lere $k = 8$ $and\ c = -3,$
lence

$$u(x, y) = 8e^{-3(4x+y)} = 8e^{-12x-3y}$$

LECTURE NO. 15

leat Conduction Equation and its Physical Interpretation:

luestion:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \quad ...(1) \qquad \therefore 0 < x < 3\ , t > 0$$

iven that (boundary values)

$$u(0, t) = u(3, t) = 0 \qquad ...(i),$$

$$u(x, 0) = 5sin4\pi x - 3sin8\pi x + 2sin10\pi x \qquad ...(ii) \therefore |u(x$$

olution: Let

$$u(x, t) = X(x).T(t) = XT \quad ...(z)$$

aking derivative w.r.to $'t'$

$$\frac{\partial u}{\partial t} = XT'$$

aking 1st derivative w.r.to $'x'$

$$\frac{\partial u}{\partial x} = X'T$$

aking 2nd derivative w.r.to $'x'$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

y putting values equation 1 becomes

$$XT' = 2X''T$$

$$\frac{X''}{X} = \frac{T'}{2T} = -\lambda^2\ (say)$$

o,

$$\frac{X''}{X} = -\lambda^2$$

$$X'' + \lambda^2 X = 0 \ ...(a)$$

$$Let\ X = e^{mx}$$

$$X'' = m^2 e^{mx} = m^2 X$$

$$(a) => m^2 X + \lambda^2 X = 0$$

$$m = \pm\lambda i \ \ \& X \neq 0$$

$$\therefore X(x) = \alpha_1 e^{\lambda i x} + \alpha_2 e^{-\lambda i x}$$

$$X(x) = \alpha_1[cos\lambda x + isin\lambda x] + \alpha_2[cos\lambda x - isin\lambda x]$$

$$X(x) = (\alpha_1 + \alpha_2)cos\lambda x + i(\alpha_1 - \alpha_2)sin\lambda x$$

$$X(x) = A_1 cos\lambda x + B_1 sin\lambda x$$

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lere we were solving both functions side by side. Now
'utting values in equation (z), we get

$$u(x, t) = XT = e^{-2\lambda^2 t}(A_1 cos\lambda x + B_1 sin\lambda x) \quad ...($$

alculating (i) (boundary value) by putting values in equaition 2,

$$u(0, t) = 0 = e^{-2\lambda^2 t}(Acos\lambda(0) + Bsin\lambda(0)) = e^{-2\lambda^2 t}.A \ => \ A$$

quation 2 becomes

$$u(x, t) = XT = B.e^{-2\lambda^2 t}sin\lambda x \qquad \therefore A = 0$$

(*ii*): For even extension of $f(x) = |x|$ in $(-2,2)$

This implies $\quad 2l = 4 \quad so \quad l = 2$

For odd expansion; $b_n = 0$ and

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

For given function,

$$a_n = \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx \;=>\; \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

By integrating and applying limits, we have

$$a_n = \ldots (1)\left\{\frac{-4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right\}\Bigg]_0^2$$

$$a_n = \ldots) \;=>\; \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

For $n = 0$,

$$a_0 = \frac{2}{l}\int_0^l x dx \;=>\; \frac{2}{2}\int_0^2 x dx \;=>\; \left[\frac{x^2}{2}\right]_0^2 \;=>\; 2$$

$$\frac{a_0}{2} = 1$$

Now putting values in Fourier Series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{2}\right) + 0 \;=>\; 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}[(-1)^n - 1]\cos\left(\frac{n\pi x}{2}\right)$$

$$f(x) = 1 + \frac{4}{\ldots}\left[-2\cos\left(\frac{\pi x}{\ldots}\right) - \frac{2}{\ldots}\cos\left(\frac{3\pi x}{\ldots}\right) - \frac{2}{\ldots}\cos\left(\frac{5\pi x}{\ldots}\right) - \ldots\right]$$

$$(a);\quad \int_{-l}^{l} A\cos\left(\frac{n\pi x}{l}\right)dx = A\frac{l}{m\pi}\int_{-l}^{l}\left(\frac{m\pi}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \frac{Al}{m\pi}\left|\sin\left(\frac{m\pi x}{l}\right)\right|_{-l}^{l} = 0$$

$$(b);\quad \int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\left\{\cos\frac{(m+n)\pi x}{l} + \cos\frac{(m-n)\pi x}{l}\right\}dx = 0 \quad ; \quad for\ m \neq n$$

∴ Here we use the trigonometric relation; $cosA\ cosB = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$

For $m = n$, we have

$$\int_{-l}^{l} \cos\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\cos^2\left(\frac{m\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\left\{1 + \cos 2\left(\frac{m\pi x}{l}\right)\right\}dx = \frac{1}{2}|x|_{-l}^{l} = l$$

$$(c);\quad \int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{m\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\left\{\sin\frac{(m+n)\pi x}{l} + \sin\frac{(m-n)\pi x}{l}\right\}dx = 0$$

∴ Here we use the trigonometric relation; $sinA\ cosB = \frac{1}{2}[\sin(A+B) + \sin(A+B)]$

LECTURE NO. 21

By putting values in equation 2, we get

$$\int_{-l}^{l} f(x)\cos\left(\frac{m\pi x}{l}\right)dx = 0 + a_m l + 0 => a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

This is our required solution for (i).

Calculating (ii):

Multiplying (equation 1) both sides by $sin\left(\frac{m\pi x}{l}\right)$ and integrating from $-l\ to\ l$, we have

$$\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx = \int_{-l}^{l} A\sin\left(\frac{m\pi x}{l}\right)dx + \sum_{n=1}^{\infty}\left\{a_n\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx + b_n\int_{-l}^{l}\sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx\right\} \ldots(3)$$

(a) (b) (c)

Now we'll calculate the values for above functions.

$$(a);\quad \int_{-l}^{l} A\sin\left(\frac{m\pi x}{l}\right)dx = 0 \quad ∴ \sin\alpha \text{ is odd } \forall\alpha \in R$$

$$(b);\quad \int_{-l}^{l} \sin\left(\frac{m\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx = 0 \quad ∴ \sin\alpha\ \cos\beta \text{ are odd functions } \forall\alpha,\beta \in R$$

$$(c);\quad \int_{-l}^{l} \sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\left\{\cos\frac{(m-n)\pi x}{l} - \cos\frac{(m+n)\pi x}{l}\right\}dx = 0 \quad ; \quad m \neq n$$

For $m = n$, we have

$$\int_{-l}^{l} \sin\left(\frac{n\pi x}{l}\right)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\sin^2\left(\frac{m\pi x}{l}\right)dx = \frac{1}{2}\int_{-l}^{l}\left\{1 - \cos 2\left(\frac{m\pi x}{l}\right)\right\}dx = \frac{1}{2}|x|_{-l}^{l} = l$$

By putting values in equation 3, we get

$$\int_{-l}^{l} f(x)\sin\left(\frac{m\pi x}{l}\right)dx = 0 + 0 + b_m l => b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

This is our required solution for (ii).

Calculating (iii):

Integrating equation 1 from $-l\ to\ l$ on both sides,

$$(ii): \quad \int_a^b \{\Phi_m(x)\}^2 = 1 \;, \quad m = 1,2,3,\ldots$$

Then the set $\{\Phi_k(x)\}, k = 1,2,3,\ldots$ is orthogonal.

