MTH401 Short Notes final term

Lec 19 to 45

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Lecture 19 Undetermined Coefficients: Annihilator Operator Approach

Undetermined Coefficients: Annihilator Operator Approach

The form of g(x): The input function g (x) has to have one of the following forms:

- A constant function k.
- A polynomial function
- An exponential function e^{x}
- The trigonometric functions $\sin(\beta\ x),\ \cos(\beta\ x)$
- Finite sums and products of these functions.

Otherwise, we cannot apply the method of undetermined coefficients.

Solution Method

Consider the following non-homogeneous linear differential equation with constantcoefficients of order n

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

If L denotes the following differential operator

$$L = a_n D^n + a_{n-1} D^{n-1} + \cdots + a_1 D + a_0$$

Then the non-homogeneous linear differential equation of order n can be written as $L(y) = g(x)$

The function g(X) should consist of finite sums and products of the proper kind of functions as already explained.

Step 1 Write the given non-homogeneous linear differential equation in the form

$$L(y) = g(x)$$

Step 2 Find the complementary solution y_c by finding the general solution of the associated homogeneous differential equation:

$$L(y) = 0$$

Step 3 Operate on both sides of the non-homogeneous equation with a differential operator L_1 that annihilates the function g(x).

Step 4 Find the general solution of the higher-order homogeneous differential equation

$$L_1 L(y) = 0$$

Step 5 Delete all those terms from the solution in step 4 that are duplicated in the complementary solution y_c, found in step 2.

Step 6 Form a linear combination y_p of the terms that remain. This is the form of a particular solution of the non-homogeneous differential equation

$$L(y) = g(x)$$

Step 7 Substitute equation y_p found in step 6 into the given non-homogeneous linear differential

$$L(y) = g(x)$$

Match coefficients of various functions on each side of the equality and solve the resulting system of equations for the unknown coefficients in y_p.

Step 8 With the particular integral found in step 7, form the general solution of the given differential equation as: $y = y_c + y_p$

Example 1

Solve $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = 4x^2$.

Solution:

Step 1 Since $\dfrac{dy}{dx} = Dy,\ \dfrac{d^2y}{dx^2} = D^2 y$

Therefore, the given differential equation can be written as

$$\left(D^2 + 3D + 2\right)y = 4x^2$$

Step 2 To find the complementary function y_c, we consider the associated homogeneous differential equation

$$\left(D^2 + 3D + 2\right)y = 0$$

The auxiliary equation is

$$m^2 + 3m + 2 = (m+1)(m+2) = 0$$
$$\Rightarrow \quad m = -1, -2$$

Therefore, the auxiliary equation has two distinct real roots.

$$m_1 = -1,\ m_2 = -2,$$

Thus, the complementary function is given by $y_c = c_1 e^{-x} + c_2 e^{-2x}$

Step 3 In this case the input function is

$$g(x) = 4x^2$$

Further $$D^3 g(x) = 4D^3 x^2 = 0$$

Therefore, the differential operator D^3 annihilates the function g. Operating on both sides of the equation in step 1, we have

$$D^3(D^2 + 3D + 2)y = 4D^3 x^2$$
$$D^3(D^2 + 3D + 2)y = 0$$

This is the homogeneous equation of order 5. Next we solve this higher order equation.

Step 4 The auxiliary equation of the differential equation in step 3 is

$$m^3(m^2 + 3m + 2) = 0$$

$$m^3(m+1)(m+2) = 0$$

$$m = 0, 0, 0, -1, -2$$

Thus its general solution of the differential equation must be

$$y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + c_5 e^{-2x}$$

Step 5 The following terms constitute y_c

$$c_4 e^{-x} + c_5 e^{-2x}$$

Therefore, we remove these terms and the remaining terms are

$$c_1 + c_2 x + c_3 x^2$$

Step 6 This means that the basic structure of the particular solution y_p is

$$y_p = A + Bx + Cx^2,$$

Where the constants c_1, c_2 and c_3 have been replaced, with A, B, and C, respectively.

Substituting into the given differential equation, we have

$$(2C)x^2 + (2B + 6C)x + (2A + 3B + 2C) = 4x^2 + 0x + 0$$

Equating the coefficients of x^2, x and the constant terms, we have

$$\begin{aligned} 2C &= 4 \\ 2B + 6C &= 0 \\ 2A + 3B + 2C &= 0 \end{aligned}$$

Solving these equations, we obtain

$$A = 7, \quad B = -6, \quad C = 2$$

Hence $$y_p = 7 - 6x + 2x^2$$

Step 7 Since $$y_p = A + Bx + Cx^2$$

$$y_p' = B + 2Cx,$$

$$y_p'' = 2C$$

Therefore $$y_p'' + 3y_p' + 2y_p = 2C + 3B + 6Cx + 2A + 2Bx + 2Cx^2$$

or $$y_p'' + 3y_p' + 2y_p = (2C)x^2 + (2B + 6C)x + (2A + 3B + 2C)$$

Step 8 The general solution of the given non-homogeneous differential equation is

$$y = y_c + y_p$$

$$y = c_1e^{-x} + c_2e^{-2x} + 7 - 6x + 2x^2$$

Lecture 20 Variations of Parameters

Variations of Parameters

- That a non-homogeneous linear differential equation with constant coefficients is an equation of the form

$$a_n\frac{d^ny}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0y = g(x)$$

- The general solution of such an equation is given by

  General Solution = Complementary Function + Particular Integral

- That the general solution of a linear first orders differential equation of the form

$$\frac{dy}{dx} + P(x)y = f(x)$$

is given by $$y = e^{-\int Pdx}. \int e^{\int Pdx} f(x)dx + c_1e^{-\int Pdx}$$

Note that

In this last equation, the 2nd term: $y_c = c_1e^{-\int Pdx}$

Is solution of the associated homogeneous equation $\frac{dy}{dx} + P(x)y = 0$

Similarly, the 1st term $y_p = e^{-\int Pdx}.\int e^{\int Pdx}.f(x)dx$

Is a particular solution of the first order non-homogeneous linear differential equation?

Therefore, the solution of the first order linear differential equation can be written in the form

$$y = y_c + y_p$$

First order equation

The particular solution y_p of the first order linear differential equation is given by

$$y_p = e^{-\int Pdx}.\int e^{\int Pdx}.f(x)dx$$

This formula can also be derived by another method, known as the variation of parameters. The basic procedure is same as discussed in the lecture on construction of a second solution.

is the solution of the homogeneous differential equation

$$\frac{dy}{dx} + P(x)y = 0,$$

and the equation is linear. Therefore, the general solution of the equation is

$$y = c_1 y_1(x)$$

The variation of parameters consists of finding a function $u_1(x)$ such that $y_p = u_1(x)\, y_1(x)$ is a particular solution of the non-homogeneous differential equation $\frac{dy}{dx} + P(x)\, y = f(x)$

Notice that the parameter c_1 has been replaced by the variable u_1. We substitute y_p in the given equation to obtain

$$u_1\left[\frac{dy_1}{dx} + P(x)y_1\right] + y_1\frac{du_1}{dx} = f(x)$$

Since y_1 is a solution of the non-homogeneous differential equation. Therefore we must have

$$\frac{dy_1}{dx} + P(x)y_1 = 0$$

So that we obtain

$\therefore$

$$y_1\frac{du_1}{dx} = f(x)$$

This is a variable separable equation. By separating the variables, we have

$$du_1 = \frac{f(x)}{y_1(x)}dx$$

Integrating the last expression $w.r.to\ x$, we obtain

$$u_1(x) = \int \frac{f(x)}{y_1}dx = \int e^{\int Pdx} \cdot f(x)dx$$

Therefore, the particular solution y_p of the given first-order differential equation is .

$$y = u_1(x)y_1$$

or

$$y_p = e^{-\int Pdx}.\int e^{\int Pdx}.f(x)\,dx$$

$$u_1 = \int \frac{f(x)}{y_1(x)}dx$$

Second Order

Equation Consider the 2nd order linear non-homogeneous differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$$

By dividing with $a_2(x)$, we can write this equation in the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

The functions $P(x)$, $Q(x)$ and $f(x)$ are continuous on some interval I. For the complementary function we consider the associated homogeneous differential equation

$$y'' + P(x)y' + Q(x)y = 0$$

Complementary function

Suppose that y_1 and y_2 are two linearly independent solutions of the homogeneous equation. Then y_1 and y_2 form a fundamental set of solutions of the homogeneous equation on the interval I. Thus the complementary function is

$$y_c = c_1 y_1(x) + c_2 y_2(x)$$

Since y_1 and y_2 are solutions of the homogeneous equation. Therefore, we have

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0$$

$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

Particular Integral

For finding a particular solution y_p, we replace the parameters c_1 and c_2 in the complementary function with the unknown variables $u_1(x)$ and $u_2(x)$. So that the assumed particular integral is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Since we seek to determine two unknown functions u_1 and u_2, we need two equations involving these unknowns. One of these two equations results from substituting the assumed y_p in the given differential equation. We impose the other equation to simplify the first derivative and thereby the 2nd derivative of y_p.

$$y_p' = u_1 y_1' + y_1 u_1' + u_2 y_2' + u_2' y_2 = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$$

To avoid 2nd derivatives of u_1 and u_2, we impose the condition

$$u_1' y_1 + u_2' y_2 = 0$$

Then $$y_p' = u_1 y_1' + u_2 y_2'$$

So that

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$$

Therefore

$$y_p'' + P\,y_p' + Q\,y_p = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + P u_1 y_1' + P u_2 y_2' + Q u_1 y_1 + Q u_2 y_2$$

Substituting in the given non-homogeneous differential equation yields

$$u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2' + P u_1 y_1' + P u_2 y_2' + Q u_1 y_1 + Q u_2 y_2 = f(x)$$

or $$u_1[y_1'' + P\,y_1' + Q\,y_1] + u_2[y_2'' + P y_2' + Q y_2] + u_1' y_1' + u_2' y_2' = f(x)$$

Now making use of the relations

$$y_1'' + P(x)y_1' + Q(x)y_1 = 0$$
$$y_2'' + P(x)y_2' + Q(x)y_2 = 0$$

we obtain

$$u_1'y_1' + u_2'y_2' = f(x)$$

Hence u_1 and u_2 must be functions that satisfy the equations

$$u_1'y_1 + u_2'y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = f(x)$$

By using the Cramer's rule, the solution of this set of equations is given by

$$u_1' = \frac{W_1}{W}, \qquad u_2' = \frac{W_2}{W}$$

Where W, W_1 and W_2 denote the following determinants

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

The determinant W can be identified as the Wronskian of the solutions y_1 and y_2. Since the solutions y_1 and y_2 are linearly independent on I. Therefore

$$W(y_1(x), y_2(x)) \neq 0, \ \forall \ x \in I.$$

Summary of the Method

Summary of the Method

To solve the 2nd order non-homogeneous linear differential equation

$$a_2y'' + a_1y' + a_0y = g(x),$$

using the variation of parameters, we need to perform the following steps:

Step 1 We find the complementary function by solving the associated homogeneous differential equation

$$a_2y'' + a_1y' + a_0y = 0$$

Step 2 If the complementary function of the equation is given by

$$y_c = c_1y_1 + c_2y_2$$

then y_1 and y_2 are two linearly independent solutions of the homogeneous differential equation. Then compute the Wronskian of these solutions.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Step 3 By dividing with a_2, we transform the given non-homogeneous equation into the standard form

$$y'' + P(x)y' + Q(x)y = f(x)$$

and we identify the function $f(x)$.

Step 4 We now construct the determinants W_1 and W_2 given by

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 5 Next we determine the derivatives of the unknown variables u_1 and u_2 through the relations

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

Step 6 Integrate the derivatives u_1' and u_2' to find the unknown variables u_1 and u_2. So that

$$u_1 = \int \frac{W_1}{W}\, dx, \quad u_2 = \int \frac{W_2}{W}\, dx$$

Step 7 Write a particular solution of the given non-homogeneous equation as

$$y_p = u_1 y_1 + u_2 y_2$$

Step 8 The general solution of the differential equation is then given by

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2 .$$

Constants of Integration

The constants of integration, when computing the indefinite integrals in step 6 to find the unknown functions of u_1 and u_2. For, if we do introduce these constants, then

$$y_p = (u_1 + a_1) y_1 + (u_2 + b_1) y_2$$

So that the general solution of the given non-homogeneous differential equation is

$$y = y_c + y_p = c_1 y_1 + c_2 y_2 + (u_1 + a_1) y_1 + (u_2 + b_1) y_2$$

or $$y = (c_1 + a_1) y_1 + (c_2 + b_1) y_2 + u_1 y_1 + u_2 y_2$$

If we replace $c_1 + a_1$ with C_1 and $c_2 + b_1$ with C_2, we obtain

$$y = C_1 y_1 + C_2 y_2 + u_1 y_1 + u_2 y_2$$

This does not provide anything new and is similar to the general solution found in step 8, namely

$$y = c_1 y_1 + c_2 y_2 + u_1 y_1 + u_2 y_2$$

Lecture 21 Variation of Parameters Method for Higher-Order Equations

The method of the variation of parameters just examined for second-order differential equations can be generalized for an nth-order equation of the type.

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

The application of the method to nth order differential equations consists of performing the following steps.

