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Binary Operation:

Let S be a non-empty set. The binary operation $*$ is a function

$$* : S \times S \longrightarrow S$$

which assigns each ordered pair (a, b) of S x S, a unique element of S.

Suppose $a, b \in S$,

$a * b \in S$

$S = \{1, 2, 3, \dots\}$

$2, 5 \in S, \quad 2+5 = 7 \in S$

$x, y \in S, \quad x+y \in S$

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Example: **Show that multiplication is a binary operation on the set of Natural numbers N.**

Sol: $N = \{1, 2, 3, 4, \dots\}$

$7, 5 \in N$

$7 \times 5 = 35 \in N$

$x, y \in N, \quad xy \in N$

So multiplication is a binary operation on N.

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Example : **Show that division is not a binary operation on the set of Natural Numbers N.**

Sol. $N = \{1, 2, 3, 4, \ldots\}$

$4, 2 \in N$, $4 \div 2 = 2 \in N$

$2 \div 4 = 0.5 \notin N$

for all $x, y \in N$, $x \div y \notin N$

So $\div$ is not a binary operation on N.

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Example : Show that subtraction is not a binary operation on N.

Sol: $N = \{1, 2, 3, 4, \cdots\}$

$2, 3 \in N, \quad 3 - 2 = 1 \in N$

$2 - 3 = -1 \notin N$

for all $x, y \in N, \quad x - y \notin N$

So '$-$' is not a binary operation on N.

$Z = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$

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Example : Show that division is a binary operation on R.

Sol :

$\frac{2}{3}, \frac{5}{7} \in R$

$\frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \times \frac{7}{5} = \frac{14}{15} \in R$

$\sqrt{2} \div \sqrt{3} = \frac{\sqrt{2}}{\sqrt{3}} \in R$

$R = Q \cup Q'$

p/q $\quad \sqrt{2}, \sqrt{3}$

$\frac{1}{2}, \frac{5}{7}$

for all $x, y \in R$, $x \div y \in R$

Hence '$\div$' is a binary operation on R.

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Example : Show that the operation $\circ$ defined by

$$x \circ y = 0 \qquad \text{for all } x, y \in Z$$

is a binary operation on Z.

Sol: $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$2, 3 \in Z, \quad 2 \circ 3 = 0 \in Z$

$x, y \in Z, \quad x \circ y = 0 \in Z$

Hence the operation '$\circ$' is a binary operation on Z.

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Example : Show that the operation $\circ$ defined by

$$x \circ y = x + y - xy \quad \text{for all } x, y \in Z$$

is a binary operation on Z.

Sol: $Z = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$3, 5 \in Z$

$3 \circ 5 = 3 + 5 - 3 \times 5$

$= 8 - 15$

$= -7 \in Z$

for $x, y \in Z$

$x + y \in Z, \quad xy \in Z$

$x + y - xy \in Z \Rightarrow x \circ y \in Z$

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GROUPOID:

A non-empty set with a binary operation is called Groupoid.

$N = \{1, 2, 3, \cdots\}$

$(N, +)$

$3, 5 \in N, \quad 3 + 5 = 8 \in N$

$x, y \in N, \quad x + y \in N$

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Example: Show that $(Z, -)$ is a groupoid under the operation of usual subtraction.

Sol: $Z = \{\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots\}$

$3, 5 \in Z,$

$3 - 5 = -2 \in Z$

for $x, y \in Z, \quad x - y \in Z$

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Example: Show that $(Z, \div)$ is not a groupoid under the operation of usual division.

Sol: $Z = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

$1, 2 \in Z.$

$1 \div 2 = 0.5 \notin Z$

$4, 2 \in Z, \quad 4 \div 2 = 2 \in Z$

for all $x, y \in Z \quad x \div y \in Z$

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A non-empty set with an associative binary operation is called semigroup.

$(S, *)$

1) for $x, y \in S$, $x * y \in S$

S is closed under *

2) Associative law must hold

for $x, y, z \in S$

$$(x * y) * z = x * (y * z)$$

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Example: Show that N = { 1, 2, 3, ... } is a semigroup under the operation of addition.

Sol: $(N, +)$ $N = \{1, 2, 3, \dots\}$

1) $11, 13 \in N$, $11 + 13 = 24 \in N$

Similarly, for all $x, y \in N$, $x + y \in N$

So N is closed under +.

2) $7, 8, 9 \in N$

$(7+8)+9 = 15+9 = 24$

$7+(8+9) = 7+17 = 24$

$(7+8)+9 = 7+(8+9)$

for any $x, y, z \in N$,

$(x+y)+z = x+(y+z)$

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Example: Show that the set $(Z, \circ)$ is a semigroup, where $\circ$ is the binary operation defined by

$$x \circ y = x + y - xy$$

Sol: $Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$2, 3 \in Z$

1) ⊂ binary operation closure property

2) Associative law.

$2 \circ 3 = 2 + 3 - 2 \times 3$

$= 5 - 6 = -1 \in Z$

for $x, y \in Z$, $x + y \in Z$, $xy \in Z$

$x + y - xy \in Z$

$x \circ y \in Z$.

So the closure property holds.

2) Associative law.

$$(x \circ y) \circ z = x \circ (y \circ z)$$

$LHS = (x \circ y) \circ z$

$= [x + y - xy] \circ z = (x + y - xy) + z - (x + y - xy)z$

$= x + y - xy + z - xz - yz + xyz$

$= x + y + z - xy - yz - xz + xyz$ — I

$RHS = x \circ (y \circ z)$

$= x \circ (y + z - yz)$

$= x + (y + z - yz) - x(y + z - yz)$

$= x + y + z - yz - xy - xz + xyz.$

$= x + y + z - xy - yz - xz + xyz$ — II

By I and II,

$$(x \circ y) \circ z = x \circ (y \circ z)$$

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MONOID:

(2) (1)
A non-empty set with an associative binary operation and an identity element (3) is called Monoid.

Let S be a non-empty set

1) Let $*$ be the binary operation on S.
   for every $x, y \in S$, $x * y \in S$
   So closure property holds

2) Associative law holds.
   for $x, y, z \in S$,
   $$(x * y) * z = x * (y * z)$$

3) Identity element exists in S
   for all $x \in S$, $e * x = x * e = x$

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Example: Show that the set W = { 0, 1, 2, 3, ...} is Monoid under the binary operation of addition.

$$W = \{0, 1, 2, \cdots\} \qquad (W, +)$$

1) Closure Property:

$13, 15 \in W$

$13 + 15 = 28 \in W$

Hence closure property holds.

2) Associative law:

$4, 5, 6 \in W$

$(4+5)+6 = 4+(5+6)$

$9+6 = 15 \qquad 4+11 = 15$

for any $x, y, z \in S$, $(x+y)+z = x+(y+z)$

3) Identity element: $e = 0$

$2+0 = 2 = 0+2$

for any $x \in W$, $x+0 = x = 0+x$

Since W satisfies all the three conditions, so W is monoid.

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MONOID:

A non-empty set with an associative binary operation and an identity element is called Monoid.

(2) (1) (3)

Let S be a non-empty set

1) Let $*$ be the binary operation on S.
   for every $x, y \in S$, $x * y \in S$
   So closure property holds
2) Associative law holds.
   for $x, y, z \in S$,
   $(x * y) * z = x * (y * z)$
3) Identity element exists in S
   for all $x \in S$, $e * x = x * e = x$

Example: Show that the set W = { 0, 1, 2, 3, ...} is Monoid under the binary operation of addition.

$W = \{0, 1, 2, \cdots\}$ $(W, +)$

1) Closure property:
   $13, 15 \in W$
   $13 + 15 = 28 \in W$
   Hence closure property holds.
2) Associative law:
   $4, 5, 6 \in W$
   $(4+5)+6 = 4+(5+6)$
   $9+6 = 15$ $4+11 = 15$
   for any $x, y, z \in S$, $(x+y)+z = x+(y+z)$
3) Identity element: $e = 0$
   $2+0 = 2 = 0+2$
   for any $x \in W$, $x+0 = x = 0+x$

Since W satisfies all the three conditions, so W is monoid.

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GROUP:

A non-empty set with an associative binary operation, an identity element and the inverses of its all elements is called Group.

$S \neq \{\}$, $(S, *)$

1) Closure property: for $x, y \in S$,

$x * y \in S$

So closure property holds.

2) Associative law:

for $x, y, z \in S$, $(x * y) * z = x * (y * z)$

3) Identity element exists.

$e \in S$, $e * x = x * e = x$ for $x \in S$

4) Inverses of all elements of S exist in S.

for all $x \in S$, $\exists\ x^{-1} \in S$ such that

$x * x^{-1} = e = x^{-1} * x$

Therefore $(S, *)$ is a group.

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Example: Show that the set {1, -1} is a group under multiplication.

Sol: $G = \{1, -1\}$

1) Closure Property

for $x, y \in G$, $xy \in G$

·	1	-1
1	1	-1
-1	-1	1

2) Associative law.

