

Q1 Show that a set X with cofinite topology is compact.

Ans: Proof:-

If X is finite. Then

$$\tau_{\text{cof}} = P(X)$$

①

$\Rightarrow (X, \tau_{\text{cof}})$ is compact.

But if X is not finite. Then

Consider an open cover D of X

Now let $\emptyset \in D \in D$ $\therefore D$ is finite

Now for every $x \in D^c$

Choose an element of D . D_x containing x .

$\{D_x \mid x \in D^c\}$ is finite

$\therefore D^c$ is finite

Consider this subcover of D

$$C = \{D\} \cup \{D_x \mid x \in D^c\}$$

$\Rightarrow C$ is a finite cover of D

(X, τ_{cof}) is compact.

Q2 Define Separated Set.

Let A and B two subsets of a topological space (X, T) . Then A and B are said to be separated sets if

and only if $A \cap B = \emptyset$

$A' \cap B = \emptyset$ and $A \cap B' = \emptyset$

Q3 Show that every Second Countable Space is first countable.

Proof:-

Let X be a Second Countable Space
then for each $x \in X$ - There is an open
basic set which is Countable. Therefore
for every $x \in X$ there is countable
neighbourhood base. Thus X is also
first countable Space.

Ex $X = \{1, 2, 3\}$

$\tau = \{\emptyset, \{1\}, \{2, 3\}, X\}$

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open Base $\Rightarrow B = \{\{1\}, \{2, 3\}\}$

we see 1st countable neighbourhood

$1 \in \{1\} \subseteq \{1\}$

$2 \in \{2, 3\} \subseteq \{2, 3\}$

$3 \in \{2, 3\} \subseteq \{2, 3\}$ countable neighbourhood base

Q4:- Give one example of Second Countable Space.

Ans A Topological space X is said to be Second countable Space if it has Countable open Base.

EXAMPLE

$$Ex \quad X = \{1, 2, 3\}$$

$$\tau = \{\emptyset, \{1\}, \{2, 3\}, X\}$$

$$\beta = \{\{1\}, \{2, 3\}\} \checkmark$$

Imp point of view

second countable sp = first countable $\checkmark$

if First countable space $\xrightarrow{\text{maybe}}$ Second countable

Q5

Define T_1 Space.

A topological space X is said to be T_1 -Space if for any pair of distinct points of X . There exist two open sets which contain one but not the other.

x	$x, y \in X \quad (x \neq y)$
$x \in U$	$y \in V$
$y \notin U$	$x \notin V$

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we see example Just concepts

$$X = \{1, 2\} \quad 1, 2 \in X$$

$$\tau = \{\emptyset, X, \underbrace{\{1\}}_U, \underbrace{\{2\}}_V\}$$

$$2 \in \{2\} \quad 1 \in \{1\}$$

$$1 \notin \{2\} \quad 2 \notin \{1\}$$

$$\begin{aligned} &1 \in \{1\} \\ &2 \notin \{1\} \end{aligned}$$

So X is a T_1 Space

Q6 Let $X = \{a, b, c\}$. write the basis for the discrete topology on X .

Ans Let $X = \{a, b, c\}$

Consider the set of all singletons i.e.

$$B = \{\{a\}, \{b\}, \{c\}\}$$

It is clear that this collection satisfies both the conditions of basis. let us

generate the topology from B

Following the topology

$$\tau = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$$

It is discrete topology.

Q7 Every Subspace of a Second Countable Space is Second Countable.

Proof:- Same as for first countable space.

Let $A \subset X$

(X is second countable space)

So $\exists$

$$B = \{B_n \mid n \in \mathbb{N}\} \rightarrow \text{Countable}$$

Now

$$B_A = \{A \cap B_n \mid n \in \mathbb{N}\} \text{ is a}$$

basis for τ_A and B_A is countable

$\Rightarrow A$ is second countable.

Q8 Let $(X, \mathcal{J})$ be a Path Connected Topological Space and $f: (X, \mathcal{J}) \rightarrow (Y, \mathcal{J}')$ be a Continuous map. Show that $f(X)$ is Path connected.