Step 1 *To find the complementary function we solve the associated homogeneous equation*

$$a_n\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0y=0$$

Step 2 *Suppose that the complementary function for the equation is*

$$y=c_1y_1+c_2y_2+\cdots+c_ny_n$$

Then $y_1, y_2, \ldots, y_n$ *are* n *linearly independent solutions of the homogeneous equation. Therefore, we compute Wronskian of these solutions.*

$$W\left(y_1,y_2,y_3,\ldots,y_n\right)=\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Step 4 *We write the differential equation in the form*

$$y^{(n)}+P_{n-1}\left(x\right)y^{(n-1)}+\cdots+P_1\left(x\right)y'+P_\circ(x)y=f\left(x\right)$$

and compute the determinants W_k*;* $k=1,2,\ldots,n$*; by replacing the kth column of W by the column*

$$\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(x) \end{matrix}$$

Step 5 *Next we find the derivatives* $u_1', u_2', \ldots, u_n'$ *of the unknown functions* $u_1, u_2, \ldots, u_n$ *through the relations*

$$u_k'=\frac{W_k}{W},\qquad k=1,2,\ldots,n$$

Note that these derivatives can be found by solving the n *equations*

$$\begin{array}{lcl} y_1u_1' + y_2u_2' + \cdots + y_nu_n' & = & 0 \\ y_1'u_1' + y_2'u_2' + \cdots + y_n'u_n' & = & 0 \\ \vdots & & \\ y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' & = & f(x) \end{array}$$

Step 6 *Integrate the derivative functions computed in the step 5 to find the functions* u_k

$$u_k=\int\frac{W_k}{W}dx,\qquad k=1,2,\ldots,n$$

Step 7 *We write a particular solution of the given non-homogeneous equation as*

$$y_p=u_1\left(x\right)y_1\left(x\right)+u_2\left(x\right)y_2\left(x\right)+\cdots+u_n\left(x\right)y_n\left(x\right)$$

Step 8 *Having found the complementary function* y_c *and the particular integral* y_p*, we write the general solution by substitution in the expression:* $y=y_c+y_p$

Example

Solve the differential equation by variation of parameters $\frac{d^3y}{dx^3}+\frac{dy}{dx}=\csc x$.

Solution:

Step1

The associated homogeneous equation is $\frac{d^3y}{dx^3}+\frac{dy}{dx}=0$

Auxiliary equation $m^3+m=0 \Rightarrow m\left(m^2+1\right)=0 \Rightarrow m=0, \quad m=\pm i$

Therefore the complementary function is $y_c=c_1+c_2\cos x+c_3\sin x$

Step 2: *Since* $y_c=c_1+c_2\cos x+c_3\sin x \Rightarrow y_1=1, \quad y_2=\cos x, \quad y_3=\sin x$

So that the Wronskian of the solutions y_1, y_2 and y_3

$$W\left(y_1,y_2,y_3\right)=\begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

By the elementary row operation R_1+R_3, *we have*

$$=\begin{vmatrix} 1 & 0 & 0 \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix}$$

$$=\left(\sin^2 x+\cos^2 x\right)=1\neq 0$$

Step 3: *The given differential equation is already in the required standard form*

$$y'''+0\,y''+\,y'+0\,y=\csc x$$

Step 4: *Next we find the determinants* W_1, W_2 and W_3 *by respectively, replacing* 1^{st}, 2^{nd} *and* 3^{rd} *column of* W *by the column* $\begin{matrix} 0 \\ 0 \\ \csc x \end{matrix}$

$$W_1=\begin{vmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \csc x & -\cos x & -\sin x \end{vmatrix}$$

$$=\csc x\,\left(\sin^2 x+\cos^2 x\right)=\csc x$$

$$W_2=\begin{vmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \csc x & -\sin x \end{vmatrix}$$

$$=\begin{vmatrix} 0 & \cos x \\ \csc x & -\sin x \end{vmatrix}=-\cos x\csc x=-\cot x$$

and $$W_3 = \begin{vmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \csc x \end{vmatrix} = \begin{vmatrix} -\sin x & 0 \\ -\cos x & \csc x \end{vmatrix} = -\sin x \csc x = -1$$

Step 5: *We compute the derivatives of the functions* u_1, u_2 and u_3 *as:*

$$u_1' = \frac{W_1}{W} = \csc x$$

$$u_2' = \frac{W_2}{W} = -\cot x$$

$$u_3' = \frac{W_3}{W} = -1$$

Step 6: *Integrate these derivatives to find* u_1, u_2 and u_3

$$u_1 = \int \frac{W_1}{W} dx = \int \csc x dx = \ln|\csc x - \cot x|$$

$$u_2 = \int \frac{W_2}{W} dx = \int -\cot x dx = \int \frac{-\cos x}{\sin x} dx = -\ln|\sin x|$$

$$u_3 = \int \frac{W_3}{W} dx = \int -1 dx = -x$$

Step 7: *A particular solution of the non-homogeneous equation is*

$$y_p = \ln|\csc x - \cot x| - \cos x \ln|\sin x| - x \sin x$$

Step 8: *The general solution of the given differential equation is:*

$$y = c_1 + c_2 \cos x + c_3 \sin x + \ln|\csc x - \cot x| - \cos x \ \ln|\sin x| - x \sin x$$

Lecture 22 Applications of Second Order Differential Equation

Applications of Second Order Differential Equation

A single differential equation can serve as mathematical model for many different phenomena in science and engineering.

Different forms of the 2nd order linear differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x)$ appear in the analysis of problems in physics, chemistry and biology.

The present and next lecture we shall focus on one application; the motion of a mass attached to a spring.

Simple Harmonic Motion

The Newton's 2nd law is combined with the Hook's Law; we can derive a differential equation governing the motion of a mass attached to spring–the simple harmonic motion.

Hook's Law

Suppose that

- A mass is attached to a flexible spring suspended from a rigid support, then
- The spring stretches by an amount's'.
- The spring exerts a restoring F opposite to the direction of elongation or stretch.

The Hook's law states that the force F is proportional to the elongation s. i.e

$$F = ks$$

Where k is constant of proportionality, and is called spring constant.

Newton's Second Law

When a force F acts upon a body, the acceleration a is produced in the direction of the force whose magnitude is proportional to the magnitude of force. i.e

$$F = ma$$

Where m is constant of proportionality and it represents mass of the body.

Weight

- The gravitational force exerted by the earth on a body of mass m is called weight of the body, denoted by W.
- In the absence of air resistance, the only force acting on a freely falling body is its weight. Thus from Newton's 2nd law of motion.

Differential Equation

When a body of mass m is attached to a spring the spring stretches by an amount s and attains an equilibrium position. At the equilibrium position, the weight is balanced by the restoring force ks. Thus, the condition of equilibrium is

$$mg = ks \Rightarrow mg - ks = 0$$

If the mass is displaced by an amount x from its equilibrium position and then released. The restoring force becomes k(s + x). So that the resultant of weight and the restoring force acting on the body is given by

$$Resultant = -k(s+x) + mg.$$

By Newton's 2nd Law of motion, we can written

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg$$

or $$m\frac{d^2x}{dt^2} = -kx - ks + mg$$

Since $$mg - ks = 0$$

Therefore $$m\frac{d^2x}{dt^2} = -kx$$

By dividing with m, the last equation can be written as:

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

or $$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Where $\omega^2 = \frac{k}{m}$. This equation is known as the equation of simple harmonic motion or as the free un-damped motion.

Initial Conditions

Associated with the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

are the obvious initial conditions

$$x(0) = \alpha, \quad x'(0) = \beta$$

These initial conditions represent the initial displacement and the initial velocity. For example

- If $\alpha > 0$, $\beta < 0$ then the body starts from a point below the equilibrium position with an imparted upward velocity.
- If $\alpha < 0$, $\beta = 0$ then the body starts from rest $|\alpha|$ units above the equilibrium position.

Solution and Equation of Motion

Consider the equation of simple harmonic motion

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

Put $$x = e^{mx}, \quad \frac{d^2x}{dt^2} = m^2 e^{mx}$$

Then the auxiliary equation is

$$m^2 + \omega^2 = 0 \quad \Rightarrow \quad m = \pm i\omega$$

Thus the auxiliary equation has complex roots.

$$m_1 = \omega i, \quad m_2 = -\omega i$$

Hence, the general solution of the equation of simple harmonic motion is

$$x(t) = c_1 \cos\omega t + c_2 \sin\omega t$$

Alternative form of Solution

It is often convenient to write the above solution in a alternative simpler form. Consider

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t$$

and suppose that $A,\ \phi \in R$ such that

$$c_1 = A \sin \phi,\ c_2 = A \cos \phi$$

Then $$A = \sqrt{c_1^2 + c_2^2}\ ,\ \tan \phi = \frac{c_1}{c_2}$$

So that

$$x(t) = A \sin \omega t\ \cos \phi + B \cos\ \omega t\ \sin \phi$$

or $$x(t) = A \sin(\omega t + \phi)$$

The number ϕ is called the phase angle;

This form of the solution of the equation of simple harmonic motion is very useful because

Amplitude of free vibrations becomes very obvious

The times when the body crosses equilibrium position are given by

$$x = 0 \Rightarrow \sin(\omega t + \phi) = 0$$

or $$\omega t + \phi = n\pi$$

Where n is a non-negative integer.

The Nature of Simple Harmonic Motion

Amplitude

- The solution of the equation of simple harmonic motion can be written as $x(t) = A \sin(\omega t + \phi)$
- This maximum distance called the amplitude of motion and is given by

$$Amplitude = A = \sqrt{c_1^2 + c_2^2}$$

A Vibration or a Cycle

In travelling from x = A to x = - A and then back to A, the vibrating body completes one vibration or one cycle.

Period of Vibration

The simple harmonic motion of the suspended body is periodic and it repeats its position after a specific time period T . We know that the distance of the mass at any time t is given by

$$x = A\sin(\omega t + \phi)$$

Since

$$A\sin\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right]$$

$$= A\sin[(\omega t + \phi + 2\pi)]$$

$$= A\sin[(\omega t + \phi\ )]$$

Therefore, the distances of the suspended body from the equilibrium position at the times t and $t + \dfrac{2\pi}{\omega}$ are same

Further, velocity of the body at any time t is given by

$$\frac{dx}{dt} = A\omega\cos(\omega t + \phi\ )$$

$$A\omega\cos\left(\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right)$$

$$= A\omega\cos[\omega t + \phi\ + 2\pi]$$

$$= A\omega\cos(\omega t + \phi\ )$$

Therefore the velocity of the body remains unaltered if t is increased by $2\pi/\omega$. Hence the time period of free vibrations described by the 2nd order differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

is given by

$$T = \frac{2\pi}{\omega}$$

Frequency

The number of vibration /cycle completed in a unit of time is known as frequency of the free vibrations, denoted by f. Since the cycles completed in time T is 1. Therefore, the number of cycles completed in a unit of time is 1/T.

Hence

$$f = \frac{1}{T} = \frac{\omega}{2\pi}$$

Lecture No. 23 Damped Motions

Damping Force

The damping forces acting on a body are considered to be proportional to a power of the instantaneous velocity dt/dx . In the hydro dynamical problems, the damping force is proportional to $(dx/dt)^2$. So that in these problems

$$Damping\ force = -\beta\left(\frac{dx}{dt}\right)^2$$

Where β is a positive damping constant and negative sign indicates that the damping force acts in a direction opposite to the direction of motion. In the present discussion, we shall assume that the damping force is proportional to the instantaneous velocity dt /dx . Thus for us

$$Damping\ force = -\beta\left(\frac{dx}{dt}\right)$$

The Differential Equation of the Motion

The differential equation of the motion with a damping force will be given by:

$$m\ddot{x} + \lambda\dot{x} + kx = 0$$

In order to obtain the leading coefficient equal to 1, we divide this equation by the mass:

$$\ddot{x} + \frac{\lambda}{m}\dot{x} + \frac{k}{m}x = 0$$

Non-conservation of energy

We may multiply the equation of motion by the velocity $\dot{x}$ in order to get an integrable form:

$$m\ddot{x}\dot{x} + \lambda\dot{x}^2 + kx\dot{x} = 0$$

Now we integrate this equation from **0** to **t** to obtain an expression for the energy:

$$m\frac{\dot{x}^2(t)}{2} + k\frac{x^2(t)}{2} = m\frac{\dot{x}^2(0)}{2} + k\frac{x^2(0)}{2} - \frac{\lambda}{2}\int_0^t \dot{x}^2 dt$$

Denoting the mechanical energy by

$$E(t) := m\frac{\dot{x}^2(t)}{2} + k\frac{x^2(t)}{2}$$

the variation of energy is given by:

$$E(t) - E(0) = -\frac{\lambda}{2}\int_0^t \dot{x}^2 dt$$

That is to say, if the damping friction force coefficient λ is not zero, or integration over square of the velocity does not vanish, the system is losing energy. Physically speaking, friction converts mechanical energy into thermal energy.

Initial condition

With the free motion equation, there are generally two bits of information one must have to appropriately describe the mass's motion.

- The starting position of the mass X^2.
- The starting direction and magnitude of motion. V
- Generally, one isn't present without the other. For simplicity, we will consider all displacement below the equilibrium point as X >0 and above as X<0.
- For upward motion V<0, and for downward motion V>0.

Solution

We look for a general solution in the following form:

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

substituting this solution into the equation, we find the quadratic equation:

$$ms^2 + \lambda s + k = 0$$

the solution of this equation is given by:

$$s_1, s_2 = \frac{-\lambda \pm \sqrt{\lambda^2 - 4mk}}{2m}$$

And A_1,A_2 are determined by initial conditions. Obviously, this solution may have real-valued or complex-valued roots. In any case, the real part of the roots is always negative (since both K and M are positive), implying stable solution. When both roots are real-valued, the system is called over-damped; whilst it has two complex roots (where one is the complex conjugate of the other) the system is called under-damped. In case λ=0, both roots has zero real-parts, and the solution is oscillating, energy-conserving.

