Lecture #22

Example :-

Find the order of permutation

Let

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$$

$$= \underset{\alpha_1}{(1\,2)}\,\underset{\alpha_2}{(3\,4)}$$

length of $\alpha_1 = 2$

length of $\alpha_2 = 2$

ORDER of permutation $= LCM(2, 2)$

Example

$= 2$ Ans. mcq

$f \circ g \neq g \circ f$

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

Sol/ LHS $f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

R.H.S $= g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$

$\therefore f \circ g \neq g \circ f$

LCM $= 2 \,|\, 2, 2$; $1, 1$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 7 & 9 & 6 & 5 & 8 & 12 & 11 & 10 \end{pmatrix}$$

Answer

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 5 & 7 & 6 & 9 & 8 \\ 7 & 6 & 9 & 8 & 5 \end{pmatrix}\begin{pmatrix} 10 & 12 \\ 12 & 10 \end{pmatrix}\begin{pmatrix} 11 \\ 11 \end{pmatrix}$$

$(1, 2, 3, 4)(5, 7, 6, 9, 8)(10, 12)$

Length of $\alpha_1 = 4$

Length of $\alpha_2 = 5$

Length of $\alpha_3 = 2$

$2 \,|\, 4, 5, 2$
$2 \,|\, 2, 5, 1$
$5 \,|\, 1, 5, 1$
$\;\;\, 1, 1, 1$

$LCM = 2 \times 2 \times 5 = 20$

$LCM(2, 5, 4) = 20$ mcq

So order of permutation $\alpha = 20$ Ans.

MTH405

Final Term Notes

Lecture #23

Introduction to Metric Spaces.

Introduction and Definitions mcq

- Metric (meter, unit of distance) mcq
- Space (set in which some relation is defined) e.g. groups (set, -)
- Metric space (X, distance function defined b/w its elements) mcq — vector space

Distance :- $d : X \times X \to \mathbb{R}$ satisfies

i) $d(x, y) \geq 0$ (non-negativity) mcq

ii) $d(x, y) = 0 \Leftrightarrow x = y$ (Reflexive property) mcq

iii) $d(x, y) = d(y, x)$ (Symmetric property) mcq

iv) $d(x, y) + d(y, z) \geq d(x, z)$ mcq (Triangular inequality) mcq

axioms (set of distance means)

Then (X, d) is called a metric space

Question :-

Can there be a unified and concise theory of distance ??

MCQ

Product of two odd permutation is an even permutation.

Ans: Yes, Metric spaces can be a unified and concise theory of distance.

metric space [illegible]

Points on $\mathbb{R}, \mathbb{R}^2$ distance points in $\mathbb{R}^n$

higher dimensions

vectors, sets, functions, sequences

Some Important Metrics and distance function:-

i) usual Metric space $(\mathbb{R}, d)$ → defined on real line

$y \quad x$ set $x = \mathbb{R}$ real no, $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

$|x-y| \quad d(x,y) = |x-y|$ distance also real

MCQ (iii) $H = \mathbb{R}^n$, $d = \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $x, y \in \mathbb{R}$

$x = (x_1, \dots, x_n), y = (y_1, y_2, \dots, y_n)$

$d(x,y) = \sum_{i=1}^{n} |x_i - y_i| \in \mathbb{R}$

$1 \le i \le n$

(iii) $x \neq \emptyset$ (Discret Metrics) distinct [illegible]

$$d_n(x,y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$ 3

[illegible]

(iv) $x = [a,b]$ = Class of all continuous function on $[a,b]$

$d : X \times X \to \mathbb{R}$

$f, g \in X = [a,b]$

$d(f,g) = \int_a^b |f(x) - g(x)|\, dx \quad \forall$

$f, g \in C[a,b]$

$x \in [a,b]$

MCQ $\alpha, \beta \in A_n$

$\alpha\beta^{-1} \in A_n$

Lec 24

then we say that $\alpha\beta$ is the subgroup of S_n

Since the inverse of even permutation is a even permutation

MCQ A_n be the set of all even permutations of S_n. we have to show that A_n is subgroup of S_n clearly

Lecture #22

Q# Show by means of an example that the product of two cyclic permutation need not be a cyclic permutation

Sol Let $\alpha = (1\ 2\ 5)$, $\beta = (2\ 1\ 4\ 5)$ be two cyclic permutations then

$\alpha\beta = (1\ 2\ 5)(2\ 1\ 4\ 5\ 6)$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 5 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 5 & 4 & 2 \end{pmatrix}$$

This is not a cyclic permutation

Hence the product of two cyclic permutation is not a cyclic permutation

Properties 4

(1) $d(x,y) \geq 0$ (Non-negativity)

(2) $d(x,y) = 0 \Rightarrow x = y$ (Reflective)

(3) $d(x,y) = d(y,x)$ (Symmetric)

(4) $d(x,y) \leq d(x,z) + d(z,y)$

Lecture # 24

Euclidean Metric / Natural

01- Usual Metric space

let $x = \mathbb{R}$ & define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$

as: $d(x,y) = |x-y|$, $x, y \in \mathbb{R}$.

Show that $(\mathbb{R}, d)$ is a metric space:-

non negative property [illegible] Absolute value $x-y$

4 properties to be proved

(i) $d(x,y) = |x-y| \geq 0 \quad \forall x, y \in \mathbb{R}$ real

(ii) if $d(x,y) = 0$

$\Rightarrow |x-y| = 0$

Conversely $x-y=0$ $x = y$ ✓

$x - y = 0$ ✓

$|x-y| = 0$ ✓

$d(x,y) = 0$ ✓

(iii) $d(x,y) = d(y,x)$ ✓

$d(x,y) = |x-y|$ ✓

$= |-(y-x)|$ ✓

$= |-1(y-x)|$ ✓ ∵ $|ab| = |a||b|$

$= |-1||y-x|$ ∵ $|-1| < 0 = -(-1)$

$\Rightarrow |y-x| = d(y,x)$ ✓

Absolute value

$f(x) = \begin{cases} +x \\ -x \end{cases}$

$|3| = +3$

$|-2| = -(-2)$

Absolute value product [illegible]

$\therefore d(x,y) < d(x,z) + d(z,y)$

(iv) $d(x,y) = |x-y|$

Add subtract z

$= |x - z + z - y|$

$\because \Rightarrow |a+b| \le |a| + |b| \quad \forall a,b \in \mathbb{R}$

$d(x,y) = |x - z + z - y|$

$\le |x-z| + |z-y| = d(x,z) + d(z,y)$

$\therefore d(x,y) \le d(x,z) + d(z,y)$

★ **Usual Metric in R_2**

Cartesian product R^2 Real $\mathbb{R} \Leftarrow$

xy-plane

If $P_1(x_1, y_1)$, $P_2(x_2, y_2) \in \mathbb{R}^2$, then show that:-

$$d(P_1, P_2) = +\sqrt{(x_1+x_2)^2 + (y_1+y_2)^2}$$

is a metric on $\mathbb{R}^2$.

xy plane

★ Properties :-

M_1 $d(x,y) \ge 0$ (Non negative

M_2 $d(x_1 - x_2)^2 \ge 0$

M_3 $d(y_1 - y_2)^2 \ge 0$

M_4 d(

Add

① $d(x,y) \ge 0$ (non negativity)

② $d(x,y) = 0 \Rightarrow x = y$ (Reflective)

③ $d(x,y) = d(y,x)$ (Symmetric)

④ $d(x,y) \le d(x,z) + d(z,y)$ integral inequality

M_1 :- $(x_1 - x_2)^2 \ge 0$ non-negative square

$(y_1 - y_2)^2 \ge 0$ non-negative

By adding :- +ve square

$(x_1 - x_2)^2 + (y_1 - y_2)^2 \ge 0$

$\Rightarrow \sqrt{(x_1 - x_2)^2 + (y_1 + y_2)^2} \ge 0$

$\Rightarrow d(P_1, P_2) \ge 0$ distance

$(d(P_1, P_2) \ge 0)$ define

M_2 :- $d(P_1, P_2) = 0$ ✓

$\sqrt{(x_1 + x_2)^2 + (y_1 - y_2)^2} = 0$

let :- zero distance

$\Leftrightarrow (x_1 - x_2)^2 + (y_1 - y_2)^2 = 0$ consedent

$\Rightarrow (x_1 - x_2)^2 = 0$ & $(y_1 - y_2)^2 = 0$

$\Leftrightarrow x_1 - x_2 = 0$ $y_1 - y_2 = 0$ square root something be zero itself is equal to zero

$\Leftrightarrow x_1 = x_2$ pair $y_1 = y_2$

$\Leftrightarrow (x_1, y_1) = (x_2, y_2)$

$\Rightarrow P_1 = P_2$

M_3 :- $d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d(P_2, P_1)$

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= d(P_2, P_1)$

$(x-y)^2 = (y-x)^2$

M_4 :- $d(P_1, P_2) \le d(P_1, P_3) + d(P_3, P_2)$

All Points are on the same line.

Case 1:- ایک ہی لائن کے اوپر ہیں

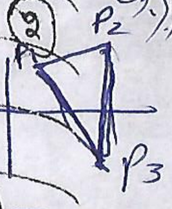
if P_1, P_2, P_3 are $\boxed{\text{Collinear.}}$

$$d(P_1,P_2) = d(P_1,P_3) + d(P_3,P_2) \rightarrow (1)$$

Case 2:- اگر یہ ایک ہی لائن کے اوپر نہیں ہیں

if P_1, P_2, P_3 are non-collinear

$$d(P_1,P_2) < d(P_1,P_3) + d(P_3,P_2) \rightarrow (2)$$



On Combining (1) and (2)

$$d(P_1,P_2) \leq d(P_1,P_3) + d(P_3,P_2)$$

Triangle inequality Property

$\therefore (\mathbb{R}^2, d)$ is a metric Space.

A B

lec 25

Lecture #25
Taxicab and Discrete Metric Spaces.

Discrete Metric Space:-

Show that any non-empty set X, can be made a metric space (X, d_0); where $d : X \times X \rightarrow \mathbb{R}$ is defined by:-

$$d_0(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

unit = 1 different

PROOF

(i) Non negativity

m_1:- $d_0(x,y) = 0$ if $x = y$ } By def
$d_0(x,y) = 1$ if $x \neq y$

On Combining both:-

$d_0(x,y) \geq 0$

m_2:- if $(x,y) = 0 \Rightarrow x = y$ } by def
if $x = y \Rightarrow d_0(x,y)$

m_3:- Case i:- $d(x,y) = d(y,x)$

if $x = y \Rightarrow d_0(x,y) = 0 = d_0(y,x)$

Case ii:- $d(x,y) = 1$ if $x \neq y$, $d(y,x)$ if $y = x$

(2) if $x \neq y$, $d(y,x) = 1$, $d_0(x,y) = 1 = d_0(x,y)$

$\therefore d_0(x,y) = d_0(y,x) \rightarrow$ if $y = x$

$d(x,y) = 1$ if $x \neq y$

$d(y,x) = 1$ if $y \neq x$

$d_0(x,y) \le d_0(x,z) + d_0(z,y)$

M4 :-

$x, y, z \in X \neq \phi$

cases

$x = y = z$, $x \neq y = z$, $x \neq y \neq z$

$x = y \neq z$.

$d(x,z) = 0$ $\quad x = z$

$d(x,y) = 0$ $\quad x = y$

$d(x,y) = 0$ $\quad z = y$

Case i:

if $x = y = z \Rightarrow d(x,y) = d(y,z)$

$= d(z,x) = 0$

$d(x,y) \le d_0(x,z) + d_0(z,y) \rightarrow (1)$

$0 = 0 + 0$

Case ii :- if $x = y \neq z \Rightarrow x \neq z$

$d_0(x,y) = 0$; $d_0(y,z) = 1$.

$d_0(x,z) = 1$

$\therefore (1) \Rightarrow 0 < 1 + 1 \rightarrow (2)$

Case (iii) :-

if $x \neq y \neq z$

$d(x,y) = d(y,z) = d(z,y) = 1$

$(1) \Rightarrow 1 < 1 + 1$

Combining all three cases;

$d_0(x,y) \le d_0(x,z) + d_0(z,y)$

$1 \le 1 + 1$

$1 < 2$

$R^2 = $ xy plane

Taxicab - 1 :-

Show that $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$ is a metric on $\mathbb{R}^2$, where $P_1(x_1, y_1)$, $P_2(x_2, y_2) \in \mathbb{R}^2$.

M1 :- non-negative

(1) $|x_1 - x_2| \ge 0 \quad \forall x_1, x_2 \in \mathbb{R}$ (non-ve)

(2) $|y_1 - y_2| \ge 0 \quad \forall y_1, y_2 \in \mathbb{R}$

Adding (1) and (2)

(1+2) $|x_1 - x_2| + |y_1 - y_2| \ge 0$

$\Rightarrow d(P_1, P_2) \ge 0$

point

(1) $d(x,y) \ge 0$ (non-ve)

(2) $d(x,y) = 0 \Leftrightarrow$ if

(3) $d(x,y) = d(y,x)$

(4) $d(x,y) = d(x,z) + \ldots$ Triangle

M2 :- Reflexive

$d(P_1, P_2) = 0$

$\Leftrightarrow |x_1 - x_2| + |y_1 - y_2| = 0$

$\Leftrightarrow |x_1 - x_2| = 0$ & $|y_1 - y_2| = 0$

$\Leftrightarrow x_1 - x_2 = 0$ & $y_1 - y_2 = 0$

$x_1 = x_2$ $\quad y_1 = y_2$

$\Leftrightarrow (x_1, y_1) = (x_2, y_2)$

$P_1 = P_2$

Taxicab - 2 :-

$d(P_1, P_2) = 0$

Symmetric property

$d(x,y) = d(y,x)$

M3 :-

$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|$

$= |-(x_2 - x_1)| + |-(y_2 - y_1)|$

$= |x_2 - x_1| + |y_2 - y_1|$

$= d(P_2, P_1)$

always -ve come for symm

4th property $d(x,y) \le d(x,z) + d(z,y)$ —> Triangular property

$(x_1,y_1)\ (x_3,y_3)\ P_3$, $P_2(x_2,y_2)$

My :- $d(P_1,P_2) = |x_1 - x_2| + |y_1 - y_3| + |y_3 - y_2|$

$= |x_1 - x_3 + x_3 - x_2| + \dots$ using

$\Rightarrow \{ |a+b| \le |a| + |b|$ ✓ $+ |y_1 - y_3| + |y_3 - y_2|$

$\le |x_1 - x_3| + |x_3 - x_2| + |y_1 - y_3| + |y_3 - y_2|$

$= (|x_1 - x_3| + |y_1 - y_3|) + (|x_3 - x_2| + |y_3 - y_2|)$

$= d(P_1,P_3) + d(P_3,P_2)$

$d(P_1,P_2) \le d(P_1,P_3) + d(P_3,P_2)$ △ property

★ Taxicabe In R_{n-1} :- R^n

$x \in R$, $(x_1,x_2) \in R^2$, $(x_1,x_2,\dots x_n) \in R^n$

let $x = (x_1,x_2,\dots,x_n)$, $y = (y_1,y_2,\dots,y_n) \in \mathbb{R}$
then Show that $d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$
is a metric on $\mathbb{R}^n$.

start $i = x_1 + x_2 \dots x_3$

$y \ne x$

1st property

My :- $d(x,y) \ge 0$

$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

$= |x_1 - y_1| + |x_2 - y_2| + \dots + |x_n - y_n| \ge 0$

2nd property ⟹ $d(x,y) \ge 0$

My :-

$d(x,y) = 0 \therefore x = y$

$d(x,y) = 0 \Longleftrightarrow x = y$

$\Rightarrow \sum_{i=1}^{n} |x_i - y_i| = 0$

$|x_1 - y_1| + |x_2 - y_2| + \dots + |x_n - y_n| = 0$ ✓ $(+) + (+) = 0$, 0, 0

$|x_1 - y_1| = 0$, $|x_2 - y_2| = 0 \dots$, $|x_n - y_n| = 0$ ✓

$x_1 - y_1 = 0$, $x_2 - y_2 = 0 \dots x_n - y_n = 0$ ✓

$x_1 = y_1$, $x_2 = y_2$, $\dots$ $x_n = y_n$

$\therefore (x_1, x_2, \dots x_n) = (y_1, y_2, \dots y_n)$

$\therefore x_i = y_i$

So $d(x,y) = 0$ $1 \le i \le n$

My :- $d(x,y) = d(y,x)$

$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$ ✓ (-1) common

$= \sum_{i=1}^{n} |y_i - x_i|$ ✓

$= d(y,x)$ ✓

My :- $d(x,y) \le d(x,z) + d(z,y)$

$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$

$= \sum_{i=1}^{n} |x_i - z_i + z_i - y_i|$

$= |(x_1 - z_1) + (z_1 - y_1)| + |(x_2 - z_2) + (z_2 - y_2)| + \dots |(x_n - z_n) + (z_n - y_n)|$

$\therefore\ |a_1 + a_2| \le |a_1| + |a_2|,\ \forall a_1, a_2 \in \mathbb{R}$.