Ans

$$f: (X, \mathcal{J}) \xrightarrow{\text{contin}} (Y, \mathcal{J}')$$

$f(X)$ is path connected

let $a, b \in f(X)$

$$\Rightarrow \exists x, y \in X \text{ s.t.}$$

$$f(x) = a, \quad f(y) = b \quad \underline{5}$$

Now $\because X$ is path connected

$$\text{So, } \exists P: I \xrightarrow{\text{cont}} X \quad P(0) = x, P(1) = y$$

$$-q = f \circ P: I \rightarrow f(X)$$

$$q(0) = f \circ P(0) = f(P(0)) = f(x) = a$$

$$q(1) = f \circ P(1) = f(P(1)) = f(y) = b$$

$\Rightarrow f(X)$ is path connected

Q9 A Discrete Space is separable if and only if it is Countable.

$$X, \mathcal{J}_{\text{dis}} - P(X)$$

We know that the only dense Subset in X is X itself. So the only Choice for A is X s.t

$$\bar{A} = X$$

$\Rightarrow X$ is separable iff X is countable

Q10 Define Connected set?

A subset A of a topological space (X, T) is said to be connected iff there exist no pair of nonempty open subset U and V of X such that

$$A \cap U \quad \& \quad A \cap V$$

are nonempty disjoint sets and 6

$$A = (A \cap U) \cup (A \cap V)$$

Q11 Define Discrete Space is locally Connected.

A Discrete Space is locally connected.

$$X \quad \tau_{dis} = P(X)$$

$$x \in X$$

$$U_x \in P(X)$$

$$\forall x = \{x\} \subset U_x$$

Q12 Show that T_1 is T_0 Space.

$$X = \{1, 2, 3\}$$

X is T_1 space

$$\tau = \{\emptyset, \{1\}, \{2\}, X\}$$

X is also

T_0 -Space $1, 2 \in X$

$$\left\{ \begin{array}{l} 1 \in \{1\} \\ 2 \notin \{1\} \end{array} \right\} \quad \left\{ \begin{array}{l} 1 \notin \{2\} \\ 2 \in \{2\} \end{array} \right\}$$

Q13- Consider $(\mathbb{R}, T_u)$ then $A = (1, 3)$ and $B = [3, 5)$ are not separated Set.

Both sets are disjoint, i.e.

$$(1, 3) \cap [3, 5) = \emptyset$$

But at the same time both sets should not contain each other's limit points. But here

$$(1, 3)' \cap [3, 5) = \{3\} \neq \emptyset \text{ Are}$$

Q14: Define Normal Space?

A topological space (X, T) is said to be "Normal" iff for every pair of disjoint closed subsets, $F_1, F_2 \subset X$ there exist open subsets U_{F_1} and U_{F_2} containing F_1 and F_2 respectively such that

$$U_{F_1} \cap U_{F_2} = \emptyset$$

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Q15 Define Metric topology.

Let X be a nonempty set with metric d .

The topology T on X generated by the set of all open balls in X with respect to d is called "Metric Topology".

We also say that a topology T on X induced by the metric d .

Q16 Show that a Set X with Topology T containing Finite number Subsets of X is Compact.

Since every open Cover C of X is a Subclass of T and T itself Finite so C is finite too and any Subcover S of C is also Finite.

Q7 ~~Define Connected Set~~.

Q17: Define Connected Component in a topological Space X .

Consider a topological Space (X, T) .

Let $a \in X$ and C be a Connected Subset of X containing a .

Then $C_a = \bigcup_{a \in C} C$

is called Connected component of X containing a .

Q18: Example of Regular Space.

$$X = \{1, 2, 3\}$$

Define $T = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$

Let Suppose $A = \{2\}$ $1 \in X$

$$U = \{1\}$$

$$1 \in U$$

$$V = \{2\}$$

$$A \subseteq V$$

$$U \cap V = \emptyset$$

Q19 Consider $X = \{a, b, c, d, e\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, d, e\}, \{a, b, d, e\}, X\}$. Show that the set $A = \{c, e\}$ is disconnected subset of X .
 Since there exist a pair of nonempty open subsets $U = \{a, b, d, e\}$ and $V = \{a, b, c\}$ of X such that

$A \cap U = \{e\}$ and $A \cap V = \{c\}$ are nonempty disjoint sets and

$$A = (A \cap U) \cup (A \cap V) = \{e\} \cup \{c\} = \{c, e\}$$

Answer

Q20 Explain in detail why an Indiscrete Space (X, τ) where X consist of more than one point is not metrizable.