Case 1 Real and distinct roots

If $\lambda^2-\omega^2 > 0$ then $\beta > k$ and the system is said to be over-damped. The solution of the equation of free damped motion is

$$x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$$

$$x(t) = e^{-\lambda t}\left[c_1 e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2 e^{-\sqrt{\lambda^2-\omega^2}\,t}\right]$$

This equation represents smooth and non-oscillatory motion.

Case 2 Real and equal roots

If $\lambda^2-\omega^2 >=0$ then $\beta = k$ and the system is said to be critically damped, because any slight decrease in the damping force would result in oscillatory motion. The general solution of the differential equation of free damped force is

$$x(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t}$$

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

Case 3 Complex roots

If $\lambda^2-\omega^2 < 0$ then $\beta < k$ and the system is said to be under-damped. We need to rewrite the roots of the auxiliary equation as:

$$m_1 = -\lambda + \sqrt{\omega^2 - \lambda^2}\,i, \quad m_2 = -\lambda - \sqrt{\omega^2 - \lambda^2}\,i$$

Thus, the general solution of the equation of free damped motion is

$$x(t) = e^{-\lambda t}\left[c_1 \cos\sqrt{\omega^2 - \lambda^2}\,t + c_2 \sin\sqrt{\omega^2 - \lambda^2}\,t\right]$$

This represents an oscillatory motion; but amplitude of vibration $\to 0$ as $t \to \infty$ because of the coefficient $e^{-\lambda t}$.

Alternative form of the Solution

When $\lambda^2 - \omega^2 < 0$ the solution of the differential equation of free damped motion

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

$$x(t) = e^{-\lambda t}\left[c_1 \cos\sqrt{\omega^2 - \lambda^2}\,t + c_2 \sin\sqrt{\omega^2 - \lambda^2}\,t\right]$$

Suppose that A and ϕ are two real numbers such that

$$\sin\phi = \frac{c_1}{A}, \quad \cos\phi = \frac{c_2}{A}$$

$$A = \sqrt{c_1{}^2 + c_2{}^2}, \quad \tan\phi = \frac{c_1}{c_2}$$

The number ϕ is known as the phase angle. Then the solution of the equation becomes:

$$x(t) = Ae^{-\lambda t}\left[\sin\sqrt{\omega^2 - \lambda^2}\,t\cos\phi + \cos\sqrt{\omega^2 - \lambda^2}\,t\sin\phi\right]$$

$$x(t) = Ae^{-\lambda t}\sin\left(\sqrt{\omega^2 - \lambda^2}\,t + \phi\right)$$

The coefficient $Ae^{-\lambda t}$ is called the damped amplitude of vibrations.

The time interval between two successive maxima of x(t)is called quasi period, and is given by the number $\frac{2\pi}{\sqrt{\omega^2 - \lambda^2}}$.

Quasi Period

Since $$x(t) = \frac{2}{3}\sqrt{10}e^{-t}\sin(3t + 4.391)$$

Therefore $$\sqrt{\lambda^2 - \omega^2} = 3$$

So that the quasi period is given by

$$\frac{2\pi}{\sqrt{\lambda^2 - \omega^2}} = \frac{2\pi}{3}\text{ seconds}$$

Hence, difference between the successive t_γ and t_γ^* is $\frac{\pi}{3}$ units.

Lecture 24 Forced Motion

Applications of second order linear differential equations

A vibrational system consisting of a body of mass m attached to a spring. The motion of the body is being driven by an external force f (t) i.e. forced motion.

Flow of current in an electrical circuit that consists of an inductor, resistor and a capacitor connected in series, because of its similarity with the forced motion.

Forced motion with damping

Suppose that we now take into consideration an external force f (t). Then, the forces acting on the system are:

- Weight of the body = mg
- The restoring force = - k(s + x)
- The damping effect = -ß (dx / dt)
- The external force = f (t).

Hence x denotes the distance of the mass m from the equilibrium position. Thus the total force acting on the mass m is given by

$$Force = mg - k(s + x) - \beta\left(\frac{dx}{dt}\right) + f(t)$$

By the Newton's 2nd law of motion, we have

$$Force = ma = m\frac{d^2x}{dt^2}$$

Therefore
$$m\frac{d^2x}{dt^2} = mg - ks - kx - \beta\left(\frac{dx}{dt}\right) + f(t)$$

But
$$mg - ks = 0$$

So that
$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\left(\frac{dx}{dt}\right) + \frac{k}{m}x = \frac{f(t)}{m}$$

or
$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

where $F(t) = \frac{f(t)}{m}$, $2\lambda = \frac{\beta}{m}$ and $\omega^2 = \frac{k}{m}$.

Transient and Steady-State Terms

Due to the presence of the factor e^{-3t} we notice that the complementary function

$$x_c(t) = e^{-3t}\left(\frac{38}{51}\cos t - \frac{86}{51}\sin t\right)$$

possesses the property that

$$\lim_{x\to\infty} x_c(t) = 0$$

Thus for large time, the displacements of the weight are closely approximated by the particular solution

$$x_p(t) = -\frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$$

Since $x_c(t) \to 0$ as $t \to \infty$, it is said to be transient term or transient solution. The particular solution $x_p(t)$ is called the steady-state solution.

Hence, when F is a periodic function, such as

$$F(t) = F_0 \sin \gamma t \quad or \quad F(t) = F_0 \cos \gamma t$$

The general solution of the equation

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = F(t)$$

consists of

$$x(t) = Transient\ solution + Steady\ State\ Solution$$

Motion without Damping

If the system is impressed upon by a periodic force and there is no damping force then there is no transient term in the solution.

Electric Circuits

Many different physical systems can be described by a second order linear differential equation similar to the differential equation of the forced motion:

$$m\frac{d^2x}{dt^2} + \beta \frac{dx}{dt} + kx = f(t)$$

The LRC Series Circuits

The LRC series circuit consist of an inductor, resistor and capacitor connected in series with a time varying source voltage E(t),

Resistor

- A resistor is an electrical component that limits or regulates the flow of electrical current in an electrical circuit.
- The measure of the extent to which a resistor impedes or resists with the flow of current through it is called resistance, denoted by R.

Clearly higher the resistance, lower the flow of current. Lower the resistance, higher the flow of current. Therefore, we conclude that the flow of current is inversely proportional to the resistance,

$$I = V.\frac{1}{R} \Rightarrow V = IR$$

Inductor

An inductor is a passive electronic component that stores energy in the form of magnetic field. This property of the coil due to which it opposes any change of current through it is called the inductance.

Suppose that I denotes the current then the rate of change of current is given by $\frac{dI}{dt}$ This produces a counter emf voltage V. Then V is directly proportional to $\frac{dI}{dt}$

$$V \alpha \frac{dI}{dt} \Rightarrow \quad V = L\frac{dI}{dt}$$

Capacitor

A capacitor is a passive electronic component of an electronic circuit that has the ability to store charge and opposes any change of voltage in the circuit. The ability of a capacitor to store charge is called capacitance of the capacitor denoted byC . If + q coulomb of a charge to the capacitor and the potential difference of V volts is established between 2 plates of the capacitor then

$$q \ \alpha \ C \Rightarrow q = CV$$
$$V = {}^{q}\!/\!{}_{C}$$

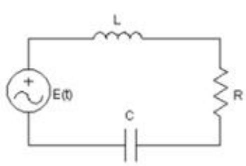
Where "C" is called constant of proportionality, which represent capacitance. The standard unit to measure capacitance is farad, denoted by F .

Kirchhoff's Voltage Law

The Kirchhoff's 2nd law states that the sum of the voltage drops around any closed loop equals the sum of the voltage rises around that loop. In other words the algebraic sum of voltages around the close loop is zero.

The Differential Equation

Now we consider the following circuit consisting of an inductor, a resistor and a capacitor in series with a time varying voltage source E(t).



If V_L, V_R and V_c denote the voltage drop across the inductor, resistor and capacitor respectively. Then

$$V_L = L\frac{dI}{dt},\ V_R = RI,\ V_c = \frac{q}{C}$$

Now by Kirchhoff's law, the sum of V_L, V_R and V_c must equal the source voltage $E(t)$ i.e

$$V_L + V_R + V_c = E(t)$$

or

$$L\frac{dI}{dt} + RI + \frac{q}{C} = E(t)$$

Since the electric current I represents the rate of flow of charge $\frac{dq}{dt}$. Therefore, we can write

$$I = \frac{dq}{dt}$$

Substituting in the last equation, we have:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

- If E(t) = 0, R = 0 then the electric vibration can be called free un-damped oscillations.
- If E(t) = 0, R ≠ 0 the electric vibration of the circuit are said to be free damped oscillation.

Solution of the differential equation

The differential equation that governs the flow of charge in an LRC-Series circuit

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = E(t)$$

The complementary function we find general solution of the associated homogeneous differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = 0$$

We put $$q = e^{mt}, \quad \frac{dq}{dt} = me^{mt}, \quad \frac{d^2q}{dt^2} = m^2e^{mt}$$

Then the auxiliary equation of the associated homogeneous differential equation is:

$$Lm^2 + Rm + \frac{1}{C} = 0$$

If R ≠ 0 then, depending on the discriminant, the auxiliary equation may have

- Real and distinct roots
- Real and equal roots
- Complex roots

Case 1 Real and distinct roots

If $Disc = R^2 - \frac{4L}{C} > 0$ Then the auxiliary equation has real and distinct roots. In this case, the circuit is said to be over damped.

Case 2 Real and equal

If $Disc = R^2 - \frac{4L}{c} = 0$ Then the auxiliary equation has real and equal roots. In this case, the circuit is said to be critically damped.

Case 3 Complex roots

If $Disc = R^2 - 4\frac{L}{c} < 0$ Then the auxiliary equation has complex roots. In this case, the circuit is said to be under damped.

Since by the quadratic formula, we know that

$$m = \frac{-R \pm \sqrt{R^2 - 4L/c}}{2L}$$

- In the under damped case when q (0) = q_o the charge on the capacitor oscillates as it decays. This means that the capacitor is charging and discharging as t $\to \infty$
- In the under damped case, i.e. when E(0) = 0, and R = 0, the electrical vibration do not approach zero ast $\to \infty$. This means that the response of the circuit is Simple Harmonic.

Lecture 25 Forced Motion Examples

Example

Find the charge $q(t)$ on the capacitor in an LRC series circuit when L=0.25 Henry, R=10 Ohms, C=0.001 farad, $E(t)=0$, $q(0)=q_o$ and $I(0)=0$.

Solution

We know that for an LRC circuit, the governing differential equation is

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{c} = E(t)$$

Since $L = 0.25 = \frac{1}{4}$, $R = 10$,$C = 0.001 = \frac{1}{1000}$

Therefore, the equation becomes:

$$\frac{1}{4}\frac{d^2q}{dt^2}+10\frac{dq}{dt}+1000q=0$$

or
$$\frac{d^2q}{dt^2}+40\frac{dq}{dt}+4000q=0$$

The initial conditions are

$$q(0)=q_o,\ \ I(0)=0$$

or
$$q(0)=q_o,\ \ q'(0)=0$$

To solve the differential equation, we put

$$q=e^{mt},\quad \frac{dq}{dt}=me^{mt},\quad \frac{d^2q}{dt^2}=m^2e^{mt}$$

Therefore, the auxiliary equation is

$$m^2+40m+4000=0$$

$$\Rightarrow m=\frac{-40\pm\sqrt{1600-16000}}{2}$$

$$\Rightarrow m=-20\pm 60\,i$$

Thus, the solution of the differential equation is

$$q(t)=e^{-20t}\left(c_1\cos 60t+c_2\sin 60t\right)$$

Now, we apply the initial conditions

$$q(0)=q_o\Rightarrow c_1.1+c_2.0=q_o$$
$$\Rightarrow c_1=q_o$$

Therefore
$$q(t)=e^{-20t}\left(q_o\cos 60t+c_2\sin 60t\right)$$

Now
$$q'(t)=-20e^{-20t}\left(q_o\cos 60t+c_2\cos 60t\right)+e^{-20t}\left(-60q_o\sin 60\,t+60\,c_2\cos 60t\right)$$

Thus
$$q'(0)=0\Rightarrow -20q_o-20c_2+60c_2.1=0$$

$$\Rightarrow c_2=\frac{q_o}{2}$$

Hence the solution of the initial value problem is

$$q(t)=q_oe^{-20t}\left(\cos 60t+\frac{1}{2}\sin 60t\right)$$

As discussed in the previous lectures, a single sine function

$$q(t)=\frac{q_o\sqrt{10}}{3}e^{-20t}\sin(60t+1.249)$$

Since $R\neq 0$ and $\lim_{t\to\infty}q(t)=0$

Note that The electric vibrations in this case are free damped oscillations as there is no impressed voltage E(t)on the circuit.

Reactance

The quantity $X = L\gamma - \dfrac{1}{C\gamma}$ is called the ***reactance*** of the circuit.

Impedance

The quantity $Z = \sqrt{X^2 + R^2}$ is called ***impedance*** of the circuit.

Ohms

Both the reactance and the impedance are measured in Ohms.

Lecture 26 Differential Equations with Variable Coefficients

Differential Equations with non-constant (variable) coefficients

These equations normally arise in applications such as temperature or potential u in the region bounded between two concentric spheres. Then under some circumstances we have to solve the differential equation $r\dfrac{d^2u}{dr^2} + 2\dfrac{du}{dr} = 0$.