$\le |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2| + \dots |x_n - z_n| + |z_n - y_n|$

$= \{|x_1 - z_1| + |x_2 - z_2| + \dots |x_n - z_n|\} + \{|z_1 - y_1| + |z_2 - y_2| + \dots |z_n - y_n|\}$

$= \sum_{i=1}^{n} |x_i - z_i| + \sum_{i=1}^{n} |z_i - y_i|$ ✓

$= d(x,z) + d(z,y)$ ✓

$d(x,y) \le (x,z) + d(z,y)$ ✓

lec 26

Lecture # 26: Metric in terms of ... Metric

non-negative $\frac{1}{x} - \frac{1}{y}$ refraction non-zero

- **Metric on $\mathbb{R}-\{0\}$**

Show that $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by: $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|,\ \forall\, x, y \in \mathbb{R}-\{0\}$ is a metric on $\mathbb{R}-\{0\}$.

M_1: $\left|\frac{1}{x} - \frac{1}{y}\right| \geq 0 \quad \forall\, x, y \in \mathbb{R}-\{0\}$ — 1st property

Given defined distance function proof; distance property $d(x,y) \geq 0$ ✓ mod is +ve non-negative — distance property

M_2: $d(x,y) = 0$ — 2nd property

$\iff \left|\frac{1}{x} - \frac{1}{y}\right| = 0$ ✓ — variable, absolute value, Quantity, possible zero

$\iff \frac{1}{x} - \frac{1}{y} = 0$ — infact value itself zero

$\iff \frac{1}{x} = \frac{1}{y} \iff x = y \quad \forall\, x, y \in \mathbb{R}-\{0\}$

excludes zero — Real Nos

M_3: $d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$

Symmetric Property

$= \left|-\left(\frac{1}{y} - \frac{1}{x}\right)\right|$ ✓ $\quad |-x| = x$

$= \left|\frac{1}{y} - \frac{1}{x}\right|$ ✓

$d(x,y) = d(y,x)$ ✓

$d(x,y) = \frac{1}{x-a} - \frac{1}{y-a} \quad \mathbb{R}$

M_4: $d(x,y) \leq d(x,z) + d(z,y)$

$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|$

$= \left|\frac{1}{x} - \frac{1}{z} + \frac{1}{z} - \frac{1}{y}\right|$ ✓ property 3 apply

$d(x,y) \leq \left|\frac{1}{x} - \frac{1}{z}\right| + \left|\frac{1}{z} - \frac{1}{y}\right|$ ✓ — Triangle inequality $|a+b| \leq |a| + |b|$

$d(x,y) \leq d(x,z) + d(z,y)$ ✓

$|x-y| \leq |x-z| + |z-y|$

British Rail distance function: is distance ... proof ... metric space

let $d'(x,y) = |x| + |y|$; $x, y \in \mathbb{R}$, Show that d' is not a metric on $\mathbb{R}$.

M_1:

$d'(x,y) = |x| + |y|$

$\because |x| \geq 0;\ |y| \geq 0 \Rightarrow d'(x,y) \geq 0$ ✓ +ve

M_2:

$d'(x,y) = 0$ ✓

$\Rightarrow |x| + |y| = 0$ ✓

$\Rightarrow |x| = 0;\ |y| = 0$ ✓

$x = 0;\ y = 0$ ✓

$x = 0 = y \Rightarrow \boxed{x = y}$

Conversely:

if $x = y$ ✓

$x = y = 0$ — Given

$x - y = 0$

$|x| \geq 0,\ |y| \geq 0$

$x = 0,\ y = 0$

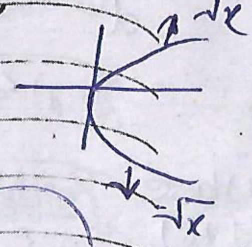
⇐ contradiction آ رہی ہے یہاں وہ ہیں یہ ہے

⇐ $x = y = 0$ کا مطلب یہ ہے کہ یہ کبھی بھی دونوں برابر نہیں ہو سکتے equal to zero نہیں ہو سکتے

$\therefore d'(x,y)$ is not a metric.

★ Absolute and radical metric:-

Show that the function $d: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by $d(x,y) = \sqrt{|x-y|}$, $\forall x,y \in \mathbb{R}$ is a metric on $\mathbb{R}$



⇒ M_1 :- $\because |x-y| \geq 0$ ✓

$\Rightarrow \sqrt{|x-y|} \geq 0$ ✓

$\Rightarrow d(x-y) \geq 0$ ✓

⇒ M_2 :- $d(x,y) = 0$ ✓ — converse is imp

$\Leftrightarrow \sqrt{|x-y|} = 0$ ✓

$\Leftrightarrow |x-y| = 0$ ✓

$\Leftrightarrow x-y = 0$ ✓

$\Leftrightarrow x = y$ ✓

Absolute value of zero possible when itself zero

$\sqrt{4} = -2$

$+\sqrt{4} = +2$

$\Rightarrow x^2 = 4$

$\Rightarrow x = \pm 2$

⇐ اگر ہم value کو square کریں تو دو values ملتی ہیں ± 2 لیکن جب square root لیں تو ایک ہی value آتی ہے

⇒ M_3 :- $d(x,y) = d(y,x)$

$= \sqrt{|x-y|} = \sqrt{|-(y-x)|}$

$= \sqrt{|y-x|} = d(y,x)$ ✓

M_4 :- ایک z introduce کروا دیں

⇒ $d(x,y) = \sqrt{|x-y|} = \sqrt{|x-z+z-y|}$

$\Rightarrow |a+b| \leq |a|+|b|$ $= \sqrt{|(x-z)+(z-y)|}$

$|a+b| \leq |a|+|b|+2\sqrt{a}\sqrt{b}$ $\quad d(x,y) \leq d(x,z)+d(z,y)$

$\sqrt{|a+b|} < \left(\sqrt{|a|}+\sqrt{|b|}\right)^2$ $\quad \forall x,y \in \mathbb{R}$

$2 < 3$

$2 < 3+2$

$\sqrt{|a+b|} \leq \sqrt{|a|} + \sqrt{|b|} \rightarrow (1)$

Put :-

$a = x-z, \quad b = z-y$

$(1) \Rightarrow \sqrt{|x-z+z-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|}$

$\Rightarrow \sqrt{|x-y|} \leq \sqrt{|x-z|} + \sqrt{|z-y|}$

Proof triangle inequality $\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

$\Rightarrow d(x,y) = \sqrt{|x-y|}$ is a metric on $\mathbb{R}$.

★ Metric in terms of defined metrics:-

Let (X,d) be metric space, show that d' defined by:- In terms of another distance function distance $d'(x,y)$

$d'(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$ is also metric on X. First d then d'

⇒ M_1 :- $\because d$ metric on X

d metric space ہے تو satisfy کرے گا 1st property $\Rightarrow d(x,y) \geq 0 \;\forall x,y \in X$

$\Rightarrow 1 + d(x,y) \geq 0$ ✓ دونوں non negative ہیں Add 1 one

$\Rightarrow \dfrac{d(x,y)}{1+d(x,y)} \geq 0 \Rightarrow d'(x,y) \geq 0 \quad \forall x,y \in \mathbb{R}$

M₃:- $d'(x,y)=0 \Rightarrow d(x,y)=0$

$\dfrac{d(x,y)}{1+d(x,y)}=0$ $\quad \because d$ is Metric on X

$\Rightarrow d(x,y)=0 \Rightarrow x=y$

Conversely: if $x=y \Rightarrow d(x,y)=0$

$\Rightarrow \dfrac{d(x,y)}{1+d(x,y)}=0 \Rightarrow \boxed{d'(x,y)=0}$

M₂:-

$d'(x,y)=\dfrac{d(x,y)}{1+d(x,y)}$

$\because d$ is Metric on 'X'

$\Rightarrow d(x,y)=d(y,x)\ \forall$

$\Rightarrow d'(x,y)=\dfrac{d(x,y)}{1+d(y,x)}=d'(y,x)$

$\dfrac{d(y,x)}{1+d(y,x)}$

M₄:-

$d'(x,y)\le d'(x,z)+d'(z,y)$

$\because$ 'd' is Metric on X.

$\Rightarrow d(x,y)\le d(x,z)+d(z,y),\ \forall x,y,z$

Add one دونوں طرف $\in X$

$\Rightarrow 1+d(x,y)\le 1+d(x,z)+d(z,d)$

$\dfrac{1}{1+d(x,z)+d(z,y)}\le\dfrac{1}{1+d(x,y)}$

$\left\{\begin{array}{l} 2<3 \\ \frac{1}{3}<\frac{1}{2}\end{array}\right.$

$a<b$

$\Rightarrow \dfrac{1}{b}<\dfrac{1}{a}$

$\Rightarrow$

$\left\{\begin{array}{l} a<b \\ \frac{1}{b}<\frac{1}{a} \\ \Rightarrow -b<- \end{array}\right.$

$1 - \dfrac{1}{1+d(x,y)} \le 1-\dfrac{1}{1+d(x,z)+d(z,y)}$

$\dfrac{1+d(x,y)-1}{1+d(x,y)}\le\dfrac{1+d(x,z)+d(z,y)-1}{1+d(x,z)+d(z,y)}$

$\therefore$

$\dfrac{d(x,y)}{1+d(x,y)}\le\dfrac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$

$d'(x,y)\le\dfrac{d(x,z)}{1+d(x,z)+d(z,y)}+\dfrac{d(z,y)}{1+d(x,z)+d(z,y)}$

excessive

~~Repeated~~

~~$\Rightarrow \dfrac{1+d(x,y)-1}{1+d(x,y)}\le\dfrac{1+d(x,z)+d(z,y)-1}{1+d(x,z)+d(z,y)}$~~

~~$\Rightarrow \dfrac{d(x,y)}{1+d(x,y)}\le\dfrac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$~~

$\Rightarrow d'(x,y)\le\dfrac{d(x,z)}{1+d(x,z)}+\dfrac{d(z,y)}{1+(z,y)}$ Right

$\Rightarrow d'(x,y)\le d'(x,z)+d'(z,y)$

is also metric space

Hence Prove

lec 27

Lecture # 27

Chebyshev Metric and Metric on Continuous Functions

(maximum norm) max distance is Answer

× Chebyshev Metric (in R_2 -1)

Show that $d(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

$\forall\ P_1(x_1, y_1);\ P_2(x_2, y_2) \in \mathbb{R}^2$

$d: R^2 \times R^2 \to R$ is a Metric space (d, R^2)

$d(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

$P_1(2,3)$, $P_2(-5,-7)$

$|2-(-5)| > |-3-(-7)|$

$= |7|\ ,\ |10|$

$= 10$

y-components, x components

maximum difference

defined distance, function

value

maximum

$\max(2,4) = 4$

$\max(2,2) = 2$

1st property

$d(P_1, P_2) \geq 0$

$\max(2,3) = 3 \geq 0$

m_1:- $d(x, y) \geq 0$

$|x_1 - x_2| \geq 0$ (I)

$|y_1 - y_2| \geq 0$ (II)

from (I) and (II)

$\max\{|(x_1 - x_2)|, |y_1 - y_2|\} \geq 0$

$d(P_1, P_2) \geq 0$

2nd property

$d(P_1, P_2) = 0 \Rightarrow P_1 = P_2$

m_2:- $d(x, y) = 0$

$|x_1 - x_2| = 0$, $|y_1 - y_2| = 0$

$x_1 - x_2 = 0$, $y_1 - y_2 = 0$

$x_1 = x_2$, $y_1 = y_2$

ordered pairs

$(x_1, y_1) = (x_2, y_2)$

$P_1 = P_2$

$d(P_1, P_2) = 0$

3rd property

$d(P_1, P_2) = d(P_2, P_1)$

m_3:- $d(x, y) = d(y, x)$

$d(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

$= \max\{|-(x_2 - x_1)|, |-(y_2 - y_1)|\}$

$= \max\{|x_2 - x_1|, |y_2 - y_1|\}$

$= d(P_2, P_1)$

$P_3(x_3, y_3) \in R^2$

We have to Proof

m_4:- $d(P_1, P_2) \leq d(P_1, P_3) + d(P_3, P_2)$

$d(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$, $\boxed{P_3(x_3, y_3) \in \mathbb{R}^2}$

$= \max\{|x_1 - x_3 + x_3 - x_2|, |y_1 - y_3 + y_3 - y_2|\}$

$= \max\{|(x_1 - x_3) + (x_3 - x_2)|, |(y_1 - y_3) + (y_3 - y_2)|\}$

$\because |a+b| \leq |a| + |b|$

$|(x_1 - x_3) + (x_3 - x_2)| \leq |x_1 - x_3| + |x_3 - x_2|$

$|(y_1 - y_3) + (y_3 - y_2)| \leq |y_1 - y_3| + |y_3 - y_2|$

$d(P_1, P_2) \leq \max\{\underbrace{|x_1 - x_3|}_{a} + \underbrace{|x_3 - x_2|}_{c}, \underbrace{|y_1 - y_3|}_{b} + \underbrace{|y_3 - y_2|}_{d}\}$ → (2)

If $a, b, c, d \in \mathbb{R}$

$a \leq \max\{a, b\}$, $c \leq \max\{c, d\}$ → (A)

$b \leq \max\{a, b\}$, $d \leq \max\{c, d\}$ → (B)

By adding (A)

$a + c \leq \max\{a, b\} + \max\{c, d\}$

By adding (B)

$b + d \leq \max\{a, b\} + \max\{c, d\}$

$\{\binom{3}{2} < \binom{4}{8}\} < 8$

$\Rightarrow \max\{a+c,\ b+d\} \leq \max\{a, b\} + \max\{c, d\}$

$\therefore (2) \Rightarrow$

$d(P_1, P_2) \leq \max\{|x_1 - x_3|, |y_1 - y_3|\} + \max\{|x_3 - x_2|, |y_3 - y_2|\}$ $d(P_1, P_3)$ $d(P_3, P_2)$

$\Rightarrow d(P_1, P_2) \leq d(P_1, P_3) + d(P_3, P_2)$

$\therefore (d, \mathbb{R}^2)$ is a metric space.

