Rightarrow x' \in L_a$ or $x' \in R_b$

Case I: If $x' \in L_a$ then $x' < a$

Let $d(x', a) = r$

$\Rightarrow |x' - a| = r$

$\Rightarrow x' - a = -r \quad \because x' < a$

$\Rightarrow x' + r = a$ ———— (1)

Now $S_r(x') = \{x \mid x \in R,\ d(x, x') < r\}$

$= \{x \mid x \in R,\ |x - x'| < r\}$

$= \{x \mid x \in R,\ x - x' < r,\ x - x' > -r\}$

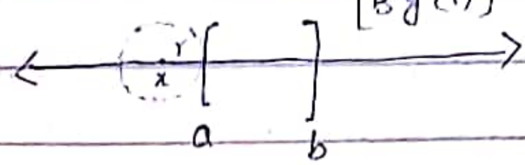
$= \{x \mid x \in R,\ x < x' + r,\ x > x' - r\}$

$= \{x \mid x \in R,\ x' - r < x < x' + r\}$

$=]x' - r,\ x' + r[$

$=]x' - r,\ a[$

[By (1)]



Thus $S_r(x') =]x' - r, a[\subseteq L_a \subseteq L_a \cup R_b = A'$

i.e. $S_r(x') \subseteq A'$.

Hence in this case A' is open.

Case II If $x' \in R_b$ then $x' > b$

Let $d(x', b) = r$

$\Rightarrow |x' - b| = r$

$\Rightarrow x' - b = r \quad \because x' > b$

$\Rightarrow x' - r = b \quad (2)$

Now $S_r(x') = \{x \mid x \in R,\ d(x, x') < r\}$

$= \{x \mid x \in R,\ |x - x'| < r\}$

$= \{x \mid x \in R,\ x - x' < r,\ x - x' > -r\}$

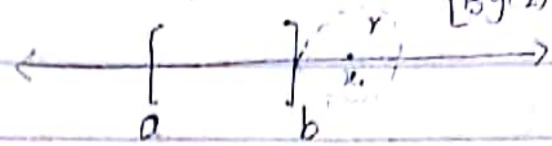
$= \{x \mid x \in R,\ x < x' + r,\ x > x' - r\}$

$= \{x \mid x \in R,\ x' - r < x < x' + r\}$

$=]x'-r, x'+r[$

$=]b, x'+r[$

[By (2)]



Thus $S_r(x') =]b, x'+r[\subseteq R_b \subseteq L_a \cup R_b = A'$

i.e. $S_r(x') \subseteq A'$

Hence in this case A' is also open.

Since in both the cases A' is open.

Therefore A is closed set.

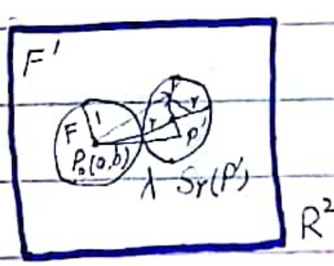
Example:

Let R^2 be the metric space.

Let $F = \{(x, y) \mid (x, y) \in R^2, (x-a)^2 + (y-b)^2 \le 1\}$

Show that F is closed set.

Solution:



Here given metric space is (R^2, d) where $d: R^2 \times R^2 \to R$ is given by

$$d[(x_1, y_1), (x_2, y_2)] = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Here $F = \{(x, y) \mid (x, y) \in R^2, (x-a)^2 + (y-b)^2 \le 1\}$

Thus $F' = \{(x, y) \mid (x, y) \in R^2, (x-a)^2 + (y-b)^2 > 1\}$

In order to prove that F is closed, we will show that F' is open.

Let $P' \in F'$. Let $d(P', P_0) = \lambda$

Let $r = \lambda - 1$, clearly $r > 0$

We shall prove that $S_r(P') \subseteq F'$

Let $P \in S_r(P') \Rightarrow d(P, P') < r$

Since d is a metric on R^2.