Where the variable r>0 represents the radial distance measured outward from the center of the spheres. Differential equations with variable coefficients such as

$$x^2y'' + xy' + (x^2 - v^2)y = 0$$

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0$$

and $$y'' - 2xy' + 2ny = 0$$

Cauchy- Euler Equation

Any linear differential equation of the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{d\ y}{dx} + a_0 y = g(x)$$

where $a_n, a_{n-1}, \cdots, a_0$ are constants, is said to be a *Cauchy-Euler* equation or equi-dimensional equation. The degree of each monomial coefficient matches the order of differentiation i.e x^n is the coefficient of nth derivative of y, x^{n-1} of (n-1)th derivative of y, etc.

For convenience we consider a homogeneous second-order differential equation

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = 0, \quad x \neq 0$$

The solution of higher-order equations follows analogously.

Also, we can solve the non-homogeneous equation

$$ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy = g(x), \quad x \neq 0$$

Method of Solution

A solution of the form y = x^m, where m is to be determined. The first and second derivatives are, respectively,

$$\frac{dy}{dx} = mx^{m-1} \quad \text{and} \quad \frac{d^2 y}{dx^2} = m(m-1)x^{m-2}$$

Consequently the differential equation becomes

$$\begin{aligned} ax^2 \frac{d^2 y}{dx^2} + bx \frac{dy}{dx} + cy &= ax^2 \cdot m(m-1)x^{m-2} + bx \cdot mx^{m-1} + cx^m \\ &= am(m-1)x^m + bmx^m + cx^m \\ &= x^m (am(m-1) + bm + c) \end{aligned}$$

Thus $y = x^m$ is a solution of the differential equation whenever m is a solution of the Auxiliary equation.

$(am(m-1) + bm + c) = 0$ or $am^2 + (b-a)m + c = 0$

The solution of the differential equation depends on the roots of the AE.

Case-I (Distinct Real Roots)

Let m_1 and m_2 denote the real roots of the auxiliary equation such that $m_1 \neq m_2$. Then $y = x^{m_1}$ and $y = x^{m_2}$ form a fundamental set of solutions.

Hence the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}.$$

Case II (Repeated Real Roots)

If the roots of the auxiliary equation are repeated, that is, then we obtain only one solution y = x^m .

To construct a second solution y^2, we first write the Cauchy-Euler equation in the form

$$\frac{d^2y}{dx^2}+\frac{b}{ax}\frac{dy}{dx}+\frac{c}{ax^2}y=0$$

Comparing with $\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=0$

We make the identification $P(x)=\frac{b}{ax}$. Thus

$$y_2=x^{m_1}\int\frac{e^{\int\frac{b}{ax}dx}}{(x^{m_1})^2}dx$$

$$=x^{m_1}\int\frac{e^{-(\frac{b}{a})\ln x}}{x^{2m_1}}dx$$

$$=x^{m_1}\int x^{-\frac{b}{a}}.x^{-2m_1}dx$$

Since roots of the AE $am^2+(b-a)m+c=0$ are equal, therefore discriminant is zero

$$\text{i.e } m_1=-\frac{(b-a)}{2a} \quad \text{or} \quad -2m_1=+\frac{(b-a)}{a}$$

$$y_2=x^{m_1}\int x^{\frac{-b}{a}}.x^{\frac{b-a}{a}}dx$$

$$y_2=x^{m_1}\int\frac{dx}{x}=x^{m_1}\ln x.$$

The general solution is then

$$y=c_1x^{m_1}+c_2x^{m_1}\ln x$$

Case III (Conjugate Complex Roots)

If the roots of the auxiliary equation are the conjugate pair $m_1=\alpha+i\beta,\quad m_2=\alpha-i\beta$

Where α and β >0 are real, then the solution is $y=c_1x^{\alpha+i\beta}+c_2x^{\alpha-i\beta}$.

In the case of equations with constant coefficients, when the roots of the auxiliary equation are complex, we wish to write the solution in terms of real functions only. We note the identity

$$x^{i\beta} = (e^{\ln x})^{i\beta} = e^{i\beta \ln x},$$

which, by *Euler's formula*, is the same as

$$x^{i\beta} = \cos(\beta \ln x) + i\sin(\beta \ln x)$$

Similarly we have

$$x^{-i\beta} = \cos(\beta \ln x) - i\sin(\beta \ln x)$$

Adding and subtracting last two results yields, respectively,

$$x^{i\beta} + x^{-i\beta} = 2\cos(\beta \ln x)$$

and $$x^{i\beta} - x^{-i\beta} = 2i\sin(\beta \ln x)$$

From the fact that $y = c_1 x^{\alpha+i\beta} + c_2 x^{\alpha-i\beta}$ is the solution of $ax^2 y'' + bxy' + cy = 0$, for any values of constants c_1 and c_2, we see that

$$y_1 = x^{\alpha}(x^{i\beta} + x^{-i\beta}), \quad (c_1 = c_2 = 1)$$

$$y_2 = x^{\alpha}(x^{i\beta} - x^{-i\beta}), \quad (c_1 = 1, c_2 = -1)$$

or $y_1 = 2x^{\alpha}(\cos(\beta \ln x))$, $y_2 = 2x^{\alpha}(\sin(\beta \ln x))$ are also solutions.

Since $W(x^{\alpha}\cos(\beta \ln x), x^{\alpha}\sin(\beta \ln x)) = \beta x^{2\alpha-1} \neq 0; \beta > 0$, on the interval $(0,\infty)$, we conclude that $y_1 = x^{\alpha}\cos(\beta \ln x)$ and $y_2 = x^{\alpha}\sin(\beta \ln x)$ constitute a fundamental set of real solutions of the differential equation. Hence the general solution is

$$y_1 = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]$$

Lecture 27 Cauchy-Euler Equation (Alternative Method of Solution)

We reduce any Cauchy-Euler differential equation to a differential equation with constant coefficients through the substitution

$$x = e^t \quad \text{or} \quad t = \ln x$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{1}{x}\cdot\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dt}\right) = \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dt}\right) - \frac{1}{x^2}\cdot\frac{dy}{dt}$$

or $$\frac{d^2y}{dx^2} = \frac{1}{x}\cdot\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx} - \frac{1}{x^2}\cdot\frac{dy}{dt}$$

or $$\frac{d^2y}{dx^2} = \frac{1}{x^2}\cdot\frac{d^2y}{dt^2} - \frac{1}{x^2}\cdot\frac{dy}{dt}$$

Therefore $$x\frac{dy}{dx} = \frac{dy}{dt},\ x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

Now introduce the notation

$$D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, \text{ etc.}$$

and $$\Delta = \frac{d}{dt}, \Delta^2 = \frac{d^2}{dt^2}, \quad \text{etc.}$$

Therefore, we have

$$xD = \Delta$$

$$x^2D^2 = \Delta^2 - \Delta = \Delta(\Delta - 1)$$

Similarly

$$x^3D^3 = \Delta(\Delta-1)(\Delta-2)$$

$$x^4D^4 = \Delta(\Delta-1)(\Delta-2)(\Delta-3) \quad \text{so on so forth.}$$

This substitution in a given *Cauchy-Euler* differential equation will reduce it into a differential equation with constant coefficients.

At this stage we suppose y= e^{mt} to obtain an auxiliary equation and write the solution in terms of y and t. We then go back to x through t =e^x.

Example

Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = 0$

Solution

The given differential equation can be written as

$$(x^2D^2 - 2xD - 4)y = 0$$

With the substitution $x = e^t$ or $t = \ln x$, we obtain

$$xD = \Delta, \quad x^2D^2 = \Delta(\Delta - 1)$$

Therefore the equation becomes:

$$[\Delta(\Delta - 1) - 2\Delta - 4]y = 0$$

or $\qquad (\Delta^2 - 3\Delta - 4)y = 0$

or $\qquad \dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} - 4y = 0$

Now substitute: $y = e^{mt}$ then $\dfrac{dy}{dt} = me^{mt}$, $\dfrac{d^2y}{dt^2} = m^2e^{mt}$

Thus $(m^2 - 3m - 4)e^{mt} = 0$ or $m^2 - 3m - 4 = 0$, which is the auxiliary equation.

$$(m+1)(m-4) = 0 \qquad m = -1, 4$$

The roots of the auxiliary equation are distinct and real, so the solution is

$$y = c_1e^{-t} + c_2e^{4t}$$

But $\quad x = e^t$, therefore the answer will be

$$y = c_1x^{-1} + c_2x^4$$

Lecture 28 Power Series (An Introduction)

A standard technique for solving linear differential equations with variable coefficients is to find a solution as an infinite series. Often this solution can be found in the form of a power series.

Power Series

A power series in(x − a) is an infinite series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots.$$

The coefficients $c_0, c_1, c_2, \ldots$ And a are constants and x represents a variable. In this discussion we will only be concerned with the cases where the coefficients, x and a are real numbers. The number a is known as the center of the power series.

Example The infinite series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} x^n = x - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \cdots$

is a power series in x. This series is centered at zero.

Convergence and Divergence

- If we choose a specified value of the variable x then the power series becomes an infinite series of constants. If, for the given x , the sum of terms of the power series equals a finite real number, then the series is said to be convergent at x .
- A power series that is not convergent is said to be a divergent series. This means that the sum of terms of a divergent power series is not equal to a finite real number.

The Ratio Test

To determine for which values of x a power series is convergent, one can often use the Ratio Test. The Ratio test states that if $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series and $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| |x-a| = L$

- The power series converges absolutely for those values of x for which L < 1.
- The power series diverges for those values of x for which L > 1 or L = ∞ .
- The test is inconclusive for those values of x for which L =1.

Interval of Convergence

The set of all real values of x for which a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges is known as the interval of convergence of the power series.

Radius of Convergence

Consider a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$

Then exactly one of the following three possibilities is true:

- The series converges only at its center x =a.
- The series converges for all values of x .
- There is a number $R > 0$ such that the series converges absolutely $\forall x$ satisfying $|x-a| < R$ and diverges for $|x-a| > R$. This means that the series converges for $x \in (a-R, a+R)$ and diverges out side this interval.

The number R is called the radius of convergence of the power series. If first possibility holds then R = 0 and in case of 2nd possibility we write R = ∞ .

From the Ratio test we can clearly see that the radius of convergence is given by

$$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

provided the limit exists.

Convergence at an Endpoint

If the radius of convergence of a power series is R > 0 , then the interval of convergence of the series is one of the following

$$(a-R, a+R),\ (a-R, a+R],\ [a-R, a+R),\ [a-R, a+R]$$

To determine which of these intervals is the interval of convergence, we must conduct separate investigations for the numbers $x = a - R\ and\ x = a + R$.

Absolute Convergence

Within its interval of convergence a power series converges absolutely. In other words, the series of absolute values $\sum_{n=o}^{\infty} |c_n| \left|(x-a)^n\right|$ converges for all values x in the interval of convergence.

Power Series Representation of Functions

A power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ determines a function f whose domain is the interval of convergence of the power series. Thus for all x in the interval of convergence, we write

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

If a function is f is defined in this way, we say that $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series representation for $f(x)$. We also say that f is represented by the power series

Theorem

Suppose that a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ has a radius of convergence $R>0$ and for every x in the interval of convergence a function f is defined by

$$f(x)=\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$$

Then

- The function f is continuous, differentiable, and integrable on the interval $(a-R,\ a+R)$.
- Moreover, $f'(x)$ and $\int f(x)dx$ can be found from term-by-term differentiation and ntegration.
  Therefore

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

$$\int f(x)\ dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \cdots$$

$$= C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

The series obtained by differentiation and integration have same radius of convergence. However, the convergence at the end points $x=a-R$ and $x=a+R$ of the interval

Analytic

At a Point A function f is said to be analytic at point a if the function can be represented by power series in (x − a) with a positive radius of convergence. The notion of analyticity at a point will be important in finding power series solution of a differential equation.

Example

Since the functions e^x, $\cos x$, and $\ln(1+x)$ can be represented by the power series.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Arithmetic of Power Series

- Power series can be combined through the operations of addition, multiplication, and division.
- The procedure for addition, multiplication and division of power series is similar to the way in which polynomials are added, multiplied, and divided.
- Thus we add coefficients of like powers of x , use the distributive law and collect like terms, and perform long division.

Example

Find the first four terms of a power series in x for the function sec x .

Solution

We know that $\sec x = \dfrac{1}{\cos x}, \quad \cos x = 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720} + \cdots$

Therefore using long division, we have

$$
\begin{array}{r|l}
 & 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \dfrac{61x^6}{720} + \cdots \\
\hline
1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720} + \cdots & 1 \\
 & 1 - \dfrac{x^2}{2} + \dfrac{x^4}{24} - \dfrac{x^6}{720} + \cdots \\
\cline{2-2}
 & \dfrac{x^2}{2} - \dfrac{x^4}{24} + \dfrac{x^6}{720} - \cdots \\
 & \dfrac{x^2}{2} - \dfrac{x^4}{4} + \dfrac{x^6}{48} - \cdots \\
\cline{2-2}
 & \dfrac{5x^4}{24} - \dfrac{7x^6}{360} + \cdots \\
 & \dfrac{5x^4}{24} - \dfrac{5x^6}{48} + \cdots \\
\cline{2-2}
 & \dfrac{61x^6}{720} - \cdots
\end{array}
$$

Hence, the power series for the function $f(x) = \sec x$ is

$$\sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$$

The interval of convergence of this series is $(-\pi/2, \pi/2)$.

Lecture 29 Power Series Solution of a Differential Equation

We know that the explicit solution of the linear first-order differential equation

$$\frac{dy}{dx} - 2xy = 0$$

is $$y = e^{x^2}$$

Also $$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

If we replace x by x^2 in the series representation of e^x, we can write the solution of the differential equation as

$$y = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$$

This last series converges for all real values of x. In other words, knowing the solution in advance, we were able to find an infinite series solution of the differential equation.