2024

Chebyshev Metric in $\mathbb{R}^n$:-

Let $X = \mathbb{R}^n$, $P_1 = (x_1, x_2, \ldots x_n)$, $P_2 = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ then prove that $d: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a metric on $\mathbb{R}^n$ defined by $d(P_1, P_2)$

$= \max\{|x_i - y_i|;\ 1 \leq i \leq n,\ n \in \mathbb{Z}^+\}$.

$P_1 = (x_1, x_2, \ldots, x_n)$, $P_2 = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$

given that $d(P_1, P_2) = \max\{|x_i - y_i|, 1 \leq i \leq n, n \in \mathbb{Z}^+\}$

$\Rightarrow d(P_1, P_2) = \max\{|x_1 - x_2|, |x_2 - y_2|, \ldots, |x_n - y_n|\} \to (1)$

M_1:- $\because |x_1 - x_2| \geq 0$, $|x_2 - y_2| \geq 0, \ldots, |x_n - y_n|$

$\max\{|x_1 - x_2|, |x_2 - y_2|, \ldots, |x_n - y_n|\} \geq 0$

$\max\{|x_i - y_i|, 1 \leq i \leq n\} \geq 0$

$\Rightarrow d(P_1, P_2) \geq 0$

M_2:-

$d(P_1, P_2) = 0$

$\Rightarrow \max\{|x_i - y_i|, 1 \leq i \leq n\} = 0$

$\Rightarrow \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\} = 0$

$\Rightarrow |x_1 - y_1| = 0,\ |x_2 - y_2| = 0, \ldots |x_n - y_n| = 0$

$\Rightarrow x_1 - y_1 = 0,\ x_2 - y_2 = 0 \ldots x_n - y_n = 0$

$\Rightarrow x_1 = y_1 \quad x_2 = y_2 \ldots x_n = y_n$

$(x_1, x_2, \ldots x_n) = (y_1, y_2, \ldots y_n)$

$\Rightarrow P_1 = P_2$

M_3:-

$d(P_1, P_2) = \max\{|x_i - y_i|;\ 1 \leq i \leq n\}$

$= \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$

$= \max\{|-(y_1 - x_1)|, |-(y_2 - x_2)|, \ldots, |-(y_n - x_n)|\}$

$\because |-a| = |a| \ \forall\ a \in \mathbb{R}$.

$= \max\{|y_1 - x_1|, |y_2 - x_2|, \ldots |y_n - x_n|\}$

$= \max\{|y_i - x_i|;\ 1 < i < n\}$

$= d(P_2, P_1)$

Proof :- Now we will prove triangular Inequality for $P=(x_1, x_2, \dots, x_n)$, $q=(y_1, y_2, \dots, y_n)$, $r=(z_1, z_2, \dots, z_n) \in \mathbb{R}^n$.

$d(P,q) \leq d(P,r) + d(r,q)$

$d(P,q) = \max\{|x_i - y_i| ;\ 1 \leq i \leq n,\ n \in \mathbb{Z}^+\}$

$= \max\{|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|\}$

$= \max\{|x_1 - y_1|, |x_2 - z_2 + z_2 - y_2|, \dots, |x_n - z_n + z_n - y_n|\}$

$= \max\{|x_1 - z_1 + z_1 - y_1|, \dots - y_n|\} \to (1)$

$2<3$
$3<4$
$\vdots$
$n_1 < n_1$

$|x_1 - z_1 + z_1 - y_1| \leq |x_1 - z_1| + |z_1 - y_1|$

similarly

$|x_2 - z_2 + z_2 - y_2| \leq |x_2 - z_2| + |z_2 - y_2|$

$\vdots$

$|x_n - z_n + z_n - y_n| \leq |x_n - z_n| + |z_n - y_n|$

Left side maximum ... ⟸

by Less maximum < Right side

$\Rightarrow \max\{|x_1 - z_1 + z_1 - y_1|, |x_2 - z_2 + z_2 - y_2|, \dots, |x_n - z_n + z_n - y_n|\} \leq \max\{|x_1 - z_1| + |z_1 - y_1|, \dots, |x_n - z_n| + |z_n - y_n|\} \to (2)$

If $a_1, a_2, \dots a_n \in \mathbb{R}^+$, $b_1, b_2, \dots b_n \in \mathbb{R}$

$\Rightarrow a_1 \leq \max\{a_1, a_2, \dots a_n\}$, $b_1 \leq \max\{b_1, b_2, \dots b_n\}$

$\Rightarrow a_2 \leq \max\{a_1, a_2, \dots a_n\}$, $b_2 \leq \max\{b_1, b_2, \dots b_n\}$

$\Rightarrow a_n \leq \max\{a_1, a_2, \dots, a_n\}$, $b_n \leq \max\{b_1, b_2, \dots b_n\}$

$a_1 + b_1 \leq \{a_1, a_2, \dots a_n\} + \max\{b_1, b_2, \dots b_n\}$

$a_2 + b_2 \leq \max\{a_1, a_2, \dots a_n\} \max\{b_1, b_2, \dots b_n\}$

$\vdots$

$a_n + b_n \leq \max\{a_1, a_2, \dots a_n\} + \max\{b_1, b_2, \dots b_n\}$

$\max\{a_1 + b_1, a_2 + b_2, \dots a_n + b_n\} \leq \max\{a_1, a_2, \dots a_n\} + \max\{b_1, b_2, \dots b_n\}$

subtract / Add ... mode ... components

$d(P,q) \leq \max\{|x_1 - z_1|, |x_2 - z_2|, \dots, |x_n - z_n|\} + \max\{|z_1 - y_1|, |z_2 - y_2|, \dots |z_n - y_n|\}$

$\leq \max\{|x_i - z_i|, 1 \leq i \leq n\} + \max\{|z_i - y_i| ; 1 \leq i \leq n\}$

$\boxed{d(P,q) \leq d(P,r) + d(r,q)}$

$\therefore (d, \mathbb{R}^n)$ is a metric space.

closed interval max, min value

C[a,b] as a Metric space-1 :-

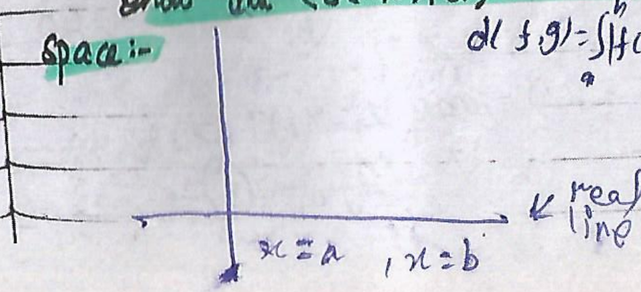
for $C[a,b]$, the set of all real Continuous functions defined on $[a,b]$ with $d(f,g) = \int_a^b |f(x) - g(x)| dx$, $(b > a)$ for all $f(x), g(x) \in C[a,b]$.

Calculus → Real Analysis → Function Analysis

imaginary line

Show that $(C[a,b], d)$ is a Metric space :-

$d(f,g) = \int_a^b |f(x) - g(x)|$



definite integral ↔ Area under curve function $\int f(x)\,dx$

m_1:- $d(f,g) = \int_a^b |f(x) - g(x)|\,dx$

$\because |f(x) - g(x)| \geq 0 \quad \forall x \in [a,b]$

$\Rightarrow \int_a^b |f(x) - g(x)|\,dx \geq 0$

$\Rightarrow d(f,g) \geq 0$

m_2:- $d(f,g) = 0$

$\Rightarrow \int_a^b |f(x) - g(x)|\,dx = 0$

$\Rightarrow |f(x) - g(x)| = 0$

$\Rightarrow f(x) = g(x)$

$\Rightarrow f = g$

m_3:- $d(f,g) = \int_a^b |f(x) - g(x)|\,dx$

$= \int_a^b |-\{g(x) - f(x)\}|\,dx$

$= \int_a^b |g(x) - f(x)|\,dx = d(g,f)$

m_4:- $d(f,g) = \int_a^b |f(x) - g(x)|\,dx$

$= \int_a^b |f(x) - h(x) + h(x) - g(x)|\,dx \to (1)$

$\because |f(x)| = |f(x) - h(x) + h(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$

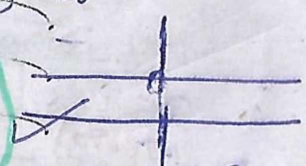
continus function at $x = a \in \mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$

e.g $f(x) = \frac{x}{x}, \; x \neq 0$

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{x} = 1$

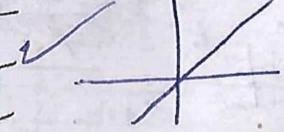
$\lim_{x\to 0} f(x) \neq f(0)$

It is not continues funtn



if $f(x) = x$

$\lim_{x\to 0} f(x) = 0$



It is continus funh

$\lim_{x\to a} f(x) = f(a)$ ← continuous

$\Rightarrow \int_a^b |f(x) - h(x) + h(x) - g(x)|\,dx \leq \int_a^b |f(x) - h(x)| + |h(x) - g(x)|\,dx$

$= \int_a^b |f(x) - h(x)|\,dx + \int_a^b |h(x) - g(x)|\,dx$

$d(f,g) \leq d(f,h) + d(h,g)$

Example:-

Determine whaer the given distance functions are the metrices on $\mathbb{R}$.

i) $d(x,y) = |x^2 - y^2|$

m_1:- Non-negativity:-

$\because |x^2 - y^2| \geq 0 \Rightarrow d(x,y) \geq 0$

m_2:- Reflexive property:-

if $d(x,y) = 0 \Rightarrow |x^2 - y^2| = 0$

$x^2 - y^2 = 0, \quad x^2 = y^2$

$\Rightarrow x = \pm y \Rightarrow d(x,y) = 0 \Rightarrow x = y$

but also $\Rightarrow \boxed{x = -y}$

$\Rightarrow$ Reflexive property is not satisfied

$\Rightarrow$ given ~~property~~ d is not a Metric on $\mathbb{R}$.

(ii) $d(x,y) = |x-y|^2$.

m_1 :- Non-Negtivity:-

$|x-y|^2 \geq 0 \Rightarrow d(x,y) \geq 0$

m_2 :- Reflexive Property:-

if $d(x,y) = 0$

$\Rightarrow |x-y|^2 = 0$, $|x-y| = 0 \Rightarrow x-y = 0$

$\Rightarrow x = y$

m_3) Symmetric Property:-

$d(x,y) = |x-y|^2 = |y-x|^2 = d(y,x)$

m_4 Triangular Inequality:-

if $x=1$, $y=2$, $z=3$ $d(x,y) \leq d(x,z) + d(z,y)$

$d(1,3) = |1-3|^2 = 4$.

$d(1,2) = |1-2|^2 = 1 = d(2,3) = 1$

$\therefore d(1,3) > d(1,2) + d(2,3)$ ($\leq$ should)

$\Rightarrow$ Triangular Inequality is not Satisfied $\therefore (d, \mathbb{R})$ is not a Metric Space.

lec 28

Mth405- Lec #28

Uniform metric space on class of continuous

Distance formula $\to$ close | bounded

If $B[a,b]$, $a<b$ is the set of all real valued bounded function defined on $[a,b]$. Then show that distance function.

$d(f,g) = \sup|f(x) - g(x)|$ $\forall f,g \in B[a,b]$

cos, sin, $y=1$, $y=-1$

Bounded function: (g) attain certain height / depth

$\Leftarrow$ Like bounded function, attain

$\because$ which is in between close interval

$n \in \mathbb{N}$ $f(n) = (-1)^n$, $f(x) = (-1)^x$

Which show $[-]$

Unbounded function: unbound $0 < e^x < \infty$

Which lie in open interval: bounded below $e^x > 0$ $(n \to -\infty, e^x \to 0)$ $0 < e^x < \infty$ Mathematically $n \in \mathbb{N}$

Superimum (S)

$\hookrightarrow$ Like maximum.

close interval

infimum: left minimum

right $\to$ maximum supremum

e.g. Real line.

What is the maximum value of closed interval $[a,b]$. max

1st number Greater $> b$ = maximum supremum

Above the real line value on the right is higher on the left

open $(3,4)$ infimum a b 3 4

$d: X \times X \to \mathbb{R}$ | $d'(x,y) = \min\{1, d(x,y)\}$ $A \subseteq B \to$ subset

Case II: $d'(x,y) = d(x,y)$ $\quad d(x,y) < 1$ d is metric

$d(y,x)$ $\quad d(y,x) < 1$ my $d(x,y) \geq 0$

$d'(x,y) = \min\{1, d(y,x)\} = d'(y,x)$ — B $\quad 1 > 0$ $\quad d(x,y) \geq 0$

from A, B $\quad d'(x,y) = d'(y,x) \quad \forall x,y \in X \neq \phi$

m_4) Triangular $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \neq \phi$

Case I $\quad d'(x,y) = \min\{1, d(x,y)\}$

$= d(x,y)$

$= d(x,z) + d(z,y)$

$= \min\{1, d(x,z) + d(z,y)\}$ — A

Case II $\quad \min\{1, d(x,z)\} = 1, \min\{1, d(z,y)\} = 1$ — II

$\min\{1, d(x,z)\} = d'(x,z)$ — III

$\min\{1, d(z,y)\} = d'(z,y)$ — IV

Add I, II $\min\{1, d(x,z)\} + \min\{1, d(z,y)\} = 1+1 = 2$ —(a)

Add I, IV $\min\{1, d(x,z)\} + \min\{1, d(z,y)\} = 1 + d'(z,y)$ —(b)

Add II, III $\min\{1, d(x,z)\} + \min\{1, d(z,y)\} = d'(x,z) + 1$ —(c)

Add III, IV $\min\{1, d(x,z)\} + \min\{1, d(z,y)\} = d'(x,z) + d'(z,y)$ —(d)

from a, b, c, d

$\min\{1, d(x,z)\} + \min\{1, d(z,y)\} = \{a, b, c, d\}$

$\min\{1, d(x,z)\} + \min\{1, d(z,y)\} = \{$ ~~1+1, 1+d(z,y)~~ $\} =$

$\{1+1, 1+d(z,y), d'(x,z)+1, d'(x,z) + d(z,y)\}$

from A, B.