$\therefore d(P', P) + d(P, P_0) \geq d(P', P_0)$

$\Rightarrow \quad r + d(P, P_0) > \lambda$

$\Rightarrow \quad d(P, P_0) > \lambda - r$

$\Rightarrow \quad d(P, P_0) > \lambda - (\lambda - 1) = 1$

$\Rightarrow \quad d(P, P_0) > 1$

$\Rightarrow \quad P \in F'$

Since $P \in S_r(P') \Rightarrow P \in F'$

$\therefore \quad S_r(P') \subset F'$

$\Rightarrow$ F' is an open set.

$\Rightarrow$ F is closed set.

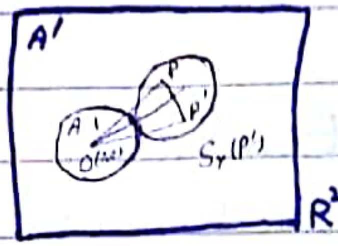
Example:

Let R^2 be the metric space.

Let $A=\{(x,y) \mid (x,y)\in R^2,\ x^2+y^2\leq 1\}$ be a subset of R^2.

Is A a closed set in R^2?

Solution:



Here given metric space is (R^2, d) where $d: R^2\times R^2 \to R$ is given by

$$d[(x_1,y_1),(x_2,y_2)] = \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$$

Here $A=\{(x,y) \mid (x,y)\in R^2,\ x^2+y^2\leq 1\}$

Thus $A'=\{(x,y) \mid (x,y)\in R^2,\ x^2+y^2>1\}$

In order to prove that A is closed, we will show that A is open.

Let $P'\in A'$. Let $d(P',O)=\lambda$

Let $r=\lambda-1$, clearly $r>0$

We shall prove that $S_r(P')\subseteq A'$

Let $P\in S_r(P') \Rightarrow d(P,P')<r$

Since d is a metric on R^2.

$\therefore d(P',P)+d(P,O)\geq d(P',O)$

$\Rightarrow r+d(P,O)>\lambda \quad \because r>d(P',P)$

$\Rightarrow d(P,O)>\lambda-r$

$\Rightarrow d(P,O)>\lambda-(\lambda-1)=1$

$\Rightarrow d(P,0) > 1$

$\Rightarrow P \in A'$

Since $P \in S_r(P') \Rightarrow P \in A'$

$\therefore S_r(P') \subseteq A'$

$\Rightarrow A'$ is an open set.

$\Rightarrow A$ is closed set.

Example:

Let R be the real line and let $A = \{x \mid x \in R, 0 \leq x < 1\}$, be a subset of R. Show that A is not closed.

Solution:

The given metric space is (R, d), where $d: R \times R \to R$ is given by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $A = \{x \mid x \in R, 0 \leq x < 1\}$

$= [0, 1[$

$\therefore A' = R - A$

$=]-\infty, 0[\cup [1, \infty[$

Note that, $1 \in A'$. We take $x_0 = 1$ and $r > 0$

Then $S_r(x_0) = \{x \mid x \in X, d(x, x_0) < r\}$

Put $x_0 = 1$ and $X = R$

$S_r(1) = \{x \mid x \in R,\ d(x,1) < r\}$

$= \{x \mid x \in R,\ |x-1| < r\}$

$= \{x \mid x \in R,\ x-1 < r,\ x-1 > -r\}$

$= \{x \mid x \in R,\ x < 1+r,\ x > 1-r\}$

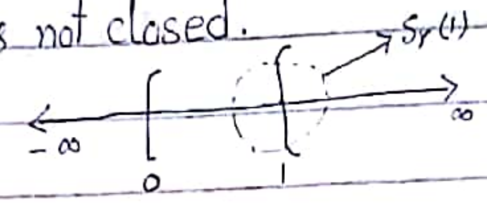
$= \{x \mid x \in R,\ 1-r < x < 1+r\}$

$= \ ]1-r, 1+r[$

But $S_r(1) =]1-r, 1+r[\not\subset A' \ \forall r > 0$

Thus A' is not open.