$(1 \cdot (-1)) \cdot (-1) = 1 \cdot ((-1) \cdot (-1))$

$(-1) \cdot (-1) = 1 \cdot (1)$

$1 = 1$

$(xy)z = x(yz)$ for all $x, y, z \in G$

3) Identity Element.

$e \cdot x = x = x \cdot e$ for all $x \in G$

$e = 1$

$1 \cdot x = x \cdot 1 = x \quad \forall x \in G.$

4) Inverse Element.

for every $x \in G$, $\exists\ \bar{x}^1 \in G$ s.t.

$x\bar{x}^1 = \bar{x}^1 x = e$

$(1)(1) = 1 \qquad x = 1,\ \bar{x}^1 = 1$

$(-1)(-1) = 1 \qquad x = -1,\ \bar{x}^1 = -1$

Hence inverse of each element of G exists in G.

So $G = \{1, -1\}$ is a group under multiplication.

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Example : Show that (Z, +) is a group under the binary operation of addition.

Sol: $Z = \{ \cdots, -2, -1, 0, 1, 2, \cdots \}$

1) $10, 15 \in Z, \quad 10 + 15 = 25 \in Z$

for every $x, y \in Z, \quad x + y \in Z$

So closure property holds.

2) Associative law.

$3, 4, 7 \in Z$

$(3+4)+7 = 3+(4+7)$

$7+7 = 3+11$

$14 = 14$

for $x, y, z \in Z$,

$(x+y)+z = x+(y+z)$

3) Identity Element: $e = 0$

for any $x \in Z, \quad 0 + x = x + 0 = x$

4) Inverse Element of each $x \in Z$.

$2 + (-2) = 0$

$-10 + (10) = 0$

for every $x \in Z, \exists -x \in Z$ such that

$x + (-x) = -x + x = 0$

So $(Z, +)$ satisfies all conditions of group, Therefore $(Z, +)$ is a group.

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Example : Show that $(Z - \{0\}, \cdot)$ is not a group under the binary operation of multiplication.

Sol: $Z-\{0\} = \{\ldots, -2, -1, 1, 2, \ldots\}$

1) Closure property :

$3, 4 \in Z, \quad 3 \times 4 = 12 \in Z$

for any $x, y \in Z, \quad xy \in Z$

2) Associative law :

$2, 5, 4 \in Z - \{0\}$

$$\begin{aligned} (2 \times 5) \times 4 &= 2 \times (5 \times 4) \\ 10 \times 4 &= 2 \times 20 \\ 40 &= 40 \end{aligned}$$

for $x, y, z \in Z - \{0\}$

$$(xy)z = x(yz)$$

3) Identity Element: $e = 1 \in Z - \{0\}$

$1 \cdot x = x \cdot 1 = x \quad \forall x \in Z - \{0\}$

4) Inverse Element of each $x \in Z - \{0\}$

$2 \times \left(\frac{1}{2}\right) = 1$ but $\frac{1}{2} \notin Z - \{0\}$

$x \bar{x}^1 = e$

for any $x \in Z - \{0\}$, there is no inverse.

i.e. $\frac{1}{x} \notin Z - \{0\}$

So $Z - \{0\}$ is not a group.

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Example: Show that $G = \{1, w, w^2\}$ is a group under multiplication, where w is cube root of unity.

Sol: $G = \{1, \omega, \omega^2\}$

$\cdot$	1	ω	ω^2
1	1	ω	ω^2
ω	ω	ω^2	1
ω^2	ω^2	1	ω

$x = \sqrt[3]{1}$

$x = (1)^{1/3}$

$x^3 = 1$

$1, \omega, \omega^2$

$\omega^3 = 1$

$\omega^2 \omega^2 = \omega^4$
$= \omega^3 \omega$
$= 1 \cdot \omega$
$= \omega$

1) It is clear from the table that closure property holds.

2) Associative law:

$1, \omega, \omega^2 \in G$

$(1 \cdot \omega) \cdot \omega^2 = 1 \cdot (\omega \cdot \omega^2)$

$\omega \cdot \omega^2 = 1 \cdot \omega^3$

$\omega^3 = \omega^3$

$1 = 1$

for any $x, y, z \in G$,

$(xy)z = x(yz)$

3) Identity element: $e = 1$

$\omega \cdot 1 = 1 \cdot \omega = \omega$

$\omega^2 \cdot 1 = 1 \cdot \omega^2 = \omega^2$

$1 \cdot 1 = 1 \cdot 1 = 1$

4) Inverse Element of each element $x \in G$.

$1 \cdot (1) = 1 \Rightarrow (1)^{-1} = 1$

$\omega (\omega^2) = 1 \Rightarrow (\omega)^{-1} = \omega^2$

$\omega^2 (\omega) = 1 \Rightarrow (\omega^2)^{-1} = \omega$

Hence G satisfies all the conditions of a group. Hence G is a group under multiplication.

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Example: Show that $G = \{ 1, -1, i, -i \}$ is a group under multiplication.

Sol:

·	1	-1	i	-i
1	1	-1	i	-i
-1	-1	1	-i	i
i	i	-i	-1	1
-i	-i	i	1	-1

1) Closure Property

It is clear from the table that the closure property holds.

2) Associative law

$i = \sqrt{-1}$

$i^2 = (\sqrt{-1})^2 = -1$

$$(i \cdot (-i))(-1) = i \cdot ((-i)(-1))$$

$$(+1)(-1) = i \cdot (i)$$

$$-1 = -1$$

for any $x, y, z \in G$, $(xy)z = x(yz)$

3) Identity Element: $e = 1$

for any $x \in G$, $1 \cdot x = x \cdot 1 = x$

4) Inverse element of each $x \in G$.

$$x \cdot x^{-1} = e = x^{-1} x$$

$(1)(1) = 1$ | $(i)(-i) = 1$

$(-1)(-1) = 1$ | $(-i)(i) = 1$

$(1)^{-1} = 1$, $(-1)^{-1} = (-1)$

$(i)^{-1} = -i$, $(-i)^{-1} = i$

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Theorem : In a group G, the identity element is unique.

Proof: Contrarily suppose that identity element is not unique.

Let e & e' be two identity elements.

for all $x \in G$,

$$xe = x = ex$$

$$e'e = e' = ee' \quad \text{—— (1)}$$

$$ee' = e = e'e \quad \text{—— (2)}$$

Now By (1) & (2),

$$e = e'$$

Hence e is unique.

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Theorem : The only idempotent element in a group G is the identity.

Proof: Let $a \in G$, be an idempotent element.

$$a^2 = a$$

$$aa = a$$

$$\bar{a}^1(aa) = \bar{a}^1 a$$

$$(\bar{a}^1 a)a = e$$

$$ea = e$$

$$a = e$$

Hence proved.

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Definition :

An element a of a group G is called idempotent if $a^2 = a$.

$a \in G$, $a^2 = a$

$1^2 = 1$, $0^2 = 0$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I^2 = I$

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Theorem : In a group G, the inverse of each element is unique.

Proof: Suppose Contrarily that the inverse of each element of G is not unique.

So let $\bar{a}^1$ & $\bar{a}^{-1}$ be the inverses of $a \in G$.

$$\bar{a}^1 a = e = a\bar{a}^1 \quad \text{—①}$$

and $$\bar{a}^{-1} a = e = a\bar{a}^{-1} \quad \text{—②}$$

Consider $a\bar{a}^1 = e$

$$\bar{a}^{-1}(a\bar{a}^1) = \bar{a}^{-1}e$$

$$(\bar{a}^{-1}a)\bar{a}^1 = \bar{a}^{-1}$$

$$e\bar{a}^1 = \bar{a}^{-1} \qquad \text{by ②}$$

$$\bar{a}^1 = \bar{a}^{-1}$$

Hence the inverse of each element is unique.

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Theorem : If G is a group, then $(a^{-1})^{-1} = a \quad \forall\, a \in G$

Proof: Since G is a group, and $a \in G$,

$$\bar{a}^{1} \in G \qquad \bar{a}^{1}a = a\bar{a}^{1} = e \quad \text{—①}$$

$$(\bar{a}^{1})^{-1} \in G \qquad (\bar{a}^{1})^{-1}\bar{a}^{1} = e = \bar{a}^{1}(\bar{a}^{1})^{-1} \text{—②}$$

from ②

$$(\bar{a}^{1})^{-1}\,\bar{a}^{1} = e$$

$$((\bar{a}^{1})^{-1}\,\bar{a}^{1})a = ea$$

$$(\bar{a}^{1})^{-1}(\bar{a}^{1}a) = a$$

$$(\bar{a}^{1})^{-1}\,e = a$$

$$(\bar{a}^{1})^{-1} = a$$

Hence the proof.