Now propose to obtain a power series solution of the differential equation directly; the method of attack is similar to the technique of undetermined coefficients.

Lecture 30 Solutions about Ordinary Points

Analytic Function

A function f is said to be analytic at a point a if it can be represented by a power series in (x-a) with a positive radius of convergence. Suppose the linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

is put into the form

$$y'' + P(x)y' + Q(x)y = 0 \qquad (2)$$

by dividing by the leading coefficient $a_2(x)$.

Ordinary and singular points

A point x_0 is said to be a ordinary point of a differential equation (1) if both P(x) and Q(x) are analytic at x_0. A point that is not an ordinary point is said to be singular point of the equation.

Polynomial Coefficients

If $a_2(x)$, $a_1(x)$ and $a_0(x)$ are polynomials with no common factors, then $x = x_0$ is

(i) an ordinary point if $a_2(x) \neq 0$ or

(ii) a singular point if $a_2(x) = 0$

Theorem (Existence of Power Series Solution)

A function f(x) is called analytic at x_0 if f(x) is equal to its power series.

Theorem

Let x_0 be an ordinary point of the differential equation

$$L(y) = y'' + p(t)y' + q(t)y = 0$$

Then the general solution can be represented by the power series

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 y_1(x) + a_1 y_2(x)$$

where a_0 and a_1 are arbitrary constants and y_1 and y_2 are analytic at x_0. The radii of convergence for y_1 and y_2 are at least as large as the minimum radii of convergences for p and q.

Example

Find a lower bound for the radius of convergence of series solutions about $x = 1$ for the differential equation

$$(x^2 + 4)y'' + \sin(x)y' + e^x y = 0$$

Solution

We have

$$p(x) = \frac{\text{Sin } x}{x^2 + 4} \qquad q(x) = \frac{e^x}{x^2 + 4}$$

Both of these are quotient of analytic functions. the roots of $x^2 + 4$ are $2i$ and $-2i$

The distance from 1 to $2i$ is the same as the distance from (1,0) to (0,2) which is $\sqrt{5}$

We get the same distance from 1 to $-2i$. Hence the radii of convergence of the solutions are both at least $\sqrt{5}$.

Non-polynomial Coefficients

"Non-polynomial coefficients" refers to coefficients that are not part of a polynomial expression. In mathematics, a polynomial is an expression made up of variables, coefficients, and exponents, combined using addition, subtraction, multiplication, and non-negative integer exponents. A polynomial expression looks like this:

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

Here, $a_n, a_{n-1}, \ldots, a_0$ are the coefficients of the polynomial.

"Non-polynomial coefficients" would refer to coefficients in equations or expressions that are not structured as polynomials. This could encompass various types of equations, functions, or models that do not follow the polynomial form. They might involve different mathematical operations, functions, or structures.

For example, if you're working with an exponential equation like $y=a \cdot e^{bx}$, the coefficients a and b are non-polynomial coefficients because they don't fit the polynomial format.

Lecture No. 31 Solutions about Singular Points

Solutions about Singular Points

If $x = x_0$ is singular point, it is not always possible to find a solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n \text{ for the equation } a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

However, we may be able to find a solution of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r} \text{ , where } r \text{ is constant to be determined.}$$

To define regular/irregular singular points, we put the given equation into the standard form

$$y'' + P(x)y' + Q(x)y = 0$$

Regular and Irregular Singular Points

Singular points come in two different forms: regular and irregular. Regular singular points are well-behaved and defined in terms of the ratio Q(x)/P(x) and R(x)/P(x), where P(x), Q(x), and R(x) are the polynomial coefficients in the differential equation you're trying to solve.

Irregular singular points are a totally different ball game and one that I don't get into in this chapter. As you work through the practice problems here, if the singular point in question doesn't appear to be regular, you know it's irregular.

Allow me to introduce you to this dainty differential equation:

$$p(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

In order for x_0 to be a regular singular point, these two relations must be true:

$\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)}$ remains finite and

$\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$ remains finite

If you define

p(x) = Q(x)/P(x)

and

q(x) = R(x)/P(x)

Then the two limits become

$\lim_{x\to x_0}(x-x_0)p(x)$ remains finite And

$\lim_{x\to x_0}(x-x_0)^2 q(x)$ remains finite

If both of these statements are true, then the point x_0 is a regular singular point.

Polynomial Coefficients

If the coefficients in the given differential equation $a_2(x)y''+a_1(x)y'+a_0(x)y=0$ are polynomials with no common factors, above definition is equivalent to the following:

Let $a_2(x_0)=0$ *Form* $P(x)$ *and* $Q(x)$ *by reducing* $\frac{a_1(x)}{a_2(x)}$ *and* $\frac{a_0(x)}{a_2(x)}$ *to lowest* terms, respectively. If the factor $(x-x_0)$ appears at most to the first powers in the denominator of P(x) and at most to the second power in the denominator of Q(x), then $x = x_0$ is a regular singular point.

Method of Frobenius

To solve a differential equation $a_2(x)y''+a_1(x)y'+a_0(x)y=0$ about a regular singular point we employ the Frobenius' Theorem.

Identify regular singular point x_0,

Substitute $y=\sum_{n=0}^{\infty}c_n(x-x_0)^{n+r}$ in the given differential equation,

Determine the unknown exponent r and the coefficients c_n.

For simplicity assume that $x_0=0$.

Frobenius' Theorem

If $x = x_0$ is a regular singular point of equation $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$, then there exists at least one series solution of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where the number r is a constant that must be determined. The series will converge at least on some interval $0 < x - x_0 < R$.

Note that the solutions of the form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ are not guaranteed.

Method of Frobenius

1. Identify regular singular point x_0,
2. Substitute $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ in the given differential equation,
3. Determine the unknown exponent r and the coefficients c_n.
4. For simplicity assume that $x_0 = 0$.

Cases of Indicial Roots

The method of Frobenius, we usually distinguish three cases corresponding to the nature of the indicial roots. For the sake of discussion let us suppose that r_1 and r_2 are the real solutions of the indicial equation and that, when appropriate, r_1 denotes the largest root.

Case I: Roots not Differing by an Integer

If r_1 and r_2 are distinct and do not differ by an integer, then their exist two linearly independent solutions of the differential equation of the form

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1} \,..\, c_0 \neq 0, \text{ and } y_2 = \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0.$$

Lecture 32 Solutions about Singular Points

Method of Frobenius (Cases II and III)

When the roots of the indicial equation differ by a positive integer, we may or may not be able to find two solutions of

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

having form $y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$ (2)

If not, then one solution corresponding to the smaller root contains a logarithmic term. When the exponents are equal, a second solution always contains a logarithm. This latter situation is similar to the solution of the Cauchy-Euler differencial equation when the roots of the auxiliary equation are equal. We have the next two cases.

Case II (Roots Differing by a Positive Integer)

If , $r_1 - r_2 = N$ where N is a positive integer, then there exist two linearly independent solutions of the form

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0 \qquad (3a)$$

$$y_2 = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, b_0 \neq 0 \qquad (3b)$$

Where C is a constant that could be zero.

Case III: Equal Indicial Roots:

If $r_1 - r_2$, there always exist two linearly independent solutions of (1) of the form

$$y_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}, c_0 \neq 0 \qquad (4a)$$

$$y_2 = y_1(x)\ln x + \sum_{n=1}^{\infty} b_n x^{n+r_1} \qquad \because r_1 = r_2 \qquad (4b)$$

Lecture 33 Bessel's Differential Equation

Bessel Equation

The equation

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0, \qquad (1)$$

Where α is a nonnegative constant, is called the Bessel equation. The point $x_0 = 0$ is a regular singular point. We shall use the method of Frobenius to solve this equation. Thus, we seek solutions of the form

Power Series Solutions to the Bessel Equation

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}, \quad x > 0, \tag{2}$$

with $a_0 \neq 0$.

Differentiation of (2) term by term yields

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}.$$

Similarly, we obtain

$$y'' = x^{r-2} \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^n.$$

Substituting these into (1), we obtain

$$\sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r} + \sum_{n=0}^{\infty} (n+r) a_n x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \sum_{n=0}^{\infty} \alpha^2 a_n x^{n+r} = 0.$$

This implies

$$x^r \sum_{n=0}^{\infty} [(n+r)^2 - \alpha^2] a_n x^n + x^r \sum_{n=0}^{\infty} a_n x^{n+2} = 0.$$

Now, cancel x^r, and try to determine a_n's so that the coefficient of each power of x will vanish.

For the constant term, we require $(r^2 - \alpha^2) a_0 = 0$. Since $a_0 \neq 0$, it follows that

$$r^2 - \alpha^2 = 0,$$

which is the indicial equation. The only possible values of r are α and $-\alpha$.

Case I. For $r = \alpha$, the equations for determining the coefficients are:

$$[(1+\alpha)^2 - \alpha^2] a_1 = 0 \text{ and,}$$
$$[(n+\alpha)^2 - \alpha^2] a_n + a_{n-2} = 0, \quad n \geq 2.$$

Since $\alpha \geq 0$, we have $a_1 = 0$. The second equation yields

$$a_n = -\frac{a_{n-2}}{(n+\alpha)^2 - \alpha^2} = -\frac{a_{n-2}}{n(n+2\alpha)}. \tag{3}$$

Since $a_1 = 0$, we immediately obtain

$$a_3 = a_5 = a_7 = \cdots = 0.$$

For the coefficients with even subscripts, we have

$$a_2 = \frac{-a_0}{2(2+2\alpha)} = \frac{-a_0}{2^2(1+\alpha)},$$

$$a_4 = \frac{-a_2}{4(4+2\alpha)} = \frac{(-1)^2 a_0}{2^4 2!(1+\alpha)(2+\alpha)},$$

$$a_6 = \frac{-a_4}{6(6+2\alpha)} = \frac{(-1)^3 a_0}{2^6 3!(1+\alpha)(2+\alpha)(3+\alpha)},$$

and, in general

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(1+\alpha)(2+\alpha)\cdots(n+\alpha)}.$$

Therefore, the choice $r=\alpha$ yields the solution

$$y(x) = a_0 x^{\alpha}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1+\alpha)(2+\alpha)\cdots(n+\alpha)}\right).$$

Note: The ratio test shows that the power series formula converges for all $x \in \mathbb{R}$.

For $x < 0$, we proceed as above with x^r replaced by $(-x)^r$. Again, in this case, we find that r satisfies

$$r^2 - \alpha^2 = 0.$$

Taking $r = \alpha$, we obtain the same solution, with x^{α} is replaced by $(-x)^{\alpha}$. Therefore, the function $y_{\alpha}(x)$ is given by

$$y_{\alpha}(x) = a_0|x|^{\alpha}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1+\alpha)(2+\alpha)\cdots(n+\alpha)}\right) \tag{4}$$

is a solution of the Bessel equation valid for all real $x \neq 0$.

Case II.For $r = -\alpha$, determine the coefficients from

$$[(1-\alpha)^2 - \alpha^2]a_1 = 0 \;\text{ and }\; [(n-\alpha)^2 - \alpha^2]a_n + a_{n-2} = 0.$$

These equations become

$$(1-2\alpha)a_1 = 0 \;\text{ and }\; n(n-2\alpha)a_n + a_{n-2} = 0.$$

If 2α is not an integer, these equations give us

$$a_1 = 0 \;\text{ and }\; a_n = -\frac{a_{n-2}}{n(n-2\alpha)}, \quad n \geq 2.$$

Note that this formula is same as (3), with α replaced by $-\alpha$. Thus, the solution is given by

$$y_{-\alpha}(x) = a_0|x|^{-\alpha}\left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} n!(1-\alpha)(2-\alpha)\cdots(n-\alpha)}\right), \tag{5}$$

which is valid for all real $x \neq 0$.

Euler's gamma function and its properties

Bessel function of the first kind

For $s \in \mathbb{R}$ with $s > 0$, we define $\Gamma(s)$ by

$$\Gamma(s) = \int_{0+}^{\infty} t^{s-1} e^{-t} dt.$$

The integral converges if $s > 0$ and diverges if $s \leq 0$.
Integration by parts yields the functional equation

$$\Gamma(s+1) = s\Gamma(s).$$

In general,

$$\Gamma(s+n) = (s+n-1)\cdots(s+1)s\Gamma(s), \text{ for every } n \in \mathbb{Z}^+.$$

Since $\Gamma(1) = 1$, we find that $\Gamma(n+1) = n!$. Thus, the gamma function is an extension of the factorial function from integers to positive real numbers. Therefore, we write

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}, \quad s \in \mathbb{R}.$$

Using this gamma function, we shall simplify the form of the solutions of the Bessel equation. With s = 1 + α, we note that

$$(1+\alpha)(2+\alpha)\cdots(n+\alpha) = \frac{\Gamma(n+1+\alpha)}{\Gamma(1+\alpha)}.$$

Choose $a_0 = \frac{2^{-\alpha}}{\Gamma(1+\alpha)}$ in (4), the solution for $x > 0$ can be written

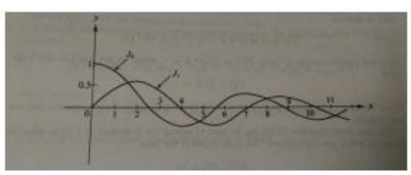
$$J_\alpha(x) = \left(\frac{x}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1+\alpha)} \left(\frac{x}{2}\right)^{2n}.$$

The function Jα defined above for x > 0 and α ≥ 0 is called the Bessel function of the first kind of order α.