$\Rightarrow$ A is not closed.



Imp

Theorem:

★ A subset U of a metric space is open if and only if $X-U$ is closed.

★ A subset F of a metric space is open if and only if F' is closed.

Proof: Let (X,d) be a metric space. We have to prove that

U is open $\Leftrightarrow$ $X-U$ is closed.

Suppose U is an open set.

Then $(X-U)' = (U')'$ $\quad \because X-U = U'$

$= U$ (open set)

Since $(X-U)'$ is an open set.

$\therefore$ $X-U$ is a closed set.

Conversely suppose that $X-U$ is a closed set.
Then $(X-U)'$ is an open set.
$\Rightarrow (U')'$ is an open set. $\because X-U = U'$
$\Rightarrow U$ is an open set.

Theorem: Let X be a metric space.
i) Intersection of any collection $\{F_\alpha : \alpha \in I\}$ of closed sets is closed.
ii) Union of finite collection $\{F_1, F_2, \dots, F_n\}$ of closed set is closed.
iii) X and ϕ are closed.

Proof:

$(A\cup B)' = A' \cap B'$
$(A\cap B)' = A' \cup B'$

i) Let $\{F_\alpha : \alpha \in I\}$ be any collection of closed sets in (X,d). Then F_α' is open. $\forall \alpha \in I$
$\Rightarrow \bigcup_{\alpha\in I} F_\alpha'$ is open ($\because$ Union of any number of open sets is open)
$\Rightarrow \left(\bigcap_{\alpha\in I} F_\alpha\right)'$ is open. $\because \bigcup_{\alpha\in I} F_\alpha' = \left(\bigcap_{\alpha\in I} F_\alpha\right)'$

$\Rightarrow \bigcap_{\alpha\in I} F_\alpha$ is closed. $\bigcap_{\alpha\in I} F_\alpha = F_1 \cap F_2 \cap F_3 \cdots$

ii) Let $\{F_\alpha : \alpha = 1, 2, \dots, n\}$ be any finite collection of closed sets in (X,d).
Then F_α' is open. $\forall \alpha = 1, 2, \dots, n$.
$\Rightarrow \bigcap_{\alpha=1}^{n} F_\alpha'$ is open ($\because$ Intersection of finite number of open sets is open).

$\Rightarrow \left(\bigcup_{\alpha=1}^{n} F_\alpha\right)'$ is open. $\because \bigcap_{\alpha=1}^{n} F_\alpha' = \left(\bigcup_{\alpha=1}^{n} F_\alpha\right)'$

$\Rightarrow \bigcup_{\alpha=1}^{n} F_\alpha$ is closed.

iii) Since $\phi' = X - \phi = X$ which is open.

And $X' = X - X = \phi$ which is open.

$\Rightarrow$ X is closed.

Question:

Is N closed in R?.

Solution.

Here $N = \{1, 2, 3, \dots\}$

$N' = R - N$

$=]-\infty, 1[\cup]1, 2[\cup]2, 3[\cup \dots$

= Union of open intervals in R.

= Union of open sets. ($\because$ An open intervals in R is an open set.)