$(i) \& (ii)$ implies,

$$\int_a^b \Phi_m(x).\Phi_n(x)dx = S_{mn} = \begin{cases} 0 & if \quad m \neq n \\ 1 & if \quad m = n \end{cases}$$

Orthogonality w.r.to Weight Function:

If

$$\int_a^b \Phi_m(x)\,\Phi_n(x)\,\omega(x)\,dx = S_{mn}$$

Where $\omega(x) \geq 0$, then $\{\Phi_k(x)\}_{k=1}^{\infty}$ is orthogonal w.r.to weight function.

LECTURE NO. 33

Obtaining Normalizing Constants From Orthogonal Sets:

Show that the set

$$\left\{1, sin\frac{\pi x}{l}, cos\frac{\pi x}{l}, sin\frac{2\pi x}{l}, cos\frac{2\pi x}{l}, \ldots\right\}$$

Is an orthogonal set. Also find its corresponding normalizing constants, so that given set is orthogonal.

Solution: As we know condition for orthogonal set condition

$$\int_{-l}^{l} \Phi_m(x).\Phi_n(x)dx = \begin{cases} 0 & if \quad m \neq n \\ 1 & if \quad m = n \end{cases}$$

Possibilities for given set (using Fourier series results;

$$(i): \int_{-l}^{l} 1 \; sin\frac{k\pi x}{l} dx = 0 = \int_{-l}^{l} 1 \; cos\frac{k\pi x}{l} dx \quad \forall\, k = 1,2,3,\ldots$$

$$(ii): \int_{-l}^{l} sin\frac{k\pi x}{l}\; cos\frac{p\pi x}{l}\; dx = 0 \qquad k \neq p$$

$$(iii): \int_{-l}^{l} sin\frac{m\pi x}{l}\; sin\frac{n\pi x}{l}\; dx = \int_{-l}^{l} cos\frac{m\pi x}{l}\; cos\frac{n\pi x}{l}\; dx = \begin{cases} 0 & if \quad m \neq n \\ l & if \quad m = n \end{cases}$$

(iii) Implies

$$\int_{-l}^{l} sin^2\frac{m\pi x}{l} dx = l \qquad \& \qquad \int_{-l}^{l} cos^2\frac{m\pi x}{l} dx = l$$

$$\int_{-l}^{l} \left(\frac{1}{\sqrt{l}} sin\frac{m\pi x}{l}\right)^2 dx = 1 \qquad \& \qquad \int_{-l}^{l} \left(\frac{1}{\sqrt{l}} cos\frac{m\pi x}{l}\right)^2 dx = 1$$

Similarly for (i):

$$\int_{-l}^{l} (1)^2\, dx = 2l \Rightarrow \int_{-l}^{l} \left(\frac{1}{\sqrt{2l}}\right)^2 dx = 1$$

Hence, Orthonormalizing constants;

1 1

to these variables. Example

$$\frac{\partial^2 u}{\partial x dy} = 2x - y \qquad order = 2$$

der: It is the order of highest derivative involved in PDE.

ution: It is a function which satisfy the given DE. Example:

$$u = x^2y - \frac{1}{2}xy^2$$

$$\frac{\partial u}{\partial y} = x^2 - xy \qquad 1st\ derivative\ w.r.to\ 'y'$$

$$\frac{\partial^2 u}{\partial x dy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 2x - y \qquad 2nd\ derivative\ w.r.to\ 'x'$$

Similarly; $u = x^2y - \frac{1}{2}xy^2 + F(x) + G(y)$ is also a solution. Here $'F(x) and\ G(x)'$ a

Particular Solution: It is obtained from the general solution by particular choice of ar

Example:

$$u = x^2y - \frac{1}{2}xy^2 + 2sinx + 3y^4 - 5$$

Singular Solution: It cannot be obtained from the general solution by choosing arbitr

LECTURE NO. 07

Example: $xy' = \sqrt{y} => y = \frac{1}{4}(lncx)^2$ (General Solution—Here $'c'$ is arbitrary

But $'y = 0'$is also a solution of given DE.As it is not obtained from general solution,

singular solution. ($\nexists\ c \in R\ such\ that\ y = 0$)

MTH647

LECTURE NO. 08

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$$I.F \times u = \int (R.H.S \times I.F)dt + F(x)$$

ues

$$t^2u = \int \left((x^2 + \frac{G(t)}{t}) \times t^2\right) dt + F(x)$$

$$t^2u = \int \left(x^2t^2 + \frac{G(t)}{t}t^2\right) dt + F(x)$$

$$t^2u = x^2\frac{t^3}{3} + \int t.G(t)dt + F(x)$$

$$t^2u = x^2\frac{t^3}{3} + H(t) + F(x)$$

LECTURE NO. 13

tion for Solving PDEs:

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial t} + 2\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{...... (1)}$$

$$u = e^{ax+by}$$

d 2nd derivatives w.r.to 'x'

$$\frac{\partial u}{\partial x} = e^{ax+by}\frac{\partial}{\partial x}(ax + by) = au$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(au) = a\frac{\partial u}{\partial x} = a^2 u$$

ng derivatives w.r.to 'y'

$$\frac{\partial^2 u}{\partial y^2} = b^2 u$$

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$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(bu) = b\frac{\partial u}{\partial x} = abu$$

$$\begin{cases} x = 0 \Rightarrow y = 1, \quad x = 1 \Rightarrow 0 \\ \therefore dx = -dy \end{cases}$$

By putting, we get

$$B(m,n) = \int_0^1 (1-y)^{m-1}\, y^{n-1}\, (-dy) \Rightarrow \int_0^1 y^{n-1}(1-y$$

$$B(m,n) = \int_0^1 x^{n-1}\,(1-x)^{m-1}\, dx$$

Question: Prove that

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta . \cos^{2n-1}\theta \, d\theta$$

Proof: Since,

$$B(m,n) = 2\int_0^{\pi/2} x^{m-1}(1-x)^{n-1}\, dx$$

Let

$$\begin{cases} x = \sin^2\theta \Rightarrow dx = 2\sin\theta \cos\theta \\ x = 0 \Rightarrow \sin^2\theta = 0, x = 1 \Rightarrow \sin^2\theta = 1 \Rightarrow \theta = \end{cases}$$

By putting, we get

$$y = e^{mx} \Rightarrow y'' = m^2 y \Rightarrow m = \pm i\sqrt{\lambda}; \quad y \neq 0$$

By superposition principle;

$$y = A\cos\sqrt{\lambda x} + B\sin\sqrt{\lambda x} \quad \dots (1)$$

For $y(0) = A = 0$. So equation (1) becomes

$$y = B\sin\sqrt{\lambda x}$$

And for

$$y(1) = B\sin\sqrt{\lambda} = 0 \Rightarrow B \neq 0 \ \& so \ \sin\sqrt{\lambda} = 0$$

This implies,

$$\sqrt{\lambda} = m\pi \Rightarrow \lambda_m = m^2\pi^2; \quad m \in Z$$

These are called Eigen values of given S-L System. And

$$y_m = B_m \sin m\pi x$$

These are called the Corresponding Eigen Functions.