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Theorem : Let G be a group, then $(ab)^{-1} = b^{-1}a^{-1} \quad \forall\, a, b \in G$

Proof: Since G is a group

and $a, b \in G \Rightarrow a^{-1} \in G, b^{-1} \in G$

then $ab \in G$

So $(ab)^{-1} \in G$

Therefore, $(ab)^{-1}ab = e$

$$((ab)^{-1}ab)b^{-1} = eb^{-1}$$

$$((ab)^{-1}a)(bb^{-1}) = b^{-1}$$

$$((ab)^{-1}a)e = b^{-1}$$

$$((ab)^{-1}a) = b^{-1}$$

$$((ab)^{-1}a)a^{-1} = b^{-1}a^{-1}$$

$$(ab)^{-1}(aa^{-1}) = b^{-1}a^{-1}$$

$$(ab)^{-1}e = b^{-1}a^{-1}$$

$$(ab)^{-1} = b^{-1}a^{-1}$$

Hence proved.

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Theorem : For any three elements a, b, c of a group G,

(i) $ab = ac \Rightarrow b = c$ (Left Cancellation Law)

(ii) $ba = ca \Rightarrow b = c$ (Right Cancellation Law)

Proof: (i) $ab = ac$

$a^{-1}(ab) = a^{-1}(ac)$ $\quad \because a \in G,\ a^{-1} \in G$

$(a^{-1}a)b = (a^{-1}a)c$

$eb = ec$

$b = c$

Hence proved.

(ii) $ba = ca$

$(ba)a^{-1} = (ca)a^{-1}$

$b(aa^{-1}) = c(aa^{-1})$

$be = ce$

$b = c$

Hence proved.

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ORDER OF A GROUP

The number of elements in a group G is called order of the group.

It is denoted be O(G) or |G|.

Finite Group:

If a group has a finite number of elements, then it is called a finite group.

In simple words, a finite group has a finite order.

Infinite Group:

If a group has an infinite number of elements, then it is called an infinite group.

In simple words, an infinite group has an infinite order.

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Example: What is the order of the group G = { 1, -1, i, -i }?

Sol: Since there are 4 elements in the group G.

So $o(G) = 4$

$|G| = 4$

Example: What is the order of the group

G = { 1, -1, i, -i, j, -j, k, -k }?

Sol: Since there are 8 elements in the group G.

So the order of G is 8

$o(G) = 8$

$|G| = 8$

Example: What is the order of the group (Z, +) ?

Sol: $Z = \{ \dots, -2, -1, 0, 1, 2, \dots \}$

Since there are infinite many elements in Z

$o(Z) = \infty$

$|Z| = \infty$

Example: What is the order of the group $\{ 2^k : k = 0, \pm 1, \pm 2, \dots \}$?

Sol: $G = \{ 2^k : k = 0, \pm 1, \pm 2, \dots \}$

Since the values of k are infinite

So there are infinite many elements of the group in form 2^k.

So order of the group is infinity

$|G| = \infty$

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ORDER OF AN ELEMENT OF A GROUP

Let G be a group and $a \in G$, then the order of 'a' is the least +ve integer n such that

$$a^n = e$$

where 'e' is identity.

$G = \{1, -1\}$ $\quad e = 1$

$(1)^1 = 1$

Order of $1 = 1$

$(1)^2 = 1$

$(1)^3 = 1$

$(-1)^1 = -1$

$(-1)^2 = 1$

order of $-1 = 2$

$(-1)^4 = 1$

$(-1)^6 = 1$

$(-1)^8 = 1$

Example: Find the order of each element of the group

$G = \{1, -1, i, -i\}$

Sol: $(G, \cdot)$ Here $e = 1$

$(1)^1 = 1 \Rightarrow e^n = 1 \Rightarrow n = 1$

$\boxed{o(1) = 1}$

$(-1)^2 = (-1)(-1) = 1 \Rightarrow \boxed{o(-1) = 2}$

$(i)^2 = -1 \quad \because i = \sqrt{-1}$

$(i)^3 = i^2 \cdot i = -1 \cdot i = -i$

$(i)^4 = i^2 \cdot i^2 = (-1)(-1) = 1 \Rightarrow \boxed{o(i) = 4}$

$(-i)^2 = i^2 = -1$

$(-i)^3 = (-i)^2 (-i)^1 = (-1)(-i) = i$

$(-i)^4 = (-i)^3 (-i)^1 = (-1)(-1) = 1$

$\Rightarrow \boxed{o(-i) = 4}$

Example: Find the order of each element of the group

$G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

of residue class modulo 5, under addition.

Sol: Here $e = \bar{0}$

$\bar{0} = \bar{0} \Rightarrow \boxed{o(\bar{0}) = 1}$

$\bar{1} + \bar{1} = \bar{2}$

$\bar{1} + \bar{1} + \bar{1} = \bar{3}$

$\bar{1} + \bar{1} + \bar{1} + \bar{1} = \bar{4}$

$\bar{1} + \bar{1} + \bar{1} + \bar{1} + \bar{1} = \bar{0} \Rightarrow \boxed{o(\bar{1}) = 5}$

$5 \overline{)5} \; 1 \; (5 \; 0$ ✓

$\bar{2} + \bar{2} = \bar{4}$

$\bar{2} + \bar{2} + \bar{2} = \bar{1}$

$\bar{2} + \bar{2} + \bar{2} + \bar{2} = \bar{3}$

$\bar{2} + \bar{2} + \bar{2} + \bar{2} + \bar{2} = \bar{0}$

$\boxed{o(\bar{2}) = 5}$

$5 \overline{)6} \; 1 \; (5 \; 1$ ✓

$5 \overline{)8} \; 1 \; (5 \; 3$ ✓

$5 \overline{)10} \; 2 \; (10 \; 0$

$\boxed{a^n = e}$ $\quad 5(\bar{1})$ $\quad \boxed{na = e}$

$\bar{3} + \bar{3} = \bar{1}$

$\bar{3} + \bar{3} + \bar{3} = \bar{4}$

$\bar{3} + \bar{3} + \bar{3} + \bar{3} = \bar{2}$

$\bar{3} + \bar{3} + \bar{3} + \bar{3} + \bar{3} = \bar{0}$

$\boxed{o(\bar{3}) = 5}$

$5 \overline{)6} \; 1 \; (5 \; 1$ ✓

$5 \overline{)9} \; 1 \; (5 \; 4$ ✓

$5 \overline{)12} \; 2 \; (10 \; 2$ ✓

$5 \overline{)15} \; 3 \; (15 \; 0$

Similarly, $\boxed{o(\bar{4}) = 5}$

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Theorem: If n is the order of an element 'a' of a group G, then for any integer m,

$a^m = e$ if and only if $m = qn$, where q is an integer.

or

$a^m = e$ if and only if n divides m.

Sol: Given $a \in G$

$10 = 5 \times 2$

$a^n = e$ ——① n is least +ve integer

Suppose $a^m = e$ ——②

By Division algorithm, there exist $q, r \in Z$, such that

$m = qn + r \quad 0 \leq r < n$

$n \overline{)m} \; q \; r$

By ②

$a^m = e$

$a^{qn+r} = e$

$a^{qn} \cdot a^r = e$

$(a^n)^q \cdot a^r = e$

$e^q \cdot a^r = e$

$e \cdot a^r = e$

$a^r = e$

$\because a^n = e$

$e^r = e$

$e \cdot a^r = a^r$

It is possible only when $r = 0$

$m = qn + r$

$= qn + 0$

$m = qn$

n divides m

Conversely, suppose that

$m = qn$

Consider $a^m = a^{qn}$

$= (a^n)^q$

$= (e)^q \quad \because a^n = e$

$a^m = e$

Hence it is proved.

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Theorem: Show that in a group, the order of an element is same as that of its inverse.

Proof: Let G be a group, $a \in G$

suppose $a^n = e$ — (1)

$(a^{-1})^m = e$ — (2)

Now we have to prove that $n = m$.

By eq (1) $a^n = e$

$a^{-n} a^n = a^{-n} e$

$a^{-n+n} = a^{-n}$

$e = a^0 = a^{-n}$

$e = (a^{-1})^n$

But the order of a^{-1} is m, so m must divide n — I

By eq (2), $(a^{-1})^m = e$

$a^{-m} = e$

$a^m a^{-m} = a^m e$

$a^{m-m} = a^m$

$a^0 = a^m$

$e = a^m$

But the order of a is n, so n must divide m — II

It is possible only when $m = n$

Hence it is proved.

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Theorem: Let G be a group and a, b ∈ G. Show that the orders of ab and ba are equal.

Proof:- let $(ab)^n = e$ ——— (1)

$(ba)^m = e$ ——— (2)

from eq(1) $(ab)^n = e$

$(ab)(ab)\cdots(ab) = e$ n-factors

$\bar{a}^1 a(bab\cdots ab) = \bar{a}^1 e$

$e(baba\cdots bab) = a^{-1}$

$ba.ba\cdots ba.b = \bar{a}^1$

$ba.ba\cdots ba.ba = \bar{a}^1 a$

$(ba)^n = e$

By (2), the order of ba is m.

Therefore m must divide n ——— (A)

From (2), $(ba)^m = e$

$ba.ba\cdots ba = e$ m-factors

$b^{-1}(ba.ba\cdots ba) = b^{-1}e$

$b^{-1}b(a.ba\cdots ba) = b^{-1}$

$e(abab\cdots ba) = b^{-1}$

$ab.ab\cdots ab.a = b^{-1}$

$(ab.ab\cdots ab.a)b = b^{-1}b$

$ab.ab\cdots ab.ab = e$

$(ab)^m = e$

But from eq(1), the order of ab is n.