$\min\{1, d(x,z) + d(z,y)\} \leq \min\{1, d(x,z)\} + \min\{1, d(z,y)\}$

$\boxed{d'(x,y) \leq d'(x,z) + d'(z,y)}$

$\sup A \leq \sup B$

$\sup|A+B| \leq \sup|A| + \sup|B|$

If $p = (x_1, x_2), q = (y_1, y_2) \in \mathbb{R}^2$ Show that $d(p,q) = \min\{|x_1 - y_1|, |x_2 - y_2|\}$ is metric on.

Q# Proof m_1) non-negative $d(x,y) \geq 0 \;\forall x,y$

$|x_1 - y_1| \geq 0$ — I $\quad |x_2 - y_2| \geq 0$ — II

from I, II $\quad \min\{|x_1 - y_1|, |x_2 - y_2|\} \geq 0$.

$d(p,q) \geq 0 \quad \forall x,y \in \mathbb{R}^2$

m_2) Reflexive $d(x,y) = 0 \quad x = y$

$d(p,q) = 0 \quad \min\{|x_1 - y_1|, |x_2 - y_2|\} = 0$

Case I $|x_1 - y_1| = 0$ $\quad$ Then $|x_2 - y_2| > 0$

$x_1 - y_1 = 0$ $\quad x_2 - y_2 > 0$

$x_1 = y_1$ $\quad x_2 > y_2$

$(x_1, y_1) \neq (x_2, y_2)$ $\quad x_2 \neq y_2$

$d(x,y) = 0 \Rightarrow p \neq q$ $\quad x_2 \neq y_2$ not true for reflexive.

Hence

$d(p,q) = \min\{|x_1 - y_1|, |x_2 - y_2|\}$ not metric.

*=

Q# If $p = (x_1, x_2), q = (y_1, y_2) \in \mathbb{R}^2$. Show that $d(p,q) = |x_1| + |x_2| + |y_1| + |y_2|$ is metric space on $\mathbb{R}^2$

Proof m_1) - non-negative $d(x,y) \geq 0 \;\forall x,y$

$|x_1| \geq 0, |x_2| \geq 0, |y_1| \geq 0, |y_2| \geq 0$

$|x_1| + |x_2| + |y_1| + |y_2| \geq 0$

$d(p,q) \geq 0$

Sn: Finite : Converges] depends on
infinite: diverges] sum of series-

2) Reflexive $d(x,y)=0 \quad x=y \quad x,y\in X\neq\phi$

$d(p,q)=0, \ |x_1|+|x_2|+|y_1|+|y_2|=0$

$|x_1|=0, |x_2|=0, |y_1|=0, |y_2|=0$

$x_1=0, x_2=0, y_1=0, y_2=0,$

$x_1=x_2, \quad y_1=y_2 = 0$

$(x_1,y_1)=(x_2,y_2)$

Conversely $\Rightarrow P=q \not\Rightarrow d(p,q)=0$

Hence reflexive property does not satisfied

$|x_1|\neq 0, |x_2|\neq 0, |y_1|\neq 0, |y_2|\neq 0$

Not a metric space

Ans So it is not metric space

Lecture # 29 missing

[illegible] mobile

lec 30

Mth 405

lec 30

$x,y\in Z$
$\Rightarrow (x_1,y_1)\in R^2$

Theorem

If $(x_1,x_2,\dots x_n),(y_1,y_2,\dots y_n)\in R^n$

Cauchy Schwarz

imp $\rightarrow \left(\sum_{r=1}^{n} x_r y_r\right)^2 \le \left(\sum_{r=1}^{n} x_r^2\right)\cdot\left(\sum_{r=1}^{n} y_r^2\right)$

Real No

Let proof R = Real plane, R^2 = x,y plane

$\vec{u}=(x_1,x_2,\dots x_n), \ \vec{v}=(y_1,y_2,\dots y_n)\in R^n$ vector

Let $\vec{u}=\binom{x_1}{x_2}, \vec{v}=\binom{y_1}{y_2}\in R^n$

$\vec{r}\binom{x}{y}$

Applying dot product

polar components

Rectangular $\leftarrow \vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\phi$ (ϕ is equal to angle b/w ...)

θ = angle b/w vectors

polar form $\rightarrow \begin{pmatrix}x_1\\x_2\\x_n\end{pmatrix}\cdot\begin{pmatrix}y_1\\y_2\\y_n\end{pmatrix}=\sqrt{x_1^2+x_2^2+\dots x_n^2}\cdot\sqrt{y_1^2+y_2^2+\dots y_n^2}$

$x_1y_1+x_2y_2+\dots x_ny_n = \sqrt{(x_1^2+x_2^2+\dots x_n^2)\cdot(y_1^2+y_2^2+\dots)}$

Taking square on both side.

$(x_1y_1+x_2y_2+\dots x_ny_n)^2 \le (x_1^2+x_2^2+\dots x_n^2)\cdot(y_1^2+\dots)$

We can write $\quad$ we can write

$\left(\sum_{r=1}^{n} x_r y_r\right)^2 \le \left(\sum_{r=1}^{n} x_r^2\right)\left(\sum_{r=1}^{n} y_r^2\right)$

Hence proved

(2)

(i) Minkoski Inequality.

$$\sqrt{\sum_{r=1}^{n}(x_r+y_r)^2} \le \sqrt{\sum_{r=1}^{n} x_r^2} + \sqrt{\sum_{r=1}^{n} y_r^2}$$

$$(\vec{u}+\vec{v})(\vec{u}+\vec{v}) = \bar{u}\cdot\bar{u} + \bar{u}\cdot\bar{v} + \bar{v}\cdot\bar{u} + \bar{v}\cdot\bar{v}$$

$$|\bar{u}+\bar{v}|^2 = |\bar{u}|^2 + |\bar{v}|^2 + \bar{u}\cdot\bar{v} + \bar{u}\cdot\bar{v}$$

$$|\bar{u}+\bar{v}|^2 = |\bar{u}|^2 + |\vec{v}|^2 + 2|\bar{u}||\bar{v}|$$

$$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos\alpha$$

$$|\vec{u}+\vec{v}|^2 = (|\vec{u}| + |\vec{v}|)^2$$

taking square root

$$|\vec{u}+\vec{v}| \le |\vec{u}| + |\vec{v}|$$

If $a, b \in R$, $|a+b| \le |a|+|b|$

$\alpha = 0$, $\vec{r}\cdot\vec{r} = |\vec{r}||\vec{r}|\cos 0$, $= |\vec{r}|^2$

$$(x_1+y_1)^2 + (x_2+y_2)^2 + \dots (x_n+y_n)^2 < \sqrt{x_1^2 + x_2^2 + \dots + x_r}\left(\sqrt{y_1^2+y_2^2+\dots y_n^2}\right)$$

$\because \cos 0 = 1$

MCQ

are proved

Minkoski $= \sqrt{\sum_{r=1}^{n}(x_r+y_r)^2} \le \sqrt{\sum_{r=1}^{n} x_r^2} + \sqrt{\sum_{r=1}^{n} y_r^2}$

Caushy Schwarz $= \left(\sum_{r=1}^{n} x_r y_r\right)^2 \le \left(\sum_{r=1}^{n} x_r^2\right)\cdot\left(\sum_{r=1}^{n} y_r^2\right)$

Lec 31

Lec # 31 Method

If $x_1, x_2, \dots x_n \in R$, then show that

(1) $\left(\sum_{k=1}^{n} x_k\right)^2 \le n \sum_{k=1}^{n} x_k^2$

Proof We know that

$$\left(\sum_{r=1}^{n} x_r y_r\right)^2 \le \sum_{r=1}^{n} x_r^2 \cdot \sum_{r=1}^{n} y_r^2$$

Put $r = k$ Then By cauchy schwarz inequality

$$\left(\sum_{k=1}^{n} x_k y_k\right)^2 \le \sum_{k=1}^{n} x_k^2 \cdot \sum_{k=1}^{n} y_k^2$$

$y_k = 1$; $\forall\ r = 1, 2, \dots n$ Put $y_k = 1$

$$\left(\sum_{k=1}^{n} x_k \cdot 1\right)^2 \le \sum_{k=1}^{n} x_k^2 \cdot \sum_{k=1}^{n} (1)^2$$

$\because x_r \cdot 1 = x_r$ $\quad r = 1, 2, \dots n$

$$\left(\sum_{k=1}^{n} x_k\right)^2 \le \sum_{k=1}^{n} x_k^2 \cdot n$$

$$\boxed{\left(\sum_{k=1}^{n} x_k\right)^2 \le n \sum_{k=1}^{n} x_k^2}$$

MCQ

Question 2

$$(x_1+x_2+x_3+x_4+x_5)^2 \le 5(x_1+x_2+x_3+x_4+x_5)$$

We know that:

$$\sum_{k=1}^{n} x_k^2 \le n \sum_{k=1}^{n} x_k^2$$

Put $n = 5$

$$\sum_{k=1}^{5} x_k^2 \le 5 \sum_{k=1}^{5} x_k^2$$

$$(x_1+x_2+x_3+x_4+x_5)^2 \le 5(x_1^2+x_2^2+x_3^2+x_4^2+x_5^2)$$

Let $x = (x_1, x_2, x_3, \dots x_n)$, $y = (y_1, y_2, y_3, \dots y_n) \in \mathbb{R}^n$

Show that $d(x,y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2}$ is metric space

$d \geq 0$

$|\vec{PQ}| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2}$

$d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2}$

m_1) Non negativity

$\because (x_1-y_1)^2 \geq 0;\ (x_2-y_2)^2 \geq 0, \dots (x_n-y_n)^2 \geq 0$

$d(x,y) \geq 0 \quad \forall\ x, y \in X \neq \phi$ absolute value $|x_1 - x_2| = \sqrt{(x_1-x_2)^2}$, $\sqrt{x^2} = |x|$

$(x_1-y_1)^2 \geq 0 + (x_2-y_2)^2 \geq 0 \dots + (x_n-y_n)^2 \geq 0$

$(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2 \geq 0$

$= \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2} \geq 0$

$= \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} \geq 0, \quad d(x,y) \geq 0$

m_2) Reflexive property

$d(x,y) = 0 \quad \boxed{x = y}$

$d(x,y) = 0, \quad \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} = 0$

$\sqrt{x} = 0 \Rightarrow x = 0$

$\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2} = 0$

$(x_1-y_1)^2 + (x_2-y_2)^2 + \dots + (x_n-y_n)^2 = 0$

Check all points

$(x_1-y_1)^2 = 0$	$(x_2-y_2)^2 = 0$	$(x_n-y_n)^2 = 0$
$x_1 - y_1 = 0$	$x_2 - y_2 = 0$	$x_n - y_n = 0$
$x_1 = y_1$	$x_2 = y_2$	$x_n = y_n$

$(x_1, x_2, \dots x_n) = (y_1, y_2, \dots y_n)$

$\boxed{x = y}$

m_3) Symmetric property

$d(x,y) = d(y,x) \quad \forall\ x, y \in X \neq$

$d(x,y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} = \sqrt{\sum_{j=1}^{n} [-(y_j - x_j)]^2} = \sqrt{\sum_{j=1}^{n}}$

$d(x,y) = d(y,x)$

m_4) Triangular property

$d(x,y) \leq d(x,z) + d(z,y) \quad \forall\ x, y, z \in X \neq$

$d(x,y) = \sqrt{\sum_{j=1}^{n} (x_j - y_j)^2} = \sqrt{\sum_{j=1}^{n} [(x_j - z_j + z_j - y_j)]^2}$ by Le M

$\sqrt{\sum_{j=1}^{n} [(x_j - z_j) + (z_j - y_j)]^2} \leq 0$

$\leq \sqrt{\sum_{j=1}^{n} (x_j - z_j)^2} + \sqrt{\sum_{j=1}^{n} (z_j - y_j)^2}$

$d(x,y) \leq d(x,z) + d(z,y)$

is a metric space.

Lec 32

Lecture 32 MTH405

collection of point

subset of X

Open sphere and close sphere.

let (X,d) be a metric space $r>0$ is positive real number x_0 for given point $x_0 \in X$

b/w x & x_0

(X,d)

open sphere

Radius is written in small brackets.

Open sphere

distance MCQ

$B_r(x_0) = S_r(x_0) = \{x \in X : d(x,x_0) < r\}$

close open

Skin or boundary is not included where r = radius

x_0 = centre, B = Ball, S = sphere

[ ] used square brackets used.

Close sphere MCQ

(X,d)

$B_r[x_0] = S_r[x_0] = \{x \in X : d(x,x_0) \le r\}$

Skin or boundary is included. MCQ

Point sphere is never empty

$d(x,x) = 0$ but $S_r[x] \ne \phi$, $S_r(x) \ne \phi$

closed sphere open sphere

Open sphere in $\mathbb{R}$ (Under usual metric)

less than Fixed point distance

$d(x,y) = |x-y|$ MCQ

put $y = x_0$

$|x| = a$

$S_r(x_0) = \{x \in \mathbb{R} : d(x,x_0) < r\}$

$y \in \mathbb{R}$

$-a < x < a$

open

$= \{x \in \mathbb{R} : |x-x_0| < r\}$

$-a \quad x \quad a$

$-r < x - x_0 < r$

add x_0 $\{x_0 - r < x < x_0 + r\}$ → open interval

$x_0 - r \quad x_0 + r$

$-r \quad x_0 \quad +r$

Length of interval $(-r, r) = r-(-r) = r+r = 2r$

big - small value

infact open sphere in Real open interval

e.g $d(x,y) =$

$y = \sin x, \cos x$

$|\sin x| \le 1$, $-1 \le \sin x \le 1$

Bounded function for given metric space (X,d).