LECTURE NO. 43

Orthogonality of Eigen Functions:

$$y_m = B_m \sin m\pi x; \quad 0 \leq x \leq 1$$

$$\int_0^1 (B_m \sin m\pi x)(B_n \sin n\pi x)\, dx$$

$$= B_m B_n \int_0^1 (\sin m\pi x)(\sin n\pi x)\, dx$$

Using relation$\{2\sin A \sin B = \cos(A - B) - \cos(A + B)\}$, we have

$$= \frac{B_m B_n}{2} \int_0^1 \{\cos(m-n)\pi x - \cos(m+n)\pi x\} dx$$

By integrating and applying limits, we get

$$= \frac{B_m B_n}{2} \left[\frac{\sin(m-n)\pi x}{(m-n)\pi} - \frac{\sin(m+n)\pi x}{(m+n)\pi} \right]_0^1 = 0$$

Hence, $\{B_m \sin m\pi x\}_{m=1}^{\infty}$ is an orthogonal set.

LECTURE NO. 44

$$x^2 = \frac{16}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right] - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \ldots\right]$$

Comparing above relation with general form;

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

We have

$$\frac{a_0}{2} \Rightarrow \frac{1}{l}\int_0^l f(x)dx = \frac{16}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right] \ldots (1)$$

Here

$$\frac{1}{l}\int_0^l f(x)dx = \frac{1}{2}\int_0^2 x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

So equation (1) becomes,

$$\frac{4}{3} = \frac{16}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

$$\frac{4}{3} = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

Required result.

LECTURE NO. 29

Problem:

Check the term by term differentiation of Fourier Series;

$$f(x) = \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \ldots\right]$$

Solution: Taking derivative,

$$f'(x) = \frac{4}{\pi}\left[\frac{\pi}{2}\cos\frac{\pi x}{2} - \frac{1}{2}\frac{2\pi}{2}\cos\frac{2\pi x}{2} + \frac{1}{3}\frac{3\pi}{2}\cos\frac{3\pi x}{2} - \ldots\right]$$

$$f'(x) = 2\left[\cos\frac{\pi x}{2} - \cos\frac{2\pi x}{2} + \cos\frac{3\pi x}{2} - \ldots\right]$$

This implies

$$a_n = (-1)^{n-1}\left(\cos\frac{n\pi x}{2}\right) \nrightarrow 0$$

$\Rightarrow$ Series does not converge $\Rightarrow$ It does not converges uniformly $\Rightarrow$ Term by term differentiation is not possible.

LECTURE NO. 30

Heat Flow Problem:

A bar of length "l" whose entire surface is insulated including its ends at $x = 0$ and $x = l$. Its initial temperature is $f(x)$, then determine the subsequent temperature of the bar.

Solution: It is a heat flow boundary value problem (B.V.P.). As we know heat equation

Here $k = 8 \quad and\ c = -3,$

Hence

$$u(x, y) = 8e^{-3(4x+y)} = 8e^{-12x-3y}$$

LECTURE NO. 15

Heat Conduction Equation and its Physical Interpretation:

Question:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \quad \ldots (1) \qquad \therefore 0 < x < 3\,, t > 0$$

Given that (boundary values)

$$u(0,t) = u(3,t) = 0 \qquad \ldots (i),$$

[illegible] $\ldots (ii) \therefore |u(x$

Solution: Let

$$u(x,t) = X(x).T(t) = XT \quad \ldots (z)$$

Taking derivative w.r.to $'t'$

$$\frac{\partial u}{\partial t} = XT'$$

Taking 1st derivative w.r.to $'x'$

$$\frac{\partial u}{\partial x} = X'T$$

Taking 2nd derivative w.r.to $'x'$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

By putting values equation 1 becomes

$$XT' = 2X''T$$

$$\frac{X''}{X} = \frac{T'}{2T} = -\lambda^2 (say)$$

So,

$$\frac{X''}{X} = -\lambda^2$$

$$X'' + \lambda^2 X = 0 \ldots (a)$$

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$$Let\ X = e^{mx}$$

$$X'' = m^2 e^{mx} = m^2 X$$

$$(a) => m^2 X + \lambda^2 X = 0$$

$$m = \pm\lambda i \ \ \& X \neq 0$$

$$\therefore X(x) = \alpha_1 e^{\lambda ix} + \alpha_2 e^{-\lambda ix}$$

$$X(x) = \alpha_1[cos\lambda x + isin\lambda x] + \alpha_2[cos\lambda x - isin\lambda x]$$

$$X(x) = (\alpha_1 + \alpha_2)cos\lambda x + i(\alpha_1 - \alpha_2)sin\lambda x$$

$$X(x) = A\ cos\lambda x + B\ sin\lambda x$$

$$I.F \times u = \int (R.H.S \times I.F)\,dt + F(x)$$

By putting values

$$t^2u = \int \left((x^2 + \frac{G(t)}{t}) \times t^2\right) dt + F(x)$$
$$t^2u = \int \left(x^2t^2 + \frac{G(t)}{t}t^2\right) dt + F(x)$$
$$t^2u = x^2\frac{t^3}{3} + \int t.G(t)dt + F(x)$$
$$t^2u = x^2\frac{t^3}{3} + H(t) + F(x)$$

LECTURE NO. 13

General Solution for Solving PDEs:

Question
$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial t} + 2\frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots\ldots (1)$$

Solution: Let
$$u = e^{ax+by}$$

Taking 1st and 2nd derivatives w.r.to 'x'
$$\frac{\partial u}{\partial x} = e^{ax+by}\frac{\partial}{\partial x}(ax+by) = au$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(au) = a\frac{\partial u}{\partial x} = a^2u$$

Similarly taking derivatives w.r.to 'y'
$$\frac{\partial^2 u}{\partial y^2} = b^2u$$

And
$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(bu) = b\frac{\partial u}{\partial x} = abu$$

Putting values in equation 1, we have

$$a^2u + 3abu + 2b^2u = 0$$
$$(a^2 + 3ab + 2b^2)u = 0$$
$$(a + 2b)(a + b) = 0$$

By solving, we get
$$a = -2b \;;\; a = -b$$

<u>$For\ a = -2b$</u>
$$u_1 = e^{-2bx+by} = e^{b(y-2x)}$$

<u>$For\ a = -b$</u>
$$u_2 = e^{-bx+by} = e^{b(y-x)}$$

Given PDE is homogeneous, therefore by superposition principle
$$u = \alpha u_1 + \beta u_2$$

By putting values
$$u = \alpha e^{b(y-2x)} + \beta e^{b(y-x)}$$

Let
$$e^{b(y-2x)} = G(y-2x) \;\; and \;\; e^{b(y-x)} = H(y-x)$$

So, above equation becomes
$$u = G(y-2x) + H(y-x)$$

Required general solution of given PDE.

LECTURE NO. 14

Solving PDEs by Separation of Variables:

4.7. Evaluate (*a*) $\Gamma(-1/2)$ (*b*) $\Gamma(-5/2)$.

We use the generalization to negative values defined by $\Gamma(n) = \frac{\Gamma(n}{}$

Short

(*a*) Letting $n = -\frac{1}{2}$, $\Gamma(-1/2) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$.