So n must divide m. ——— (B)

By (A) & (B), $m = n$

$o(ba) = o(ab)$

Hence the proof.

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Abelian Group:

Let (G, $*$) be a group, then G is called an Abelian group or Commutative group if the commutative law holds in G with respect to $*$, i.e.

$$x * y = y * x \qquad \text{for all } x, y \text{ in G}$$

Non-Abelian Group:

Let (G, $*$) be a group, then G is called a Non-Abelian group or Non-Commutative group if the commutative law does not hold in G with respect to $*$, i.e.

$$x * y \neq y * x \qquad \text{for all } x, y \text{ in G}$$

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Set

Example: Show that the group (Z, +) is an Abelian Group.

Sol: $Z = \{ \dots, -1, 0, 1, \dots \}$

$10 + 21 = 31$

$21 + 10 = 31$

for all $x, y \in Z$,

$x + y = y + x$

Z is an abelian group.

3 − 2 = 1
2 − 3 = −1

1) Closure property
2) Associative law
3) Identity
4) Inverse
5) Commutative law

Set

Example: Show that the group (R, +) is an Abelian Group.

Sol:

$2.5 + 3.5 = 6.0$

$3.5 + 2.5 = 6.0$

for $x, y \in R$,

$x + y = y + x$

So Commutative law holds.

So (R, +) is an abelian group.

Example: Show that the group G = { *1*, *w*, w^2 }, where *w* is the cube root of unity, is an Abelian Group.

Sol: $G_1 = \{1, w, w^2\}$

$x = (1)^{1/3}$

$x^3 = 1$

$\{1, w, w^3\}$

$w^3 = 1$

·	1	w	w^2
1	1	w	w^2
w	w	w^2	1
w^2	w^2	1	w

$w \cdot w^2 = w^3 = 1$

$w^2 \cdot w = w^3 = 1$

$w^4 = w^3 w = 1 \cdot w = w$

for all $x, y \in G_1$,

$xy = yx$ So Commutative law holds

So G_1 is an abelian group.

Example: Show that the group { *1*, *-1*, *i*, *-i* } is an Abelian Group.

$G_1 = \{1, -1, i, -i\}$

·	1	−1	i	−i
1	1	−1	i	−i
−1	−1	1	−i	i
i	i	−i	−1	1
−i	−i	i	1	−1

$i(-i) = 1$

$(-1)\, i = 1$

$-1 \cdot i = -i$

$i \cdot (-1) = -i$

for any $x, y \in G_1$,

$xy = yx$

So G_1 is an abelian group.

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Set

Example: Show that the group { *1, -1, i, -i, j, -j, k, -k* } is a Non-Abelian Group.

Sol:

$ij = -ji$ $\quad ij \neq ji$

$jk = -kj$ $\quad jk \neq kj$

$ki = -ik$ $\quad ki \neq ik$

1) Closure property

2) Associative law

3) Identity

4) Inverse exists.

5) Commutative law

So the given is not abelian.

Example: Show that the set of non-singular matrices of order 2 is a Non-Abelian Group.

Sol: G = Set of non-singular matrices of order 2.

1) closure property.

Let $A, B \in G$, $\quad AB = C \in G$

2) Associative law.

$(AB)C = A(BC)$

3) Identity Element exists

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ such that

$AI = A = IA \quad \forall A \in G$

4) Inverse of each element in G.

Since G has non-singular matrices, so for all $A \in G$,

$\exists\ \bar{A}^1 \in G$, $\quad \bar{A}^1 A = I = A\bar{A}^1$

5) Commutative law.

In general,

$AB \neq BA$

So commutative law does not hold,

So G is a non-abelian group.

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Theorem:

Show that a group G is abelian if and only if $(ab)^2 = a^2 b^2$ for all a, b in G.

Proof: Let G be an abelian,

$ab = ba$ —— ① for all $a, b \in G$

Consider $(ab)^2 = (ab)(ab)$

$= a(ba)b$

$= a(ab)b$ by ①

$= (aa)(bb)$

$= a^2 b^2$ Hence proved

Now Conversely suppose that

$(ab)^2 = a^2 b^2$

$(ab)(ab) = (aa)(bb)$

$a(ba)b = a(ab)b$

$\bar{a}^1(a(ba)b) = \bar{a}^1(a(ab)b)$

$\bar{a}^1 a(ba)b = \bar{a}^1 a(ab)b$

$e(ba)b = e(ab)b$

$(ba)b = (ab)b$

$((ba)b)b^{-1} = ((ab)b)b^{-1}$

$(ba)(bb^{-1}) = (ab)(bb^{-1})$

$(ba)e = (ab)e$

$ba = ab$

So the Commutative law holds, so G is an abelian group.

Hence the proof.

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Theorem:

If a group G is such that

$$a.a = e \qquad \text{for all } a \text{ in } G$$

where e is identity, then G is an Abelian Group.

Proof: Let $a, b \in G$,

$$a \cdot a = e \quad \text{——} (1)$$

$$b \cdot b = e \quad \text{——} (2)$$

By (1)

$$a^{-1}(a \cdot a) = a^{-1}e$$
$$(a^{-1}a)\,a = a^{-1}$$
$$e\,a = a^{-1}$$
$$a = a^{-1} \quad \text{——} (3)$$

By (2)

$$b \cdot b = e$$
$$b^{-1}(b \cdot b) = b^{-1}e$$
$$(b^{-1}b)\,b = b^{-1}$$
$$e\,b = b^{-1}$$
$$b = b^{-1} \quad \text{——} (4)$$

Since $a, b \in G$, $ab \in G$

$$ab = (ab)^{-1}$$
$$= b^{-1}a^{-1} \qquad \because (ab)^{-1} = b^{-1}a^{-1}$$
$$ab = ba$$

Since Commutative law holds,
So G is an abelian group.

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Theorem:

If each element of a group G is its own inverse, then show that G is an Abelian Group.

Proof: Let $a, b \in G$,

$$a = a^{-1} \quad \text{---} (1)$$

$$b = b^{-1} \quad \text{---} (2)$$

Since $a, b \in G$, $ab \in G$

$$(ab) = (ab)^{-1}$$

$$ab = b^{-1}a^{-1}$$

$$ab = ba$$

$\because (ab)^{-1} = b^{-1}a^{-1}$

So G is an abelian group.

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Theorem:
If a group G has three elements, then show that G is an Abelian Group.

Proof: Let $G=\{e,a,b\}$

From the table, it is clear that $a^2 \in G$

.	e	a	b
e	e	a	b
a	a	a^2	ab
b	b	ba	b^2

There are three possibilities

(i) $a^2=e$
(ii) $a^2=a$
(iii) $a^2=b$

Case(i): Suppose $a^2=e$ ✗

Since $ab \in G$, then

1) $ab=e \Rightarrow a(ab)=ae \Rightarrow a^2b=a$
$eb=a$
$b=a$
It is not possible.

2) $ab=a \Rightarrow b=e$ Not

3) $ab=b$
$(ab)b^{-1}=bb^{-1}$
$ae=e$
$a=e$ Not

$a \neq b \neq e$

Case(ii): Suppose $a^2=a$

$aa=a$
$a^{-1}(aa)=a^{-1}a$
$(a^{-1}a)a=e$
$ea=e$
$a=e$ Not possible.

Case(iii)

$G=\{e,a,a^2\}$

Clearly the elements of the group G satisfy commutative law.
So G is an abelian group.

$a^2=e$ ✗
$a^2=a$ ✗
$a^2=b$

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Subgroups

Definition: Let (G, *) be a group and H be a subset of G.
If H is itself a group under the same binary operation * ,
then H is called a Subgroup of G

Examples: (1) $H = \{1, -1\}$ is a subgroup of $\{1, -1, i, -i\}$ under the binary operation of '.'

(2) $(\mathbb{Z}, +)$ is a subgroup of $(\mathbb{Q}, +)$

(3) $(\mathbb{Q}, +)$ is a subgroup of $(\mathbb{R}, +)$

(4) $(\mathbb{R}, +)$ is a subgroup of $(\mathbb{C}, +)$

Remarks: If H is a subset of a group G and H is a group under different binary operation than that of group G, then H is not called a subgroup of G.

$H = \{1, -1\}$ $\quad (H, \cdot)$

$Z = \{\cdots, -1, 0, 1, \cdots\}$, $\quad (Z, +)$

$H \subset Z$

H is not a subgroup of Z.

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Trivial Subgroups:

Every group G has at least two subgroups {e} and G itself. These two subgroups {e} and G are called trivial subgroups of G.

Example : $G = \{1, -1, i, -i\}$

$\{1\}$, $\{1, -1, i, -i\}$

Non-Trivial Subgroups:

Any subgroups of a group G, other than {e} and G are called non-trivial subgroups of G.