If In which exists a positive real number M

$d(x,y) \le M \quad \forall x,y \in X \ne \phi$ then (X,d) is bounded metric space. MCQ

① e.g $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \ne y \end{cases}$ $\quad \frac{a}{1+a} < 1$ $\forall a \in \mathbb{R}^+$

② e.g $d'(x,y) = \frac{kd(x,y)}{1+kd(x,y)}$ $\quad d_0(x,y) \le 1 \quad \forall x,y \in (X_0,d)$

$0 < 1$ (true)

$d''(x,y) = \min(1, d(x,y)) \Rightarrow d''(x,y) \le 1 \quad \forall x,y \in \mathbb{R}$

$d''(x,y) \le 1$

MCQ

If $\sup\{d(x,y) \; x,y \in X\}$ is finite then

unbounded metric space

e.g $\quad d(x,y) \ge n$

$d(x,y) = |x-y|$

unbounded x, y are infinite

finite supremum → bounded

Bounded

A metric space is bound if $\sup\{d(x,y);\; x,y \in X\}$ is finite

unbounded

how much larger it is

$\forall M \ge 0$, $d(x,y) > M$, for least $x,y \in X$

open sphere on $\mathbb{R}$ (other metric)

on $\mathbb{R}$ $d: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ as:

$d(x,y) = \sqrt{|x-y|}$ $\quad y = x_0$

collection of all those points on a real line i.e $x \in \mathbb{R}$ such that whose distance from fixed point less than r i.e $d(x, x_0) < r$

$S_r(x_0) = \{x \in \mathbb{R} : d(x, x_0) < r\}$

$= \{x \in \mathbb{R} : \sqrt{|x-x_0|} < r\}$

$\sqrt{|x-x_0|} < r$

$|x-x_0| < r^2$

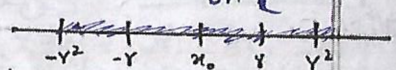
$= \{x \in \mathbb{R} \; -r^2 < x - x_0 < r^2\}$

add x_0

$\{x \in \mathbb{R} \; x_0 - r^2 < x < x_0 + r^2\}$

when $r > 1$ $\quad \{|x| < a\} \; -a < x < a$ on $\mathbb{R}$

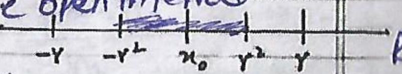
$2 < 3$, $2^2 < 3^2$, $4 < 9$



when $0 < r < 1$ $\quad r^2 < r < 1$

i.e open interval

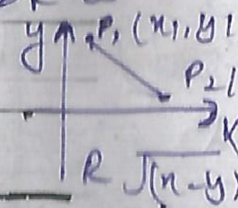
$r = \frac{1}{2}$, $r^2 = \frac{1}{4}$



Open sphere on $\mathbb{R}^2$ (Taxicab metric space)

on $\mathbb{R}^2$; $d: \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$ as

$d(P_1, P_2) = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$



let $P_1 = P = (x,y)$, $P_2 = P_0 = (0,0)$

$d(P, P_0) = \sqrt{(x-0)^2 + (y-0)^2}$

$= \sqrt{x^2 + y^2}$

$\mathbb{R}$ $\sqrt{(x-y)^2} = |x-y|$

MCQs

open sphere $S_r(P_0) = \{P \in \mathbb{R}^2 : d(P, P_0) < r\}$ $\quad \boxed{r=1}$

$= \{P \in \mathbb{R}^2 : \sqrt{(x-0)^2 + (y-0)^2} < 1\} \because r = 1$

$\sqrt{(x-0)^2} + \sqrt{(y-0)^2} < 1$

$\sqrt{x^2+y^2} < \sqrt{x^2} + \sqrt{y^2}$

$\sqrt{x^2} + \sqrt{y^2} < 1$

$\sqrt{x^2} = |x|$

$|x| + |y| = 1$

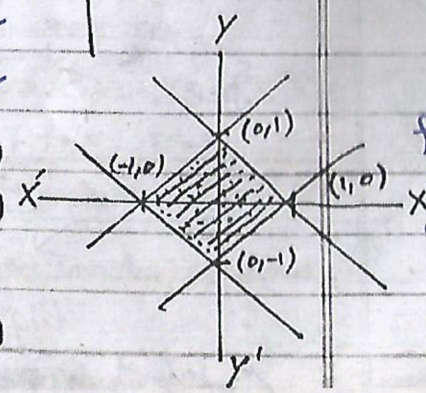
$\pm x \pm y = 1$

$x + y = 1 \quad (1,0)$

$x - y = 1 \quad (0,-1)$

$-x + y = 1 \quad (0,1)$

$-x - y = 1 \quad (-1,0)$



functions ays < 1 so bounday not included.

open sphere in Chebyshev metric on $\mathbb{R}^2$

about max

on $\mathbb{R}^2$ it is defined and given by

If $P_1(x_1, y_1)$, $P_2(x_2, y_2) \in \mathbb{R}^2$ $d: \mathbb{R}^2\times\mathbb{R}^2 \to \mathbb{R}$

$d(P_1, P_2) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$

let $P_1 = P = (x, y)$ $\quad P_2 = P_0 = (0,0)$

$S_r(P_0) = \{P \in \mathbb{R}^2 : d(P, P_0) < r\}$

$= \{P \in \mathbb{R}^2 \; \max\{|x-0|, |y-0|\} < 1\}$ $\quad r = 1$

$= \{P \in \mathbb{R}^2 \; \max\{|x|, |y|\} \leq 1\}$

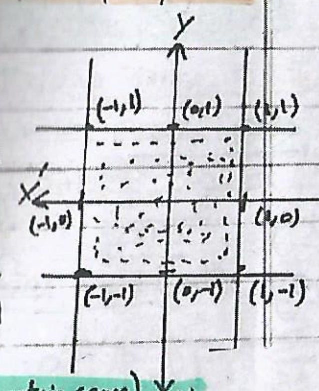
$S(P_r) = \{(x,y) \in \mathbb{R}^2 \; \max\{\pm x, \pm y\} \leq 1\}$

$(1,0) = 1$ $\quad (0,-1) \leq 1$

$(0,1) = 1$ $\quad (-1,0) < 1$

$(-1,1) = 1$ $\quad (1,1) \leq 1$

$(1,-1) = 1$ $\quad (-1,-1) < 1$



open sphere in (discrete metric space)

$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

$S_r(x_0) = \{x \in X : d(x, x_0) < r\}$ $\quad x_0 = y$

MCQs

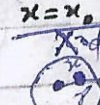
case I $r \leq 1$

$S_r(x_0) = \{x \in X : d(x, x_0) < 1\}$

$= \{x \in X ; d(x, x_0) = 0\}$ $\quad x = x_0$

distance $= 0$

$= \{x_0\}$ only centre

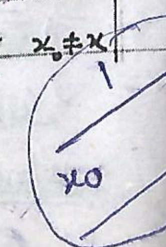


case II $r > 1$

let $x \in X$ be an arbitrary point $x_0 \neq x$

(centre of open sphere)

$\Rightarrow d(x_0, x) = 1$

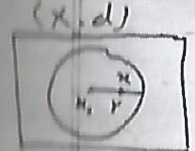


$x \in S_r(x_0)$ $(r>1)$, $U\{x\} \subseteq S_r(x_0)$

$S_r(x_0) \subseteq X$ → sphere ⊆ X

$U\{x\} \subseteq S_r(x_0) \subseteq X$ ∃ $x \in A$, $\{x\} \subseteq A$

$\{x\} U\{x\} U\{x\} \dots = X$



$X \subseteq S_r(x_0) \subseteq X$

$\boxed{A \subseteq B \subseteq A \;,\; A = B}$

$S_r(x_0) = X$ MCQ

open sphere in complete metric space when $r>1$

Open sphere in $C[a,b]$ continuous function

$d(f,g) = \max|f(x)-g(x)|$, $\forall x \in [a,b]$ MCQ

$g(x) = f_0(x)$ centre $r = 1$

define

$S_r(f_0) = \{f \in C[a,b] : d(f,f_0) < 1\}$ less than one

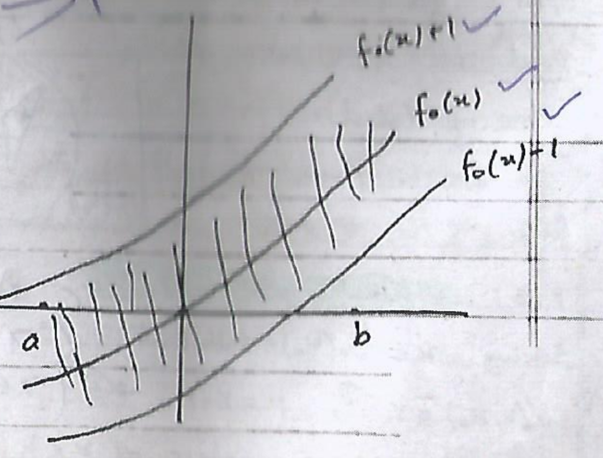
$= \{f \in C[a,b] : \max|f(x)-f_0(x)| < 1\}$

$|f(x)-f_0(x)| < 1$

$-1 < f(x)-f_0(x) < 1$ open sphere

add $f_0(x)$ ⇒ $f_0(x)-1 < f(x) < f_0(x)+1$

MCQ (i) (2) Redious



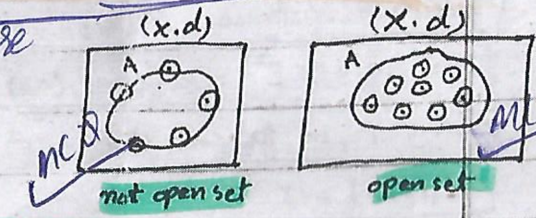
Lec 33

MTH 405 lectur 33

Open set in metric space

Let $(X.d)$ be a metric space $A \neq \phi$ such That $A \subseteq X$, if $\forall x \in A$ and There exist $r>0$ such That $S_r(x) \subseteq A$, Then 'A' is open set. MCQ

centre



1 Usual metric space

$d(x_1,x_2) = |x_1 - x_2|$

$S_r(x_0) = \{x \in \mathbb{R} : d(x,x_0) < r\}$ MCQ

$= \{x \in \mathbb{R} : |x - x_0| < r\}$ MCQ $x_1 = x$, $x_2 = x_0$

$= \{x \in \mathbb{R} : -r < x - x_0 < r\}$

Add x_0, $x_0 - r < x < x_0 + r$

put $x_0 = 0.999$, $r = 0.0001 > 0$

$S_r(x_0) = \{x \in \mathbb{R} : x_0 - r < x < x_0 + r$

$S_{0.0001}(0.999) = \{x \in \mathbb{R}: 0.999 - 0.0001 < x < 0.999 + 0.0001\}$

$= \{x \in \mathbb{R} : 0.9989 < x < 0.9991\}$

$]0.9989, 0.9991[\subseteq]0,1[$ which hold $\forall x \in \mathbb{R}$

$x \in]0,1[$ as well $]0,1[$ is open set in $(\mathbb{R}.d)$

② Open set in $\mathbb{R}^2$ (under usual metric space)

Prove That The set $U = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$ is an open set in $(d, \mathbb{R}^2)$

Proof

$U = \{(x,y) \in R^2 : x^2 + y^2 < 1\}$

$= \{(x,y) \in R^2 : (x-0)^2 + (y-0)^2 < 1\}$ centre MCQ

$x < y$

$\sqrt{x} < \sqrt{y}$

$= \{(x,y) \in R^2 : \sqrt{(x-0)^2 + (y-0)^2} < 1\}$ MCQ

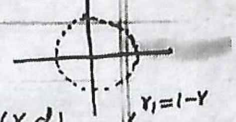
$d(P_1, P_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ is a

Euclidean distance $P_1(x_1, y_1)$, $P_2(x_2, y_2)$

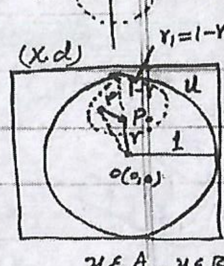
$U = \{(x,y) \in R^2 : d[(x,y),(0,0)] < 1\}$

Taking P_0 in 'U' such that

$d(P_0, 0) = r$



$r_1 = 1 - r$



Now prove That $S_{r_1}(P_0) \subseteq U$ — 1

let $P \in S_{r_1}(P_0)$

an applying triangular inequality

for $P, P_0, O(0,0)$.

$x \in A$, $x \in B$

$A \subseteq B$

$d(P,0) < d(P,P_0) + d(P_0,0)$

$d(P,0) < r_1 + r \Rightarrow 1 - r + r = 1$

$d(P,0) < 1$, $P \in U$ — ②

from ① and ② $\boxed{S_{r_1}(P_0) \subseteq U}$ which holds

arbitrary for all points of 'U'

So 'U' is a open set in (R^2, d) MCQ

Clopen Theorem

In a metric space $(X.d)$, The empty set and The full space 'X' are open sets.

MCQ

Proof Case 1 $\phi = \{x, x \neq x_0\}$

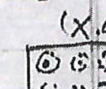
Empty set.

let "ϕ" is not an empty set in $(X.d)$. There exist a point say $x \in \phi$ and $r > 0$; $S_r(x) \not\subseteq \phi$, but bel

spheres not

ϕ is empty and does not have any point 'x'

a contradiction occurs "ϕ" is an empty set and is an open set in $(X.d)$ MCQ.



Case II 'X' is open set

$\forall x \in X$ and There exist $r > 0$ such That

$S_r(x) \subseteq X$, So 'X' is an open set. MCQ

Imp

Deductions

characteristic

clopen

if 'A' is open set in $(X.d)$ Then A' is closed.

if 'ϕ' is open set in $(X.d)$ Then ϕ' is closed. $\phi' = X$

if 'X' is open set in $(X.d)$ Then X' is closed $X' = \phi$

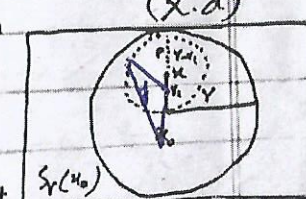
Open sphere forming metric space

each open sphere is any metric space is an open set. However, its converse is not true

$[0,1] = X$

both open & close

Proof let $S_r(x_0)$ be an open sphere in $(X.d)$.



we are to prove

$\forall x \in S_r(x_0)$, and There exist $r_1 > 0$ such that $S_{r_1}(x) \subseteq S_r(x_0)$

$A \subseteq B$

$x \in A$

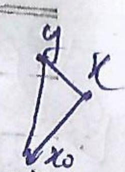
$x \in B \Rightarrow A \subseteq B$

taking $x \in S_r(x_0)$ such That

$d(x, x_0) = r_1$, $p = r - r_1$ → ②

Now draw an open sphere at $x \in X$ of size

$p = r - r_1$ $S_p(x)$ such That if $y \in S_p(x)$ Then



$d(y,x) < p$ — ① — [illegible] proof [illegible]

Now applying triangular inequality for, $x, y, x_0 \in X$

$d(y,x_0) < d(y,x) + d(x,x_0)$

$< p + r_1$, $= r - r_1 + r_1 = r$

$d(y,x_0) < r$, so, $y \in S_r(x_0)$

$y \in S_p(x)$ and $y \in S_r(x_0)$

So, $S_p(x) \subseteq S_r(x_0)$ $\quad x \in A, x \in B, A \subseteq B$

which holds $\forall$ $x \in S_r(x_0)$ is an open set in $(X.d)$

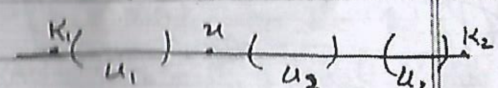
Conversly

Let U_1, U_2, U_3 are open sets in $\mathbb{R}$

$U_1 \cup U_2 \cup U_3$ is also open set in $\mathbb{R}$

But it is not open interval MCQ

{ if $(a,b) \in \mathbb{R}$ such That $a < x < b \Rightarrow x \in (a,b)$



Here although $k_1 < x < k_2$

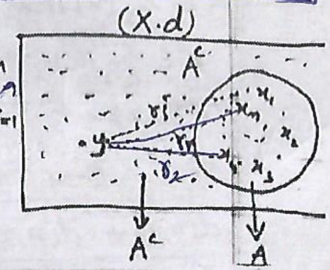
But $x \notin U_1 \cup U_2 \cup U_3$

$U_1 \cup U_2 \cup U_3$ is not an open sphere in $\mathbb{R}$

Lec 34 Imp lecture & MCQs

Nom. lecture 34 MTH-405

Theorem ; The compliment of finite set in a metric space is open set ;



Proof Let $A = \{x_1, x_2, \dots x_n\} = \{x_i\}_{i=1}^{n}$ be a finite subset of "X"

$A \subseteq X$

Let $y \in A^c$, $y \notin A$

$y \neq x_i$ $(x_1, x_2, \dots x_n)$ $i = 1, 2, \dots n$

$d(x_i, y) \neq 0$

Let $d(x_i, y) = r_i$ $\forall$ $i = 1, 2, 3, \dots n$ and also

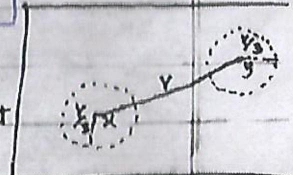
Let $r_i = \min\{r_1, r_2 \cdots r_n\} = \{r_i\}_{i=1}^{n}$

Let an open sphere of radius "r" with center at "y" does not contain an point of set "A" so,

$S_r(y) \cap A = \phi$ MCQ and $A^c \cap A = \phi$ so, $S_r(y) \subseteq A^c$ MCQ

it holds for all MCQ $y \in A^c$, A^c is an open set in $(X.d)$ MCQ

Housdorff property every two distinct points in a metric space are separated by pair of disjoint open sphere containing the respective points.