(*b*) Letting $n = -3/2$, $\Gamma(-3/2) = \frac{\Gamma(-1/2)}{-3/2} = \frac{-2\sqrt{\pi}}{-3/2} = \frac{4\sqrt{\pi}}{3}$, using (*a*).

Then $\Gamma(-5/2) = \frac{\Gamma(-3/2)}{-5/2} = -\frac{8}{15}\sqrt{\pi}$.

Proof: I)

$$\int_{-\infty}^{\infty} F(\alpha)\, G(-\alpha) d\alpha = \int_{-\infty}^{\infty} F(\alpha)\left\{\int_{-\infty}^{\infty} g(u) e^{-i(-\alpha)u}\, du\right\}$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} F(\alpha) e^{i\alpha u}\right\} g(u)\, du = \int_{-\infty}^{\infty} f(u)\, g(u)\, du$$

II): Putting $f = g$

$$\int_{-\infty}^{\infty} F(\alpha) G(-\alpha) d\alpha = \int_{-\infty}^{\infty} F(\alpha)\overline{G(\alpha)} d\alpha = \int_{-\infty}^{\infty} F(\alpha) G(\alpha) d\alpha = \int_{-\infty}^{\infty} f(u)\, g(u)\, du$$

$$\Rightarrow \int_{-\infty}^{\infty} |G(\alpha)|^2 d\alpha = \int_{-\infty}^{\infty} |g(u)|^2\, du \; Or \; \int_{-\infty}^{\infty} |f(u)|^2\, du = \int_{-\infty}^{\infty} |F(\alpha)|^2\, d\alpha$$

Required result proved.

LECTURE NO. 68

Convolution (Definition and Related Theorem):

$$F(f) = F(\alpha) \quad , \quad G(g) = G(\alpha)$$

$$F(f \,.\, g) \neq F(\alpha) G(\alpha)$$

$$F(f * g) = F(\alpha)\, G(\alpha)$$

Convolution: of function $f(x)$ and $g(x)$ is defined and given,

$$f * g = \int_{-\infty}^{\infty} f(u) g(x - u) du$$

Now we have to prove some theorems.

1) Commutative $(f * g = g * f)$: By definition

$$f * g = \int_{-\infty}^{\infty} f(u) g(x - u) du$$

'ut $\quad x - u = t\,, u = x - t \Rightarrow du = -dt \quad and\ as \begin{cases} u \to \infty & \Rightarrow t \to \pm\infty \\ u \to -\infty & \Rightarrow t \to \infty \end{cases}$

), we get

$$f * g = \int_{+\infty}^{-\infty} f(x - t) g(t)(-dt) \Rightarrow \int_{-\infty}^{\infty} g(t) f(x - t)(dt) =$$

:e proved

$$\sum_{n=1}^{\cdot} (c_n)^2 \; will \; converge \;\; \Rightarrow \;\; c_n \rightarrow 0 \;\; as$$

$$\lim_{n\to\infty} c_n = 0 \;\; \Rightarrow \;\; \lim_{n\to\infty} \int_a^b f(x)\Phi_n(x)\, d\!:$$

reslut.

LECTURE NO. 40

:em:

·oblem of the form:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda\, r(x)]y = 0 \qquad a \le$$

$$\alpha_1\, y(a) + \alpha_2\, y'(a) = 0 \;\; , \;\; \beta_1\, y(b) + \beta_2\, y'(\!b$$

are given constants and $p(x), q(x)\; and\; r(x)$ are

ıeter independent of x.

quence of eigon value λ_n and corresponding eigen

$$\lambda y = 0 \quad ; \qquad B.V.P: \;\; y(0) = y(1) = 0$$

$$\frac{d}{dx}\left[1 \cdot \frac{dy}{dx}\right] + [(0 + \lambda.1)y] = 0 \qquad 0 \le$$

ı be written as,

$$y(0) = 1\, y(0) + 0\, y'(0) = 0 = \alpha_1\, y(a) +$$

$$y(1) = 1\, y(1) + 0\, y'(1) = 0 = \beta_1\, y(b) +$$

Sturm-Liouville System, we have

LECTURE NO. 71

Equations Solution by Fourier Sine Transform:

integral equation

$$\int_0^\infty f(x)\, sin\alpha x\, dx = \begin{cases} 1-\alpha & 0 \le \alpha \le 1 \\ 0 & \alpha > 1 \end{cases}$$

The given function can be writen as

$$F_s\{f(\alpha)\} = F_s(\alpha) = \int_0^\infty f(x)\, sin\alpha x\, dx = \begin{cases} 1-\alpha & 0 \le \\ 0 & \alpha \end{cases}$$

or given function,

$$F_s^{-1}\{f(x)\} = f(x) = \frac{2}{\pi}\int_0^\infty F_s(\alpha)\, sin\alpha x\, d\alpha$$

$$f(x) = \frac{2}{\pi}\int_0^1 (1-\alpha)\, sin\alpha x\, d\alpha$$

ing and applying limits, we have

$$= \frac{2}{\pi}\left[\left\{(1-\alpha)\left(-\frac{cos\alpha x}{x}\right)\right\}_0^1 + \int_0^1 \frac{cos\alpha x}{x}(-1)\, d\alpha\right]$$

$$= \frac{2}{\pi}\left[0-(1)\left(-\frac{1}{x}\right) - \frac{1}{x}\left|\frac{sin\alpha x}{x}\right|_0^1\right] \Rightarrow \frac{2}{\pi}\left[\frac{1}{x} - \frac{sinx}{x^2}\right]$$

$$\Gamma(-3/2) = \frac{\Gamma(-3/2+1)}{-3/2} = \frac{\Gamma(-1/2)}{-3/2} = \frac{-2\sqrt{\pi}}{-3/2} = \frac{4\sqrt{\pi}}{3}$$

3)

$+ k + 1)$

$$\Gamma(-5/2)$$

$$n = -5/2$$

$$\Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma(-5/2) = \frac{\Gamma(-5/2+1)}{-5/2}$$

$$= \frac{\Gamma(-3/2)}{-5/2}$$

$\overline{5} = -2\sqrt{\pi}$

$$\frac{4\sqrt{\pi}}{3} \div \frac{2}{5} = -\frac{8}{} \sqrt{\pi}$$

$^{/2} e^{-x}\, dx$

$$\frac{8}{3} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right)$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

Proved. Now let

$$S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots$$

$$S = \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots\right)$$

$$S = \left(\frac{\pi^4}{96}\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots\right)$$

$$S = \left(\frac{\pi^4}{96}\right) + \frac{1}{2^4}(S)$$

$$S = \frac{\pi^4}{90}$$

Deduced.

LECTURE NO. 28

Finding a Fourier Series by Integration:

... $f(x)$ can be integrated by term form a to x and the resulting

$= -b$

PDE is homogeneous, therefore by superpo...

$$u = \alpha u_1 + \beta u_2$$

$$u = \alpha e^{b(y-2x)} + \beta e^{b(y-x)}$$

ing values

$$e^{b(y-2x)} = G(y-2x) \text{ and } e^{b(y-x)} = H(y-x)$$

ve equation becomes

$$u = G(y-2x) + H(y-x)$$

d general solution of given PDE.