Example : $G = \{1, -1, i, -i\}$

Trivial Subgroups : $\{1\}$, $\{1, -1, i, -i\}$

Non-Trivial Subgroups : $\{1, -1\}$

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Theorem :

Let $(G, *)$ be a group. A non-empty subset H of G is a subgroup of G if and only if

(i) $a, b \in H \Rightarrow a * b \in H$

(ii) $a \in H \Rightarrow \bar{a}^1 \in H$, where $\bar{a}^1$ is the inverse of a.

Proof:

Suppose H is subgroup of G.

Since H is a group,

So closure property holds $\Rightarrow$ for any $a, b \in H$, $a * b \in H$

and inverse of each element of H exists $\Rightarrow$ for all $a \in H$, $\bar{a}^1 \in H$

Hence the proof.

Conversely, suppose that

for $a, b \in H$, $a * b \in H$ — (1)

for $a \in H$, $\bar{a}^1 \in H$ — (2)

G_1: closure property holds by (1)

G_2: Associative law:

Since $H \subset G$ and associative law holds in G, so the associative law holds in H, too.

G_3: By (2), for $a \in H$, $\bar{a}^1 \in H$, so inverse of every element of H exists

G_4: By (1), $a, \bar{a}^1 \in H$, $a * \bar{a}^1 \in H$

$e \in H$

So identity exists in H.

So H is the subgroup of G.

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Theorem:
The intersection of any collection of subgroups of a group $(G, *)$ is a subgroup of G.

Proof: Let $(G, *)$ be a group.

$\{H_\alpha, \alpha \in I\}$ be a collection of subgroups of G. $\bigcap_{\alpha \in I} H_\alpha$ is the intersection of all H_α, $\alpha \in I$.

Suppose $a, b \in \bigcap_{\alpha \in I} H_\alpha$

$\Rightarrow a, b \in H_\alpha$, for all $\alpha \in I$

$\Rightarrow a * b^{-1} \in H_\alpha$, for all $\alpha \in I$

$\because H_\alpha$ are groups.

$\Rightarrow a * b^{-1} \in \bigcap_{\alpha \in I} H_\alpha$

| let $a, b \in H$
| $a * b^{-1} \in H$
| So H is a subgroup.

Hence $\bigcap_{\alpha \in I} H_\alpha$ is a subgroup of G.

Hence the proof.

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Theorem:
The union of two subgroups H and K of a group G is a subgroup of G if and only if either $H \subset K$ or $K \subset H$.

Proof: HUK is a subgroup of G if and only if either $H \subset K$ or $K \subset H$

Suppose either $H \subset K$ or $K \subset H$ —— (1)

When $H \subset K \Rightarrow HUK = K$
Since K is a subgroup of G
so HUK is also a subgroup of G

When $K \subset H \Rightarrow HUK = H$
Since H is a subgroup of G,
So HUK is also a subgroup of G
Hence the proof.

Now suppose that HUK is a subgroup of G, then we have to show that either $H \subset K$ or $K \subset H$.

Suppose Contrarily, $H \not\subset K$ or $K \not\subset H$

If $a \in H$, then $a \notin K$ —— (A)
If $b \in K$, then $b \notin H$ —— (B)

then $a, b \in HUK$
$\Rightarrow a * b \in HUK$ $\because HUK$ is a subgroup of G
$\Rightarrow a * b \in H$ or $a * b \in K$

When $a * b \in H$, then $a^{-1} * (a * b) \in H$
$(a^{-1} * a) * b \in H$
$e * b \in H$
$b \in H$
which contradicts to (B)

When $a * b \in K$, then $(a * b) * b^{-1} \in K$
$a * (b * b^{-1}) \in K$
$a * e \in K$
$a \in K$
which contradicts (A)

So our supposition that $H \not\subset K$ or $K \not\subset H$ is wrong.

Therefore either $H \subset K$ or $K \subset H$
Hence the proof.

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Theorem:
The union of two subgroups H and K of a group G is a subgroup of G if and only if either $H \subset K$ or $K \subset H$.

Proof: HUK is a subgroup of G if and only if either $H \subset K$ or $K \subset H$

Suppose either $H \subset K$ or $K \subset H$ —— (1)

When $H \subset K \Rightarrow HUK = K$
Since K is a subgroup of G
so HUK is also a subgroup of G

When $K \subset H \Rightarrow HUK = H$
Since H is a subgroup of G,
So HUK is also a subgroup of G
Hence the proof.

Now suppose that HUK is a subgroup of G, then we have to show that either $H \subset K$ or $K \subset H$.

Suppose Contrarily, $H \not\subset K$ or $K \not\subset H$

If $a \in H$, then $a \notin K$ —— (A)
If $b \in K$, then $b \notin H$ —— (B)

then $a, b \in HUK$
$\Rightarrow a * b \in HUK$ $\because HUK$ is a subgroup of G
$\Rightarrow a * b \in H$ or $a * b \in K$

When $a * b \in H$, then $a^{-1} * (a * b) \in H$
$(a^{-1} * a) * b \in H$
$e * b \in H$
$b \in H$
which contradicts to (B)

When $a * b \in K$, then $(a * b) * b^{-1} \in K$
$a * (b * b^{-1}) \in K$
$a * e \in K$
$a \in K$
which contradicts (A)

So our supposition that $H \not\subset K$ or $K \not\subset H$ is wrong.

Therefore either $H \subset K$ or $K \subset H$
Hence the proof.

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Definition: Let G be a group under multiplication, then G is called Cyclic group, if each element of G can be written as a power of an element $a \in G$

let $a \in G$,

for all $x \in G$,

$$x = a^n \qquad \text{for } n \in \mathbb{N}.$$

$$G = \langle a : a^n = e \rangle$$

Example: $G = \{1, -1\}$

$$(-1)^1 = -1$$
$$(-1)^2 = 1$$

So G is a cyclic group generated by -1.

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Example : Let G = { 1, -1, i, -i } . Show that G is a cyclic group.

Sol : $G = \{1, -1, i, -i\}$

$(1)^1 = 1$

$(1)^2 = 1$

$(-1)^1 = -1$

$(-1)^2 = 1$

$(-1)^3 = -1$

$(-1)^4 = 1$

$(i)^1 = i$ ✓

$(i)^2 = -1$ ✓

$(i)^3 = i^2 \cdot i$

$= -i$ ✓

$(i)^4 = i^2 \cdot i^2$

$= (-1)(-1)$

$= 1$ ✓

$i = \sqrt{-1}$

$i^2 = (\sqrt{-1})^2$

$i^2 = -1$

$G = \langle i : i^4 = 1 \rangle$

$(-i)^1 = -i$ ✓

$(-i)^2 = i^2 = -1$ ✓

$(-i)^3 = -i^3 = -i^2 \cdot i$

$= -(-1)\, i$

$= i$ ✓

$(-i)^4 = i^4 = i^2 \cdot i^2 = (-1)(-1)$

$= 1$ ✓

$G = \langle -i : (-i)^4 = 1 \rangle$

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Theorem : Every cyclic group is an abelian group.

Proof: Let G be a cyclic group with the generator a.

Let $x, y \in G$ such that

$$x = a^m \quad , \quad y = a^n \qquad m, n \in \mathbb{Z}$$

Consider $$xy = a^m a^n = a^{m+n} = a^{n+m} = a^n a^m$$

$$xy = yx \quad \text{for } x, y \in G$$

Hence G is an abelian group.

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Remark 1: A cyclic group may have two generators.

$$G = \{1, -1, i, -i\} \qquad |G| = 4$$

$$(i)^1 = i \qquad (-i)^1 = -i \qquad i^4 = (-i)^4 = 1$$

$$(i)^2 = -1 \qquad (-i)^2 = -1 \qquad |i| = |-i| = 4$$

$$i^3 = -i \qquad (-i)^3 = i$$

$$i^4 = 1 \qquad (-i)^4 = 1$$

$$G = \langle i \rangle \quad , \quad G = \langle -i \rangle$$

Remark 2: $a \in G$

$$\boxed{a^n = e} \checkmark$$

$$aaa\cdots a = a^n \qquad n\text{-times}$$

$$aa = a^2$$

$$aaa = a^3$$

$$G = \{1, -1, i, -i\}$$

$(G', +)$ $\qquad a \in G'$ $\qquad e' \in G'$

$$\boxed{na} = a + a + \cdots + a = e'$$

$$a + a = 2a \quad , \quad a + a + a = 3a$$

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Theorem : Every subgroup of a cyclic group is cyclic.

Proof: Let G be a cyclic group generated by $a \in G$.

Let H be a subgroup of G.

$a^k \in H$ $\quad$ k is least +ve integer

Let $b \in H$ be an arbitrary element of H

So $b = a^m \in H$ for some integer m.

$$k \leq m \quad \text{—(1)}$$

By division algorithm, $\exists\ q, r$ such that

$$m = qk + r \quad , \ 0 \leq r < k$$

Consider $a^m = a^{qk+r}$ —(2)

$$a^m = a^{qk} \cdot a^r$$

$$a^m \cdot a^{-qk} = a^{qk}\, a^r\, a^{-qk}$$

$$= a^{qk-qk}\, a^r$$

$$= a^0\, a^r$$

$$a^m \cdot (a^k)^{-q} = a^r \quad , \ 0 \leq r < k$$

Since $a^k \in H$, $(a^k)^{-q} \in H$ $\quad \therefore H$ is subgroup.