Proof let $(X.d)$ be an arbitrary metric space containing Two distinct Points. x and y such That $d(x,y) = r$, Now taking Two open sphere at x and y of radius : $\frac{r}{3}$ MCQ

we are to prove That $S_{r/3}(x) \cap S_{r/3}(y) = \phi$

Let if $S_{r/3}(x) \cap S_{r/3}(y) \neq \phi$ There exist z point

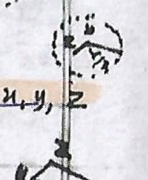
$z \in S_{r/3}(x) \cap S_{r/3}(y)$

$z \in S_{r/3}(x)$, $z \in S_{r/3}(y)$

$d(x,z) < r/3$, $d(y,z) < r/3$

Now appling Triangular inequality on x, y, z

$d(x,y) \leq d(x,z) + d(y,z)$ mcQ



$d(x,y) < \frac{r}{3} + \frac{r}{3}$

$r < \frac{r+r}{3}$, $r < \frac{2r}{3}$, $1 < \frac{2}{3}$ is false

so, our soppositian $S_{r/3}(x) \cap S_{r/3}(y) \neq \phi$ is wrong mcQ

Hence $\boxed{S_{r/3}(x) \cap S_{r/3}(y) = \phi \text{ . } x, y \text{ distinct point}}$ → intersection is empty

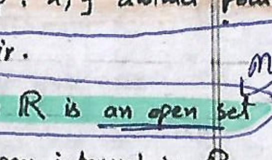
open sphere are dijoint pair.

Theorem Any open interval on $\mathbb{R}$ is an open set mcQ imp

Proof Let $(-1,1) \subseteq \mathbb{R}$ be an open interval in $\mathbb{R}$.

$x \in (-1,1)$ be its an arbitrary point

Taking $r = \min\{|x+1|, |1-x|\}$



mcQ $S_r(x) \subseteq (-1,1)$, so, $(-1,1)$ is an open set

Theorem every Non-empty subset in a discrete metric space is open set. imp

$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

Proof Let $A \neq \phi$ bet a subset of $\mathbb{R}$ (x,d) a discrete metric space

Boundry open spare metric space Centre open spare

for $r \geq 1$, $S_r(x) = A = X$ $\forall x \in X = A$ mcQ

for $r \leq 1$ $S_r(x) \subseteq \{x\}$ $\forall x \in (A = X)$ mcQ

from $r \geq 1$ and $r \leq 1$ is a open set in metric space.

distance zero Spare

boundary include open sphere

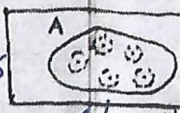
Land Mark Theorem for open sets (Arbitrary Union)

mcQ Arbitrary union of open set in a metric space is open set.

2021

Proof open sphere, $S_r(x) = \{y \in X : d(x,y) < r\}$

open set 'A' is open in $(X.d)$ if all of its centre of some open sphere



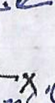
mcQ Arbitrary Union many open spheres are called open set.

finite — start point — countable — infinite — Uncountable

$A_1 \cup A_2 \cup A_3 \cdots A_n$ mcQ $A_1 \cup A_2 \cup A_3 \cdots = \bigcup_{i=1}^{\infty} A_i$ mcQ $\bigcup_{\alpha \in I} A_\alpha$

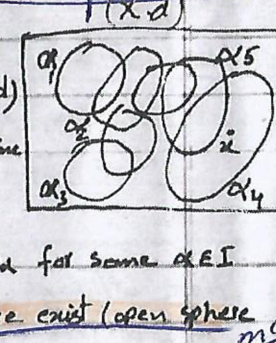
e.g uncountable $A_x = \{(x,y) : y \in \mathbb{R}\}$

$x \in \mathbb{R}$ $A_{x \in \mathbb{R}} = \{(x,y) : y \in \mathbb{R}\}$



Union of $(\bigcup A_{x \in r}) = \cup\{(x,y) : y \in \mathbb{R}\} = \mathbb{R} \times \mathbb{R} = XY$-Plane $(X.d)$

Theorem Let $\{A_\alpha\}_{\alpha \in I}$ be the arbitrary collection in metric space $(X.d)$ mcQ



1. Then if $\bigcup_{\alpha \in I} A_\alpha = \phi$ Then by define "ϕ" is open set in $(X.d)$ mcQ

2. If $\bigcup_{\alpha \in I} A_\alpha \neq \phi$, There exist $x \in A_\alpha$ for some $\alpha \in I$

A_α is open set in $(X.d)$, There exist (open sphere $r > 0$ at 'x') such That $S_r(x) \subseteq A_\alpha$ ~~for some $\alpha \in I$~~ mcQ

$\boxed{S_r(x) \subseteq A_\alpha \subseteq \bigcup_{\alpha \in I} A_\alpha}$ as x is arbitrary

$\boxed{\bigcup_{\alpha \in I} A_\alpha \text{ is open set in } (X.d)}$ mcQ

Land Mark Theorem for open sets (Finite intersection)

Finite intersection of open sets in a metric space is open set

Proof infinite $\quad -1 \quad -\frac{1}{2} \quad -\frac{1}{3} \quad \{0\} \quad \frac{1}{3} \quad \frac{1}{2} \quad 1$

$\bigcap_{n=1}^{\infty}]-\frac{1}{n},\frac{1}{n}[=]-1,1[\cap]-\frac{1}{2},\frac{1}{2}[\cap]-\frac{1}{3},\frac{1}{3}[\cap \ldots \infty$

if $\lim_{n\to\infty} \bigcap_{n=\infty}^{\infty}]-\frac{1}{n},\frac{1}{n}[= \{0\}$ $\boxed{\text{It is singleton set is a close set}}$ mcq

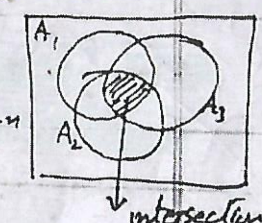
finite

let $\{A_1, A_2, \ldots A_n\} = \{A_\alpha\}_{\alpha=1}^{n}$ be the finite collection of open set mcq) their intersection is open set)

we are to prove $A_1 \cap A_2 \cap \cdots A_n = \bigcap_{\alpha=1}^{n} A_\alpha$ is open mcq

Case I if $\bigcap_{\alpha=1}^{n} A_\alpha = \phi, X$, which is open sets (empty set)

Case II



let $\bigcap_{\alpha=1}^{n} A_\alpha \neq \phi$ Then

$x \in \bigcap_{\alpha=1}^{n} A_\alpha$, so, $x \in A_\alpha$, $\alpha = 1,2,\ldots n$

$x \in A_1$, $x \in A_2 \ldots x \in A_\alpha$

A_α is open set $\forall \alpha = 1,2,3,\ldots n$

There exist $r_1, r_2 \ldots r_n > 0$ such That

$S_{r_1}(x) \subseteq A_1$, $S_{r_2}(x) \subseteq A_2 \ldots S_{r_n}(x) \subseteq A_n$

let $r = \{\min\{r_1, r_2, \ldots r_n\}$

$S_r(x) \subseteq \{A_1, A_2, \ldots A_n\}$

$S_r(x) \subseteq A_1 \cap A_2 \cap \ldots \cap A_n$

$S_r(x) \subseteq \bigcap_{\alpha=1}^{n} A_\alpha$, where x is arbitrary point mcq

So, $\bigcap_{\alpha=1}^{n} A_\alpha$ finite intersection is open set in metric space

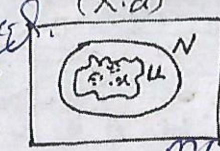
Lec 35

MTH 405 Lectur 35

Neighbourhood of a point mcq

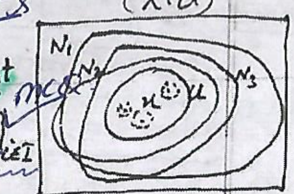
Neighbourhood means subset: A subset $(X.d)$ 'N' is said to be The Neighbourhood of a point 'x' mcq If There exist an open set 'U' containing such That $x \in U \subseteq N$ stand for Nghd $(x) \to N_x$ mcq

imp mcq



Neighbourhood system of point mcq

If $x \in U \subseteq N_1, N_2, N_3, \ldots = \{N_i\}_{i\in I}$ is said to be Neighbourhood system of point 'x'



e.g on $\mathbb{R}$ $\quad [\,-1 \;(\; -0.5 \;\; 0 \;\; 0.5 \;)\; 1\,]$ $\quad 0 \in (-0.5, 0.5) \subseteq [-1,1]$ mcq

So, $[-1,1]$ is Nghd of 0

$0 \in (-0.5, 0.5) \subseteq [-1,1), (-1,1), (-1,1]$ $\quad [\,-1 \; (-1 \; (-1 \; (-0.5 \;\; 0 \;\; 0.5) \; 1) \; 1)\; 1\,]$

are Nghd system of '0' mcq $\quad n_1 \quad n_2$

eg In discrete metric space every subset containg 'x' is Nghd of 'X' (X, d_0), we have $A \subseteq X$ such That $x \in A$, means $\boxed{x \in A \subseteq X \quad (X_x) \text{ Nghd of 'x'}}$ mcq

both points equal

In (x, d_0), every singleton set is open $\boxed{x \in \{x\} \subseteq A}$ mcq

A is Nghd of 'x' $\boxed{\forall x \in A, \text{ A is Nghd of all its points.}}$

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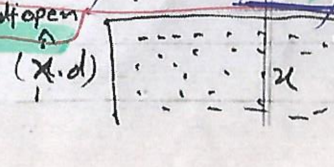
Theorem the metric space itself is a Nghd mcq of each of its point. $\quad$ Auto open

$x \in X \subseteq X$ mcq



Proof ~~In a metric space $(X.d)$, let 'x' be an arbitrary~~ Point, Then $x \in X \subseteq X$, X is open set and X is also Nghd of 'x' MCQ

Theorem A subset of metric space is open if and only if it is the Neighbourhood of each of its point. MCQ

Proof let 'A' is subset of 'X'

Given i/p: A is an open set MCQ

To prove Q: 'A' is Nghd of each of its point

$x \in U_1 \subseteq U$

U_1 is open set

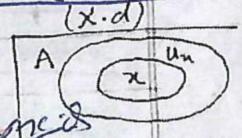
U is Nghd of x

'A' is open set in X ($\forall x \in A$ — I)

$A \subseteq A$ every set is subset of itself — II

So, $x \in A \subseteq A$, 'x' is arbitrary point of 'A' so 'A' is Nghd of all its point. MCQ

(X.d)



Conversly. Given Q: A is Nghd of each its point. To prove: 'A' is an open set MCQ

$x \in B$, $\{x\} \subseteq B$ MCQ

$x \in U_x \subseteq A$, $\{x\} \subseteq U_x \subseteq A$ MCQ

A

Taking Union for all such 'x'

$\bigcup_{x \in A} \{x\} \subseteq \bigcup_{x \in A} U_x \subseteq A$; $\bigcup_{x \in A} U_x$ is union of open set

$A \subseteq \bigcup_{x \in A} U_x \subseteq A$ MCQ

$A \subseteq B \subseteq A$, $A = B$

$A = \bigcup_{x \in A} U_x$ MCQ

Hence A is a open set, being arbitrary union of open set MCQ

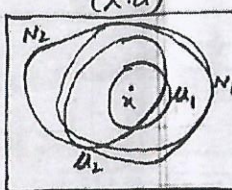
imp

Lec 36

MTH 405 Lecture 39 Nomi

Theorem Finite intersection of the Neighbourhood of a point, is also its Neighbourhood MCQ

(X.d)



$x \in U \subseteq N$

Proof let $N_1, N_2, \ldots N_r = \{N_j\}_{j=1}^{r}$ be the Nghd system point 'x' in $(X.d)$ there exist $U_1, U_2, \ldots U_r = \{U_j\}_{j=1}^{r}$ the collection of open sets

$x \in U_1 \subseteq N_1$, $x \in U_2 \subseteq N_2$, ... $x \in U_j \subseteq N_j$, $j = 1, 2, 3, \ldots r$

$x \in U_1 \cap U_2 \cap \ldots \cap U_r \subseteq N_1 \cap N_2 \cap \ldots \cap N_r$

$x \in \bigcap_{j=1}^{r} U_j \subseteq \bigcap_{j=1}^{r} N_j$ MCQ

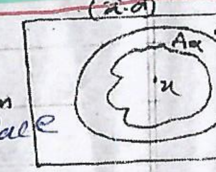
$x \in A$, $x \in B$

$x \in A \cap B$

$\bigcap_{j=1} U_j$ = finit intersection of open set so, U_j = is an open set and $\bigcap_{j=1}^{r} N_j$ is Nghd of 'x' MCQ

Theorem Arbitrary union Neighbourhood of a point is also its Neighbourhood. MCQ

(X.d)



20 Marks

Proof let $\{N_\alpha\}_{\alpha \in I}$ be the arbitrary collection of Nghd of point 'x' in $(X.d)$ metric space

each N_α is Nghd of 'x' where $\forall \alpha \in I$ MCQ

There exist an open set say A_α such that $x \in A_\alpha \subseteq N_\alpha$

$\forall \alpha \in I$ So, $x \in \bigcup_{\alpha \in I} A_\alpha \subseteq \bigcup_{\alpha \in I} N_\alpha$ MCQ

$x \in A_1$

$x \in A_1 \cap A_2$

$\bigcup_{\alpha \in I} A_\alpha$ being arbitrary union of open sets

So, $\bigcup_{\alpha \in I} N_\alpha$ is Nghd of 'x' MCQ

22/7

Lec 31

set of integers has no limit point

MTH 405 lecture 31 Nomi

Limit point of set In a metric space

(X, d), let $A \subseteq X$ and any point $x \in X$ is said to be the limit point of 'A' if every open sphere at 'x' contain at least one point of 'A' other than 'x' mcqs

