SOLVED EXERCISE MTH-632

Complex Analysis & Differential Geometry

Instructor

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Midterm's Syllabus Solved Exercises

Book Name: Brown-Churchill-Complex Variables and Application 8th edition

Chapter 1 Section 2,3,4,8,10,

Chapter 2 Section 12,18,20,**23**,25,26

Section 2 *Basic Algebraic Operations*

Question No.1: **Verify that**

a): $\left(\sqrt{2}-i\right)-i\left(1-\sqrt{2}\right)=-2i$

proof:

$$
\begin{aligned}
L.H.S &= \left(\sqrt{2}-i\right)-i\left(1-\sqrt{2}i\right) \\
&= \sqrt{2}-i-i+\sqrt{2}i^2 \\
&= \sqrt{2}-2i+\sqrt{2}\left(-1\right) \\
&= \sqrt{2}-2i-\sqrt{2} \\
&= -2i = R.HS
\end{aligned}
$$

b):

$$
\begin{aligned}
L.H.S &= (2,-3)(-2,1) \\
&= (2-3i)(-2+i) \\
&= 2(-2+i)-3i(-2+i) \\
&= \text{-}4+2i+6i\text{ - }3i^2 \\
&= \text{-}4+8i\text{ - }3(-1) \\
&= \text{-}4+8i+3 \\
&= \text{-}1+8i = R.HS
\end{aligned}
$$

c):

$$
\begin{aligned}
L.H.S &= (3,1)(3,-1)\left(\frac{1}{5},\frac{1}{10}\right) \\
&= (3+i)(3-i)\left(\frac{1}{5}+\frac{1}{10}i\right) \\
&= \left(3(3-i)+i(3-i)\right)\left(\frac{1}{5}+\frac{1}{10}i\right) \\
&= \left(9-3i+3i-i^2\right)\left(\frac{1}{5}+\frac{1}{10}i\right) \\
&= \left(9-(-1)\right)\left(\frac{1}{5}+\frac{1}{10}i\right) \\
&= 10\left(\frac{1}{5}+\frac{1}{10}i\right) \\
&= 2+i \\
&= (2,1) = R.HS
\end{aligned}
$$

Question No.2:

a): Show that *Re(iz)=-ImgZ;*

Proof:

$$
\begin{aligned}
L.H.S &= \operatorname{Re}(iz) \\
&= \operatorname{Re}\left(i(x+iy)\right) \\
&= \operatorname{Re}\left(ix+i^2y\right) \\
&= \operatorname{Re}(ix-y) \\
&= -y \\
&= -\operatorname{Im}(z) = R.H.S
\end{aligned}
$$

b): **Show that Img(iz)=Re z;**

Proof:

$$
\begin{aligned}
L.H.S &= \operatorname{Im}(iz) \\
&= \operatorname{Im}\left(i\left(x+iy\right)\right) \\
&= \operatorname{Im}\left(ix+i^2y\right) \\
&= \operatorname{Im}(ix-y) \\
&= \operatorname{Re}(z) = R.H.S
\end{aligned}
$$

Question No.3: **Show that** $(1+z)^2 = 1+2z+z^2$

Ans:

Question No.4: verify that each of the two numbers $z = 1 \pm i$ satisfies the equation $z^2 - 2z + 2 = 0$

Proof:

a): when $z = 1+i$ then

$$
\begin{aligned}
L.H.S &= z^2 - 2z + 2 \\
&= (1+i)^2 - 2(1+i) + 2 \\
&= (1)^2 + (i)^2 + 2(1)(i) - 2 - 2i + 2 \\
&= 1 + (-1) + 2i - 2 - 2i + 2 \\
&= 0 = R.H.S
\end{aligned}
$$

b): when $z = 1-i$ then

$$
\begin{aligned}
L.H.S &= z^2 - 2z + 2 \\
&= (1-i)^2 - 2(1-i) + 2 \\
&= (1)^2 + (i)^2 - 2(1)(i) - 2 + 2i + 2 \\
&= 1 + (-1) - 2i - 2 + 2i + 2 \\
&= 0 = R.H.S
\end{aligned}
$$

Question No.5: Prove that multiplication of complex numbers is commutative.

Proof:

Let z_1 and z_2 be two complex numbers such that $z_1 = x_1 + iy_1$ *and* $z_2 = x_2 + iy_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$

$$
\begin{aligned}
z_1 z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\
&= (x_1)(x_2 + iy_2) + (iy_1)(x_2 + iy_2) \\
&= (x_1)(x_2) + (x_1)(iy_2) + (iy_1)(x_2) + (iy_1)(iy_2) \\
&= (x_1)(x_2) + i(x_1)(y_2) + i(y_1)(x_2) + (i)(i)(y_1)(y_2) \\
&= (x_2)(x_1) + i(y_2)(x_1) + i(x_2)(y_1) + (i)(i)(y_2)(y_1)
\end{aligned}
$$

As real numbers are commutative

$$
\begin{aligned}
&= (x_2)(x_1) + i(x_2)(y_1) + i(y_2)(x_1) + (i)(i)(y_2)(y_1) \\
&= (x_2)(x_1 + iy_1) + (iy_2)(x_1 + iy_1) \\
&= (x_2 + iy_2).(x_1 + iy_1) \\
&= z_2 z_1
\end{aligned}
$$

Hence complex numbers are commutative.

Question No.6: Verify the associative law for addition of complex numbers

Proof:

Let z_1, z_2 and z_3 be three complex numbers such that

$z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ *and* $z_3 = x_3 + iy_3$ where $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$

$$
\begin{aligned}
(z_1 + z_2) + z_3 &= ((x_1 + iy_1) + (x_2 + iy_2)) + (x_3 + iy_3) \\
&= ((x_1 + x_2) + i(y_1 + y_2)) + (x_3 + iy_3) \\
&= ((x_2 + x_1) + i(y_2 + y_1)) + (x_3 + iy_3)
\end{aligned}
$$

As real numbers are commutative

$$
\begin{aligned}
&= \left(x_1 + x_2\right) + x_3 + i\left(y_1 + y_2\right) + iy_3 \\
&= x_1 + \left(x_2 + x_3\right) + i\left(\left(y_1 + y_2\right) + y_3\right) \\
&= x_1 + \left(x_2 + x_3\right) + i\left(y_1 + \left(y_2 + y_3\right)\right) \\
&= x_1 + \left(x_2 + x_3\right) + iy_1 + i\left(y_2 + y_3\right) \\
&= x_1 + iy_1 + \left(x_2 + x_3\right) + i\left(y_2 + y_3\right) \\
&= z_1 + \left(z_2 + z_3\right)
\end{aligned}
$$

Question No.7:

Ans:

Question No.8:

a): Write $(x, y) + (u, v) = (x, y)$ and point out how it follows that the complex number 0=(0,0) is unique as an additive identity.

Ans:

$$
\begin{aligned}
&(x, y) + (u, v) = (x, y) \\
&(x + iy) + (u + iv) = (x + iy) \\
&(x + u) + i(y + v) = (x + iy)
\end{aligned}
$$

Comparing the real and imaginary numbers on both sides,

$$x + u = x \quad and \quad y + v = y$$

$$
\Rightarrow \quad
\begin{aligned}
&x + u = x \quad and \quad y + v = y \\
&u = x - x \quad and \quad v = y - y \\
&u = 0 \qquad and \quad v = 0
\end{aligned}
$$

Hence $(u, v) = (0, 0) = 0$

Suppose there is another additive $(c + id)$ identity then

$$
\begin{aligned}
&(x, y) + (c, d) = (x, y) \\
&(x + iy) + (c + i) = (x + iy) \\
&(x + c) + i(y + d) = (x + iy)
\end{aligned}
$$

$$
\Rightarrow \quad
\begin{aligned}
&x + c = x \quad and \quad y + d = y \\
&c = x - x \quad and \quad d = y - y \\
&c = 0 \qquad and \quad d = 0
\end{aligned}
$$

Hence $(c,d)=(0,0)=0=(u,v)$ proved that additive identity is a unique number.

Question No.9: use -1=(-1,0) and z=(x,y) to show that (-1)z=-z

Ans:

Given that $-1=(-1,0)=-1+i0$ and $z=(x,y)=x+iy$

$$
\begin{aligned}
L.H.S &= (-1)z \\
&= (-1,0)(x,y) \\
&= (-1+i0)(x+iy) \\
&= (-1)(x+iy)+i0(x+iy) \\
&= -x-iy+i0-0 \\
&= -(x+iy) \\
&= -z = R.H.S
\end{aligned}
$$

Question No.10: use $i=(0,1)$ *and* $y=(y,0)$ to verify that –(iy)=(-i)y. Thus show that additive inverse of a complex number z=x+iy can be written –z=-x-iy without ambiguity.

Proof:

$$i=(0,1) \text{ and } y=(y,0)$$

$$
\begin{aligned}
L.H.S &= -(iy) \\
&= -\left((0,1)(y,0)\right) \\
&= -\left((0+i)(y+i0)\right) \\
&= -\left((0)(y+i0)+(i)(y+i0)\right) \\
&= -\left(0+i0+iy-0\right) \\
&= -\left((0-0)+(i)(y+0)\right) \\
&= (-i)y = R.H.S
\end{aligned}
$$

Additive inverse gives the zero number in complex numbers

If $z=x+iy$ then let $w=u+iv$ be the additive inverse of the z , so

$$
\begin{aligned}
&z+w=0+i0 \\
&(x+iy)+(u+iv)=0+i0 \\
&(x+u)+i(y+v)=0+i0
\end{aligned}
$$

Comparing real and imaginary parts

$$x+u=0 \quad and \quad y+v=0$$
$$u=-x \quad and \quad v=-y$$

As

$$\begin{aligned} w &= u+iv \\ &= -x-iy \\ &= -(x+iy) \\ &= -z \end{aligned}$$

Hence additive inverse of a complex number z=x+iy can be written –z=-x-iy

Question No.11:solve the equation $z^2+z+1=0 \;\; for \;\; z=(x,y)$ by writing

$$(x,y)(x,y)+(x,y)+(1,0)=(0,0)$$

Ans:

$$\begin{aligned} z^2+z+1 &= (x,y)(x,y)+(x,y)+(1,0) \\ &= (x^2-y^2, yx+xy)+(x,y)+(1,0) \\ &= (x^2-y^2, yx+xy)+(x,y)+(1,0) \\ &= (x^2-y^2+x+1, yx+xy+y+0) \end{aligned}$$

Given $z^2+z+1=0$

Comparing real and imaginary parts

$$(x^2-y^2+x+1, yx+xy+y+0)=(0,0)$$

The real part is $x^2-y^2+x+1=0$ …..……..1 and

The imaginary part is $yx+xy+y+0=0$ ………..…2

Solving equation no. 2

$$\begin{aligned} &2xy+y=0 \\ &y(2x+1)=0 \\ &y=0 \quad or \quad 2x+1=0 \\ &\qquad\qquad\qquad x=\frac{-1}{2} \end{aligned}$$

When y=0 then this equation does not satisfied.