When α is a nonnegative integer, say $\alpha = p$, the Bessel function $J_p(x)$ is given by

$$J_p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+p)!} \left(\frac{x}{2}\right)^{2n+p}, \quad (p = 0, 1, 2, \ldots).$$

This is a solution of the Bessel equation for $x < 0$.



If $\alpha \notin \mathbb{Z}^+$, define a new function $J_{-\alpha}(x)$ (replacing α by $-\alpha$)

$$J_{-\alpha}(x) = \left(\frac{x}{2}\right)^{-\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1-\alpha)} \left(\frac{x}{2}\right)^{2n}.$$

With $s = 1 - \alpha$, we note that

$$\Gamma(n+1-\alpha) = (1-\alpha)(2-\alpha)\cdots(n-\alpha)\Gamma(1-\alpha).$$

Thus, the series for $J_\alpha(x)$ is the same as that for $y_{-\alpha}(x)$ in (5) with $a_0 = \frac{2^\alpha}{\Gamma(1-\alpha)}$, $x > 0$. If α is not positive integer, $J_{-\alpha}$ is a solution of the Bessel equation for $x > 0$.

If $\alpha \notin \mathbb{Z}^+$, $J_\alpha(x)$ and $J_{-\alpha}(x)$ are linearly independent on $x > 0$. The general solution of the Bessel equation for $x > 0$ is

$$y(x) = c_1 J_\alpha(x) + c_2 J_{-\alpha}(x).$$

Example   Find the general solution of the equation $x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0$ on $(0, \infty)$

Solution The Bessel differential equation is

$$x^2y'' + xy' + \left(x^2 - v^2\right)y = 0 \qquad (1)$$

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y = 0 \qquad (2)$$

Comparing (1) and (2), we get $v^2 = \frac{1}{4}$, therefore $v = \pm\frac{1}{2}$

So general solution of (1) is $y = C_1 J_{1/2}(x) + C_2 J_{-1/2}(x)$

Example   Find the general solution of the equation $x^2y'' + xy' + \left(x^2 - \frac{1}{9}\right)y = 0$

Solution: We identify $v^2 = \frac{1}{9}$, therefore $v = \pm\frac{1}{3}$

So general solution is $y = C_1 J_{1/3}(x) + C_2 J_{-1/3}(x)$

Lecture 34 Legendre's Differential Equation

Legendre Differential Equation

The Legendre differential equation is the second-order ordinary differential equation

$$(1 - x^2)\frac{d^2 y}{d x^2} - 2x\frac{d y}{d x} + l(l+1)y = 0, \qquad \text{.....................(1)}$$

which can be rewritten

$$\frac{d}{dx}\left[\left(1-x^2\right)\frac{dy}{dx}\right]+l(l+1)y=0.$$ …………………(2)

The above form is a special case of the so-called "associated Legendre differential equation" corresponding to the case m=0. The Legendre differential equation has regular singular points at -1, 1, and ∞.

If the variable X is replaced by cos θ, then the Legendre differential equation becomes

$$\frac{d^2 y}{d\theta^2}+\frac{\cos\theta}{\sin\theta}\frac{dy}{d\theta}+l(l+1)y=0,$$ …………………(3)

derived below for the associated (m≠0) case.

Since the Legendre differential equation is a second-order ordinary differential equation, it has two linearly independent solutions. A solution $P_l(x)$ which is regular at finite points is called a Legendre function of the first kind, while a solution $Q_l(x)$ which is singular at ± 1 is called a Legendre function of the second kind. If is an integer, the function of the first kind reduces to a polynomial known as the Legendre polynomial. The Legendre differential equation can be solved using the Frobenius method by making a series expansion with k = 0,

$$y=\sum_{n=0}^{\infty} a_n x^n$$ …………………(4)

$$y'=\sum_{n=0}^{\infty} n\, a_n x^{n-1}$$ …………………(5)

$$y''=\sum_{n=0}^{\infty} n(n-1)\, a_n x^{n-2}.$$ …………………(6)

Plugging in,

$$\left(1-x^2\right)\sum_{n=0}^{\infty} n(n-1)\, a_n x^{n-2}-2x\sum_{n=0}^{\infty} n\, a_n x^{n-1}+l(l+1)\sum_{n=0}^{\infty} a_n x^n=0$$ …………………(7)

so each term must vanish and

$$\sum_{n=0}^{\infty}\{(n+1)(n+2)a_{n+2}+[-n(n-1)-2n+l(l+1)]a_n\}=0,$$

z

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty} n(n-1)a_n x^n \quad (8)$$

$$-2x\sum_{n=0}^{\infty} n a_n x^{n-1}+l(l+1)\sum_{n=0}^{\infty} a_n x^n=0 \quad (9)$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}-\sum_{n=0}^{\infty} n(n-1)a_n x^n \quad (10)$$

$$-2\sum_{n=0}^{\infty} n a_n x^n+l(l+1)\sum_{n=0}^{\infty} a_n x^n=0 \quad (11)$$

$$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n-\sum_{n=0}^{\infty} n(n-1)a_n x^n \quad (12)$$

$$-2\sum_{n=0}^{\infty} n a_n x^n+l(l+1)\sum_{n=0}^{\infty} a_n x^n=0 \quad (13)$$

$$\sum_{n=0}^{\infty}\{(n+1)(n+2)a_{n+2}+[-n(n-1)-2n+l(l+1)]a_n\}=0,$$

…………………(14)

so each term must vanish and

$$(n+1)(n+2)a_{n+2}+[-n(n+1)+l(l+1)]a_n=0$$

$$a_{n+2}=\frac{n(n+1)-l(l+1)}{(n+1)(n+2)}a_n$$

$$=-\frac{[l+(n+1)](l-n)}{(n+1)(n+2)}a_n.$$

Therefore,

$$a_2=-\frac{l(l+1)}{1\cdot 2}a_0$$

$$a_4=-\frac{(l-2)(l+3)}{3\cdot 4}a_2$$

$$=(-1)^2\frac{[(l-2)l][(l+1)(l+3)]}{1\cdot 2\cdot 3\cdot 4}a_0$$

so the even solution is

$$y_1(x)=1+\sum_{n=1}^{\infty}(-1)^n\frac{[(l-2n+2)\cdots(l-2)l][(l+1)(l+3)\cdots(l+2n-1)]}{(2n)!}x^{2n}.$$

Similarly, the odd solution is

$$y_2(x) = x + \sum_{n=1}^{\infty} (-1)^n \frac{[(l-2n+1)\cdots(l-3)(l-1)][(l+2)(l+4)\cdots(l+2n)]}{(2n+1)!} x^{2n+1}.$$

$$P_n(x) = c_n \begin{cases} y_1(x) & \text{for } l \text{ even} \\ y_2(x) & \text{for } l \text{ odd} \end{cases}$$

$$= c_n \begin{cases} {}_2F_1\left(-\frac{1}{2}, \frac{1}{2}(l+1); \frac{1}{2}, x^2\right) & \text{for } l \text{ even} \\ x\, {}_2F_1\left(\frac{1}{2}(l+2), \frac{1}{2}(1-l); \frac{3}{2}; x^2\right) & \text{for } l \text{ odd} \end{cases}$$

where c_n is chosen so as to yield the normalization $P_n(1) = 1$ and ${}_2F_1(a, b; c; z)$ is a is a hypergeometric function.

A generalization of the Legendre differential equation is known as the associated Legendre differential equation.

$$(1-x^2)y'' - 2xy' - \left[k^2 a^2 (x^2-1) - p(p+1) - \frac{q^2}{x^2-1}\right] y = 0$$

Lecture 35 Systems of Linear Differential Equations

Systems of Linear Differential Equations

A **system of linear differential equations** is a set of linear equations relating a group of functions to their derivatives. Because they involve functions and their derivatives, each of these linear equations is itself a differential equation. For example, $f'(x)=f(x)+g(x)$ is a linear equation relating $'f'$ to f and g, but $f'=fg$ is not, because the fg term is not linear. These equations can be solved by writing them in matrix form, and then working with them almost as if they were standard differential equations.

Systems of differential equations can be used to model a variety of physical systems, such as predator-prey interactions, but linear systems are the only systems that can be consistently solved explicitly.

An nth order linear differential equation with constant coefficients a_0, a_1, a_2, a_3,…….. a_n is an equation of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

If we write $D = \frac{d}{dx}, D^2 = \frac{d^2}{dx^2}, \cdots, D^n = \frac{d^n}{dx^n}$ then this equation can be written as follows

$$\left(a_n D^n + a_{n-1} D^{(n-1)} + \cdots + a_1 D + a_0\right) y = g(t)$$

Solve the system of differential equations f′(x)=g(x) and g′(x)=f(x), where f(0)=0 and g(0)=1.

In matrix form, this system is

$$\begin{bmatrix} f \\ g \end{bmatrix}' = \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} f \\ g \end{bmatrix}}_{v}.$$

It remains to compute e^{Ax}.

If we compute some small powers of A, we find that $A^{2n}=I$ and $A^{2n+1}=A$ i.e. the even powers are the identity and the odd powers are just A. Then,

$$e^{Ax} = \sum_{n=0}^{\infty} \frac{(Ax)^n}{n!} = \sum_{n=0}^{\infty} \frac{(Ax)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(Ax)^{2n+1}}{(2n+1)!} = A\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + I\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}.$$

Now, we can recognize those two sums as the Taylor series for the hyperbolic sine and hyperbolic cosine, so we have

$$e^{Ax} = A\sinh x + I\cosh x = \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix}.$$

Finally, we find

$$v(x) = e^{Ax}v(0) = \begin{bmatrix} \cosh x & \sinh x \\ \sinh x & \cosh x \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sinh x \\ \cosh x \end{bmatrix},$$

so the solution is $f(x) = \sinh x$ and $g(x) = \cosh x$.

Lecture 36 Systems of Linear Differential Equations

Systems of Linear Differential Equations

If L_1 , L_2 , L_3 and L_4 denote linear differential operators with constant coefficients, then a system of linear differential equations in two variables x and y can be written as

$$L_1x + L_2y = g_1(t)$$
$$L_3x + L_4y = g_2(t)$$

To eliminate y , we operate on the first equation with L_4 and on the second equation with L_2 and then subtracting, we obtain : $(L_1L_4 - L_2L_3)x = L_4g_1 - L_2g_2$

Similarly, operating on the first equation with L_3 and second equation with $L_1$and then subtracting, we obtain: $(L_1L_4 - L_2L_3)y = L_1g_2 - L_3g_1$

$$\because L_1L_4 - L_2L_3 = \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} \Rightarrow L_4g_1 - L_2g_2 = \begin{vmatrix} g_1 & L_2 \\ g_2 & L_4 \end{vmatrix}$$

And $L_1g_2 - L_3g_1 = \begin{vmatrix} L_1 & g_1 \\ L_3 & g_2 \end{vmatrix}$

Hence, the given system of differential equations can be decoupled into *nth* order differential equations. These equations use determinants similar to those used in Cramer's rule:

$$\begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} x = \begin{vmatrix} g_1 & L_2 \\ g_2 & L_4 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} L_1 & L_2 \\ L_3 & L_4 \end{vmatrix} y = \begin{vmatrix} L_1 & g_1 \\ L_3 & g_2 \end{vmatrix}$$

The uncoupled differential equations can be solved in the usual manner.

- The determinant on left hand side in each of these equations can be expanded in the usual algebraic sense. This means that the symbol D occurring in L_i is to be treated as an algebraic quantity. The result of this expansion is a differential operator of order n , which is operated on x and y .
- However, some care should be exercised in the expansion of the determinant on the right hand side. We must expand these determinants in the sense of the internal differential operators actually operating on the functions g_1 and g_2 . Therefore, the symbol D occurring in L_i is to be treated as an algebraic quantity.

Lecture 37 Systems of Linear First-Order Equation

Systems of Linear First Order Ordinary Differential Equations

General linear equations, differential equations can also be written as a system of linear differential equations. Similarly, we can define the **systems of linear first order ordinary differential equations**. To learn how to write the systems of linear 1st order ODEs, we require a basic understanding of what are ordinary differential equations, first order ODEs and systems of linear differential equations. These terms can be understood with the help of a brief introduction given here.

Systems of First Order Linear Ordinary Differential Equations

The first order linear system of ordinary differential equations is of the form,

$x'_1 = a_{11}(t)x_1 + a_{12}(t)x_2 + \dots + a_{1n}(t)x_n + b_1(t)$

$x'_2 = a_{21}(t)x_1 + a_{22}(t)x_2 + \dots + a_{2n}(t)x_n + b_2(t)$

$x'_3 = a_{31}(t)x_1 + a_{32}(t)x_2 + \dots + a_{3n}(t)x_n + b_3(t)$

: : : :

$x'_n = a_{n1}(t)x_1 + a_{n2}(t)x_2 + \dots + a_{nn}(t)x_n + b_n(t)$

This system of equations can be expressed in the form of matrices as:

$\mathbf{x}' = A(t)\,\mathbf{x} + \mathbf{b}(t)$

Here,

$A(t) = [a_{ij}\,(T)]$ is the coefficient matrix.

This can be written as:

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

And

$$B(t) = \begin{bmatrix} b_1(t) \\ b_2(t) \\ \vdots \\ b_n(t) \end{bmatrix}, \; X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \; X' = \begin{bmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{bmatrix}$$

This can be written in more simplified form as:

$$\vec{x}' = \frac{d\vec{x}}{dt} = A(t)\vec{x} + \vec{b}(t)$$

This is a non-homogeneous system. If it is free of b(t), i.e.,

$$\vec{x}' = A(t)\vec{x}$$

, then it is a homogeneous system.

Solution of a Systems of Linear First Order Ordinary Differential Equations

The solution of a system of linear first-order ordinary differential equations is the column vector $\mathbf{x}(t)$ subjected to the IVP.