LECTURE NO. 14

2nd method. → 9 → p and po

PDEs by Separation of Variables:

n:

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \quad \textit{subject to boundry conditio } u(0,y) = 8e^{-}$$

: Given equation

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y} \quad \dots(1)$$

$$u(x,y) = X(x)\,Y(y) = XY$$

Be its

erivative w.r.to x & y

$$\frac{\partial u}{\partial x} = X'Y \quad \text{and} \quad \frac{\partial u}{\partial y} = XY'$$

alues in equation 1, we have

$$X'Y = 4\,XY'$$

$$\frac{X'}{4X} = \frac{Y'}{Y} \qquad \dots(2)$$

$$X(x) = X\,, Y(y) = Y$$

nd Y are independent variables. Each side of equation 2 must be a con

$\frac{X'}{4X} = c$	$\frac{Y'}{Y} = c$
$X' = 4cX$	$Y' = cY$
$\frac{dX}{dx} = 4cX$	$\frac{dY}{dy} = cy$

$$e^{-mx} = \frac{\quad}{\pi}\int_{0} \frac{\quad}{m^2+\alpha^2}\,d\alpha$$

$$\frac{\pi e^{-mx}}{2m} = \int_{0}^{\infty} \frac{\cos\alpha u}{m^2+\alpha^2}\,d\alpha$$

LECTURE NO. 71

...ution by Fourier Sine Transform:

...ation

$$\int_{0}^{\infty} f(x)\,\sin\alpha x\,dx = \begin{cases} 1-\alpha & 0 \le \alpha \le 1 \\ 0 & \alpha > 1 \end{cases}$$

...nction can be writen as

$$F_s\{f(\alpha)\} = F_s(\alpha) = \int_{0}^{\infty} f(x)\,\sin\alpha x\,dx = \begin{cases} 1-\alpha & 0 \le \alpha \le \\ 0 & \alpha > 1 \end{cases}$$

...ction,

$$F_s^{-1}\{f(x)\} = f(x) = \frac{2}{\pi}\int_{0}^{\infty} F_s(\alpha)\,\sin\alpha x\,d\alpha$$

$$f(x) = \frac{2}{\pi}\int_{0}^{1} (1-\alpha)\,\sin\alpha x\,d\alpha$$

...plying limits, we have

$$\frac{2}{\pi}\left[\left\{(1-\alpha)\left(-\frac{\cos\alpha x}{x}\right)\right\}_0^1 + \int_0^1 \frac{\cos\alpha x}{x}(-1)\,d\alpha\right]$$

$F(\alpha) = F(-\alpha)$

LECTURE NO. 63

Fourier Transforms of Even and Odd Functions:

Let $f(x)$ be an odd function

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(u)e^{-i\alpha u}\, du = F(\alpha)$$

Here put $u = -t \Rightarrow du = -dt$

As $u \to \infty$, $t \to \infty$ and $u \to -\infty \Rightarrow t \to \infty$

$$F(\alpha) = -\int_{-\infty}^{\infty} f(-t)e^{-i\alpha(-t)}\, dt \Rightarrow -\int_{-\infty}^{\infty} f(t)e^{-i\alpha(-t)}\, dt = -\int_{-\infty}^{\infty} f(t)e^{-i(-\alpha)t}$$

Hence proved that

$$F(\alpha) = -[illegible] = -\overline{F(\alpha)}$$

LECTURE NO. 64

Attenuation Property of Fourier Transforms:

$$y = e^{mx} \;\Rightarrow\; y'' = m^2 y \;\Rightarrow\; m = \pm i\sqrt{\lambda}\,;\; y \neq 0$$

By superposition principle;

$$y = A\cos\sqrt{\lambda x} + B\sin\sqrt{\lambda x} \quad \dots (1)$$

For $y(0) = A = 0$. So equation (1) becomes

$$y = B\sin\sqrt{\lambda x}$$

And for

$$y(1) = B\sin\sqrt{\lambda} = 0 \;\Rightarrow\; B \neq 0 \;\;\&\, so \;\; \sin\sqrt{\lambda} = 0$$

This implies,

$$\sqrt{\lambda} = m\pi \;\Rightarrow\; \lambda_m = m^2\pi^2\,;\;\; m \in Z$$

These are called Eigen values of given S-L System. And

$$y_m = B_m \sin m\pi x$$

These are called the Corresponding Eigen Functions.

LECTURE NO. 43

Orthogonality of Eigen Functions:

$$y_m = B_m \sin m\pi x\,;\quad 0 \le x \le 1$$

$$\int_0^1 (B_m \sin m\pi x)(B_n \sin n\pi x)\,dx$$

$$= B_m B_n \int_0^1 (\sin m\pi x)(\sin n\pi x)\,dx$$

Using relation $\{2\sin A \sin B = \cos(A-B) - \cos(A+B)\}$, we have

$$= \frac{B_m B_n}{2}\int_0^1 \{\cos(m-n)\pi x - \cos(m+n)\pi x\}dx$$

By integrating and applying limits, we get

$$= \frac{B_m B_n}{2}\left[\frac{\sin(m-n)\pi x}{(m-n)\pi} - \frac{\sin(m+n)\pi x}{(m+n)\pi}\right]_0^1 = 0$$

Hence, $\{B_m \sin m\pi x\}_{m=1}^{\infty}$ is an orthogonal set.

LECTURE NO. 44

Normalization of Eigen Functions:

Given set $\{\Phi_m(x)\}_{m=1}^{\infty}$ is orthonormal if

$$\int^b \Phi_m \Phi_n \,dx = S_{mn} = \begin{cases} 0\,; & m \neq n \\ 1\,; & m = n \end{cases}$$

$$x^2 = \frac{16}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right] - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \ldots\right]$$

Comparing above relation with general form;

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

We have

$$\frac{a_0}{2} \Rightarrow \frac{1}{l}\int_0^l f(x)dx = \frac{16}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right] \ldots (1)$$

Here

$$\frac{1}{l}\int_0^l f(x)dx = \frac{1}{2}\int_0^2 x^2 dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

So equation (1) becomes,

$$\frac{4}{3} = \frac{16}{\pi^2}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

$$\frac{4}{3} = \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

Required result.

LECTURE NO. 29

Problem:

Check the term by term differentiation of Fourier Series;

$$f(x) = \frac{4}{\pi}\left[\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \ldots\right]$$

Solution: Taking derivative,

$$f'(x) = \frac{4}{\pi}\left[\frac{\pi}{2}\cos\frac{\pi x}{2} - \frac{1}{2}\frac{2\pi}{2}\cos\frac{2\pi x}{2} + \frac{1}{3}\frac{3\pi}{2}\cos\frac{3\pi x}{2} - \ldots\right]$$

$$f'(x) = 2\left[\cos\frac{\pi x}{2} - \cos\frac{2\pi x}{2} + \cos\frac{3\pi x}{2} - \ldots\right]$$

This implies

$$a_n = (-1)^{n-1}\left(\cos\frac{n\pi x}{2}\right) \nrightarrow 0$$

$\Rightarrow$ Series does not converge $\Rightarrow$ It does not converges uniformly $\Rightarrow$ Term by term differentiation is not possible.

LECTURE NO. 30

Heat Flow Problem:

A bar of length "l" whose entire surface is insulated including its ends at $x = 0$ and $x = l$. Its initial temperature is $f(x)$, then determine the subsequent temperature of the bar.