Put this value of x in equation 1

$$\left(\frac{-1}{2}\right)^2 - y^2 + \left(\frac{-1}{2}\right) + 1 = 0$$

$$\frac{1}{4} - y^2 - \frac{1}{2} + 1 = 0$$

$$\frac{1-2+4}{4} - y^2 = 0$$

$$\frac{3}{4} - y^2 = 0$$

$$\sqrt{y^2} = \sqrt{\frac{3}{4}}$$

$$y = \pm\frac{\sqrt{3}}{4}$$

Hence $x = \frac{-1}{2}$ and $y = \pm\frac{\sqrt{3}}{4}$

$$z = (x, y) = \left(-\frac{1}{2}, \pm\frac{\sqrt{3}}{4}\right)$$

When y=0 then this equation does not satisfied.

Complex Numbers: Trig Identities: 1

De Moivre's Theorem states that for whole number n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

We can use this fact to derive certain trig identites: an example of a use of complex numbers to do real calculations that would oth
we express $\cos 5 \theta$ in terms of powers of $\cos \theta. $

$$\begin{aligned}\cos 5\theta + i\sin 5\theta &= (\cos\theta + i\sin\theta)^5\\ &= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta +\\ &\quad 10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta\\ &= [\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta] +\\ &\quad i[5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta].\end{aligned}$$

Comparing real parts,

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta.$$

Using the substitution $\sin^2\theta = 1 - \cos^2\theta$,

$$\begin{aligned}\cos 5\theta &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2\\ &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.\end{aligned}$$

Section No. 3

Solved Exercise

1. Reduce each of these quantities to a real number.

$$\frac{1+2i}{3-4i}+\frac{2-i}{5i}=\left(\frac{1+2i}{3-4i}\right).\left(\frac{3+4i}{3+4i}\right)+\frac{(2-i)i}{(5i)i}$$

$$=\frac{3+4i+6i-8}{(3)^2-(4i)^2}+\frac{2i-i^2}{5i^2}$$

$$=\frac{-5+10i}{9-16(-1)}+\frac{2i-(-1)}{5(-1)}$$

$$=\frac{-5+10i}{25}-\frac{1+2i}{5}$$

$$=\frac{(-5+10i)-5(1+2i)}{25}$$

$$=\frac{-5+10i-5-10i}{25}=\frac{-10}{25}=-\frac{2}{5}$$

b):

$$\frac{5i}{(1-i)(2-i)(3-i)}=\frac{5i(1+i)(2+i)}{\left((1)^2-(i)^2\right)\left((2)^2-(i)^2\right)(3-i)}$$

$$=\frac{5i(2+i+2i+i^2)}{\left(1-(-1)\right)\left(4-(-1)\right)(3-i)}$$

$$=\frac{5i(2+3i-1)}{(1+1)(4+1)(3-i)}$$

$$=\frac{5i(1+3i)}{10(3-i)}$$

$$=\frac{i+3i^2}{2(3-i)}$$

$$=\frac{i-3}{2(3-i)}$$

$$=\frac{-(3-i)}{2(3-i)}=-\frac{1}{2}$$

c): $(1-i)^4$ =?

$$(1-i)^2 = (1)^2 + (i)^2 - 2(1)(i)$$
$$= 1\text{-}1\text{-}2i = 2i$$
$$(1-i)^4 = (1-i)^2(1-i)^2$$
$$= (2i)(2i)$$
$$= 4i^2 = 4(-1) = -4$$

2. Show that

$$L.H.S = \frac{1}{1/z} = \frac{1}{z^{-1}} \qquad \because \frac{1}{z} = z^{-1}$$
$$= \frac{1}{z^{-1}} \cdot \frac{z}{z} = \frac{1.z}{z^{-1}.z}$$
$$= \frac{z}{1} = z = R.H.S$$

3. use the associative and commutative laws for multiplication to show that

$$L.H.S = (z_1z_2)(z_3z_4)$$
$$= (z_2z_1)(z_3z_4) \qquad \because (z_1z_2) = (z_2z_1) \quad \textit{commutative law}$$
$$= (z_2)(z_1z_3)(z_4) \qquad \because (z_1)(z_3z_4) = (z_1z_3)(z_4) \quad \textit{associative law}$$
$$= (z_1z_3)(z_2)(z_4) \qquad \because (z_2)(z_1z_3) = (z_1z_3)(z_2) \quad \textit{commutative law}$$
$$= (z_1z_3)(z_2z_4) \qquad \because (z_2)(z_4) = (z_2z_4)$$
$$= R.H.S$$

5. Drive expression

$$\frac{z_1}{z_2} = \frac{x_1x_2 - y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2} \quad (z_2 \neq 0)$$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ where $(z_2 \neq 0)$

$$L.H.S=\frac{z_1}{z_2}$$

$$=\frac{x_1+iy_1}{x_2+iy_2}\cdot\frac{x_2-iy_2}{x_2-iy_2}$$

$$=\frac{x_1x_2+i^2y_1y_2+iy_1x_2-ix_1y_2}{\left(x_2+iy_2\right)\left(x_2-iy_2\right)}$$

$$=\frac{x_1x_2+(-1)y_1y_2+i(y_1x_2-x_1y_2)}{\left(x_2\right)^2-\left(iy_2\right)^2}$$

$$=\frac{x_1x_2-y_1y_2+i(y_1x_2-x_1y_2)}{{x_2}^2-\left(-1\right){y_2}^2}$$

$$=\frac{\left(x_1x_2-y_1y_2\right)+i\left(y_1x_2-x_1y_2\right)}{{x_2}^2+{y_2}^2}$$

$$=\frac{x_1x_2-y_1y_2}{x_2^2+y_2^2}+i\,\frac{y_1x_2-x_1y_2}{x_2^2+y_2^2}=R.H.S$$

6. Drive the identity

$$\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right)=\frac{z_1z_2}{z_3z_4}$$

$$L.H.S=\left(\frac{z_1}{z_3}\right)\left(\frac{z_2}{z_4}\right)$$

$$=\left(z_1\right)\left(\frac{1}{z_3}\right)\left(z_2\right)\left(\frac{1}{z_4}\right)\qquad\because\left(\frac{z_1}{z_2}\right)=\left(z_1\right)\left(\frac{1}{z_2}\right)$$

$$=\left(z_1\right)\left(z_3^{-1}\right)\left(z_2\right)\left(z_4^{-1}\right)\qquad\because\left(z_3^{-1}\right)\left(z_2\right)=\left(z_2\right)\left(z_3^{-1}\right)\ \text{commutative law}$$

$$=\left(z_1\right)\left(z_2\right)\left(z_3^{-1}\right)\left(z_4^{-1}\right)$$

$$=\left(z_1z_2\right)\left(z_3\right)^{-1}\left(z_4\right)^{-1}$$

$$=\left(z_1z_2\right)\left(z_3\right)^{-1}\left(z_4\right)^{-1}$$

$$=\left(z_1z_2\right)\left(z_3z_4\right)^{-1}$$

$$=\frac{z_1z_2}{z_3z_4}=R.H.S$$

7. Use the identity to drive the cancellation law,

$$\frac{z_1 z}{z_2 z} = \frac{z_1}{z_2}$$

$$L.H.S. = \frac{z_1 z}{z_2 z}$$

$$= \frac{z_1 z}{z_2 z} \cdot \frac{z^{-1}}{z^{-1}}$$ multiplying numerator and denomerator by z^{-1}

$$= \frac{z_1}{z_2} \frac{zz^{-1}}{zz^{-1}}$$ associative law

$$= \frac{z_1}{z_2} \frac{1}{1}$$ $\because zz^{-1} = 1$

$$= \frac{z_1}{z_2} = R.H.S$$

Section 8

Exponential form

1. Find the principal argument Arg z when

a): $z = \dfrac{i}{-2-2i}$

Answer:

Definition: (principle value of arg of z):-

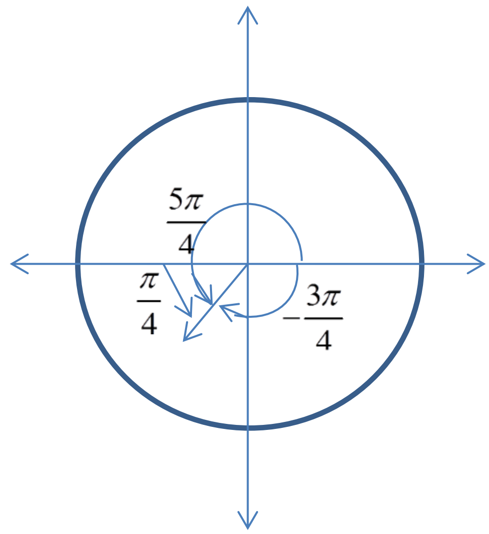
Let $z \neq 0$ be a complex number. Then principle value of argument of z, denoted by $Arg\ z$ is a unique value of θ such that

$$z = r\left(\cos\theta + i\sin\theta\right) \qquad -\pi < \theta \leq \pi$$

Principle value of argument of z is also referred as **"The argument of z"**

$$\arg z = \text{Arg}\, z + 2n\pi \qquad n = 0, \pm 1, \pm 2, \pm 3, - - - - - -$$

$$z=\frac{i}{-2-2i}$$
$$=\frac{i}{-2(1+i)}.\frac{(1-i)}{(1-i)}$$
$$=\frac{i-i^2}{-2\left((1)^2-(i)^2\right)}$$
$$=\frac{i-(-1)}{-2\left(1-(-1)\right)}$$
$$=\frac{1+i}{-2\left(1+1\right)}$$
$$=\frac{1+i}{-4}=-\frac{1}{4}-i\frac{1}{4}$$



$$z=x+iy=-\frac{1}{4}-i\frac{1}{4}$$
$$x=-\frac{1}{4}\quad,\quad y=-\frac{1}{4}$$
$$\theta=\tan^{-1}\left(\frac{y}{x}\right)=\tan^{-1}\left(\frac{-1/4}{-1/4}\right)=\tan^{-1}(1)=\frac{\pi}{4}$$

As x and y are negative so angle lies in the 3rd quadrant, so we add π to $\frac{\pi}{4}$.

$$Arg\ \text{z}=\pi+\frac{\pi}{4}=\frac{5\pi}{4}$$

this angle in anti-clockwise direction.