Also $a^m \in H$, $\quad a^m \cdot (a^k)^{-q} \in H$

$$a^r \in H$$

It is possible when $r = 0$

So $a^m = a^{qk+0}$

$$b = a^m = (a^k)^q$$

Hence a^k is the generator of H

Therefore H is a cyclic group.

Hence the proof.

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Theorem: The order of a cyclic group is equal to the order of its generator.

Proof: Let G be a cyclic group generated by a.

and $|G| = n$

So its elements are $a, a^2, a^3, \dots, a^n$.

We have to show that $|a| = n$

i.e. $a^n = e$

Suppose that the order of a is m, $m < n$

$$a^m = e$$
$$a^m a = ea$$
$$a^{m+1} = a$$
$$a^{m+1} a = aa$$
$$a^{m+2} = a^2$$
$$\vdots$$

$$a, a^2, a^3, \dots, a^m, a^{m+1}, a^{m+2}, \dots, a^n$$
$$a, a^2, a^3, \dots, e, a, a^2, \dots, a^n$$

Which is a contradiction that there are n elements in G.

$a^m = e \qquad m < n$ is wrong.

$$a = e$$
$$a, a^2, a^3, \dots, a^n$$
$$a^r = a^s \qquad 0 < s < r < n$$
$$a^r a^{-s} = a^s a^{-s}$$
$$a^{r-s} = a^{s-s} = a^0 = e$$
$$a^{r-s} = e \qquad r - s < n$$

which is the contradiction

So $a^r \neq a^s$

$$a^n = e$$

Hence $|a| = n$

$\Rightarrow$ $|G| = |a| = n$

Hence proved.

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Theorem: Show that the group (Q, +) is abelian but not cyclic.

Proof:

$$Q = \left\{ \frac{p}{q} \mid p, q \in Z, \; q \neq 0 \right\}$$

Since Commutative law holds w.r.t. + , (rational)

$$a + b = b + a \qquad \forall \; a, b \in Q$$

So $(Q, +)$ is an abelian group.

Suppose contrarily that Q is cyclic and can be generated by $a \in Q$.

$$a = \frac{p}{q} \qquad p, q \in Z$$

for $x \in Q$, $\quad x = na$

$$\frac{1}{2q} \in Q, \quad \text{So} \quad \frac{1}{2q} = na$$

$$\frac{1}{2q} = n\frac{p}{q}$$

2X3=6

$$\frac{1}{2} = np \qquad \text{—(1)}$$

But the product of two integers is integer.
So (1) is wrong.

Hence every rational number cannot be written in form generator. So $(Q, +)$ is not a cyclic.

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Theorem: Let G be a cyclic group of order n generated by a, then for each positive divisor d of n, there is a unique subgroup (of G) of order d.

Proof: $G = \langle a : a^n = e \rangle$

Suppose d divides n, $n = qd$

let $b \in G$, $b = a^q$

$$(b)^d = (a^q)^d$$

$$b^d = a^{qd} = a^n = e$$

$$H = \langle b : b^d = e \rangle$$

which is the required subgroup of G

Suppose Contrarily that subgroup of order d is not unique, there exists another subgroup K of order d. K is generated by an element $c = a^k$.

Since order of K is d, so $a^k = c$

$$(a^k)^d = c^d$$

$$a^{kd} = c^d$$

$$a^n = c^q$$

$$e = c^d$$

$kd = n$, $qd = n$

$kd = qd$

$k = q$

$$b = a^q = a^k$$

$K = H$

Hence H is a unique subgroup of order d.

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Theorem: If G is a cyclic group of even order, then prove that there is only one subgroup of order 2 in G.

Proof: let G be a cyclic group with even order

$$|G| = 2n \qquad n \text{ is +ve integer.}$$

By the previous theorem, if a +ve integer d divides order of a group G, then there will be a subgroup of order d.

Since 2 divides 2n, So there must exist a subgroup of order 2.

Hence the proof.

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Left Coset : Let H be a subgroup of a group G and $a \in G$, then the set

$$aH = \{ ah : h \in H \}$$

is called a left coset of H in G determined by a.

Example : $G = \{1, -1, i, -i\}$

$H = \{1, -1\}$

$i \in G$, $iH = \{i \cdot 1, i(-1)\} = \{i, -i\}$

Right Coset : Let H be a subgroup of a group G and $a \in G$, then the set

$$Ha = \{ ha : h \in H \}$$

is called a Right coset of H in G determined by a.

Example : $G = \{1, -1, i, -i\}$

$H = \{1, -1\}$

$i \in G$, $Hi = \{1 \cdot i, -1 \cdot i\} = \{i, -i\}$

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Remark 1 : The left (or right) cosets of H in G determined by the identity element e is H itself.

$$eH = \{ eh : h \in H \} = \{ h : h \in H \} = H$$

$$He = \{ he : h \in H \} = \{ h : h \in H \} = H$$

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Remark 2 : If the binary operation of the group G is '+', then the left (or right) cosets of H in G determined by a is given by

$$a + H = \{ a + h : h \in H \}$$

$$H + a = \{ h + a : h \in H \}$$

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Example : Find all the left costs of H = { 1, -1 } in G = { 1, -1, i, -i }.

Sol: $G = \{1, -1, i, -i\}$

$H = \{1, -1\}$ $\quad a \in G, \quad aH$

$1 \in G, \quad 1H = \{1 \cdot 1, 1 \cdot (-1)\} = \{1, -1\}$

$-1 \in G, \quad -1H = \{-1 \cdot 1, -1 \cdot (-1)\} = \{-1, 1\}$

$i \in G, \quad iH = \{i \cdot 1, i \cdot (-1)\} = \{i, -i\}$

$-i \in G, \quad -iH = \{-i \cdot 1, -i \cdot (-1)\} = \{-i, i\}$

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Example : Find all the right costs of $H = \{1, -1\}$ in $G = \{1, -1, i, -i\}$.

Sol:

$$G = \{1, -1, i, -i\}, \quad H = \{1, -1\}$$

$$H1 = \{1 \cdot 1, -1 \cdot 1\} = \{1, -1\}$$

$$H(-1) = \{1 \cdot (-1), -1 \cdot (-1)\} = \{-1, 1\}$$

$$Hi = \{1 \cdot i, -1 \cdot i\} = \{i, -i\}$$

$$H(-i) = \{1 \cdot (-i), -1(-i)\} = \{-i, i\}$$

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Example : Let $G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$ be the group of residue classes modulo 6 under addition, and $H = \{\bar{0}, \bar{2}, \bar{4}\}$ is the subgroup of G, then find all left and right cosets of H in G.

$$6\overline{)6}\ 1,\quad 6$$
$$0$$

Sol: $G = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$, $H = \{\bar{0}, \bar{2}, \bar{4}\}$

$$6\overline{)7}\ 1,\quad 6$$
$$1$$

$\bar{0} \in G$, $\bar{0} + H = \{\bar{0}+\bar{0}, \bar{0}+\bar{2}, \bar{0}+\bar{4}\} = \{\bar{0}, \bar{2}, \bar{4}\}$ ✓

$\bar{1} \in G$, $\bar{1} + H = \{\bar{1}+\bar{0}, \bar{1}+\bar{2}, \bar{1}+\bar{4}\} = \{\bar{1}, \bar{3}, \bar{5}\}$ =

$\bar{3}+\bar{4} = \bar{1}$

$\bar{3}+\bar{2} = \bar{5}$

$\bar{2} \in G$, $\bar{2} + H = \{\bar{2}+\bar{0}, \bar{2}+\bar{2}, \bar{2}+\bar{4}\} = \{\bar{2}, \bar{4}, \bar{0}\}$ ✓

$$6\overline{)8}\ 1,\quad 6$$
$$2$$

$\bar{3} \in G$, $\bar{3} + H = \{\bar{3}+\bar{0}, \bar{3}+\bar{2}, \bar{3}+\bar{4}\} = \{\bar{3}, \bar{5}, \bar{1}\}$ =

$\bar{4} \in G$, $\bar{4} + H = \{\bar{4}+\bar{0}, \bar{4}+\bar{2}, \bar{4}+\bar{4}\} = \{\bar{4}, \bar{0}, \bar{2}\}$ ✓

$$6\overline{)9}\ 1,\quad 6$$
$$3$$

$\bar{5} \in G$, $\bar{5} + H = \{\bar{5}+\bar{0}, \bar{5}+\bar{2}, \bar{5}+\bar{4}\} = \{\bar{5}, \bar{1}, \bar{3}\}$ =

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Partition of a set A : The collection $\{A_\alpha : \alpha \in I\}$ of subsets of a set A is called the partition of A if

(i) $A = \bigcup_{\alpha \in I} A_\alpha$

(ii) $A_\alpha \cap A_\beta = \phi$ for $\alpha \neq \beta$

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Index of H in G:

The number of distinct left (or right) cosets of a subgroup H of a group G is called the Index of H in G, and is denoted by [G : H].