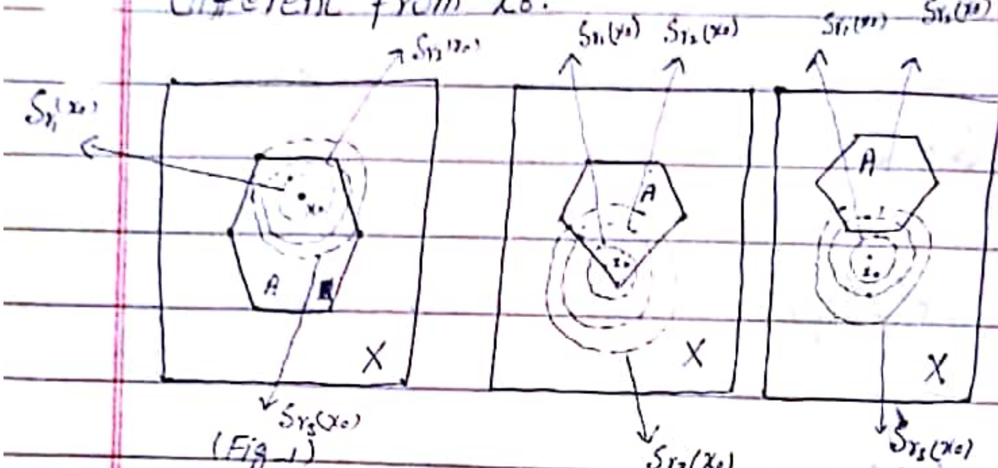
= Open set. ($\because$ Union of any number of open sets is an open set).

Since N' is an open set.

$\Rightarrow$ N is a closed set.

Limit Point :

Let (X, d) be a metric space. let $A \subseteq X$ and $x_0 \in X$. Then x_0 is called limit point of A if each open sphere centered at x_0 contains at least one point of A different from x_0.



(Fig-1) (Fig-2) (Fig-3)

In Fig-1, x_0 is a limit point of A.

In Fig-2, x_0 is also a limit point of A.

In Fig-3, x_0 is not a limit point of A.

Theorem:

Let (X, d) be a discrete metric space. Let $A \subseteq X$. Then A has no limit point.

Proof:

Consider the discrete metric space (X, d_0).

Here $d_0 : X \times X \longrightarrow R$ is defined by

$$d_0(x_1, x_2) = \begin{cases} 0 & \text{if } x_1 = x_2 \\ 1 & \text{if } x_1 \neq x_2 \end{cases}$$

We have to prove that $A \subseteq X$ has no limit point.

We shall prove it by contradiction method.

Suppose $x_0 \in X$ such that x_0 is a limit point of A.

Let $0 < r < 1$ then $S_r(x_0) = \{x_0\}$ ——(1)

$\because$ In a discrete metric space the open sphere with radius less than 1 is always singleton.

Here (1) shows that $S_r(x_0)$ contains no point of A different from x.

Thus x_0 is not a limit point of A.

Hence A has no limit point.

Question:

Let R be the metric space. Let $A=\{x \mid x \in R,\ x=\frac{1}{n},\ n \in N\}$ be subset of R. Show that "0" is a limit point of A.

Solution:

Here metric space is (R,d), where $d: R \times R \to R$ be defined by

$$d(x_1,x_2)=|x_1-x_2|$$

Here $A=\{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\ldots\}$

Then $S_r(0)=\{x \mid x \in R,\ d(x,0)<r\}$; $r>0$

$$=\{x \mid x \in R,\ |x-0|<r\}$$

$$=\{x \mid x \in R,\ |x|<r\}$$

$$=\{x \mid x \in R,\ x<r,\ x>-r\}$$

$$=\{x \mid x \in R,\ -r<x<r\}$$

$$=\ ]-r,+r[$$

Clearly for every $r>0$, $S_r(0)=\ ]-r,+r[$ contains a point of A different from "0".

Thus "0" is the limit point of A.

Question:

Let R be the metric space. Let
$A=\{x \mid x\in R,\ x=1 \text{ or } x=1+\frac{1}{n},\ n\in N\}$
be a subset of R. Show that "1" is a limit of A.