For clockwise direction from positive axis the angle is

$$Arg\ \text{z}=\frac{5\pi}{4}-2\pi=-\frac{3\pi}{4}$$

See example *z=-1-i* on page 23 in handouts of MTH 632 written by Fiddling writer.

Second method.

$$z = \frac{i}{-2-2i}$$

$Arg\ z = Arg\ (i) - Arg(-2-2i)$

For $Arg(i) = (0+i)$

If $z \neq 0 \ \ and \ \ x = 0$, then we use the following rule:

$$Arg\, z = \frac{\pi}{2} \qquad If \ \ \text{Im}\ z > 0$$
$$Arg\, z = -\frac{\pi}{2} \qquad If \ \ \text{Im}\, z < 0$$

So, for $Arg(i) = (0+i) = \frac{\pi}{2}$

For Arg(-2-2i)=$\tan^{-1}\left(\frac{-2}{-2}\right) = \tan^{-1}(1) = \frac{\pi}{4}$

As x and y are negative so angle lies in the 3rd quadrant, so we add π to $\frac{\pi}{4}$.

$Arg(-2-2i) = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

$$Arg\ z = Arg\ (i) - Arg(-2-2i)$$
$$Arg\ z = \frac{\pi}{2} - \frac{5\pi}{4}$$
$$Arg\ z = \frac{2\pi - 5\pi}{4} = -\frac{3\pi}{4}$$

b): $z = (\sqrt{3} - i)^6$

$$r = |z| = \sqrt{x^2 + y^2}$$
$$= \sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = \sqrt{3+1} = 2$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

To write in rectangular form.

$$(\sqrt{3}-i) = re^{i\theta} = 2e^{i\frac{\pi}{6}}$$

$$z = (\sqrt{3}-i)^6 = \left(2e^{i\frac{\pi}{6}}\right)^6$$

$$= 2^6 e^{i\left(\frac{\pi}{6}\right)6} = 64e^{i\pi}$$

$$\theta = \pi$$

Check.(For principal angle we subtract angle $\frac{\pi}{6}$ from π and here we just find the angle so we do not subtract.)

2. show that $\left|e^{i\theta}\right| = 1$

Solution:

$$L.H.S = \left|e^{i\theta}\right| = \left|\cos\theta + i\sin\theta\right|$$
$$= \sqrt{(\cos\theta)^2 + (\sin\theta)^2}$$
$$= \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1}$$
$$= 1 = R.H.S$$

show that $\overline{e^{i\theta}} = e^{-i\theta}$

Solution:

$$L.H.S = \overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta}$$
$$= \cos\theta - i\sin\theta = \cos\theta + (-i)\sin\theta$$
$$= e^{-i\theta} = R.H.S$$

4. show that $\left|e^{i\theta}-1\right|=2$

Solution:

$$
\begin{aligned}
&L.H.S=\left|e^{i\theta}-1\right| \\
&=\left|\cos\theta+i\sin\theta-1\right| \qquad 0\le\theta\le 2\pi \\
&when \quad \theta=\pi \\
&=\left|\cos\pi+i\sin\pi-1\right| \\
&=\left|-1+0-1\right| \\
&=\left|-2\right|=2=R.H.S
\end{aligned}
$$

5. By writing the individual factors on the left in exponential form, performing the needed operations and changing back to rectangular coordinates, show that

a): $i(1-\sqrt{3})(\sqrt{3}+i)=2(1+\sqrt{3}i)$

Solution:

Exponential form are

$$i=e^{i\frac{\pi}{2}}=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i$$

$$(1-\sqrt{3}i)=2\cos\frac{\pi}{3}-2i\sin\frac{\pi}{3}=2e^{-i\frac{\pi}{3}}$$

$$(\sqrt{3}+i)=2\cos\frac{\pi}{6}+2i\sin\frac{\pi}{6}=2e^{i\frac{\pi}{6}}$$

$$
\begin{aligned}
&L.H.S=i(1-\sqrt{3}i)(\sqrt{3}+i) \\
&=e^{i\frac{\pi}{2}}\left(2e^{-i\frac{\pi}{3}}\right)\left(2e^{i\frac{\pi}{6}}\right) \\
&=4e^{i\frac{\pi}{2}-i\frac{\pi}{3}+i\frac{\pi}{6}} \\
&=4e^{i\frac{3\pi-2\pi+\pi}{6}} \\
&=4e^{i\frac{\pi}{3}} \\
&=2(2e^{i\frac{\pi}{3}}) \\
&=2(1+\sqrt{3}i) \\
&=R.H.S
\end{aligned}
$$

b): $L.H.S = \frac{5i}{2+i} = \frac{5e^{i\frac{\pi}{2}}}{\sqrt{3}e^{i}}$

8. Prove: $\exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right) = \exp(i\theta_1)$

Solution:

$$L.H.S = \exp\left(i\frac{\theta_1+\theta_2}{2}\right)\exp\left(i\frac{\theta_1-\theta_2}{2}\right)$$

$$= \left(\cos\left(\frac{\theta_1+\theta_2}{2}\right)+i\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right)\left(\cos\left(\frac{\theta_1-\theta_2}{2}\right)+i\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)$$

$$= \left(\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)$$

$$+\mathrm{i}\left(\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)+\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)$$

$$= \cos\left(\frac{\theta_1+\theta_2}{2}+\frac{\theta_1-\theta_2}{2}\right)+i\sin\left(\frac{\theta_1+\theta_2}{2}+\frac{\theta_1-\theta_2}{2}\right)$$

$$= \cos(\theta_1)+i\sin(\theta_1)$$

$$= e^{i\theta_1} = \exp(i\theta_1) = R.H.S$$

b): $L.H.S = \exp\left(i\frac{\theta_1+\theta_2}{2}\right)\overline{\exp\left(i\frac{\theta_1-\theta_2}{2}\right)}$

$$= \left(\cos\left(\frac{\theta_1+\theta_2}{2}\right)+i\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right)\overline{\left(\cos\left(\frac{\theta_1-\theta_2}{2}\right)+i\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)}$$

$$= \left(\cos\left(\frac{\theta_1+\theta_2}{2}\right)+i\sin\left(\frac{\theta_1+\theta_2}{2}\right)\right)\left(\cos\left(\frac{\theta_1-\theta_2}{2}\right)-i\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)$$

$$= \left(\cos\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)+\sin\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)$$

$$+\mathrm{i}\left(\sin\left(\frac{\theta_1+\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right)-\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)\right)$$

$$= \cos\left(\frac{\theta_1+\theta_2}{2}-\frac{\theta_1-\theta_2}{2}\right)+i\sin\left(\frac{\theta_1+\theta_2}{2}-\frac{\theta_1-\theta_2}{2}\right)$$

$$= \cos(\theta_2)+i\sin(\theta_2)$$

$$= e^{i\theta_2} = \exp(i\theta_2) = R.H.S$$

10. use the de Moivre’s Formula to drive

a): $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$

Proof:

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$L.H.S = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$$
$$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$
$$= 4\cos^3\theta - 3\cos\theta = \cos 3\theta = R.H.S$$

$$\left(\cos\theta + i\sin\theta\right)^3 = (cos\theta + isin\theta)(cos\theta + isin\theta)(cos\theta + isin\theta)$$
$$= \left(cos^2\theta - sin^2\theta + i\sin\theta cos\theta + i\cos\theta sin\theta\right)(cos\theta + isin\theta)$$
$$= cos^3\theta + icos^2\theta\sin\theta - sin^2\theta cos\theta - i\sin^3\theta + icos^2\theta\sin\theta - sin^2\theta\cos\theta + i\cos^2\theta\sin\theta - \sin^2\theta\cos\theta$$
$$= cos^3\theta - 3sin^2\theta cos\theta + 3icos^2\theta\sin\theta - i\sin^3\theta$$
$$= cos^3\theta - 3sin^2\theta cos\theta + i(3cos^2\theta\sin\theta - \sin^3\theta)$$

using de Moivre’s law

$$\cos 3\theta + i\sin 3\theta = cos^3\theta - 3cos\theta sin^2\theta + i(3cos^2\theta\sin\theta - \sin^3)$$

Hence

$$\cos 3\theta + i\sin 3\theta = cos^3\theta - 3cos\theta sin^2\theta$$

And

$$\sin 3\theta = 3cos^2\theta\sin\theta - \sin^3\theta$$

Section 10

Examples:

1. Find the square root of 2i, Topic 12 time 15 mint

Answer:

We can write square root of 2i as $(2i)^{\frac{1}{2}}$

$$2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = 2\exp\left[i\left(\frac{\pi}{2} + 2k\pi\right)\right] \qquad k=0,1$$

$$=2e^{i\left(\frac{\pi}{2}+2k\pi\right)}$$

Taking square root

$$(2i)^{\frac{1}{2}} = \left(2e^{i\left(\frac{\pi}{2}+2k\pi\right)}\right)^{\frac{1}{2}} = \sqrt{2}e^{i\left(\frac{\pi}{4}+k\pi\right)}$$

When k=0

$$\sqrt{2}e^{i\left(\frac{\pi}{4}+k\pi\right)} = \sqrt{2}e^{i\left(\frac{\pi}{4}\right)} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1+i$$

When k=1

$$\sqrt{2}e^{i\left(\frac{\pi}{4}+(1)\pi\right)} = \sqrt{2}e^{i\left(\frac{5\pi}{4}\right)} = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -1-i = -(1+i)$$

So the roots of 2i are $\pm(1+i)$

1. Find the square root of $1-\sqrt{3}i$, Topic 12 time 15 mint

Solution:

$$1-\sqrt{3}i = \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right) = 2e^{i\left(-\frac{\pi}{6}+2k\pi\right)}$$