$(X.d)$

A

$S_r(x) \cap A - \{x\} \neq \phi$ where $r > 0$ limit point

e.g. 0,1 are the limit points of $]0,1[$ — $\mathbb{R}$

$]0,1[$ because any open interval at $0,1 \notin]0,1[$ "0" and "1" contain at least one point of $]0,1[$ mcq

$S_r(0) \cap]0,1[- \{0\} \neq \phi$, $S_r(1) \cap]0,1[- \{1\} \neq \phi$

$(0-\varepsilon, 0+\varepsilon) \cap]0,1[- \{0\} \neq \phi$, $(1-\varepsilon, 1+\varepsilon) \cap]0,1[- \{1\} \neq \phi$

$(x-\varepsilon, x+\varepsilon) \cap]0,1[- \{x\} \neq \phi \quad \forall \varepsilon > 0, \forall x \in]0,1[$

e.g. limit points of the set of integers $\mathbb{Z} \subseteq \mathbb{R}$ mcqs

$\mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \dots\}$ — $-3\ -2\ -1\ 0\ 1\ 2\ 3$ $\mathbb{R}$

$\forall x \in \mathbb{R}$ There exist $r > 0$ such that $S_r(x) \cap \mathbb{Z} - \{x\} = \phi$ mcq

Z has no limit point because all integer's number's has some distance between each other

e.g; Limit point of $\{\frac{1}{n}\}_{n \in N}$; $\lim_{n \to \infty} [\frac{1}{\infty}] = \{0\}$

$\{\frac{1}{n}\}_{n \in N} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ — $0\ \frac{1}{5}\ \frac{1}{4}\ \frac{1}{3}\ \frac{1}{2}\ 1$ $\mathbb{R}$

so, $\frac{1}{2} \notin S_r(1), r < \frac{1}{2}$

By induction $\frac{1}{n+1} \notin S_{\frac{1}{n+1}}(0)$ if $n \in N$ Then $n+1 \in N$

0 is a limit point of $\{\frac{1}{n}\}_{n \in N}$ mcqs

Closed set / Derived set

A subset 'A' of metric space (X, d) $A \subseteq X$ is closed if it contains every limit point of itself. The set of all limit points of A is called The derived set of A and denoted by A', A^c, A^d

e.g $0,1 \notin]0,1[$ $A' = A^c = A^d =]0,1[^d = [0,1]$ is closed set

$\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, $\mathbb{Z}^d = \{\ \} = \phi$ mcq → drived

$0 \notin \{\frac{1}{n}\}_{n \in N} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, $\{\frac{1}{n}\}^d_{n \in N} = \{0\}$ it is drived

empty

Derived set of Rational (Q) and Irrational (Q')

Rational $(Q) = \{x = \frac{p}{q}, q \neq 0, p, q \in Z \subset R\}$ mcq

Irrational $(Q') = \{x \neq \frac{p}{q}, q \neq 0, p, q \in Z \subset R\}$ mcq

e.g $Q: \frac{1}{2}, 7, \frac{7}{2}$, e.g $Q' = \pi, \sqrt{29}, e$ mcq

Q is infinite on $\mathbb{R}$ line, Q' is infinite on $\mathbb{R}$ Line

either $x \in Q$ or $x \in Q'$ mcq

$S_r(x) \cap Q - \{x\} \neq \phi$ when $x \in Q$ mcq

$S_r(x) \cap Q' - \{x\} \neq \phi$ when $x \in Q'$ mcq

So, derived set of $= Q^d = (Q^c)^d = \mathbb{R}$ mcq

irrational Rationals drived set of a

read

Lec 38

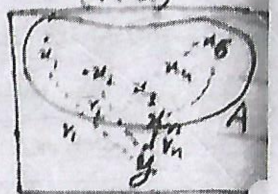
tracing scraper

MTH 405 — Lecture-38 — Notes

Limit point

Theorem Finite subset of a metric space has no limit point.

(X, d)



Proof let $A \neq \phi$, $A \subseteq X$ be a subset of 'X' such that

$A = \{x_i\}_{i=1}^{n} = \{x_1, x_2, \dots x_n\}$

let y is limit point of A, $S_r(y) \cap A - \{x\} = \{y\}$ MCQ

Let $d(x_1, y) = r_1$, $d(x_2, y) = r_2$ .... $d(x_n, y) = r_n$

$r = \min\{r_1, r_2, \dots r_n\} = \min\{r_i\}_{i=1}^{n}$

$S_r(y) = \{y \in X \ d(y, x_n) < r_n\}$ → open sphere draw

So, $S_r(y)$ has no point of A, other than 'y'

A has no limit point in 'X'

Theorem If 'x' is a limit point of a subset 'A' of a metric space (X, d) Then every open sphere centered at 'x' contains infinite number of points of 'A' different from 'x'. MCQ

Theorem The derived set of a non-empty set of discrete metric space is empty. MCQ

Proof $d_0 : X_0 \times X_0 \to \mathbb{R}$; $d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ discrete metric space

$A \subseteq X$, $A^d = \{x \in X ; x \text{ is limit point of } A\}$

$= \{x \in X ; S_r(x) \cap A - \{x\} \neq \phi\}$ MCQ

Let, $A \neq \phi$, $A \subseteq X$ and $A^d \neq \phi$ There exist $x : x \in A^d$, x is a limit point of A. MCQ

$\exists\ r > 0$ such that $S_r(x)$ should at least contain one point of 'A' other than 'x'

if $r < 1$ Then, $S_r(x) = \{y \in X_0 : d(y, x) < r\}$

$= \{y \in X_0 : d(y, x) = 0\}$ MCQ

$= \{x\}$ $\quad y = x$

equal to zero

$x \notin A^d$, 'x' is not a limit point of 'A'

MCQ $A \cap S_r(x) - \{x\} = \phi$, Hence, $A^d = \phi$.

Set Theoretic operations in Derived sets:

For any non-empty subset 'A' and 'B' of a metric space (X, d)

MCQ (i) if $A \subseteq B$ then $A^d \subseteq B^d$ (ii) $(A \cap B)^d \subseteq A^d \cap B^d$

MCQ (iii) $(A \cup B)^d = A^d \cup B^d$

Proof (i) $A^d = \{x \in X \mid A \cap S_r(x) - \{x\} \neq \phi, \forall r > 0\}$

give $A \subseteq B$ and to prove $A^d \subseteq B^d$

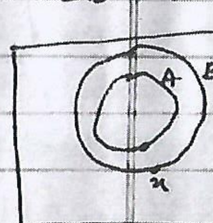
Let $x \in A^d \mid A \cap S_r(x) - \{x\} \neq \phi \quad \forall\ r > 0$ — I

$B \cap S_r(x) - \{x\} \neq \phi \quad \because A \subseteq B$

So, x is a limit point of 'B' as well $x \in B^d$ — II

So, $A^d \subseteq B^d$ from I and II

(ii) $(A \cap B)^d \subseteq A^d \cap B^d$



$A \cap B \subseteq A$ $\wedge$, $A \cap B \subseteq B$

$(A \cap B)^d \subseteq A^d$ $\wedge$, $(A \cap B)^d \subseteq B^d$

$(A \cap B)^d \subseteq A^d \cap B^d$

Counter example $A^d \cap B^d =]1,2[$

$(A \cap B)^d =]1,3[$

$]1,2[^d = [1,2]$, $]2,3[^d = [2,3]$

$]1,2[^d \cap]2,3[^d = [1,2] \cap [2,3] = \{2\}$ MCQ

$\boxed{]1,2[\cap]2,3[= \phi}$ MCQ

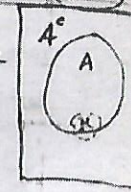
$\boxed{(]1,2[\cap]2,3[)^d = \phi^d = \phi}$ MCQ

$\boxed{Q \cap Q^c}$ ϕ MCQ

$\boxed{R \cap R = R}$ $\phi \neq R$ MCQ $\phi \neq \{2\}$

Lec 39

MTH-405 Lecture 39 Name (X.d)



Closed set A subset 'A' of a metric space $(X.d)$, $A \subseteq X$ is closed set if it contains every limit point of itself MCQ

اسکے جتنے لمٹ پوائنٹس ہیں وہ سب اسکے اندر موجود ہوں

e.g 'o' is The limit point of $\{\frac{1}{n}\}_{n=1}^{\infty}$ but 'o' $\notin \{\frac{1}{n}\}_{n=1}^{\infty}$ MCQ

So, it is not closed set. MCQ

e.g. 'o' and '1' are The limit point of open Interval $]0,1[$

but $0,1 \notin]0,1[$ is not closed set. MCQ

e.g 'o' and '1' are The limit point of close interval $[0,1]$ and $0,1 \in [0,1]$ is closed set. MCQ

e.g $R = Q \cup Q^c$ MCQ, $Q^d = R = Q \cup Q^c$ MCQ, $(Q^c)^d = R = Q \cup Q^c$ M

if $x \in Q^c$, Then its limit point is Q and $x \notin Q$ MCQ

if $x \in Q$, Then its limit point is Q^c and $x \notin Q^c$ MCQ

So, Q and Q^c are not closed in R MCQ

Example close interval on R line are closed set

$[x_1, x_2]^c = (-\infty, x_1) \cup (x_2, \infty)$ MCQ $(-\infty, x_1)$ x $[x_1$ $x_2]$ (x_2, ∞) R

we porove $(-\infty, x_1) \cup (x_2, \infty)$ is open MCQ

Let $x \in (-\infty, x_1) \cup (x_2, \infty)$, $x \in (-\infty, x_1)$ or $x \in (x_2, \infty)$

if $x \in (-\infty, x_1)$ and taking $r = |x_1 - x| > 0$ such That

$S_r(x) = \{y \in R : d(x,y) < r\} =]x, y[\subseteq (-\infty, x_1)$

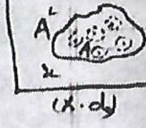
$S_r(x) \subseteq (-\infty, x_1)$, $S_r(x) \subseteq (-\infty, x_1) \cup (x_2, \infty)$

So, $(-\infty, x_1) \cup (x_2, \infty)$ is open set closed جس کا complement open

$[(-\infty, x_1) \cup (x_2, \infty)]^c = [x_1, x_2]$ is closed set. MCQ

Example every subset of discrete metric space is closed set. MCQ

$d_o(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases} \rightarrow$ discret metric space



let $\phi \neq A \subseteq (X, d_o)$, To prove 'A' is closed set and 'A^c' is open set MCQ

let $x \in A^c$, for $r = 1$, Taking open sphere $r = 1$ at $x \in A^c$, $S_r(x) = \{y \in X , d_o(x,y) < 1\}$

$= \{y \in X , d_o(x,y) = 0\}$

$= \{x\}$, $x = y$

MCQ $\{x\} \subseteq A^c$, $S_r(x) \subseteq A^c$ as 'x' is arbitrary point of A^c, so, A^c is open set, $(A^c)^c = A$ is closed set MCQ

lec 40

MTH-405 Lecture-40

Theorem The union of finite metric/number of closed sets is closed. MCQ

Proof Finite collection of closed sets $= \{A_1, A_2, \dots A_n\} = \{A_i\}_{i=1}^n$ MCQ

$\{A_1^c, A_2^c, \dots A_n^c\} = \{A_i^c\}_{i=1}^n$ is open set | Finite intersection of open set is open MCQ

$A_1^c \cap A_2^c, \dots \cap A_n^c\} = \bigcap_{i=1}^n A_i^c$ is open sets

MCQ $A^c \cap B^c = (A \cup B)^c$ (de morgan's Law)

$\{A_1^c \cap A_2^c \dots \cap A_n^c\} = \bigcap_{i=1}^n A_i^c = \{A_1 \cup A_2 \cup \dots \cup A_n\}^c = \left(\bigcup_{i=1}^n A_i\right)^c$

is open, Then $\left[\left(\bigcup_{i=1}^n A_i\right)^c\right]^c = \bigcup_{i=1}^n A_i$ is closed set MCQ

Infinitely

Let $A_n = [\frac{1}{n}, 1]$

$A_1 = [1,1]$, $A_2 = [\frac{1}{2}, 1]$, ...

as $n \to \infty$ Then $\frac{1}{n} = 0$, $A_n = (0,1]$ as $n \to \infty$

$A_1 \cup A_2 \cup \dots A_\infty = (0,1]$ is not closed sets MCQ

infinite

Theorem The union of arbitrary number of closed sets is closed. MCQ

Theorem The intersection of finite number of closed set is closed. MCQ

Closeness of a finite subset of 'A' metric space.

Theorem: A finite subset of a metric space is closed set. MCQ

Proof Let $\phi \neq A \subseteq X$ be a non-empty subset of X

p ; 'A' is finite q ; 'A' is closed

MCQ $p \to q \equiv \sim q \to \sim p$

$\sim q$; 'A' is not closed set MCQ

There exist $x \in X$; $x \in A^d$ but $x \notin A$ | $\sim p$ = 'A' is infinite, $\sim q$ = 'A' is not closed set

$x \in A^d$, 'x' is a limit point of 'A'

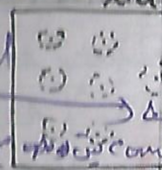
every open sphere at 'x' will contain infinite many points of 'A'

So, 'A' is infinit ($\sim p$) MCQ

So, $\sim q \to \sim p = p \to q$ MCQ

Theorem In a metric space a subset is closed set if and only if its compliment is open. MCQ

Proof Let (X,d) be a metric space. MC-①



(i) $\phi \subseteq X$, ϕ is finite, ϕ is closed

$\phi^c = X$ is open. MC-①

Finit سیٹ empty

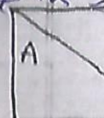
closed compact

(ii) $X \subseteq X$, X is closed set as $X^d = X$ MC-①

$X^c = \phi$ is open trivially. MC-①

finite

(iii) Let $A \subseteq X$ such that $A \neq \phi$, $A = X$



closed

drived

equal

Suppose 'A' is closed (Given)

A^c is open (to prove)

Let $x \in A^c$ Then $x \notin A$, so, A is closed set $A = A^d$ MC-①.

$x \notin A^d$; 'x' is not a limit point of 'A' Then

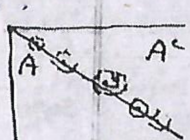
There exist $r > 0$ such that $A \cap S_r(x) = \phi$

$S_r(x) \subseteq A^c$; $\forall x \in A^c$, as 'x' is arbitrary point

A^c is open set. MCQ

Conversly Let A^c is open (Given) to prove A^c is closed

Let A is not closed, $A \neq A^d$ MCQ.



$y \in X$ such that $y \in A^d$ but $y \notin A$

if $y \in A^d$ Then 'y' is limit point

of 'A' Taking a open sphere at 'y' $\forall \ell > 0$

$S_\ell(y) \cap A \neq \phi$ — I

$y \notin A$ Then $y \in A^c$, A^c is open set, There exist

$p > 0$ such that $S_p(y) \subseteq A^c$, $S_p(y) \notin A$ Then

$S_p(y) \cap A = \phi$ — II So, our contradiction

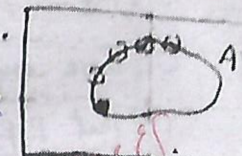
is wrong.

lec 41

MTH-405 lecture 41 Noml

Closure of a set in metric space.