Solution:

Here metric space is (R,d), where $d: R\times R\to R$ be defined by $d(x_1,x_2)=|x_1-x_2|$

Here $A=\{x \mid x\in R,\ x=1 \text{ or } x=1+\frac{1}{n},\ n\in N\}$

$=\{x \mid x\in R,\ x=1\}\cup\{x \mid x\in R,\ x=1+\frac{1}{n},\ n\in N\}$

$=\{1\}\cup\{2,\frac{3}{2},\frac{4}{3},\dots\}$

$=\{1,2,\frac{3}{2},\frac{4}{3},\dots\}$

Now $S_r(1)=\{x \mid x\in R,\ d(x,1)<r\}$

$=\{x \mid x\in R,\ |x-1|<r\}$

$=\{x \mid x\in R,\ x-1<r,\ x-1>-r\}$

$=\{x \mid x\in R\ \ x<1+r,\ x>1-r\}$

$=\{x \mid x\in R,\ 1-r<x<1+r\}$

$=\ ]1-r,\ 1+r[$

Clearly for every $r>0$, $S_r(1)=]1-r<x<1+r[$

$=]1-r,\ 1+r[$

contains a point of A different from "1".

Thus "1" is a limit point of A.

Question: Let R be the metric space. Let $A = \{x \mid x \in R, 0 < x < 1\}$ be a subset of R. Show that "0" and "1" are the Limit point of A.

Solution:

Here metric space is (R, d), where $d: R \times R \to R$ be defined by

$$d(x_1, x_2) = |x_1 - x_2|$$

Here $A = \{x \mid x \in R,\ 0 < x < 1\}$

$$=]0, 1[$$

(i) First we shall prove that "0" is the limit pt. of A.

Now $S_r(0) = \{x \mid x \in R, d(x, 0) < r\}$; $r > 0$

$$= \{x \mid x \in R, |x - 0| < r\}$$

$$= \{x \mid x \in R, |x| < r\}$$

$$= \{x \mid x \in R, x < r, x > -r\}$$

$$= \{x \mid x \in R, -r < x < r\}$$

$$=]-r, +r[$$

Clearly for every $r > 0$, $S_r(0) =]-r, +r[$ contains a point of A different from "0". Thus "0" is a limit point of A.

(ii) Now we shall prove that "1" is the limit point of A.

Now $S_r(1) = \{x \mid x \in R, d(x,1) < r\}$

$= \{x \mid x \in R, |x-1| < r\}$

$= \{x \mid x \in R, x-1 < r, x-1 > -r\}$

$= \{x \mid x \in R, x < 1+r, x > 1-r\}$

$= \{x \mid x \in R, 1-r < x < 1+r\}$

$=]1-r, 1+r[$

Clearly for every $r > 0$, $S_r(1) =]1-r, 1+r[$ contains a point of A different from "1". Thus "1" is a limit point of A.

Question:

Let R be the metric space. Describe the limit points of the followings.

(a) N (b) Z

Solution:

Here metric space is (R, d).

where $d: R \times R \to R$ be defined by

$d(x_1, x_2) = |x_1 - x_2|$

(a) Here $N = \{1, 2, 3, \dots\}$

Then $a \in N$ or $a \notin N$.

Case-I: When $a \in N$

Then $S_r(a) = \{x \mid x \in R, d(x,a) < r\}$; $r > 0$

$= \{x \mid x \in R, |x-a| < r\}$

$= \{x \mid x \in R, x-a < r, x-a > -r\}$

$= \{x \mid x \in R, x < a+r, x > a-r\}$

$= \{x \mid x \in R, a-r < x < a+r\}$

$=]a-r, a+r[$

Clearly for every $r > 0$, $S_r(a) =]a-r, a+r[$ contains no point of N different from "a".

Thus "a" is not the limit point of N.

★ **Case II** When $a \notin N$, we can also prove that "a" is not a limit point of N.

(Same +) Thus N has no limit point.

★ **((b))** When $a \notin N$, we can also prove that "a" is not a limit point of N.

Thus N has no limit point.

Here $Z = \{\dots, -3, -2, -1, 0, 1, 2, \dots\}$

let $a \in R$ be a limit point of Z.

Then $a \in Z$ or $a \notin Z$.

Case - I When $a \in Z$.