Taking square root

$$\left(1-\sqrt{3}i\right)^{\frac{1}{2}} = \left(2e^{i\left(-\frac{\pi}{6}+2k\pi\right)}\right)^{\frac{1}{2}} = \sqrt{2}e^{i\left(-\frac{\pi}{12}+k\pi\right)} \qquad k=0,1$$

When k=0

$$\left(1-\sqrt{3}i\right)^{\frac{1}{2}} = \sqrt{2}e^{i\left(-\frac{\pi}{12}+k\pi\right)}$$

$$= \sqrt{2}e^{i\left(-\frac{\pi}{12}\right)} = \sqrt{2}\left(\cos\left(-\frac{\pi}{12}\right)+i\sin\left(-\frac{\pi}{12}\right)\right)$$

$$= \sqrt{2}\left(\cos\left(\frac{\pi}{12}\right)-i\sin\left(\frac{\pi}{12}\right)\right) = -\sqrt{2}\left(\frac{\sqrt{3}-i}{2}\right) = -\frac{\sqrt{3}-i}{\sqrt{2}}$$

When k=1

$$\left(1-\sqrt{3}i\right)^{\frac{1}{2}} = \sqrt{2}e^{i\left(-\frac{\pi}{12}+(1)\pi\right)}$$

$$= \sqrt{2}e^{i\left(\frac{11\pi}{12}\right)} = \sqrt{2}\left(\cos\left(\frac{11\pi}{12}\right)+i\sin\left(\frac{11\pi}{12}\right)\right)$$

$$= \sqrt{2}\left(\frac{\sqrt{3}-i}{2}\right) = +\frac{\sqrt{3}-i}{\sqrt{2}}$$

So the roots of $1-\sqrt{3}i$ are $\pm\frac{\sqrt{3}-i}{\sqrt{2}}$

2. $(-16)^{\frac{1}{4}} = (-1\times16)^{\frac{1}{4}} = \left(i^2\times2^4\right)^{\frac{1}{4}} = 2\left(i\right)^{\frac{1}{2}} = 2\sqrt{i}$

We find the square root of i.

Topic 25 examples:

Complex Limits

Nonexistence of limits

Example: Show that the limit

$$\lim_{z \to 0} \frac{z}{\bar{z}}$$

Does not exist.

Solution: First we calculate the limit when z approaches to 0 along the real axis

$$\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{x \to 0} \frac{x + i0}{x - i0} = \lim_{x \to 0} 1 = 1.$$

If z approaches 0 along the imaginary axis.

$$\lim_{z \to 0} \frac{z}{\bar{z}} = \lim_{y \to 0} \frac{0 + iy}{0 - iy} = \lim_{x \to 0} -1 = -1.$$

As 1 is not equal to -1 so, limit does not exist.

Complex Limits

Example: Show that

$$\lim_{z \to z_0} \bar{z} = \bar{z_0}$$

Solution:

$\varepsilon > 0$

$|f(z) - \bar{z}_0| < \varepsilon$

$\Leftrightarrow |\bar{z} - \bar{z}_0| < \varepsilon$

$\Leftrightarrow |\overline{z - z_0}| < \varepsilon$

$\Leftrightarrow |z - z_0| < \varepsilon$

$\delta = \varepsilon$

$|z - z_0| < \delta = \varepsilon$

$\Rightarrow |f(z) - \bar{z}_0| < \varepsilon$

$|f(z) - \bar{z}_0| < \varepsilon$ whenever $|z - z_0| < \delta$

Example: Show that

$$\lim_{z \to z_o} \overline{z} = \overline{z_o}$$

Solution:

Let ε be any real non-negative number. $\varepsilon > 0$.

$$\left| f(z) - \overline{z_o} \right| < \varepsilon$$

$$\Leftrightarrow \quad \left| \overline{z} - \overline{z_o} \right| < \varepsilon$$

$$\Leftrightarrow \quad \left| \overline{z - z_o} \right| < \varepsilon$$

$$\Leftrightarrow \quad \left| z - z_o \right| < \varepsilon$$

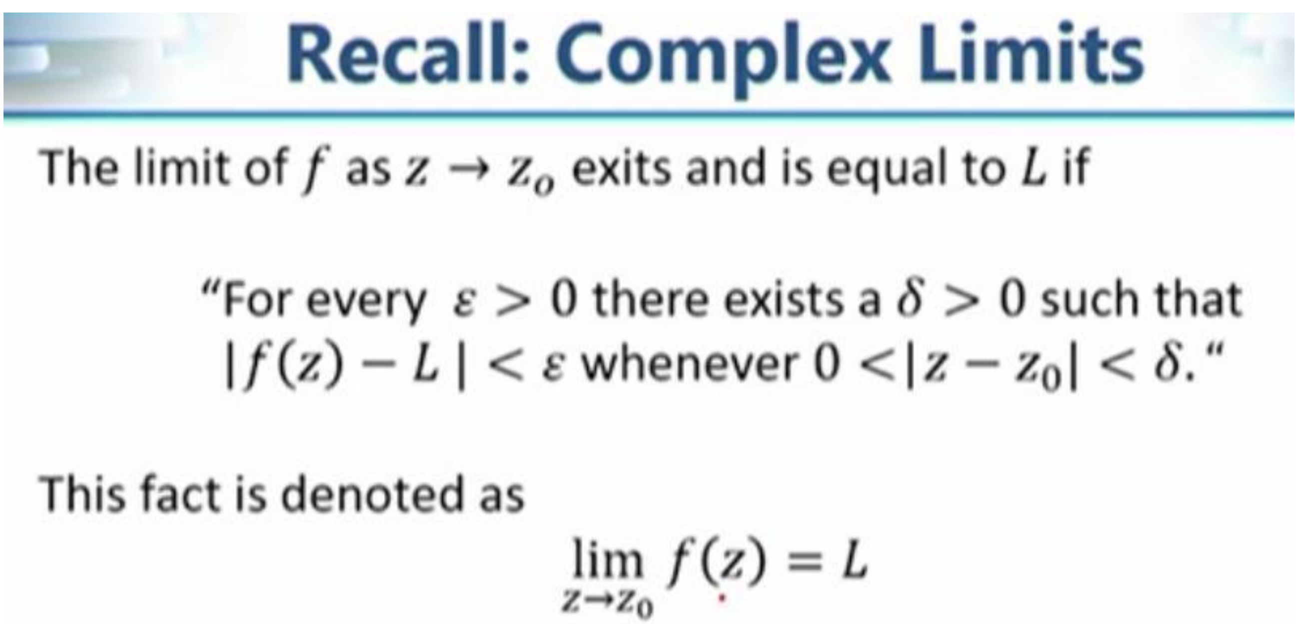
If $\delta = \varepsilon$

$$\left| z - z_o \right| < \delta = \varepsilon$$

$$\Rightarrow \quad \left| f(z) - \overline{z_o} \right| < \varepsilon$$

$\left| f(z) - z_o \right| < \varepsilon$ whenever $\left| z - z_o \right| < \delta s$

Topic 26



Complex Limits

A complex function is of the form

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

We find a relation between the complex limit of $f(z)$ and the real limits of the multivariable real function $u(x,y)$ and $v(x,y)$.

Relation between Real and Complex Limits

Main motivation

Theorem

Let $f(z) = u(x,y) + iv(x,y)$ be a complex function that is defined in some neighborhood of z_0, except perhaps at $z_0 = x_0 + iy_0$. Then

$$\lim_{z \to z_0} f(z) = u_0 + iv_0$$

iff

$$\lim_{(x,y)\to(x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y)\to(x_0,y_0)} v(x,y) = v_0 .$$

Example: if $u(x,y) = \dfrac{2x^3}{(x^2+y^2)}$ then show that.

$$\lim_{(x,y)\to(0,0)} u(x,y) = 0$$

Solution:

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$u(r\cos\theta,\ r\sin\theta) = \frac{2r^3\cos^3\theta}{r^2\cos^2\theta + r^2\sin^2\theta} = 2r\cos^3\theta$$

In polar form:

$$|u(x,y)-0|$$
$$|2r\cos^3\theta| = 2r|\cos^3\theta| \le 2r$$
$$2r < \varepsilon \quad \Rightarrow \quad r < \frac{\varepsilon}{2}$$

To find $\delta > 0$ such that $|u(x,y)-0| < \varepsilon$

$$|u(x,y)-0| \le 2r < \varepsilon \text{ when ever } 0 < \sqrt{(x-0)^2+(y-0)^2} < \delta$$

Exampe: Show that the function

$$u(x,y) = \frac{xy}{x^2+y^2}$$

Does not have a limit as (x,y) approaches (0,0).

Solution:

For y=-x

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim_{x\to 0} \frac{-x^2}{x^2+y^2}$$
$$= \lim_{x\to 0} \frac{-x^2}{x^2+(-x)^2} = \lim_{x\to 0} \frac{-x^2}{2x^2}$$
$$= \lim_{x\to 0} \frac{-1}{2} = -\frac{1}{2}$$

Topic 29

Real Limits of Multivariable functions

Properties of real limits of functions of two variables

Theorem: If $\lim\limits_{(x,y)\to(x_0,y_0)} F(x,y) = A$

and

$$\lim_{(x,y)\to(x_0,y_0)} G(x,y) = B$$

then

1. $\lim\limits_{(x,y)\to(x_0,y_0)} cF(x,y) = cA$, where c is a real constant
2. $\lim\limits_{(x,y)\to(x_0,y_0)} F(x,y) \pm G(x,y) = A \pm B$
3. $\lim\limits_{(x,y)\to(x_0,y_0)} F(x,y).G(x,y) = A\,.B$
4. $\lim\limits_{(x,y)\to(x_0,y_0)} \frac{F(x,y)}{G(x,y)} = \frac{A}{B}, B \neq 0$

Example:

Calculate the limit

$$\lim_{(x,y)\to(1,2)} 3xy^2 - y$$

Solution:

$$\lim_{(x,y)\to(1,2)} 3xy^2 - y$$

$$= \lim_{(x,y)\to(1,2)} 3xy^2 - \lim_{(x,y)\to(1,2)} y$$

$$= \left(\lim_{(x,y)\to(1,2)} 3x\right)\left(\lim_{(x,y)\to(1,2)} y^2\right) - \left(\lim_{(x,y)\to(1,2)} y\right)$$

$$= 3(1)(2)^2 - 2 = 12 - 2 = 10$$

Example: ***Compute the limit***

$$\lim_{z \to i} z^2 + z + 1$$

Solution:

$$f(z) = z^2 + z + 1$$
$$f(x+iy) = (x+iy)^2 + (x+iy) + 1$$
$$= (x^2 - y^2) - 2ixy + (x+iy) + 1$$
$$= (x^2 - y^2 + x + 1) - i(-2xy + y)$$

$$\lim_{(x,y)\to(0,1)} x^2 - y + x + 1 = -1 + 1 = 0$$
$$\lim_{(x,y)\to(0,1)} -2xy + y = 1$$
$$\Rightarrow \quad \lim_{z \to i} f(z) = i$$

Properties of Complex Limits

Theorem: Suppose that

$$\lim_{z\to z_0} f(z) = A$$

and

$$\lim_{z\to z_0} g(z) = B$$

then

1. $\lim_{z\to z_0} c\,f(z) = cA$, c a complex constant.
2. $\lim_{z\to z_0} (f(z) \pm g(z)) = A \pm B.$
3. $\lim_{z\to z_0} (f(z) \,.\, g(z)) = A.B$
4. $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{A}{B}$ provided $B \neq 0.$

Example:

Compute the limit

$$\lim_{z\to i} z^2+z+1$$
$$=\left(\lim_{z\to i} z\right)\left(\lim_{z\to i} z\right)+\left(\lim_{z\to i} z\right)+\left(\lim_{z\to i} 1\right)$$
$$=(i)(i)+i+1$$
$$=-1+i+1=i$$

Topic 34

The mean value theorem for complex numbers does not hold.

Derivatives of Real Functions

The derivative of f at x_0, written $f'(x_0)$, is

$$f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

provided the limit exists.

OR

$$f'(x_0)=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

Derivatives

Example: If $f(z)=z^2$ then show that $f'(z)=2z$.

Solution:

$$f'(z)=\lim_{\Delta z\to 0}\frac{f(z+\Delta z)-f(z)}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\frac{(z+\Delta z)^2-z^2}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\frac{z^2+2z\Delta z+\Delta z^2-z^2}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\frac{(2z+\Delta z)\Delta z}{\Delta z}=\lim_{\Delta z\to 0}(2z+\Delta z)=2z$$

Derivatives

Example: Show that the function $f(z) = \text{Re}(z)$ is differentiable nowhere.

Solution: First we approach $z_0 = x_0 + iy_0$ along a line parallel to x −axis. In this case $z = x + iy_0$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{\Delta z}$$

$$= \lim_{(x+iy_0)\to(x_0+iy_0)} \frac{\text{Re}(x+iy) - \text{Re}(x_0+iy_0)}{(x+iy_0)-(x_0+iy_0)}$$

$$= \lim_{(x+iy_0)\to(x_0+iy_0)} \frac{x - x_0}{x - x_0}$$

$$= 1$$

Derivatives

Solution (cont.): Next we approach z_0 along the line parallel to y −axis by forcing z to be of the form $z = x_0 + iy$.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{\Delta z}$$

$$= \lim_{(x_0+iy)\to(x_0+iy_0)} \frac{Re(x_0+iy) - Re(x_0+iy_0)}{(x_0+iy)-(x_0+iy_0)}$$

$$= \lim_{(x+iy_0)\to(x_0+iy_0)} \frac{x_0 - x_0}{i(y - y_0)}$$

$$= 0$$

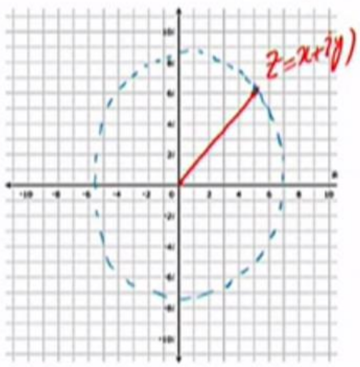
Hence the function f(z)=Re(z) is differentiable nowhere.

Derivatives

Example: Consider the real-valued function $f(z) = |z|^2$.
Find points where $f(z)$ is differentiable.

Solution: If $z = x+iy$, $f(z) = f(x+iy) = |x+iy|^2 = x^2+y^2$

Here $f(z)$ is real-valued function.



$$f(z) - |z|^2 = z\bar{z}$$

Derivatives

Example: Consider the real-valued function $f(z) = |z|^2 = z\bar{z}$
Find points where $f(z)$ is differentiable.

Solution: We know that $f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{(z+\Delta z)\overline{(z+\Delta z)} - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z+\Delta z)(\bar{z}+\overline{\Delta z}) - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z\bar{z} + z\overline{\Delta z} + \bar{z}\Delta z + \Delta z\overline{\Delta z} - z\bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(z\frac{\overline{\Delta z}}{\Delta z} + \bar{z} + \overline{\Delta z}\right)$$

Path 1

Derivatives

Example: Consider the real-valued function $f(z) = |z|^2$.
Find points where $f(z)$ is differentiable.

Solution: Path 1: z moves parallel to x-axis

$$\lim_{\Delta z \to 0} \bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z}$$

In this case $\overline{\Delta z} = \Delta z$

$$= \lim_{\Delta z \to 0} \bar{z} + \Delta z + z \frac{\Delta z}{\Delta z}$$

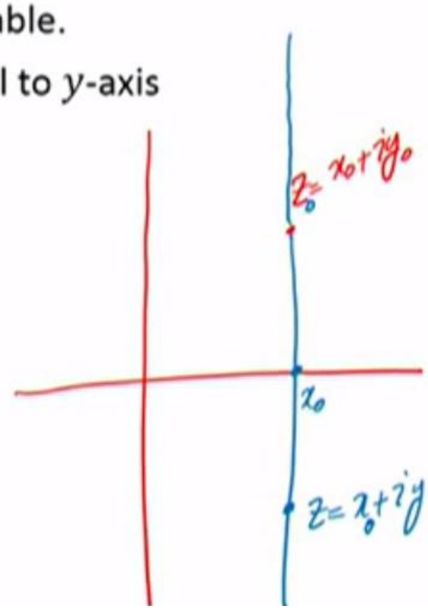
$$= \lim_{\Delta z \to 0} \left(\bar{z} + \Delta z + z\right) = \bar{z} + z$$

Derivatives

Example: Consider the real-valued function $f(z) = |z|^2$.
Find points where $f(z)$ is differentiable.

Solution: Path 2: z moves parallel to y-axis

$$\Delta z = z - z_0$$
$$= (x_0 + iy) - (x_0 + iy_0)$$
$$= i(y - y_0)$$
$$\Rightarrow \overline{\Delta z} = -\Delta z$$



Derivatives

Example: Consider the real-valued function $f(z) = |z|^2$. Find points where $f(z)$ is differentiable.

Solution: Path 2: z moves parallel to y-axis

$$\lim_{\Delta z \to 0} \bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z}$$

In this case $\overline{\Delta z} = -\Delta z$

$$= \lim_{\Delta z \to 0} \bar{z} - \Delta z + z \frac{(-\Delta z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\bar{z} - \Delta z - z\right) = \bar{z} - z.$$

Derivatives

Example: Consider the real-valued function $f(z) = |z|^2$. Find points where $f(z)$ is differentiable.

Solution: Along path 1: parallel to x-axis

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = z + \bar{z}$$

along path 2: parallel to y-axis

$$\lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \bar{z} - z$$

$$\Rightarrow \quad z + \bar{z} = \bar{z} - z \Rightarrow z = 0.$$

Derivatives

Observations

Example: Consider the real-valued function $f(z) = |z|^2$.
Find points where $f(z)$ is differentiable.

① $f(z) = f(x+iy) = (x^2+y^2) + i(0)$

$u(x,y) = x^2+y^2$, $u: R^2 \to R$
$v(x,y) = 0$, $v: R^2 \to R$

② $f(z)$ is cont but not diff.

Differentiation Formulas

The proofs are very similar to differentiation formulas for real-valued functions

$$\frac{d}{dz}C = 0$$

where C is a constant.

$$\frac{d}{dz}(10) = 0,$$

$$\frac{d}{dz}(i) = 0$$

Differentiation Formulas

$$\frac{d}{dz} z^n = n\, z^{n-1}$$

where n is a positive integer.

Example: $\frac{d}{dz}(z^{15})$, $n=15$

$= 15 z^{15-1}$

$= 15 z^{14}$

Differentiation Formulas

$$\frac{d}{dz}[Cf(z)] = C\frac{d}{dz}f(z)$$

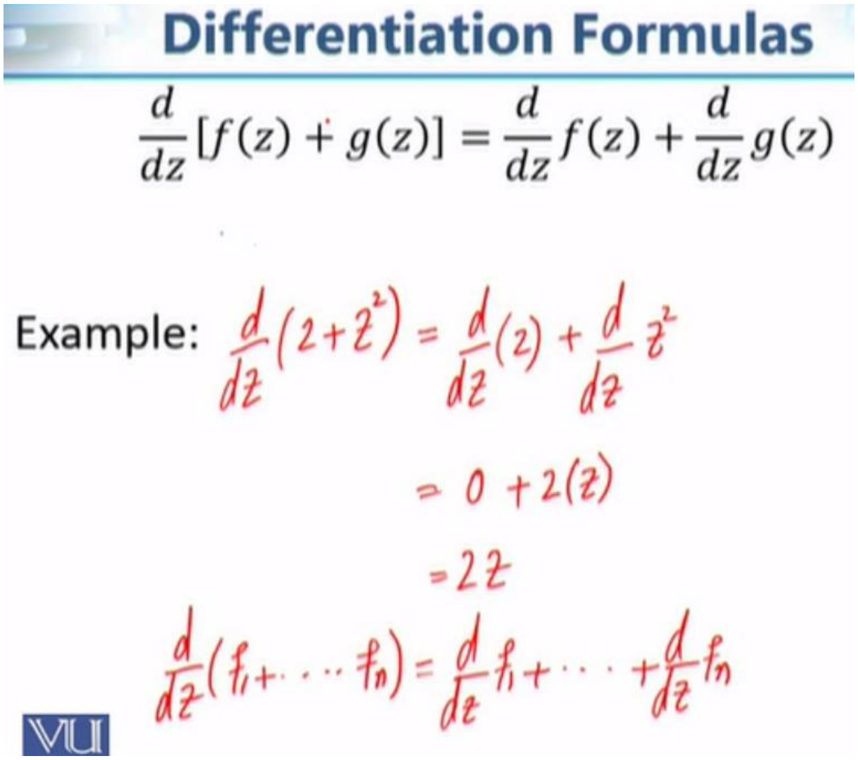
Example:

$\frac{d}{dz}(10 f(z)) = 10\frac{d}{dz}(f(z))$

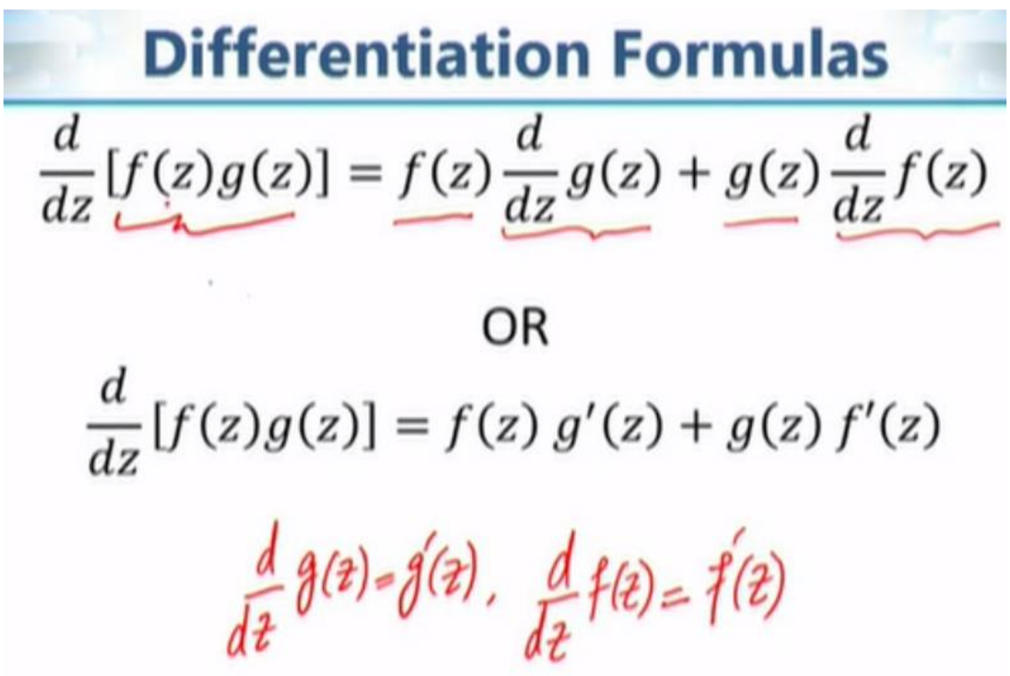
$\frac{d}{dz}(10 z^3) = 10\frac{d}{dz}z^3 = 10(3z^2)$

$= 30z^2.$

The Sum Rule



The product Rule



Differentiation Formulas

$$\frac{d}{dz}[f(z)g(z)] = f(z)\, g'(z) + g(z)\, f'(z)$$

Proof: Let $w = f(z)g(z)$

$$w'(z_0) = \lim_{\Delta z \to 0} \frac{w(z + \Delta z) - w(z)}{\Delta z}$$

Now

$$w(z + \Delta z) = f(z + \Delta z)g(z + \Delta z)$$

$$w(z + \Delta z) - w(z) = f(z + \Delta z)g(z + \Delta z) - f(z)\, g(z)$$

Differentiation Formulas

$$w(z + \Delta z) = f(z + \Delta z)g(z + \Delta z)$$

$$w(z + \Delta z) - w(z) = f(z + \Delta z)g(z + \Delta z) - f(z)\, g(z)$$

Adding and subtracting $f(z)g(z + \Delta z)$, we get

$$w(z + \Delta z) - w(z) = f(z + \Delta z)g(z + \Delta z) - f(z)g(z + \Delta z) + f(z)g(z + \Delta z) - f(z)\, g(z)$$

$$w(z + \Delta z) - w(z) = g(z + \Delta z)\,[f(z + \Delta z) - f(z)] + f(z)[g(z + \Delta z) - g(z)]$$

Differentiation Formulas

$$w(z + \Delta z) - w(z) = g(z + \Delta z)\,[f(z + \Delta z) - f(z)] + f(z)[g(z + \Delta z) - g(z)]$$

Dividing by Δz

$$\frac{w(z + \Delta z) - w(z)}{\Delta z} = \frac{g(z + \Delta z)\,[f(z + \Delta z) - f(z)]}{\Delta z} + \frac{f(z)[g(z + \Delta z) - g(z)]}{\Delta z}$$

Taking limit $\Delta z \to 0$

$$\lim_{\Delta z \to 0} \frac{w(z + \Delta z) - w(z)}{\Delta z} = \lim_{\Delta z \to 0} g(z + \Delta z)\,\frac{[f(z + \Delta z) - f(z)]}{\Delta z} + \lim_{\Delta z \to 0} f(z)\,\frac{[g(z + \Delta z) - g(z)]}{\Delta z}$$

Differentiation Formulas

$$\lim_{\Delta z \to 0} \frac{w(z + \Delta z) - w(z)}{\Delta z} = \lim_{\Delta z \to 0} g(z + \Delta z)\,\frac{[f(z + \Delta z) - f(z)]}{\Delta z} + \lim_{\Delta z \to 0} f(z)\,\frac{[g(z + \Delta z) - g(z)]}{\Delta z}$$

$$= \lim_{\Delta z \to 0} g(z+\Delta z) \lim_{\Delta z \to 0} \frac{f(z+\Delta z)-f(z)}{\Delta z} + \lim_{\Delta z \to 0} f(z) \lim_{\Delta z \to 0} \frac{g(z+\Delta z)-g(z)}{\Delta z}$$

Hence proved the following

$$w'(z) = g(z)f'(z) + f(z)g'(z)$$

Differentiation Formulas

Example: Given $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ then find

$$\frac{d}{dz} f(z)$$

Solution:

$$\frac{d}{dz}(a_0 + a_1 z + \cdots + a_n z^n) = \frac{d}{dz}a_0 + \frac{d}{dz}a_1 z + \cdots + \frac{d}{dz}a_n z^n$$