The initial value problem (IVM) for the system of a linear first order ODEs, i.e.,

$$\vec{x}' = A(t)\vec{x} + \vec{b}(t)$$

is to find the vector function $\mathbf{x}(t)$ in C^1 that satisfies the system on an interval I and the initial conditions given by $\mathbf{x}(t) = \mathbf{x}_0 = (x_{1,0}, x_{2,0}, \ldots, x_{n,0})^T$ such that $t_0 \in I$ and $\mathbf{x}_0 \in R^n$.

Then, we can find the solution using the concepts of eigenvalues and eigenvectors.

Solved Example Question:

Consider the system of two first order linear ODEs:

x′ = 3x – 2y

y′ = 2x – y

Solution:

Given system of equations is:

x′ = 3x – 2y ……..(1)

y′ = 2x – y ……..(2)

First, we need to reduce these equations.

This reduction can be made by differentiating the equations and successive replacement of the unknown functions until we get a differential equation for only one unknown function.

Consider the equation (2).

y′ = 2x – y

From this, we can write x as:

y′ + y = 2x

x = (½)y′ + (½)y……(3)

Differentiating equation equation (2), we get;

$y'' = 2x' - y'$

This can be rearranged as:

$x' = (½)y'' + (½)y'$…….(4)

Substituting equations (3) and (4) in the equation (1), we get;

$x' = 3x - 2y$

$[(½)y'' + (½)y'] = 3[(½)y' + (½)y] - 2y$

$(½)(y'' + y') = (½)[3(y' + y) - 4y]$

$y'' + y' = 3y' + 3y - 4y$

$y'' + y' - 3y' + y = 0$

$y'' - 2y' + y = 0$

This is the second-order linear ODE and is homogeneous with constant coefficients.

This can be solved using the standard method.

Thus, get the auxiliary equation as:

$m^2 - 2m + I = 0$ such that $m_{1,2} = I$

Therefore, we get the general solution to this equation as:

$y = c_1e^t + c_2te^t$…….(5)

Now, substituting equation (5) in equation (3), we get;

$x = (½)y' + (½)y = (½)(c_1e^t + c_2te^t)' + (½)(c_1e^t + c_2te^t)$

$= (½)[(c_1e^t + c_2te^t)' + (c_1e^t + c_2te^t)]$

$= (½)[c_1e^t + c_2(te^t + e^t) + c_1e^t + c_2te^t]$

$= (½)[2c_1e^t + 2c_2te^t + c_2e^t]$

$= (½)2e^t[c_1 + c_2\{(½) + t\}]$

$= e^t [c_1 + c_2\{(½) + t\}]$

Hence, the general solution of the given system of equations is:

$y(t) = c_1e^t + c_2te^t$

$x(t) = e^t [c_1 + c_2\{(½) + t\}]$

Lecture 38 Introductions to Matrices

Matrix

A matrix is a collection of numbers or functions arranged into rows and columns.

Elements

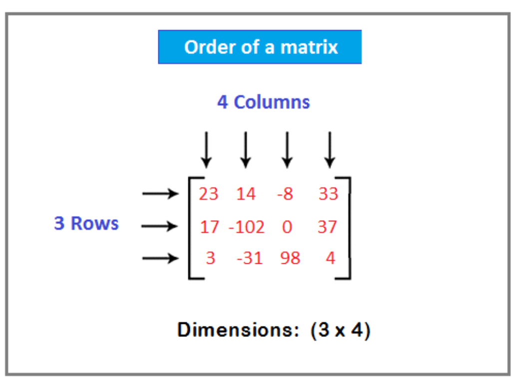
Matrices are denoted by capital letters A, B,K,Y, Z . The numbers or functions are called elements of the matrix. The elements of a matrix are denoted by small letters a,b,K, y,z.

Rows and Columns

The horizontal and vertical lines in a matrix are, respectively, called the rows and columns of the matrix.

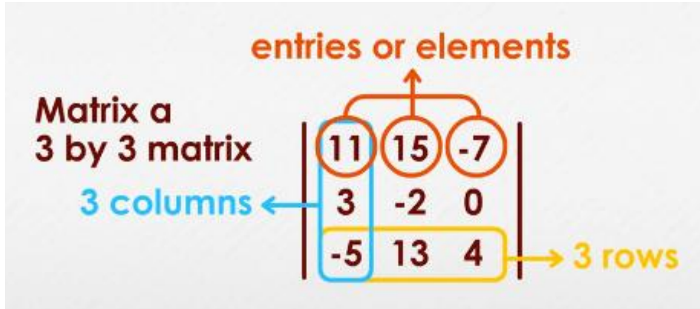
Order of a Matrix

The size (or dimension) of matrix is called as order of matrix. Order of matrix is based on the number of rows and number of columns. It can be written as r c ×; r means no. of row and c means no. of columns.



Square Matrix

A matrix with equal number of rows and columns is called square matrix.



Equality of matrices

Two matrices are said to be equal if: Both the matrices are of the same order i.e., they have the same number of rows and columns an m × n = B m × n.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \qquad B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}_{2\times 2}$$

$$A = B$$

$$\Rightarrow \begin{aligned} a &= 6 \\ b &= 7 \\ c &= 8 \\ d &= 9 \end{aligned}$$

Multiple of matrix

A multiple of a matrix A by a nonzero constant k is defined to multiplication of a matrix by a scalar mathematically as: If $A = [a_{ij}]_{m \times n}$ is a matrix and k is a scalar, then kA is another matrix obtained by multiplying each element of A by the scalar k.

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix} = [ka_{ij}]_{m\times n}$$

Addition of Matrices

Only matrices of the same order may be added by adding corresponding elements. If A = [aij] and B = [ij] b are two m× n matrices then A + B = [aij + bij] .Obviously order of the matrix A + B is m× n

$$A = \begin{pmatrix} -4 & 4 & 4 \\ 1 & -4 & 4 \\ -4 & -3 & 4 \\ -5 & 5 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -5 & 5 & 1 \\ -1 & -5 & -1 \\ 5 & 4 & -3 \\ 2 & -4 & 5 \end{pmatrix}$$

$$A + B = \begin{pmatrix} (-4 + -5) & (4 + 5) & (4 + 1) \\ (1 + -1) & (-4 + -5) & (4 + -1) \\ (-4 + 5) & (-3 + 4) & (4 + -3) \\ (-5 + 2) & (5 + -4) & (-3 + 5) \end{pmatrix}$$

$$= \begin{pmatrix} -9 & 9 & 5 \\ 0 & -9 & 3 \\ 1 & 1 & 1 \\ -3 & 1 & 2 \end{pmatrix}$$

Difference of Matrices

The difference of two matrices A and B of same order m× n is defined to be the matrix $A - B = A + (-B)$.

Multiplication of Matrices

We can multiply two matrices if and only if, the number of columns in the first matrix equals the number of rows in the second matrix. Otherwise, the product of two matrices is not possible.

$A_{m\times n} B_{n\times p} = C_{m\times p}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \times \begin{bmatrix} 10 & 11 \\ 20 & 21 \\ 30 & 31 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times10 + 2\times20 + 3\times30 & 1\times11 + 2\times21 + 3\times31 \\ 4\times10 + 5\times20 + 6\times30 & 4\times11 + 5\times21 + 6\times31 \end{bmatrix}$$

$$= \begin{bmatrix} 10+40+90 & 11+42+93 \\ 40+100+180 & 44+105+186 \end{bmatrix} = \begin{bmatrix} 140 & 146 \\ 320 & 335 \end{bmatrix}$$

Zero Matrix or Null matrix

A matrix whose all entries are zero is called zero matrix or null matrix and it is denoted by O. $O = \begin{pmatrix} 0 \\ 0 \end{pmatrix};$

Associative Law

The matrix multiplication is associative. This means that if A, B and C are $m \times p$, $p \times r$ and $r \times n$ matrices, then A(BC) = (AB)C

Distributive Law

If B and C are matrices of order $r \times n$ and A is a matrix of order $m \times r$, then the distributive law states that

- A(B + C) = AB + AC
- Furthermore, if the product (A + B)C is defined,
- then (A+ B)C = AC + BC

Determinant of a Matrix

Associated with every square matrix A of constants, there is a number called the determinant of the matrix,

Which is denoted by det (A) or |A |.

Find the determinant of the following matrix $A = \begin{pmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{pmatrix}$

Solution The determinant of the matrix A is given by

$$\det(A) = \begin{vmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{vmatrix}$$

We expand the $\det(A)$ by first row, we obtain

$$\det(A) = \begin{vmatrix} 3 & 6 & 2 \\ 2 & 5 & 1 \\ -1 & 2 & 4 \end{vmatrix} = 3\begin{vmatrix} 5 & 1 \\ 2 & 4 \end{vmatrix} - 6\begin{vmatrix} 2 & 1 \\ -1 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & 5 \\ -1 & 2 \end{vmatrix}$$

or $$\det(A) = 3(20 - 2) - 6(8 + 1) + 2(4 + 5) = 18$$

Transpose of a Matrix

The transpose of m× n matrix A is denoted by A^{tr} and it is obtained by interchanging rows of A into its columns. In other words, rows of A become the columns of. A^{tr}. Clearly A^{tr} is m× n matrix.

If $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$, then $A^{tr} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$

Properties of the Transpose

The following properties are valid for the transpose;

The transpose of the transpose of a matrix is the matrix itself:`
The transpose of a matrix times a scalar (k) is equal to the constant times the transpose of the matrix: $(\underline{A}\underline{B}\underline{C})^T = \underline{C}^T \underline{B}^T \underline{A}^T$ $(k\underline{A})^T = k\underline{A}^T$
The transpose of the sum of two matrices is equivalent to the sum of their transposes: $(\underline{A}+\underline{B})^T = \underline{A}^T + \underline{B}^T$
The transpose of the product of two matrices is equivalent to the product of their transposes in reversed order: $(\underline{A}\underline{B})^T = \underline{B}^T \underline{A}^T$
The same is true for the product of multiple matrices: $(\underline{A}\underline{B}\underline{C})^T = \underline{C}^T \underline{B}^T \underline{A}^T$

Multiplicative Inverse

Suppose that A is a square matrix of order n × n. If there exists an n × n matrix B such that AB = BA = I , then B is said to be the multiplicative inverse of the matrix A and is denoted by $B = A^{-1}$.

Singular and Non-Singular Matrices

Singular matrix

Singular matrix: A square matrix whose determinant is 0 is called singular matrix.

Nonsingular matrix

Nonsingular matrix: A square matrix that is not singular, i.e. one that has matrix inverse. Nonsingular matrices are sometimes also called regular matrices. A square matrix is nonsingular iff its determinant is non-zero.

Minor of Matrix

The minor of matrix is for each element of matrix and is equal to the part of the matrix remaining after excluding the row and the column containing that particular element. The new matrix formed with the minors of each element of the given matrix is called the minor of matrix.

The derivative matrix

The definition of differentiability in multivariable calculus is a bit technical. There are subtleties to watch out for, as one has to remember the existence of the derivative is a more stringent condition than the existence of partial derivatives.

$$Df(a) = \left[\frac{\mathrm{d}f}{\mathrm{d}x}(a)\right].$$

$f : \mathbf{R}^n \to \mathbf{R}$, viewed as a $f(\mathbf{x})$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the $1 \times n$ matrix of partial derivatives at $\mathbf{x} = \mathbf{a}$ is

$$Df(\mathbf{a}) = \left[\frac{\partial f}{\partial x_1}(\mathbf{a}) \; \frac{\partial f}{\partial x_2}(\mathbf{a}) \; \ldots \; \frac{\partial f}{\partial x_n}(\mathbf{a})\right].$$

Lecture 39 The Eigenvalue problem

The Eigenvalue problem

A rectangular arrangement of numbers in the form of rows and columns is known as a matrix. In this article, we will discuss **Eigenvalues and Eigenvectors Problems and Solutions**.

Consider a square matrix n × n. If X is the non-trivial column vector solution of the matrix equation AX = λX, where λ is a scalar, then X is the eigenvector of matrix A, and the corresponding value of λ is the eigenvalue of matrix A.

Suppose the matrix equation is written as A X – λ X = 0. Let I be the n × n identity matrix.

If I X is substituted by X in the equation above, we obtain

A X – λ I X = 0.

The equation is rewritten as (A – λ I) X = 0.

The equation above consists of non-trivial solutions if and only if the determinant value of the matrix is 0. The characteristic equation of A is Det (A – λ I) = 0. 'A' being an n × n matrix, if (A – λ I) is expanded, (A – λ I) will be the characteristic polynomial of A because its degree is n.

Properties of Eigenvalues

Let A be a matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.

The following are the properties of eigenvalues.

(1) The trace of A, defined as the sum of its diagonal elements, is also the sum of all eigenvalues,

$$tr(A) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i = \lambda_1 + \lambda_2 + \cdots + \lambda_n.$$

(2) The determinant of A is the product of all its eigenvalues,

$$\det(A) = \prod_{i=1}^{n} \lambda_i = \lambda_1 \lambda_2 \cdots \lambda_n.$$

(3) The eigenvalues of the k^{th} power of A, that is, the eigenvalues of A^k, for any positive integer k, are

$$\lambda_1^k, \ldots, \lambda_n^k.$$

(4) The matrix A is invertible if and only if every eigenvalue is nonzero.

(5) If A is invertible, then the eigenvalues of A^{-1} $\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$ are and each eigenvalue's geometric multiplicity coincide. The characteristic polynomial of the inverse is the reciprocal polynomial of the original; the eigenvalues share the same algebraic multiplicity.