Solution: It is a heat flow boundary value problem (B.V.P.). As we know heat equation

Here $k = 8 \quad and\ c = -3,$

Hence

$$u(x,y) = 8e^{-3(4x+y)} = 8e^{-12x-3y}$$

LECTURE NO. 15

Heat Conduction Equation and its Physical Interpretation:

Question:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \quad \dots(1) \qquad \therefore 0 < x < 3\,, t > 0$$

Given that (boundary values)

$$u(0,t) = u(3,t) = 0 \qquad \dots(i),$$

[redacted]. $(ii) \therefore |u(x$

Solution: Let

$$u(x,t) = X(x).T(t) = XT \quad \dots(z)$$

Taking derivative w.r.to $'t'$

$$\frac{\partial u}{\partial t} = XT'$$

Taking 1st derivative w.r.to $'x'$

$$\frac{\partial u}{\partial x} = X'T$$

Taking 2nd derivative w.r.to $'x'$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

By putting values equation 1 becomes

$$XT' = 2X''T$$

$$\frac{X''}{X} = \frac{T'}{2T} = -\lambda^2 (say)$$

So,

$$\frac{X''}{X} = -\lambda^2$$

$$X'' + \lambda^2 X = 0 \dots (a)$$

7

$$Let\ X = e^{mx}$$

$$X'' = m^2 e^{mx} = m^2 X$$

$$(a) => m^2 X + \lambda^2 X = 0$$

$$m = \pm \lambda i \ \ \& X \neq 0$$

$$\therefore X(x) = \alpha_1 e^{\lambda i x} + \alpha_2 e^{-\lambda i x}$$

$$X(x) = \alpha_1[cos\lambda x + isin\lambda x] + \alpha_2[cos\lambda x - isin\lambda x]$$

$$X(x) = (\alpha_1 + \alpha_2)cos\lambda x + i(\alpha_1 - \alpha_2)sin\lambda x$$

$$X(x) = A\ cos\lambda x + B\ sin\lambda x$$

By putting values

$$t^2u = \int \left((x^2 + \frac{G(t)}{t}) \times t^2\right) dt + F(x)$$

$$t^2u = \int \left(x^2t^2 + \frac{G(t)}{t} t^2\right) dt + F(x)$$

$$t^2u = x^2\frac{t^3}{3} + \int t.G(t)dt + F(x)$$

$$t^2u = x^2\frac{t^3}{3} + H(t) + F(x)$$

LECTURE NO. 13

General Solution for Solving PDEs:

Question

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial t} + 2\frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots\ldots (1)$$

Solution: Let

$$u = e^{ax+by}$$

Taking 1st and 2nd derivatives w.r.to 'x'

$$\frac{\partial u}{\partial x} = e^{ax+by}\frac{\partial}{\partial x}(ax+by) = au$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(au) = a\frac{\partial u}{\partial x} = a^2u$$

Similarly taking derivatives w.r.to 'y'

$$\frac{\partial^2 u}{\partial y^2} = b^2u$$

And

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(bu) = b\frac{\partial u}{\partial x} = abu$$

Putting values in equation 1, we have

$$a^2u + 3abu + 2b^2u = 0$$

$$(a^2 + 3ab + 2b^2)u = 0$$

$$(a + 2b)(a + b) = 0$$

By solving, we get

$$a = -2b \ ; \ a = -b$$

For $a = -2b$

$$u_1 = e^{-2bx+by} = e^{b(y-2x)}$$

For $a = -b$

$$u_2 = e^{-bx+by} = e^{b(y-x)}$$

Given PDE is homogeneous, therefore by superposition principle

$$u = \alpha u_1 + \beta u_2$$

By putting values

$$u = \alpha e^{b(y-2x)} + \beta e^{b(y-x)}$$

Let

$$e^{b(y-2x)} = G(y-2x) \ and \ e^{b(y-x)} = H(y-x)$$

So, above equation becomes

$$I.F \times u = \int (R.H.S \times I.F)dt + F(x)$$

ues

$$t^2 u = \int \left((x^2 + \frac{G(t)}{t}) \times t^2 \right) dt + F(x)$$

$$t^2 u = \int \left(x^2 t^2 + \frac{G(t)}{t} t^2 \right) dt + F(x)$$

$$t^2 u = x^2 \frac{t^3}{3} + \int t.G(t)dt + F(x)$$

$$t^2 u = x^2 \frac{t^3}{3} + H(t) + F(x)$$

LECTURE NO. 13

tion for Solving PDEs:

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial t} + 2\frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{...... (1)}$$

$$u = e^{ax+by}$$

d 2nd derivatives w.r.to $'x'$

$$\frac{\partial u}{\partial x} = e^{ax+by}\frac{\partial}{\partial x}(ax+by) = au$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(au) = a\frac{\partial u}{\partial x} = a^2 u$$

ng derivatives w.r.to $'y'$

$$\frac{\partial^2 u}{\partial y^2} = b^2 u$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(bu) = b\frac{\partial u}{\partial x} = abu$$

$$\partial n$$

$$Heat\ flow\ across\ plane\ I = \left\{-k\frac{\nabla u}{\nabla r}\right\}$$

(distance between planes approaches to zero), so we c vi

LECTURE NO. 06

al Equations (Definition and Related Terms):

ı equation containing unknown functions of two or more va ıbles. Example

$$\frac{\partial^2 u}{\partial x dy} = 2x - y \qquad order = 2$$

der of highest derivative involved in PDE.

ınction which satisfy the given DE. Example:

$$u = x^2y - \frac{1}{2}xy^2$$

$$\frac{\partial u}{\partial y} = x^2 - xy \qquad 1st\ derivative\ w.r.$$

$$\frac{\partial^2 u}{\partial x dy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 2x - y \qquad 2nd\ derivative\ w.r$$

$y - \frac{1}{2}xy^2 + F(x) + G(y)$ is also a solution. Here $'F(x) and$

ɔn: It is obtained from the general solution by particular choi

$$u = x^2y - \frac{1}{2}xy^2 + 2sinx + 3y^4 - 5$$

: It cannot be obtained from the general solution by choosin

LECTURE NO. 07

$= \sqrt{y} => y = \frac{1}{4}(lncx)^2$ (General Solution—Here $'c'$ is

ɔ a solution of given DE.As it is not obtained from general s

$(\nexists\ c \in R\ such\ that\ \ y = 0)$

5.6. Solve the integral equation

$$\int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1-\alpha & 0 \leqq \alpha \leqq 1 \\ 0 & \alpha > 1 \end{cases}$$

If we write

$$F_S(\alpha) = \int_0^\infty f(x) \sin \alpha x \, dx = \begin{cases} 1-\alpha & 0 \leqq \alpha \leqq 1 \\ 0 & \alpha > 1 \end{cases}$$

then, by *(10)*, page 81,

$$f(x) = \frac{2}{\pi}\int_0^\infty F_S(\alpha) \sin \alpha x \, d\alpha$$

Long question

$$= \frac{2}{\pi}\int_0^1 (1-\alpha) \sin \alpha x \, d\alpha$$

$$= \frac{2(x - \sin x)}{\pi x^2}$$

$\sqrt{2L}$ $\sqrt{L}$ $-$ $\sqrt{L}$ $-$ $\sqrt{L}$ $\sqrt{L}$

. Let $\{\phi_n(x)\}$ be a set of functions which are mutually orthonormal in (a, b). that if $\sum_{n=1}^{\infty} c_n \phi_n(x)$ converges uniformly to $f(x)$ in (a, b), then

$$c_n = \int_a^b f(x)\,\phi_n(x)\,dx$$

Multiplying both sides of

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

by $\phi_m(x)$ and integrating from a to b, we have

$$\int_a^b f(x)\,\phi_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b \phi_m(x)\,\phi_n(x)\,dx$$

where the interchange of integration and summation is justified by the fact that the series c uniformly to $f(x)$. Now since the functions $\{\phi_n(x)\}$ are mutually orthonormal in (a, b), we

$$\int_a^b \phi_m(x)\,\phi_n(x)\,dx = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

so that (*2*) becomes

$$\int_a^b f(x)\,\phi_m(x)\,dx = c_m$$

as required.