$$= a_1 \frac{d}{dz} z + a_2 \frac{d}{dz} z^2 + \cdots + a_n \frac{d}{dz} z^n$$

$$= a_1 + 2a_2 z + 3a_3 z^2 + \cdots + n a_n z^{n-1}$$

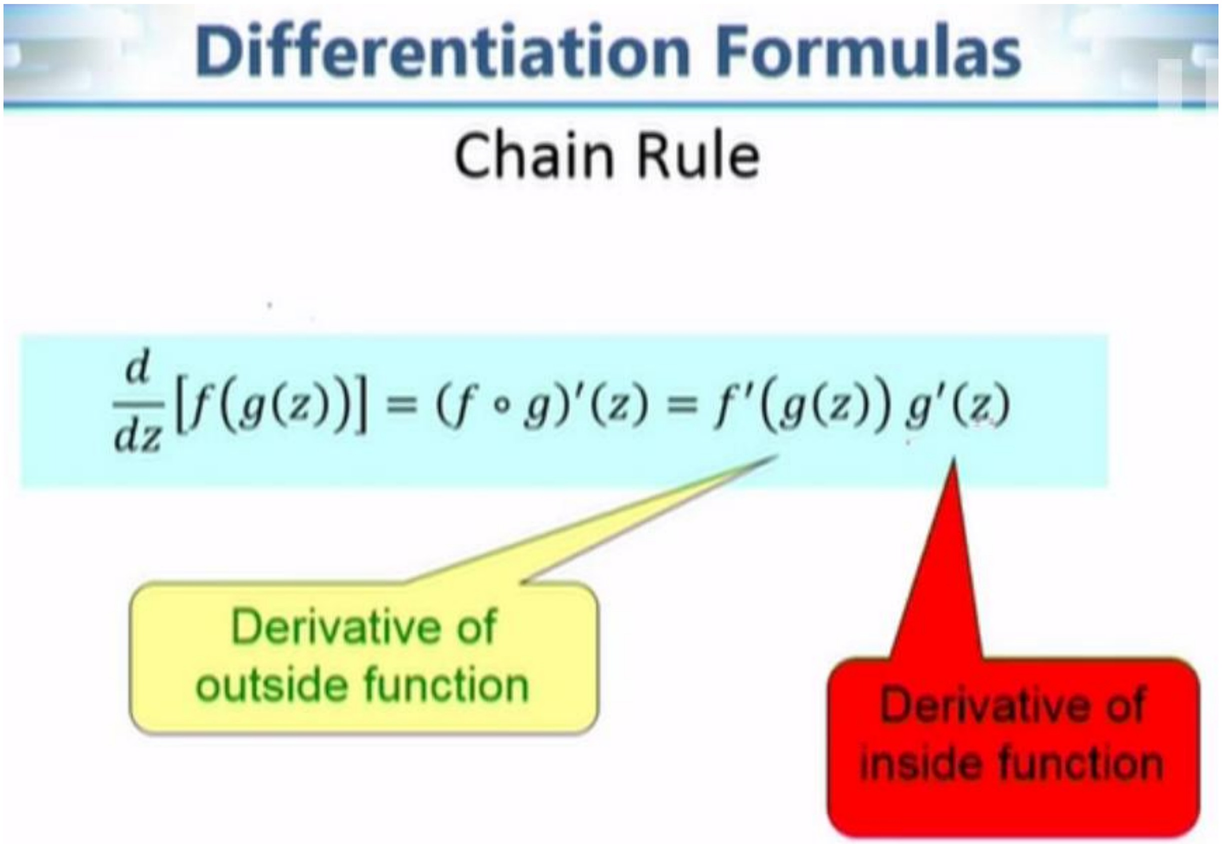
Differentiation Formulas

Chain Rule

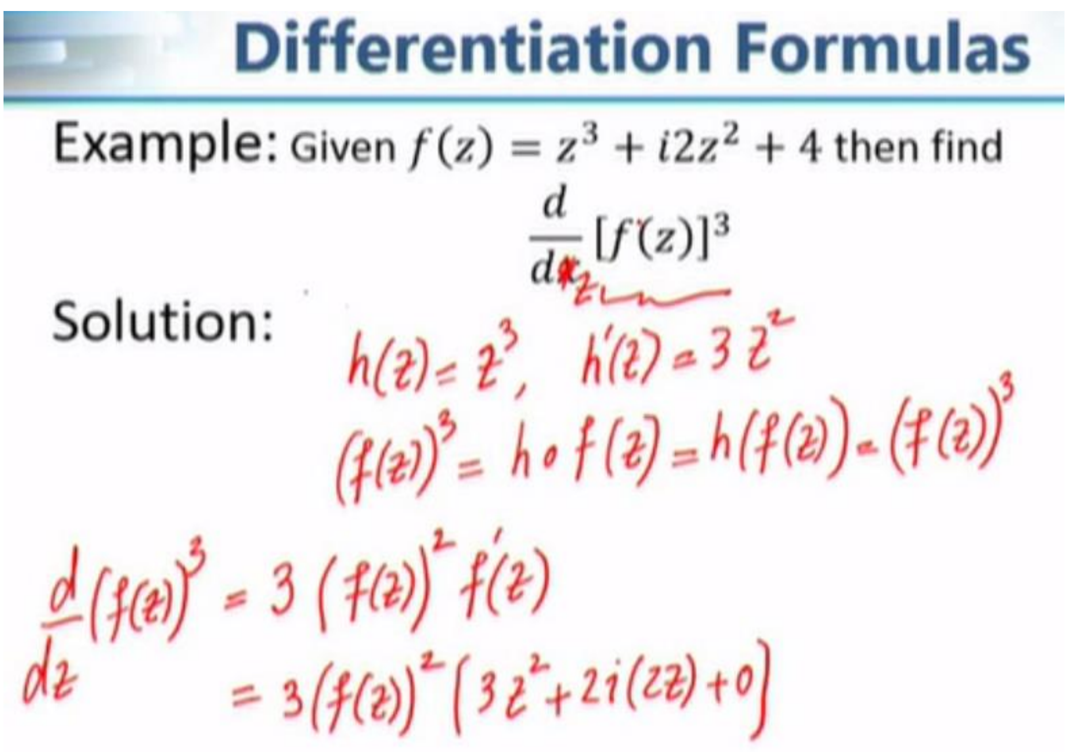
Chain rule is useful when we want to differentiate composition of two functions.

Given two functions $f(z)$ and $g(z)$ composition of f and g is given by

$$f \circ g(z) = f\big(g(z)\big)$$



Using Chain Rule



The Cauchy-Riemann Equations

Recall

Theorem: Suppose that

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

If $f(z)$ differentiable at a point $z_0 = x_0 + iy_0$ then u and v satisfy **Cauchy-Riemann equations**

$$u_x(x_0, y_0) = v_y(x_0, y_0) \text{ and } u_y(x_0, y_0) = -v_x(x_0, y_0).$$

The Cauchy-Riemann Equations

Example: We know $f(z) = z^2$ is differentiable and $f'(z) = 2z$. Check whether Cauchy-Riemann equations are satisfied by $f(z)$.

Solution: If $z = x + iy$, $f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + i(2xy)$

Here $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$

check Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$

$u_x = 2x$, $u_y = -2y$, $v_x = 2y$, $v_y = 2x$

$u_x = 2x = v_y$ ✓ $u_y = -2y = -v_x$ ✓

Hence Cauchy Riemann Equations are satisfied.

The Cauchy-Riemann Equations

Example: We Show that $f(z) = \bar{z}$ is nowhere differentiable.

Solution: If $z = x + iy$, $f(z) = f(x+iy) = \overline{(x+iy)} = x + i(-y)$

$\Rightarrow u(x,y) = x, \quad v(x,y) = -y$

check Cauchy-Riemann equations $\boxed{u_x = v_y}$, $u_y = -v_x$

$u_x = 1, \; u_y = 0, \; v_x = 0, \; v_y = -1$

$u_x = 1, \; v_y = -1$. If $u_x = v_y$ is satisfied then $1 = -1$

$\Rightarrow$ Cauchy-Riemann are nowhere satisfied.

C-R equations are not satisfied.

The Cauchy-Riemann Equations

Solution: We can rewrite $f(z)$ in the following form

$$f(z) = \begin{cases} \dfrac{(\bar{z})^2}{z} = \dfrac{x^3 - 3xy^2}{x^2 + y^2} + i\,\dfrac{y^3 - 3x^2y}{x^2 + y^2} & when\ z \neq 0 \\ 0 & when\ z = 0 \end{cases}$$

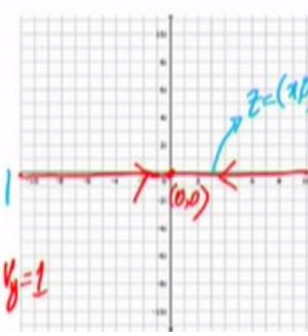
$u(x,y)$

Step1: Checking CR equations

u_x, u_y, v_x, v_y

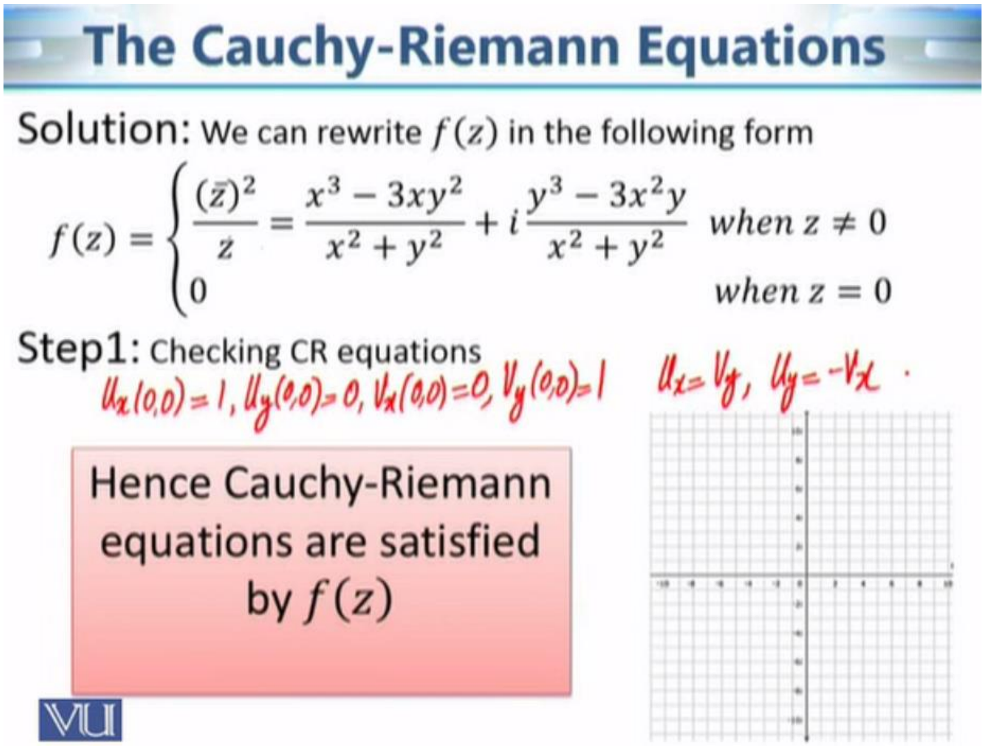
$$u_x(0,0) = \lim_{(x,0)\to(0,0)} \frac{u(x,0) - u(0,0)}{x - 0}$$

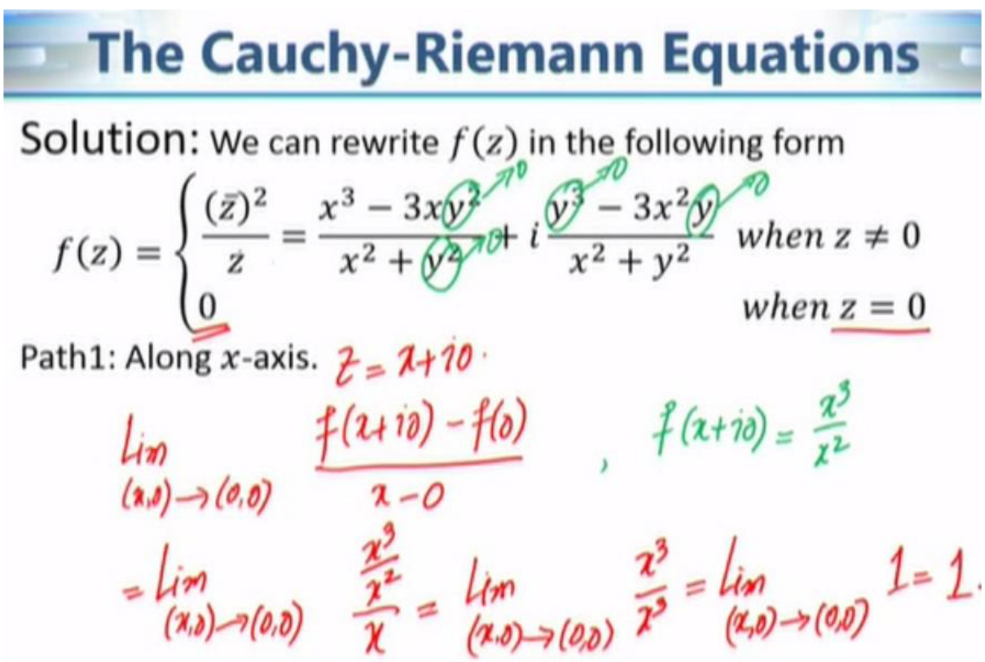
$$= \lim_{x\to 0} \frac{\frac{x^3}{x^2}}{x} = \lim_{x\to 0} \frac{x^3}{x^3} = \lim_{x\to 0} 1 = 1$$



$\Rightarrow u_x = 1$. Similarly $u_y(0,0) = 0$, $v_x(0,0) = 0$, $v_y = 1$

Uy is equal to vy so it satisfied the equation.





The function is not differentiable at the origin.