Reason:- It is bcz of the fact that in a metric space all the finite set are closed. But here only closed sets are $\emptyset$ and X . So τ_{ind} cannot be induced by a metric.

Q21 Consider $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$. (X, τ) is Regular Space.

$$\tau = \{\emptyset, \{a\}, \{b, c\}, X\}$$

$$F = \{b, c\}$$

$$F^c = \{a\}$$

$$U_a = \{a\}$$

$$U_b = \{b, c\}$$

$$U_a \cap U_b = \emptyset$$

Q22 Give an example of Second Countable Space.

Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$$

be a topology defined on X . A basis for $\mathcal{T}$ is

$$B = \{\{a\}, \{b\}, \{c, d\}\}$$

Remark: Any finite set with any $\mathcal{T}$ is Second Countable.

Q23 Homeomorphism of topological spaces is an equivalence Relation -

Proof:- $X \approx X$ $I_X: X \rightarrow X$ 10

Let $X \approx Y$

$f: X \rightarrow Y$ $X \rightarrow X$

$f^{-1}: Y \rightarrow X$

Bijection + Bicontinuous

Bijection + Cont + f is conti.

$$\Rightarrow X \approx Y \Leftrightarrow Y \approx X$$

$$X \approx Y, Y \approx Z$$

$f: X \rightarrow Y$

$g: Y \rightarrow Z$

Bijection + Bicontinuous

Bijection + Bicontinuous

$g \circ f$ is Continuous

$$f \circ g^{-1} \text{ is continuous} \Rightarrow X \approx Z$$

Q24 Let X be a Hausdorff Space then every convergent sequence of points has a unique limit point.

Solution: — Proof: — Consider a convergent sequence (x_i) in a Hausdorff Space X . Let (x_i) converges to $a, b \in X$ and $a \neq b$

Since X is Hausdorff. So $a, b \in X \exists$ a pair of open subsets, U_a and V_b of X , such that $U_a \cap V_b = \emptyset$

Now if a is the limit of (x_i) then $\exists N \in \mathbb{N}$ such that $\forall n \geq N \Rightarrow x_n \in U_a$

And if b is the limit point of (x_i) then $\exists M \in \mathbb{N}$ such that $\forall n \geq M \Rightarrow x_n \in V_b$

So it means that $\forall n \geq \max(M, N)$ we have

$$x_n \in U_a \cap V_b$$

which is contradiction of the fact

$$U_a \cap V_b = \emptyset$$

So $a=b$ i.e. in a Hausdorff Space then every convergent sequence of points has a unique limit point.

Q25 Show that a discrete space X is separable if and only if X is countable - repeat

Proof: X is $T_{\text{dis}} = P(X)$

We know that the only dense subset in X is X itself.

So the only choice for A is X

$$\text{as } \bar{A} = X$$

$\Rightarrow X$ is separable iff X is countable.

Q26 Fixed point map -

A point $x \in X$ is said to be a fixed point of a map

$$f: X \rightarrow X \text{ iff } f(x) = x$$

Ex 1) $f: \mathbb{R} \rightarrow \mathbb{R}$, defined as $f(x) = x^2$. 12

Here 0 is a fixed point of f . 1 is also a fixed point of f .

2) $g: \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$g(x) = x + 1$$

g has no fixed point.

Q27:- Show that T_0 space that is not T_1 .

A T_0 space that is not T_1

consider $\mathbb{R}$ with T generated by

$$\{(a, \infty) \mid a \in \mathbb{R}\}$$

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M T W T F S

$$a, b \in \mathbb{R} \quad a > b$$

$$a \in U_a = (c, \infty) \not\ni b \quad b < c < a$$

So T_0 -space

Q28 Every Discrete Space is T_0 ?

Consider $X = \{a, b, c, d\}$ with

$$\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$$

(X, τ) is T_0