We call the coefficients c_m given by (*3*) the *generalized Fourier coefficients* correspo $f(x)$ even though nothing may be known about the convergence of the series in (*1*). As in of Fourier series, convergence of $\sum_{n=1}^{\infty} c_n \phi_n(x)$ is then investigated using the coefficients (conditions of convergence depend of course on the types of orthonormal functions used. remainder of this book we shall be concerned with many examples of orthonormal funct series.

4.7. Evaluate (*a*) $\Gamma(-1/2$ (*b*) $\Gamma(-5/2)$.

We use the generalization to negative v..ues defined by $\Gamma(n) = \frac{\Gamma(n}{}$

Short

(*a*) Letting $n = -\frac{1}{2}$, $\Gamma(-1/2) = \frac{\Gamma(1/2)}{-1/2} = -2\sqrt{\pi}$.

(*b*) Letting $n = -3/2$, $\Gamma(-3/2) = \frac{\Gamma(-1/2)}{-3/2} = \frac{-2\sqrt{\pi}}{-3/2} = \frac{4\sqrt{\pi}}{3}$, using (*a*).

Then $\Gamma(-5/2) = \frac{\Gamma(-3/2)}{-5/2} = -\frac{8}{15}\sqrt{\pi}$.

THE CONVOLUTION THEOREM FOR FOURIER TRANSFORMS

The convolution of the functions $f(x)$ and $g(x)$ is defined by

Mcqs

$$f * g = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

An important theorem, often referred to as the *convolution theorem*, states tha transform of the convolution of $f(x)$ and $g(x)$ is equal to the product of the F forms of $f(x)$ and $g(x)$. In symbols,

$$\mathcal{F}\{f * g\} = \mathcal{F}\{f\}\,\mathcal{F}\{g\}$$

Mcqs

Th her important properties. For example, we have f, g, and ...

$$f * g = g * f, \quad f * (g * h) = (f * g) * h, \quad f * (g + h) = f * g + f * h$$

i.e., the convolution obeys the commutative, associative and distributive laws

2.4. If the series $A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ converges uniformly to $f(x)$ in $(-L, L)$, show that for $n = 1, 2, 3, \ldots,$

(a) $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$, (b) $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx$, (c) $A = \frac{a_0}{2}$.

Long question

(a) Multiplying

$$f(x) = A + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \tag{1}$$

by $\cos\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 2.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\cos\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\cos\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= a_m L \quad \text{if } m \neq 0 \end{aligned}\tag{2}$$

Thus
$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(b) Multiplying (1) by $\sin\frac{m\pi x}{L}$ and integrating from $-L$ to L, using Problem 2.3, we have

$$\begin{aligned}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx &= A\int_{-L}^{L}\sin\frac{m\pi x}{L}\,dx \\ &\quad + \sum_{n=1}^{\infty}\left\{a_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx + b_n\int_{-L}^{L}\sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx\right\} \\ &= b_m L\end{aligned}$$

Thus
$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{m\pi x}{L}\,dx \qquad \text{if } m = 1, 2, 3, \ldots$$

(c) Integrating (1) from $-L$ to L, using Problem 2.2, gives

$$\int_{-L}^{L} f(x)\,dx = 2AL \qquad \text{or} \qquad A = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

Putting $m = 0$ in the result of part (a), we find $a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$ and so $A = \frac{a_0}{2}$.

The above results also hold when the integration limits $-L, L$ are replaced by $c, c + 2L$.

Note that in all parts above, interchange of summation and integration is valid because the series is *assumed* to converge uniformly to $f(x)$ in $(-L, L)$. Even when this assumption is not warranted, the coefficients a_m and b_m as obtained above are called *Fourier coefficients* corresponding to $f(x)$, and the corresponding series with these values of a_m and b_m is called the *Fourier series* corresponding to $f(x)$. An important problem in this case is to investigate conditions under which this series actually converges to $f(x)$. Sufficient conditions for this convergence are the *Dirichlet conditions* established below in Problems 2.18–2.23.

LAPLACE'S EQUATION

2.30. Suppose that the square plate of Problem 2.29 has three sides kept at temperature zero, while the fourth side is kept at temperature u_1. Determine the steady-state temperature everywhere in the plate.

Choose the side having temperature u_1 to be the one where $y = 1$, as shown in Fig. 2-16. Since we wish the steady-state temperature u, which does not depend on time t, the equation is obtained from (*1*) of Problem 2.29 by setting $\partial u/\partial t = 0$; i.e. Laplace's equation in two dimensions:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (1)$$

The boundary conditions are

$$u(0,y) = u(1,y) = u(x,0) = 0, \quad u(x,1) =$$

and $|u(x,y)| < M$.

To solve this boundary value problem let $u = XY$ in (*1*) to obt

$$X''Y + XY'' = 0 \quad \text{or} \quad \frac{X''}{X} = -\frac{Y''}{Y}$$

Setting each side equal to $-\lambda^2$ yields

$$X'' + \lambda^2 X = 0 \qquad Y'' - \lambda^2 Y = 0$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2-1} + \frac{\cos 4x}{4^2-1} + \frac{\cos 6x}{6^2-1} + \cdots\right)$$

2.12. Expand $f(x) = x$, $0 < x < 2$, in a half-range (*a*) sine series, (*b*) cosine series.

(*a*) Extend the definition of the given function to that of the odd function of period 4 shown Fig. 2-12 below. This is sometimes called the *odd extension* of $f(x)$. Then $2L = 4$, $L = 2$.