The Cauchy-Riemann Equations

Cauchy-Riemann Conditions for differentiability

Theorem: Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be defined in some neighborhood of the point $z_0 = x_0 + iy_0$ and suppose that

1. If all the partial derivatives u_x, v_y, u_y and v_x exist everywhere in the neighbourhood.
2. those partial derivatives are continuous at (x_0, y_0) and satisfy Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

Then f is differentiable at z_0, and the derivative $f'(z_0)$ can be computed with either of the following equations

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0),$$

and also

$$f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

CR Equations in Polar Coordinates

Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

Let us write

$$u_x, u_y, v_x, v_y$$

in terms of

$$u_r, u_\theta, v_r, v_\theta.$$

CR Equations in Polar Coordinates

Chain Rule

Suppose that $u(x, y)$ is a differentiable function of x and y, where $x = x(r, \theta)$ and $y = y(r, \theta)$ are differentiable functions of r and θ. Then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

and

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

CR Equations in Polar Coordinates

Given $u(x, y)$ and $v(x, y)$ be two component functions of complex valued function

$$f(x + iy) = u(x, y) + iv(x, y)$$

Then

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$$

where $x = r\cos\theta$, $y = r\sin\theta$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$u_r = u_x \cos\theta + u_y \sin\theta$$

CR Equations in Polar Coordinates

So we have

$$u_r = u_x \cos\theta + u_y \sin\theta \qquad v_r = v_x \cos\theta + v_y \sin\theta$$

$$u_\theta = -u_x r \sin\theta + u_y r \cos\theta \qquad v_\theta = -v_x r \sin\theta + v_y r \cos\theta$$

$$u_\theta = -u_x r \sin\theta + u_y r \cos\theta = -r(u_x \sin\theta - u_y \cos\theta)$$

$$= -r(v_y \sin\theta + v_x \cos\theta)$$

$$= -r v_r$$

$$\Rightarrow \boxed{u_\theta = -r v_r, \quad r u_r = v_\theta} \longleftrightarrow u_x = v_y,\ u_y = -v_x$$

CR Equations in Polar Coordinates

Theorem: Let the function

$$f(z) = u(r,\theta) + iv(r,\theta)$$

be defined throughout some neighbourhood of a non-zero point $z_0 = r_0\, e^{i\theta_0}$ and suppose that

1. the first order partial derivates of the functions u and v with respect to r and θ exist everywhere in the neighbourhood
2. those partial derivatives are continuous at (r_0, θ_0) and satisfy polar form of Cauchy-Riemann equations

$$ru_r = v_\theta, \qquad u_\theta = -rv_r$$

Then $f'(z_0)$ exists and its value is

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)$$

where the right side is to be evaluated at (r_0, θ_0)

CR Equations in Polar Coordinates

Example: Show that if f is the principal cube root function given by

$$f(re^{i\theta}) = f(z) = z^{\frac{1}{3}} = r^{\frac{1}{3}} \cos\frac{\theta}{3} + i\, r^{\frac{1}{3}} \sin\frac{\theta}{3},$$

where the domain is restricted to be

$$\{re^{i\theta} : r > 0 \text{ and } -\pi < \theta < \pi\}$$

then the derivative is given by

$$f'(z) = \frac{1}{3z^{\frac{2}{3}}} = \frac{1}{3} r^{-\frac{2}{3}} \cos\frac{\theta}{3} + i \frac{1}{3} r^{-\frac{2}{3}} \sin\frac{\theta}{3}$$

for every point in the domain.

CR Equations in Polar Coordinates

Solution:

$$f(re^{i\theta}) = r^{\frac{1}{3}} \cos\frac{\theta}{3} + i\, r^{\frac{1}{3}} \sin\frac{\theta}{3},$$

be defined throughout some neighbourhood of a non-zero point $z_0 = r_0\, e^{i\theta_0}$ and suppose that

CR Equations in Polar Coordinates

Solution:

$$f\left(re^{i\theta}\right) = r^{\frac{1}{3}}\cos\frac{\theta}{3} + i\, r^{\frac{1}{3}}\sin\frac{\theta}{3},$$

the first order partial derivates of the functions u and v with respect to r and θ exist everywhere in the neighbourhood

$$u(r,\theta) = r^{1/3}\cos\frac{\theta}{3}, \quad v(r,\theta) = r^{1/3}\sin\frac{\theta}{3}$$

$$u_r = \frac{1}{3} r^{-2/3}\cos\theta/3, \quad v_r = \frac{1}{3} r^{-2/3}\sin\frac{\theta}{3}$$

$$u_\theta = \frac{-1}{3} r^{1/3}\sin\frac{\theta}{3}, \quad v_\theta = \frac{1}{3} r^{1/3}\cos\frac{\theta}{3}$$

CR Equations in Polar Coordinates

Solution:

$$f\left(re^{i\theta}\right) = r^{\frac{1}{3}}\cos\frac{\theta}{3} + i\, r^{\frac{1}{3}}\sin\frac{\theta}{3},$$

those partial derivatives are continuous at (r_0, θ_0) and satisfy polar form of Cauchy-Riemann equations

$$r u_r = v_\theta, \checkmark \quad u_\theta = -r v_r$$

$$u_r = \frac{1}{3} r^{-2/3}\cos\theta/3, \quad v_\theta = \frac{1}{3} r^{1/3}\cos\frac{\theta}{3} = r\left(\frac{1}{3} r^{-2/3}\cos\theta/3\right)$$

$$v_\theta = \frac{1}{3} r^{1/3}\cos\frac{\theta}{3} \qquad = \frac{1}{3} r^{1/3}\cos\frac{\theta}{3}$$

Similarly $u_\theta = -r v_r$ is satisfied.

First C-R equation is satisfied

CR Equations in Polar Coordinates

Solution:

$$f(re^{i\theta}) = r^{\frac{1}{3}}\cos\frac{\theta}{3} + i\,r^{\frac{1}{3}}\sin\frac{\theta}{3},$$

Then $f'(z_0)$ exists and its value is

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)$$

where the right side is to be evaluated at (r_0, θ_0)

$$u_r = \frac{1}{3} r^{-2/3}\cos\frac{\theta}{3}, \quad v_r = \frac{1}{3} r^{-2/3}\sin\frac{\theta}{3}$$

$$f'(z_0) = e^{-i\theta_0}\left(\frac{1}{3} r_0^{-2/3}\cos\frac{\theta_0}{3} + i\frac{1}{3} r_0^{-2/3}\sin\frac{\theta_0}{3}\right)$$

CR Equations in Polar Coordinates

Solution:

$$f(re^{i\theta}) = r^{\frac{1}{3}}\cos\frac{\theta}{3} + i\,r^{\frac{1}{3}}\sin\frac{\theta}{3},$$

$$f'(z) = e^{-i\theta}\left[\frac{1}{3}r^{-\frac{2}{3}}\cos\frac{\theta}{3} + i\frac{1}{3}r^{-\frac{2}{3}}\sin\frac{\theta}{3}\right]$$

$$= e^{-i\theta}\left[\frac{1}{3}r^{-\frac{2}{3}}e^{\frac{i\theta}{3}}\right]$$

$$= \frac{1}{3}r^{-\frac{2}{3}}e^{-i\frac{2}{3}}$$

$$f'(z) = \frac{1}{3z^{\frac{2}{3}}}$$

$$z^{2/3} = r^{2/3}e^{i\frac{2}{3}\theta}$$

$$f(z) = z^{1/3}$$

The Cauchy-Riemann Equations

Cauchy-Riemann Conditions for differentiability

Theorem: Let $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ be defined in some neighborhood of the point $z_0 = x_0 + iy_0$ and suppose that

1. If all the partial derivatives u_x, v_y, u_y and v_x exist everywhere in the neighbourhood.
2. those partial derivatives are continuous at (x_0, y_0) and satisfy Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

Then f is differentiable at z_0, and the derivative $f'(z_0)$ can be computed with either of the following equations

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0),$$

and also

$$f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0).$$

Exmaples are

The Cauchy-Riemann Equations

Example: Show that the function

$$f(z) = e^{-y}\cos x + i\,e^{-y}\sin x$$

is differentiable for all z and find its derivative.

Solution: Let (x_0, y_0) be an arbitrary point.

> defined in some neighborhood of the point $z_0 = x_0 + iy_0$ ✓
> and suppose that

$f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$

$\Rightarrow u(x,y) = e^{-y}\cos x, \quad v(x,y) = e^{-y}\sin x$

$\Rightarrow u(x,y)$ & $v(x,y)$ are defined for all (x,y).

The Cauchy-Riemann Equations

Example: Show that the function

$$f(z) = e^{-y}\cos x + i\,e^{-y}\sin x$$

is differentiable for all z and find its derivative.

Solution:

> If all the partial derivatives u_x, v_y, u_y and v_x exist everywhere in the neighbourhood.

$u(x,y) = e^{-y}\cos x$	$v(x,y) = e^{-y}\sin x$
$u_x(x,y) = -e^{-y}\sin x$	$v_x(x,y) = e^{-y}\cos x$
$u_y(x,y) = -e^{-y}\cos x$	$v_y(x,y) = -e^{-y}\sin x$

The Cauchy-Riemann Equations

Example: Show that the function

$$f(z) = e^{-y}\cos x + i\, e^{-y}\sin x$$

is differentiable for all z and find its derivative.

Solution:

> those partial derivatives are continuous at (x_0, y_0) and satisfy Cauchy-Riemann equations
>
> $$u_x = v_y, \quad u_y = -v_x$$

$u_x = -e^{-y}\sin x$, $v_x = e^{-y}\cos x$

$u_y = -e^{-y}\cos x$, $v_y = -e^{-y}\sin x$

$u_x = -e^{-y}\sin x = v_y$ ✓, $u_y = -e^{-y}\cos x = -v_x$ ✓

The Cauchy-Riemann Equations

Example: Show that the function

$$f(z) = e^{-y}\cos x + i\, e^{-y}\sin x$$

is differentiable for all z and find its derivative.

Solution:

> Then f is differentiable at z_0, and the derivative $f'(z_0)$ can be computed with either of the following equations
>
> $$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0), ✓$$
>
> and also
>
> $$f'(z_0) = v_y(x_0, y_0) - i u_y(x_0, y_0). ✓$$

$u_x = -e^{-y}\sin x$, $v_x = e^{-y}\cos x$

$f'(z_0) = -e^{-y_0}\sin x_0 + i\left(e^{-y_0}\cos x_0\right)$