Example: $G = \{1, -1, i, -i\}$

$H = \{1, -1\}$

$1 \in G, \quad 1H = \{1 \cdot 1, 1 \cdot (-1)\} = \{1, -1\}$ ✓

$-1 \in G, \quad -1H = \{-1(1), -1(-1)\} = \{-1, 1\}$

$i \in G, \quad iH = \{i \cdot 1, i(-1)\} = \{i, -i\}$ ✓

$-i \in G, \quad -iH = \{-i(1), -i(-1)\} = \{-i, i\}$

Number of distinct left Cosets of H in G = 2

$[G : H] = 2$

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Example : **Find the distinct left or (right) cosets of**

$$E = \{ 0, \pm 2, \pm 4, \dots \} = \{ 2n : n \in Z \}$$

in the group (Z, +).

Sol: $Z = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \}$

$E = \{ \dots, -4, -2, 0, 2, 4, \dots \}$

$0 \in Z$, $0+E = \{ \dots, 0+(-4), 0+(-2), 0+0, 0+2, 0+4, \dots \}$

$= \{ \dots, -4, -2, 0, 2, 4, \dots \}$ ✓

$1 \in Z$, $1+E = \{ \dots, 1+(-4), 1+(-2), 1+0, 1+2, 1+4, \dots \}$

$= \{ \dots, -3, -1, 1, 3, 5, \dots \}$ ✓

Number of distinct left cosets = 2

$[Z : E] = 2$

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Lagrange's Theorem: **Both the order and index of a subgroup of a finite group divide the order of the group.**

Proof: Let G be a group and H be its subgroup.

such that $|G| = n$, $|H| = m$

Let $a_i \in G$, $a_1H, a_2H, \cdots, a_kH$ be distinct left cosets of H in G.

Define $\varphi : H \longrightarrow a_iH$, $a_i \in G$

by $\varphi(h) = a_i h$ $h \in H$

Since each element $a_ih \in a_iH$ for each $h \in H$, so φ is onto.

Consider $\varphi(h_1) = \varphi(h_2)$ $h_1, h_2 \in H$

$a_ih_1 = a_ih_2$

$h_1 = h_2$ $\because a_i \in G$

Hence a one-one function.

So φ is bijective function.

Therefore the no. of elements in H is equal to the no. of elements in a_iH.

$|H| = m = |a_1H| = |a_2H| = \cdots = |a_kH|$

Since the cosets define the partition of G

So $a_1H \cap a_2H = \varphi$

$a_iH \cap a_jH = \varphi$ for $i \neq j$

and $G = a_1H \cup a_2H \cup \cdots \cup a_kH$

$|G| = |a_1H| + |a_2H| + \cdots + |a_kH|$

$n = m + m + \cdots + m$

$n = mk$

$6 = 2 \times 3$

$\Rightarrow$ m divides n & k divides n

$\Rightarrow$ $|H|$ divides $|G|$ & $[G:H]$ divides $|G|$.

Hence the proof.

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Theorem : **The order of an element of a finite group divides the order of the group.**

Proof : let G be a group & $x \in G$.

$$|G| = n$$

$$|x| = m$$

We show that m divides n.

$x \in G$, $H = \langle x \rangle \subset G$.

But $|x| = m$, So $|H| = m$

By Lagrange's theorem, the order of a subgroup H divides order of the group G.

$|H|$ divides $|G|$.

m divides n

Hence the proof.

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Theorem : A group G whose order is a prime number is necessarily cyclic.

Proof: Let G be a group.

$|G| = p$, p is prime number.

Let $x \in G$ be a non-identity element.

$H = \langle x \rangle \subset G$

By lagrange's theorem, $|H|$ divides $|G|$

But $|G| = p$, so $|H|$ can divide $|G|$ only when $|H| = p$ or $|H| = 1$

Therefore $|H| = p$, but $|G| = p$

It means $H = G$.

So G must be a cyclic group.

Hence the proof.

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Example : Let G be a group of order 89. Can G have a subgroup

(i) of order 12
(ii) of order 16
(iii) of order 24

Justify your anwer.

Sol: $|G| = 89$

(i) Let H be a subgroup of order 12.

$|H| = 12$

Since 89 is a prime number, and 12 does not divide it, so by Lagrange's theorem, H is not a subgroup of G.

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PERMUTATION: Let X be non-empty set. A bijecive mapping $\alpha : X \longrightarrow X$ is called the permutation on X.

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IDENTITY PERMUTATION: Let X be non-empty set.

A bijecive mapping $\alpha : X \longrightarrow X$ is called the identity permutation on X, if

$$\alpha(x) = x \qquad \text{for all } x \in X$$

$$X = \{1, 2, 3\}$$

$$\alpha(1) = 1$$
$$\alpha(2) = 2$$
$$\alpha(3) = 3$$

$$\begin{pmatrix} 1 & 2 & 3 \\ \alpha(1) & \alpha(2) & \alpha(3) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$X = \{x_1, x_2, \ldots, x_n\}$$

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ \alpha(x_1) & \alpha(x_2) & \ldots & \alpha(x_n) \end{pmatrix}$$

$$= \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ x_1 & x_2 & \ldots & x_n \end{pmatrix}$$

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REMARK : If X has n elements, then total number of permutations on X is n!

1) $X = \{1\}$, $n = 1$, Total permutations $= 1! = 1$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

2) $X = \{1, 2\}$, $n = 2$, Total permutations $= 2! = 2 \times 1 = 2$

$$\alpha_1 = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

3) $X = \{1, 2, 3\}$, $n = 3$, Total permutations $= 3! = 3 \times 2 \times 1 = 6$

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \alpha_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\alpha_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad \alpha_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\alpha_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad \alpha_6 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

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INVERSE PERMUTATION : Let X be a non-empty set and $\alpha : X \longrightarrow X$ be a permutation on X such that

$$\alpha(x) = y \qquad \forall\, x, y \in X$$

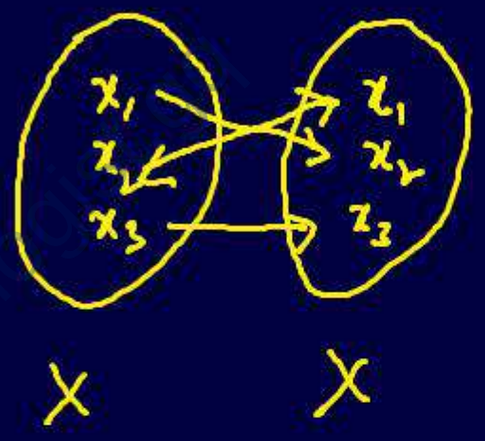
Then the inverse permutation $\alpha^{-1} : X \longrightarrow X$ of α is defined as

$$\alpha^{-1}(y) = x \qquad \forall\, x,\ y \in X$$

$\alpha(x_1) = x_3$

$\alpha(x_2) = x_1$

$\alpha(x_3) = x_3$



$\alpha(x_2) = x_1$

$\alpha^{-1}(\alpha(x_2)) = \alpha^{-1}(x_1)$

$x_2 = \alpha^{-1}(x_1)$

$\alpha(x_1) = x_2$

$x_1 = \alpha^{-1}(x_2)$

$\alpha^{-1}(x_3) = x_3$

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

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EXAMPLE : Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 5 & 4 & 1 \end{pmatrix}$

Find α^{-1}.

Sol: $\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 5 & 4 & 3 \end{pmatrix}$

Ans

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EXAMPLE : Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 6 & 2 & 4 \end{pmatrix}$

Find β^{-1}.

Sol: $\beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 2 & 6 & 3 & 4 \end{pmatrix}$

Ans

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Q: Write the inverse permutation of $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$.

Sol:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Ans

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VuBioTechnologist

Q: Write the inverse permutation of $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$.

Sol:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$1 \rightarrow 1$
$2 \rightarrow 2$
$3 \rightarrow 3$

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Ans

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Q: Write the inverse permutation of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$.

Sol:

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

$$\beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$$

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Q: Write the inverse permutation of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$.

Sol:

$$\gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$$

$$\gamma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$$

Ans

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Q: Write the inverse permutation of $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 5 & 4 & 3 \end{pmatrix}$.

উঃ:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 5 & 4 & 3 \end{pmatrix}$$

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 5 & 4 & 1 \end{pmatrix}$$

Ans

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Q: Write the inverse permutation of $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 2 & 6 & 3 & 4 \end{pmatrix}$.