Short

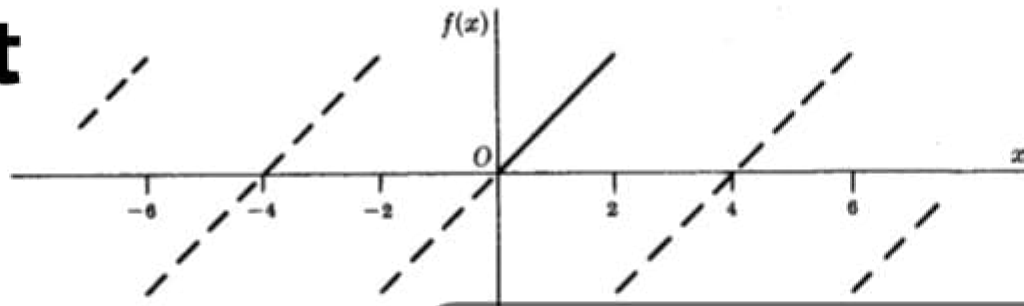

Fig. 2-12

Odd function find bn

Thus $a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{2}\int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \left\{(x)\left(\frac{-2}{n\pi}\cos\frac{n\pi x}{2}\right) - (1)\left(\frac{-4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}\Bigg|_0^2 = \frac{-4}{n\pi}\cos n\pi$$

Then
$$f(x) = \sum_{n=1}^{\infty}\frac{-4}{n\pi}\cos n\pi\sin\frac{n\pi x}{2}$$

$$= \frac{4}{\pi}\left(\sin\frac{\pi x}{2} - \frac{1}{2}\sin\frac{2\pi x}{2} + \frac{1}{3}\sin\frac{3\pi x}{2} - \cdots\right)$$

$$\sum_{n=1}^{\cdot} (c_n)^2 \; will \; converge \;\; \Rightarrow \;\; c_n \to 0 \;\; as$$

$$\lim_{n\to\infty} c_n = 0 \;\; \Rightarrow \;\; \lim_{n\to\infty} \int_a^b f(x)\Phi_n\,(x)\, d$$

reslut.

LECTURE NO. 40

em:

oblem of the form:

$$\frac{d}{dx}\left[p(x)\,\frac{dy}{dx}\right] + [q(x) + \lambda\, r(x)]y = 0 \qquad a \leq$$

$$\alpha_1\, y(a) + \alpha_2\, y'\,(a) = 0 \;\;,\;\; \beta_1\, y(b) + \beta_2\, y'\,($$

are given constants and $p(x), q(x)$ *and* $r(x)$ are

eter independent of x.

quence of eigon value λ_n and corresponding eigen

$\lambda y = 0 \quad ; \qquad B.V.P: \;\; y(0) = y(1) = 0$

$$\frac{d}{dx}\left[1\,.\frac{dy}{dx}\right] + [(0 + \lambda.1)y] = 0 \qquad 0 \leq$$

be written as,

$$y(0) = 1\, y(0) + 0\, y'\,(0) = 0 = \alpha_1\, y(a) +$$

$$y(1) = 1\, y(1) + 0\, y'\,(1) = 0 = \beta_1\, y(b) +$$

Sturm-Liouville System, we have

lere $k = 8 \quad and\ c = -3,$

lence

$$u(x,y) = 8e^{-3(4x+y)} = 8e^{-12x-3y}$$

LECTURE NO. 15

leat Conduction Equation and its Physical Interpretation:

Question:

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \quad \dots (1) \qquad \therefore 0 < x < 3\,, t > 0$$

iiven that (boundary values)

$$u(0,t) = u(3,t) = 0 \qquad \dots (i),$$

$$u(x,0) = 5sin4\pi x - 3sin8\pi x + 2sin10\pi x \qquad \dots (ii) \therefore |u(x$$

olution: Let

$$u(x,t) = X(x).T(t) = XT \quad \dots (z)$$

'aking derivative w.r.to $'t'$

$$\frac{\partial u}{\partial t} = XT'$$

'aking 1st derivative w.r.to $'x'$

$$\frac{\partial u}{\partial x} = X'T$$

'aking 2nd derivative w.r.to $'x'$

$$\frac{\partial^2 u}{\partial x^2} = X''T$$

ly putting values equation 1 becomes

$$XT' = 2X''T$$

$$\frac{X''}{X} = \frac{T'}{2T} = -\lambda^2\ (say)$$

o,

$$\frac{X''}{X} = -\lambda^2$$

$$X'' + \lambda^2 X = 0 \dots (a)$$

$$Let\ X = e^{mx}$$

$$X'' = m^2 e^{mx} = m^2 X$$

$$(a) => m^2 X + \lambda^2 X = 0$$

$$m = \pm \lambda i \ \ \&\ X \neq 0$$

$$\therefore X(x) = \alpha_1 e^{\lambda i x} + \alpha_2 e^{-\lambda i x}$$

$$X(x) = \alpha_1[cos\lambda x + isin\lambda x] + \alpha_2[cos\lambda x - isin\lambda x]$$

$$X(x) = (\alpha_1 + \alpha_2)cos\lambda x + i(\alpha_1 - \alpha_2)sin\lambda x$$

7

∂n

$$Heat\ flow\ across\ plane\ I = \left\{-k\frac{\nabla u}{\nabla r}\right\}$$

(distance between planes approaches to zero), so we c

LECTURE NO. 06

al Equations (Definition and Related Terms):

ı equation containing unknown functions of two or more va
ıbles. Example

$$\frac{\partial^2 u}{\partial x dy} = 2x - y \qquad order = 2$$

ˈder of highest derivative involved in PDE.

ınction which satisfy the given DE. Example:

$$u = x^2 y - \frac{1}{2}xy^2$$

$$\frac{\partial u}{\partial y} = x^2 - xy \qquad 1st\ derivative\ w.r.$$

$$\frac{\partial^2 u}{\partial x dy} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 2x - y \qquad 2nd\ derivative\ w.r$$

$y - \frac{1}{2}xy^2 + F(x) + G(y)$ is also a solution. Here $'F(x) and$

ɔn: It is obtained from the general solution by particular choi

$$u = x^2 y - \frac{1}{2}xy^2 + 2sinx + 3y^4 - 5$$

: It cannot be obtained from the general solution by choosin

LECTURE NO. 07

$= \sqrt{y} => y = \frac{1}{4}(lncx)^2$ (General Solution—Here $'c'$ is

ɔ a solution of given DE.As it is not obtained from general s

$(\nexists\ c \in R\ such\ that\ \ y = 0)$

***(ii)*:** For even extension of $f(x) = |x|$ in $(-2,2)$

This implies $\quad 2l = 4 \quad so \quad l = 2$

For odd expansion; $b_n = 0$ and

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

For given function,

$$a_n = \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx \;=>\; \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

By integrating and applying limits, we have

$$a_n \; \ldots \; (1)\left\{\frac{-4}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right\}\Big]_0^2$$

$$a_n = \ldots\,) \;=>\; \frac{4}{n^2\pi^2}[(-1)^n - 1]$$

For $n = 0$,

$$a_0 = \frac{2}{l}\int_0^l x\,dx \;=>\; \frac{2}{2}\int_0^2 x\,dx \;=>\; \left[\frac{x^2}{2}\right]_0^2 \;=>\; 2$$

$$\frac{a_0}{2} = 1$$

Now putting values in Fourier Series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{l}\right) + b_n\sin\left(\frac{n\pi x}{l}\right)\right]$$

$$f(x) = 1 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{2}\right) + 0 \;=>\; 1 + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}[(-1)^n - 1]\cos\left(\frac{n\pi x}{2}\right)$$

$$f(x) = 1 + \frac{4}{\pi^2}\left[-2\cos\left(\frac{\pi x}{2}\right) - \frac{2}{3^2}\cos\left(\frac{3\pi x}{2}\right) - \frac{2}{5^2}\cos\left(\frac{5\pi x}{2}\right) - \ldots\right]$$