Then $S_r(a) = \{x | x \in R, d(x,a) < r\}$; $r > 0$.

$= \{x | x \in R, |x-a| < r\}$

$= \{x | x \in R, x-a < r, x-a > -r\}$

$= \{x | x \in R, x < a+r, x > a-r\}$

$= \{x | x \in R, a-r < x < a+r\}$

$= \,]a-r, a+r[$

Clearly for every $r > 0$, $S_r(1) =]a-r, a+r[$ contains no point of Z different from "a".

Thus "a" is not the limit point of Z.

Case-II:

When $a \notin Z$, we can also prove that "a" is not a limit point of Z.

Thus Z has no limit point.

"NEIGHBOURHOOD"

Neighbourhood:

let (X,d) be a metric space let $x_0 \in X$.

let $N \subseteq X$. Then N is called a neighbourhood of x_0, if $\exists$ an open sphere $S_r(x_0)$ such that $x_0 \in S_r(x_0) \subseteq N$.

Example: Let R be the usual metric space. Let $x_0 = 0 \in R$. Show that $]-r, r[$, $]-r, r]$, $[-r, r[$ and $[-r, r]$, $(r > 0)$ is a neighbourhood of 0.

Solution: We know that

In a usual metric space R, the open sphere is an open interval.

(i) Now $0 \in]-r, r[\subseteq]-r, r[$ where $]-r, r[$ is an open sphere in R.

$\Rightarrow]-r, r[$ is a neighbourhood of "0".

(ii) Now $0 \in]-r, r[\subseteq]-r, r]$ where $]-r, r[$ is an open sphere in R.

$\Rightarrow]-r, r]$ is a neighbourhood of "0".

(iii)

Now $0 \in]-r, r[\subseteq [-r, r[$ where $]-r, r[$ is an open sphere in R.

$\Rightarrow [-r, r[$ is a neighbourhood of "0".

(iv) Now $0 \in]-r, r[\subseteq [-r, r]$ where $]-r, r[$ is an open sphere in R.

$\Rightarrow [-r, r]$ is neighbourhood of "0".

Theorem:

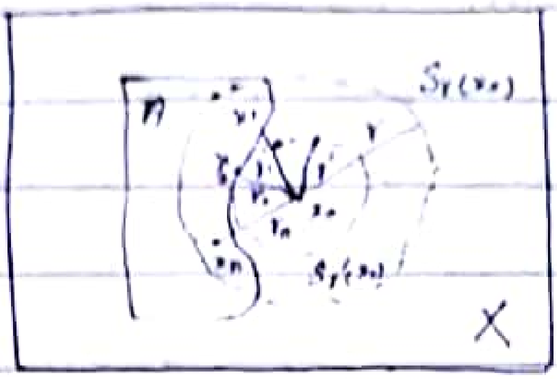
Let (X, d) be a metric space. Let $A \subseteq X$.
Let x_0 be a limit point of A.
Then every neighbourhood of x_0, contains infinitely many points of A.

Proof:

Let N be a neighbourhood of x_0.
then $\exists$ an open sphere $S_r(x_0)$ (where $r>0$) such that

$$x_0 \in S_r(x_0) \subseteq N \text{ ——— } (1)$$

We are to prove that N contains infinite points of A.
We prove it by contradiction method.
Suppose N contains finite points of A.
Then by (1) $S_r(x_0)$ also contains finite points of A.
Suppose $S_r(x_0)$ contains n points $x_1, x_2, x_3, \ldots, x_n$ of A.
Then $A \cap S_r(x_0) = \{x_1, x_2, x_3, \ldots, x_n\}$
let $d(x_0, x_i) = r_i$, $i = 1, 2, 3, \ldots, n$
Let $r' = \min(r_1, r_2, r_3, \ldots, r_n)$



Clearly $S_r(x_0)$ contains no point of A different from x_0.

This shows that x_0, is not a limit point of A. This is a contradiction.

Hence N contains infinitely many points of A.