It is collection of all those points which are arbitrary closed to "A"



MCQ

$\boxed{\text{Closure of } A = \bar{A} = A \cup A^d}$

Imp

$Q^d = R$, $\bar{Q} = Q \cup Q^d = Q \cup R = \mathbb{R}$ MCQ

$(Q^c)^d = R$, $\overline{Q^c} = Q^c \cup (Q^c)^d = Q^c \cup R = \mathbb{R}$ MCQ

Closure of Harmonic Sequence

$A = \{\frac{1}{n}\}_{n \in N} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$, $A^d = (\{\frac{1}{n}\}_{n \in N})^d = \{0\}$

$\bar{A} = A \cup A^d = \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \cup \{0\} = \{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\}$

Closure of an open set in R

$A =]0,1[$ open, $A^d = (0,1)$; $0,1 \notin]0,1[$, $0,1 \in [0,1]$ close

$\bar{A} = A \cup A^d = (0,1) \cup (0,1)^d = [0,1]$ MCQ

lec 42

lecture - 42

Closeness is a closure Theorem

$\boxed{\text{A subset 'F' of a metric space, is closed if and only if } F = \bar{F}}$ MCQ

Let p: F is a closed set, q: $F = \bar{F} = F \cup F^d$

if 'F' is closed, Then 'F' contain all of its limit point

$F^d \subseteq F$ — I, $F \cup F^d = \bar{F} \cup F$, $\bar{F} \subseteq F$ — II

[redacted] By Defination $\boxed{\bar{F} = F \cup F^d}$

$F \subseteq \bar{F}$ — III from II and III

$\bar{F} \subseteq F \subseteq \bar{F}$, $F = \bar{F}$

$A \subseteq A \cup B$

$A \subseteq B \subseteq A$

$A = B$

Conversly

Let $F = \bar{F}$ (given) to prove q; F is closed sets

$(\sim q \to \sim p)$, $\sim q$ = F is not closed set Then There exist a limit points say 'p' of F such That $p \notin F$ Then $p \notin \bar{F}$ because $F = \bar{F}$

$p \notin F \cup F^d$ $\because$ $\bar{F} = F \cup F^d$, $p \notin F^d$ so,

p is not a limit point of 'F'

contradiction is wrong so, F is closed

2024

Set Theoretic operation in closures.

Theorem: If 'A' and 'B' are two subset of a metric space (X, d), Then prove That

(i) $A \subseteq B$ Then $\bar{A} \subseteq \bar{B}$ (ii) $(\overline{A \cup B}) = \bar{A} \cup \bar{B}$

(iii) $(\overline{A \cap B}) = \bar{A} \cap \bar{B}$ (iv) $\bar{A}$ is largest closed super set of A

MCQ's

Proof (i) given $\boxed{A \subseteq B}$ Then $\boxed{A^d \subseteq B^d}$ taking a union $\boxed{A \cup A^d \subseteq B \cup B^d}$ Then $\boxed{\bar{A} \subseteq \bar{B}}$

(ii) L.H.S $= \overline{A \cup B}$

$= (A \cup B) \cup (A \cup B)^d$ $\quad \bar{A} = A \cup A^d$

$= (A \cup B) \cup (A^d \cup B^d)$

$= (A \cup A^d) \cup (B \cup B^d)$

$= \bar{A} \cup \bar{B}$

(iii) $(\overline{A \cap B}) = \bar{A} \cap \bar{B}$, $A \cap B \subseteq A$ and $A \cap B \subseteq B$

$\overline{A \cap B} \subseteq \bar{A}$ and $\overline{A \cap B} \subseteq \bar{B}$

taking intersection: $(\overline{A \cap B}) \cap (\overline{A \cap B}) \subset \bar{A} \cap \bar{B}$

$(\overline{A \cap B}) \subseteq \bar{A} \cap \bar{B}$

e.g Let $A = (1,2)$, $B = (2,3)$

$\bar{A} = [1,2]$ $\quad \bar{B} = [2,3]$

L.H.S $= (\overline{A \cap B}) = \overline{(1,2) \cap (2,3)} = \bar{\Phi} = \phi$ $\because F = \bar{F}$

R.H.S $= \bar{A} \cap \bar{B} = [1,2] \cap [2,3] = \{2\}$

So, $\phi \subseteq \{2\}$

$(\overline{A \cap B}) \subseteq \bar{A} \cap \bar{B}$

IV For any $\phi \neq A \subseteq X$, $\bar{A}$ is closed

$\bar{A} = A \cup A^d$, $A \subseteq \bar{A}$ — I

Let $\phi \neq B \subseteq X$, be arbitrary any other closed set containing 'A', $A \subseteq B$, $\bar{A} \subseteq \bar{B}$, $\bar{A} \subseteq B$ $\because \bar{B} = B$

from I and II $A \subseteq \bar{A} \subseteq B$

$\bar{A}$ is smallest closed super set of A

Let $\phi \neq B \subseteq X$ be any other arbitrary closed set contain in 'A' $B \subseteq A$ Then $\bar{B} \subseteq \bar{A}$, $B \subseteq \bar{A}$ — III

from I, III $B \subseteq A \subseteq \bar{A}$, $\bar{A}$ is largest closed super set of A

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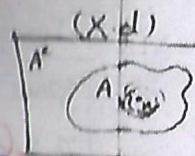
MTH-405 lecture-43 Name:

Interior, exterior and Boundary of a set in metric space (X.d)

Interior $x \in S_r(x) \subseteq A$

$Int(A) = \{x \in A,\ A \subseteq X,\ S_r(x) \subseteq A$ for some $r > 0\}$ MCQ



exterior $Ext(A) = \{x \in A^c,\ A^c \subseteq X,\ S_r(x) \subseteq A^c$ for some $r>0\}$ MCQ

$ext(A) = Int(A^c)$ MCQ

Boundary / **Frontier** $Fr(A) = \{x \in X;\ Int(A) \cap S_r(x) \neq \phi,\ ext(A) \cap S_r(x) \neq \phi,\ \forall r > 0\}$

e.g $A = [0,1]$ MCQ $(-\infty, 0)$ $[\ \ ]$ $(1, \infty)$

$Int(A) = (0,1)$, $ext(A) = (-\infty, 0) \cup (1, \infty)$ MCQ

$Fr(A) = \{0,1\}$ MCQ $A = \{(x,y) \in R^2,\ x^2+y^2=1\}$

e.g $Int\{1\} \neq \phi$ MCQ, $Int\ Z = \phi$ MCQ

e.g Discrete metric space (X.d)

Say $A \subseteq X$, $Int(A) = \bigcup_{x \in A}\{x\} = A$ MCQ

$ext(A) = \bigcup_{x \in A^c}\{x\} = A^c$ MCQ

$Fr(A) = \phi$

$Int(A),\ x^2+y^2<1$

$ext(A),\ x^2+y^2>1$

$Fr(A),\ x^2+y^2=1$

Theorem $\boxed{Int(A) \cup ext(A) \cup Fr(A) = X}$ MCQ

44

MTH-405 lecture 44

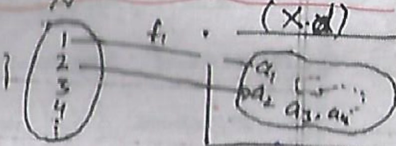
Sequences in a metric space.

Sequence infinit list of numbers.

" It is a function whose domain is set of Natural numbers and whose range is subset of a metric space

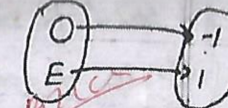
$\{a_n\}_{n=1}^{\infty}$, $(a_n) = \{a_1, a_2, \ldots\}$

$\{a_n\}$, $a_n \to n$th term



e.g $f(n) = (-1)^n \quad n \in N$

$f(1) = -1$, $f(2) = 1 \ldots\ldots$



e.g $a_n = \frac{1}{n}$, $a_n = n^2$ Infinit

Montonicity of sequences.

i) increasing sequences $a_{n+1} \geq a_n \quad \forall\ n \in N$ $\quad a_n = \frac{1}{n}$

ii) Decreasing sequences $a_{n+1} \leq a_n \quad \forall\ n \in N$ $\quad a_n = n^2$

Constant sequences.

It is not increasing and decreasing e.g $a_n = 3 \quad n \in N$

$a_n = a_\infty$ = No such term exist $\infty \notin N$

Bounded sequences let (X.d) be a metric space an $\{x_n\}$ be a sequence in X, Then it is said to be bounded if There exist $m \in \mathbb{R}$, $m < \infty$ such That $d(x_m, x_n) < M \quad \forall\ m,n \in N$

e.g $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$: $d(x,y) = |x-y|$, $\forall\ x,y \in \mathbb{R}$

Ex $\left\{x_n = \frac{1}{n}\right\}_{n \in N}^{\infty}$: $d(x_m, x_n) = \left|\frac{1}{m} - \frac{1}{n}\right| < 1$ MCQ

it is bounded sequence

unbo

Bounded

e.g $\{a_n = (-1)^n\}_{n=1}^{\infty}$, $n \in N$ is bounded sequences

because $-1 \le a_n \le 1 \quad \forall n \in N$

-1 is lower bound and 1 is upper bound

$d(x_m, x_n) \le 2$

$x_m \quad 0 \quad x_n$

e.g $\{a_n = n\}_{n=1}^{\infty}$ $n \in N$ it is unbounded as There not exist any $K \in \mathbb{R}$ such That $|a_n - a_m| < K$

Convergent Sequence in a metric space.

A sequence $\{x_n\}$ in a metric space $(X.d)$ is said to be convergence to limit 'l' if $\forall$ open spheres at 'l' There exist $n_0 \in N$ such That $x_n \in S_\varepsilon(l)$ $\forall n \ge n_0$

Bounded

(X.d)

Symbolically $\{x_n\} \to l$, $x_n \to l$ $\lim\limits_{n \to \infty} x_n = l$

e.g $\{a_n = \frac{1}{n}\}$ Show That $\lim\limits_{n \to \infty} \frac{1}{n} = 0$ $n \to \infty = 0$

$\forall \varepsilon > 0$ taking $|a_n - 0| < \varepsilon$, $|\frac{1}{n} - 0| < \varepsilon$, $\frac{1}{n} < \varepsilon \to \varepsilon$

$n > 0$, $|\frac{1}{n}| = \frac{1}{n}$, so, $\frac{1}{n} < \varepsilon$, $n > \frac{1}{\varepsilon}$

if m_0 is such a natural n_0

$\varepsilon \quad \frac{1}{\varepsilon} \quad \vec{m_0} \quad R$

Just $\ge \frac{1}{\varepsilon}$, Then $|a_n - 0| < \varepsilon$, $\forall n \ge m_0$ Just $\ge \frac{1}{\varepsilon}$

$a_n \to 0$

Convergent is bounded

e.g $\{\frac{1}{5^n}\} \to 0$ let the given $\varepsilon > 0$, taking $|a_n - 0| < \varepsilon$

$|\frac{1}{5^n} - 0| < \varepsilon$, $\frac{1}{5^n} < \varepsilon$, $5^n < \frac{1}{\varepsilon}$

$\log_5 5^n \le \log_5(\frac{1}{\varepsilon})$, $n \log_5 5 \le \log_5(\frac{1}{\varepsilon})$

$n < \log_5(\frac{1}{\varepsilon})$; if "$m_0$" is a natural n_0

Just $\ge \log_5(\frac{1}{\varepsilon})$ Then $\{\frac{1}{5^n}\} \to 0$

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infinite sequence

Divergent Sequence

A sequence $\{x_n\}$ in a metric space $(X.d)$. However large $L > 0$, There exist $n_0 \in N$ such That $n \ge n_0$. $\forall n \in N$, $a_n \notin S_L(l)$

e.g $\{a_n = n\}_{n=1}^{\infty}$ $n \in N$ $a_n \to \infty$

e.g $\{a_n = 5^n\}_{n=1}^{\infty}$ $n \in N$ $a_n \to \infty$

Uniqueness of a limit of a sequence in a metrice space.

Theorem limit of a convergent sequence in a metric space is unique.

Proof Let $\{x_n\} \to x$ and $\{x_n\} \to y$ in a metric space $(X.d)$ such That $x \ne y \Rightarrow d(x,y) \ne 0$

For any $\varepsilon > 0$ There exist $x \in N$ such That

$\frac{d(x,y)}{n} < \varepsilon$ $\quad \varepsilon > 0$

$\frac{d(x,y)}{n} = 0 < \varepsilon$ $\quad x \quad y$ $\quad t < \varepsilon$, $t = 0$

it is not possible so, $x = y$

Theorem Every convergent sequence in a metric space is bounded but converse may not true.

Proof in $(X.d)$ let $\{x_n\}$ be a ~~metric space~~ a convergent sequence such That $x_n \to x \in N$

$\forall \varepsilon > 0$, There exist $n_0 \in N$ such That $\forall n \ge n_0$

$d(x_n, x_0) < \varepsilon$ —(1)

Let $K = \max\{d(x_1, x) + \varepsilon, d(x_2, x) + \varepsilon, \ldots d(x_{n-1}, x)$

$d(x_n, x) \leq k$, $\forall n \in N$, $\{x_n\}_{n=1}^{\infty} \subseteq S_k(x)$

$d(x_m, x_n) \leq k$, $\forall m, n \in N$

So, $\{x_n\}_{n=1}^{\infty}$ is bounded sequence

Conversely in (R, d)

$\{x_n\} = \{(-1)^n\}_{n=1}^{\infty}$ is a bounded sequence

Here $|a_n| = 1$ $\forall n \in N$

$a_n \to 1$ if n is even, $a_n \to -1$ if n is odd

limit of the sequence is not unique and $\{x_n\}$ is not convergence

convergence $\Rightarrow$ bounded MCQ

Bounded $\not\Rightarrow$ convergence. MCQ

Theorem: In a metric space (X, d), if $\{x_n\} \to x$, $\{y_n\} \to y$ Then $d(x_n, y_n) \to d(x, y)$

Proof: $\{x_n\} \to x$ $\forall \varepsilon > 0$, There exist $n_1 \in N$ such That $\forall n \geq n_1$, $d(x_n, x) < \varepsilon$ — I

$\{y_n\} \to y$ $\forall \varepsilon > 0$, There exis $n_2 \in N$ such That $\forall n \geq n_2$ $d(y_n, y) < \varepsilon$ — ②

Let $n_0 = \max\{n_1, n_2\}$

from ① and ② $\left.\begin{array}{l} d(x_n, x) < \frac{\varepsilon}{2} \\ d(y_n, y) < \frac{\varepsilon}{2} \end{array}\right\} \forall n \geq n_0$

$$|d(x_n, y_n) - d(x, y)| \leq |d(x_n, x) + d(x, y) + d(y, y_n) - d(x, y)|$$

$$\leq |d(x_n, x) + d(y_n, y)|$$

$$\leq d(x_n, x) + d(y_n, y)$$

$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \frac{2\varepsilon}{2} = \varepsilon$$

$|d(x_n, y_n) - d(x, y)| < \varepsilon$ $\forall n \geq n_0$

$\therefore d(x_n, y_n) \to d(x, y)$