(6) If A is equal to its conjugate transpose, or equivalently if A is Hermitian, then every eigenvalue is real. The same is true for any real symmetric matrix.

(7) If A is not only Hermitian but also positive-definite, positive-semidefinite, negative-definite, or negative-semidefinite, then every eigenvalue is positive, non-negative, negative, or non-positive, respectively.

(8) If A is unitary, every eigenvalue has absolute value $|\lambda_i| = 1$.

(9) If A is a n×n matrix and $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$ are its eigenvalues, then the eigenvalues of the matrix I + A (where I is the identity matrix) are $\{\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_k + 1\}$.

Lecture 40 Matrices and Systems of Linear First-Order Equations

Matrices are used to represent and manipulate data in various fields such as mathematics, physics, computer graphics, and more. They play a fundamental role in linear algebra and are used to solve systems of equations, transform vectors, and perform other mathematical operations.

Systems of Linear First-Order Equations:

A system of linear first-order equations involves a set of equations where each equation is linear (the highest power of any variable is 1) and contains only first-order derivatives. These systems are commonly encountered in fields like physics, engineering, and economics to model relationships between different variables.
For example, consider the following system of linear first-order equations:

dx/dt = 3x - 2y

dy/dt = x + y

Here, `x` and `y` are the variables, and `dx/dt` and `dy/dt` are their respective first-order derivatives with respect to time `t`. The coefficients (like 3, -2, 1) in the equations determine how each variable affects the rate of change of the other variables.

Matrix Representation of Systems:

Systems of linear first-order equations can be represented using matrices. The above system can be written in matrix form as:

$$\begin{aligned}
\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\
\frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\
&\vdots \\
\frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)
\end{aligned}$$

Suppose that X, $A(t)$ and $F(t)$, respectively, denote the following matrices

$$X = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \quad \mathrm{F(t)} = \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Then the system of differential equations can be written as

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

or simply

$$\frac{dX}{dt} = A(t)X + F(t)$$

If the system of differential equations is homogenous, then $F(t) = 0$ and we can write

$$\frac{dX}{dt} = A(t)X$$

Both the non-homogeneous and the homogeneous systems can also be written as

$$X' = AX + F, \quad X' = AX$$

Initial –Value Problem

Let t_0 denote any point in some interval denoted by I and

$$X(t_o) = \begin{pmatrix} x_1(t_o) \\ x_2(t_o) \\ \vdots \\ x_n(t_o) \end{pmatrix}, \quad X_o = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix}$$

$\gamma_i; i = 1, 2, \ldots, n$ are given constants. Then the problem of solving the system of differential equations

$$\frac{dX}{dt} = A(t)X + F(t)$$

Subject to the initial conditions

$$X(t_0) = X_0$$

is called an initial value problem on the interval I .

Existence of a unique Solution

Suppose that the entries of the matrices A t( ) and F t( ) in the system of differential equations

$$\frac{dX}{dt} = A(t)X + F(t)$$

Being considered in the above mentioned initial value problem, are continuous functions on a common interval I that contains the point t_0 . Then there exist a unique solution of the initial–value problem on the interval I.

Superposition Principle

Suppose that X1 ,X2 ,……. , Xn be a set of solution vectors of the homogenous system $\frac{dX}{dt} = A(t)X$

on an interval I . Then the principle of superposition states that linear combination

$$X = c_1X_1 + c_2X_2 + \cdots + c_kX_k$$

$c_{i;j}$=1,2,3,4,…….. ,K being arbitrary constants, is also a solution of the system on the same interval I .

Linear Dependence of Solution Vectors

Suppose that X1 ,X2 ,……. , Xn be a set of solution vectors of the homogenous system $\frac{dX}{dt} = AX$ We say that the set is linearly dependent on I if there exist constants C1 , C2, C3, not all zero such that

$$X(t) = c_1X_1(t) + c_2X_2(t) + \cdots + c_kX_k(t) = 0, \quad \forall\ t \in I$$

- Any two solution vectors X1 and X2 are linearly dependent if and only if one of the two vectors is a constant multiple of the other.
- For k > 2 if the set of k solution vectors is linearly dependent then we can express at least one of the solution vectors as a linear combination of the remaining vectors.

Linear Independence of Solution Vectors

Suppose that X1 ,X2 ,……. , X_k is a set of solution vectors, on an interval I, of the homogenous system of differential equations $\frac{dX}{dt} = AX$.

Then the set of solution vectors is said to be linearly independent if it is not linearly dependent on the interval I . This means that

$$X(t) = c_1X_1(t) + c_2X_2(t) + \cdots + c_kX_k(t) = 0$$

only when each $c_i = 0$.

Lecture 41 Matrices and Systems of Linear 1st-Order Equations (Continued)

Fundamental set of solution

Suppose that { X1 ,X2 ,..., Xn } is a set of n solution vectors, on an interval I , of a homogenous system / X AX = . The set is said to be a fundamental set of solutions of the system on the interval I if the solution vectors X1 ,X2 ,..., Xn are linearly independent.

General solution

Suppose that { X1 ,X2 ,..., Xn } is a fundamental set of solution of the homogenous system $X^{/}$ = AX on an interval / . Then any linear combination of the solution vectors

$$X = c_1X_1 + c_2X_2 + \cdots + c_nX_n$$

$c_i; i = 1, 2, \ldots, n$ being arbitrary constants is said to be the general solution of the system on the interval I.

Non-homogeneous Systems

As stated earlier in this lecture that a system of differential equations such as $\frac{dX}{dt} = A(t)X + F(t)$

Particular Integral

A particular solution, on an interval I , of a non-homogeneous system is any vector Xp free of arbitrary parameters, whose entries are functions that satisfy each equation of the system.

Complementary function

Let { X1 ,X2 ,..., Xn } be solution vectors of the homogenous system $X' = AX$ on an interval I , then the general solution $X = c_1X_1 + c_2X_2 + \ldots + c_nX_n$

of the homogeneous system is called the complementary function of the nonhomogeneous system

$X' = A(t)X + f(t)$

Fundamental Matrix

Suppose that the a fundamental set of n solution vectors of a homogeneous system $X' = AX$, on an interval I , consists of the vectors

$$X_1 = \begin{pmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{pmatrix}, X_2 = \begin{pmatrix} x_{12} \\ x_{22} \\ \vdots \\ x_{n2} \end{pmatrix}, \ldots, X_n = \begin{pmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{nn} \end{pmatrix}$$

Then a fundamental matrix of the system on the interval I is given by

$$\phi(t) = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

- The fundamental matrix φ(t) of a homogenous system $X' = A(t)X$ is nonsingular because the determinant det(( )) φ t coincides with the Wronskian of the solution vectors of the system and linear independence of the solution vectors guarantees that det(φ (t)) ≠ 0.

- Let φ (t) be a fundamental matrix of the homogenous system $X' = A(t)X$ on an interval I . Then, in view of the above mentioned observation, the inverse of the matrix $\varphi^{-1}(t)$ exists for every value of t in the interval I .

Lecture 42 Homogeneous Linear Systems

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Lecture 43 Real and Repeated Eigenvalues

Real and Repeated Eigenvalues

A system of linear differential equations having a coefficient matrix that has real distinct and complex eigenvalues.

$$X' = AX$$

in which some of the n eigenvalue $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda n$ K of the n × n coefficient matrix A are repeated.

Eigenvalue of multiplicity m

Suppose that m is a positive integer and $(\lambda - \lambda 1)^m$ is a factor of the characteristic equation

$$\det(A - \lambda I) = 0$$

Further, suppose that $(\lambda - \lambda 1)^{m+1}$ is not a factor of the characteristic equation. Then the number λ1 is said to be an eigenvalue of the coefficient matrix of multiplicitym .

Method of solution

Consider the following system of n linear differential equations in n unknowns

$$X' = AX$$

Suppose that the coefficient matrix has an eigenvalue of multiplicity of m . There are two possibilities of the existence of the eigenvectors corresponding to this repeated eigenvalue:

- For the n× n coefficient matrix A, it may be possible to find m linearly independent eigenvectors $K_1, K_2, \ldots\ldots\ldots\ldots, K_m$ corresponding to the eigenvalue λ_1 of multiplicity $m \leq n$. In this case the general solution of the system contains the linear combination;

$$c_1 K_1 e^{\lambda_1 t} + c_2 K_2 e^{\lambda_1 t} + \cdots + c_n K_n e^{\lambda_1 t}$$

- If there is only one eigenvector corresponding to the eigenvalue λ1 of multiplicity m , then m linearly independent solutions of the form

$$X_1 = K_{11} e^{\lambda_1 t}$$

$$X_2 = K_{21} e^{\lambda_1 t} + K_{22} e^{\lambda_1 t}$$

$$\vdots$$

$$X_m = K_{m1} \frac{t^{m-1}}{(m-1)!} e^{\lambda_1 t} + K_{m2} \frac{t^{m-2}}{(m-2)!} e^{\lambda_1 t} + \cdots + K_{mm} e^{\lambda_1 t}$$

Where the column vectors Kij can always be found.

Eigenvalue of Multiplicity Two

W the systems of differential equations X ′ = AX in which the coefficient matrix A has an eigenvalue λ1 of multiplicity two. Then there are two possibilities;

The case of the possibility of us being able to find two linearly independent eigenvectors K1, K2, corresponding to the eigenvalue λ1 is clear. In this case the general solution of the system contains the linear combination

$$c_1 K_1 t e^{\lambda_1 t} + c_2 K_2 e^{\lambda_1 t}$$

That there is only one eigenvector K1 associated with this eigenvalue and hence only one solution vector X1. Then, a second solution can be found of the following form:

$$X_2 = Kte^{\lambda_1 t} + Pe^{\lambda_1 t}$$

In this expression for a second solution, K and P are column vectors

$$K = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}, \quad P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix}$$

We substitute the expression for X 2 into the system X ′ = AX and simplify to obtain

$$\left(AK - \lambda_1 K\right) t\, e^{\lambda_1 t} + \left(AP - \lambda_1 P - K\right) e^{\lambda_1 t} = 0$$

Since this last equation is to hold for all values of t, we must have:

$$(A - \lambda_1 I)K = 0, \quad (A - \lambda_1 I)P = K$$

$$X_1 = Ke^{\lambda_1 t}$$

To find the second solution X2 , we only need to solve, for the vector P , the additional system

$$(A - \lambda_1 I)P = K$$

Eigenvalues of Multiplicity Three

When a matrix A has only one eigenvector associated with an eigenvalue λ_1 of multiplicity three of the coefficient matrix A, we can find a second solution X_2 and a third solution X_3 of the following forms

$$X_2 = Kte^{\lambda_1 t} + Pe^{\lambda_1 t}$$
$$X_3 = K\frac{t^2}{2}e^{\lambda_1 t} + Pte^{\lambda_1 t} + Qe^{\lambda_1 t}$$

The K, P and Q are vectors given by

$$K = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}, \quad P = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_n \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix}$$

By substituting X_3 into the system $X' = AX$, we find the column vectors K, P and Q must satisfy the equations

$$(A - \lambda_1 I)K = 0$$
$$(A - \lambda_1 I)P = K$$
$$(A - \lambda_1 I)Q = P$$

The solutions of first and second equations can be utilized in the formulation of the solution X_1 and X_2.

Lecture 44 Non-Homogeneous System

Non-Homogeneous System

A non-homogeneous system refers to a system of linear equations that is not homogeneous. In mathematics, a system of linear equations consists of multiple linear equations with the same variables. A linear equation can be represented in the general form:

$a_1x_1 + a_2x_2 + ... + a_nx_n = b$

Here, $x_1, x_2, ..., x_n$ are the variables, $a_1, a_2, ..., a_n$ are the coefficients of these variables, and b is the constant term.

A system of linear equations is considered homogeneous if all the constant terms (b values) are zero. In other words, a homogeneous system has equations of the form:

$a_1x_1 + a_2x_2 + \ldots + a_nx_n = 0$

On the other hand, a non-homogeneous system is one in which at least one equation has a nonzero constant term:

$a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ (where $b \neq 0$)

Matrix Notation

In the matrix notation we can write the above system of differential can be written as

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \ldots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) \ldots & a_{2n}(t) \\ \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) \ldots & a_{nn}(t) \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Or $X' = AX + F(t)$

Method of Solution

To find general solution of the non-homogeneous system of linear differential equations, we need to find:

- The complementary function X_c , which is general solution of the corresponding homogeneous System
$$X' = A X.$$
- Any particular solution X_p of the non-homogeneous system $X' = AX + F(t)$ by the method of undetermined coefficients and the variation of parameters.

The general solution X of the system is then given by sum of the complementary function and the particular solution.

$$X = X_c + X_p$$

Method of Undetermined Coefficients

The form of F (t)

As mentioned earlier in the analogous case of a single nth order non-homogeneous linear differential equations. The entries in the matrix F (t) can have one of the following forms:

- Constant functions
- Polynomial functions
- Exponential functions
- $\sin(\beta x)$, $\cos(\beta x)$
- Finite sums and products of these functions.

Duplication of Terms

The assumption for the particular solution X p has to be based on the prior knowledge of the complementary function Xc to avoid duplication of terms between Xc and Xp.

In the above example the entries of the matrix F (t) were constants and the complementary function Xc did not involve any constant vector. Thus there was no duplication of terms between Xc and Xp.

However, if F(t) were a constant vector and the coefficient matrix had an eigenvalue λ = 0 . Then Xc contains a constant vector. In such a situation the assumption for the particular solution Xp would be

$$X_p = \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} t + \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

instead of

$$X_p = \begin{pmatrix} a_1 \\ b_1 \end{pmatrix}$$

Lecture 45

THIS END