Sol:

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 2 & 6 & 3 & 4 \end{pmatrix}$$

$$\beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 6 & 2 & 4 \end{pmatrix}$$

Ans

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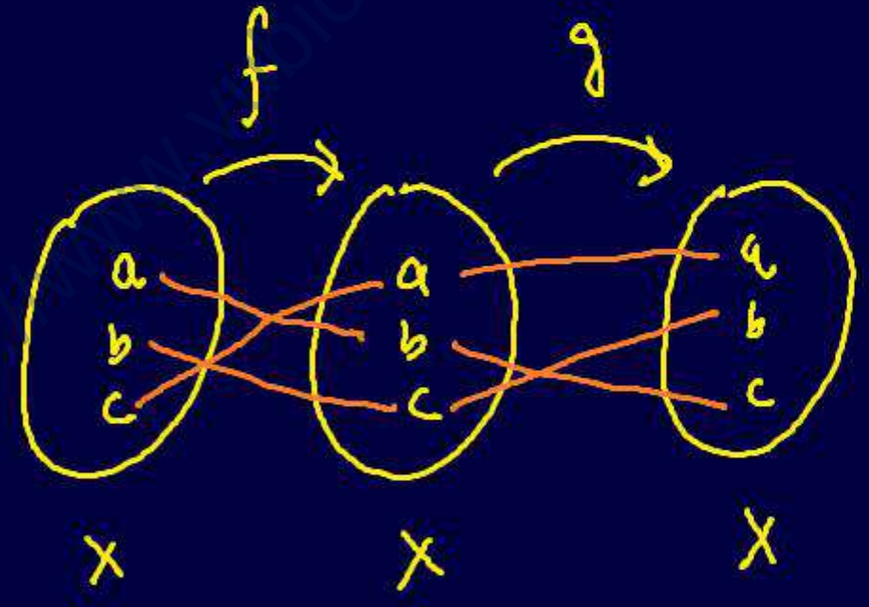
PRODUCT or COMPOSITION OF PERMUTATIONS:

Let $f : X \longrightarrow X$ and $g : X \longrightarrow X$ be two permutations on a non-empty set X, then the permutation $fg : X \longrightarrow X$ on X defined as

$$(x)\,fg = ((x)\,f)g \qquad \forall\, x \in X$$

is called the Product or Composition of permutations f and g.

It is also denoted by $f \circ g$.



$$f \circ g(x) = \begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}$$

$$= \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} \qquad \text{Ans}$$

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PRODUCT or COMPOSITION OF PERMUTATIONS:

Let $f : X \longrightarrow X$ and $g : X \longrightarrow X$ be two permutations on a non-empty set X, then the permutation $fg : X \longrightarrow X$ on X defined as

$$(x)\, f\, g = ((x)\, f)g \qquad \forall\, x \in X$$

is called the Product or Composition of permutations f and g.

It is also denoted by $f \circ g$.

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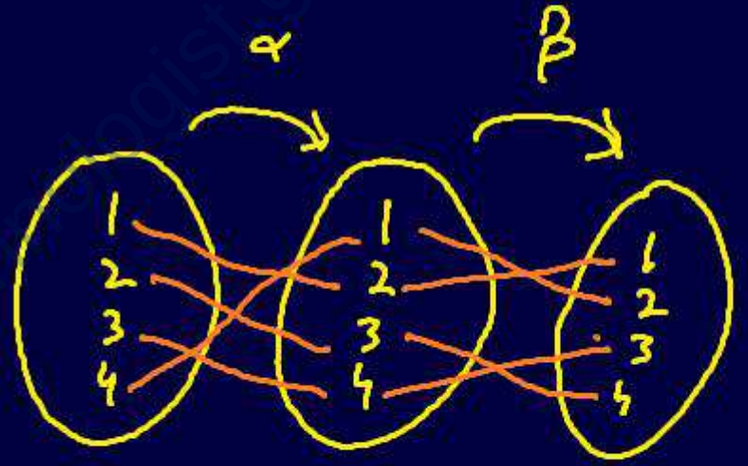
EXAMPLE : Find the product of the following permutations

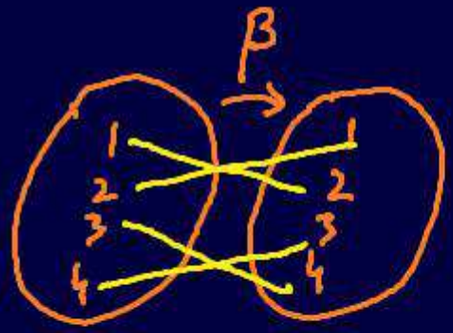
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$



Sol:

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$



$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$$

Ans

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EXAMPLE : Find the product or composition $\alpha \circ \alpha$ of the permutation

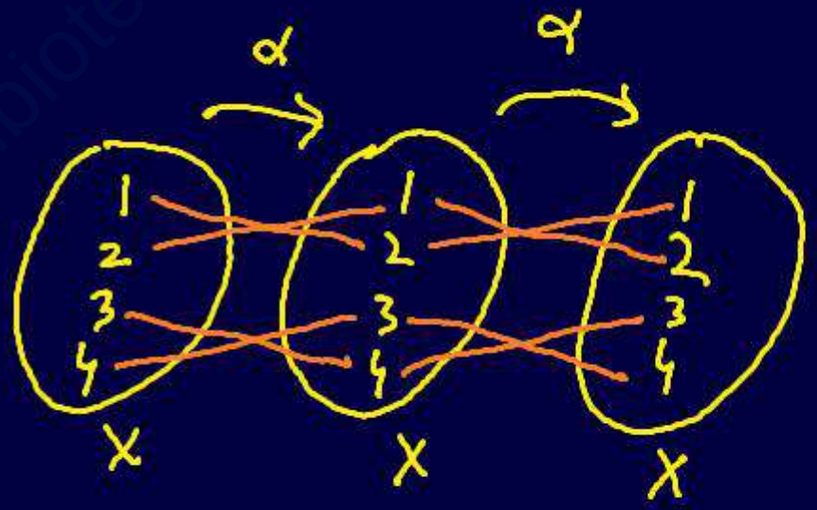
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Sol:

$$\alpha \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Ans



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EXAMPLE : Find the product or composition $\alpha \circ \alpha$ of the permutation

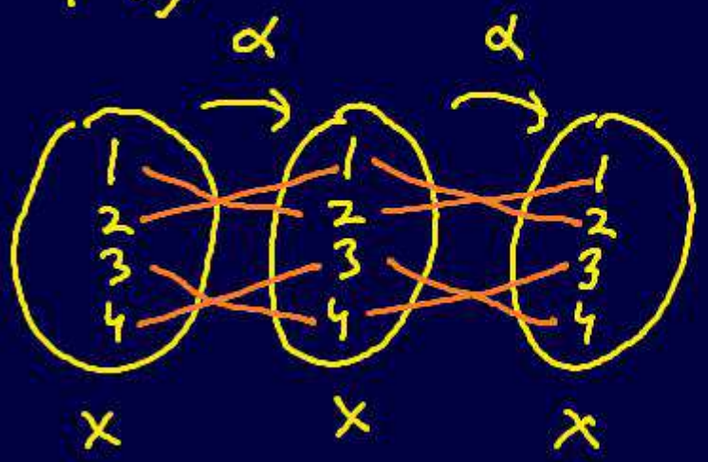
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Sol:

$$\alpha \circ \alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

Ans



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EXAMPLE : Let

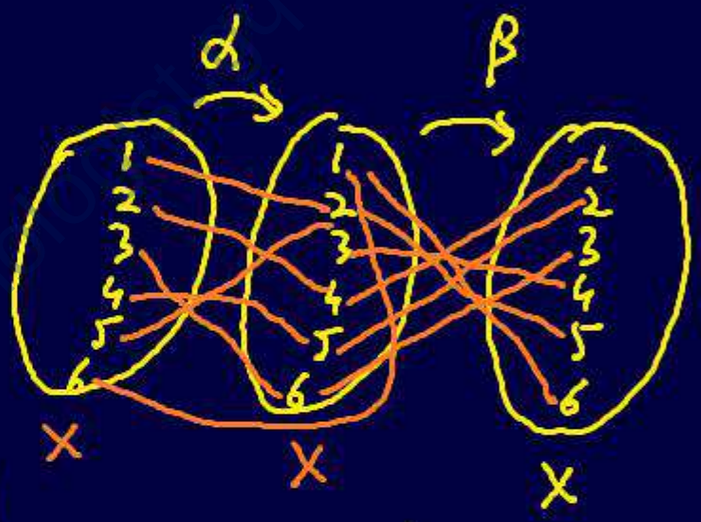
$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$$

Find $\alpha\circ\beta$, $\alpha\circ\alpha$, $\beta\circ\beta$, $\beta\circ\alpha$.

Sol: $\alpha\circ\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 2 & 4 & 6 \end{pmatrix}$



$\alpha\circ\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 1 & 3 & 6 & 2 \end{pmatrix}$ Ans

$\beta\circ\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 6 & 5 & 4 \end{pmatrix}$ Ans.

$\beta\circ\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 4 & 6 \end{pmatrix}$ Ans

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EXAMPLE : Let

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix}$$

Show that $\alpha\beta \neq \beta\alpha$

Sol:

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 4 & 6 \end{pmatrix}$$

$$\beta\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 5 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 1 & 3 & 2 & 4 & 6 \end{pmatrix}$$

$$\alpha\beta \neq \beta\alpha$$

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Example : Multiply the following permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

Sol:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 4 & 2 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 5 & 2 & 3 \end{pmatrix}$$

Ans

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Example : Multiply the following permutations

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 4 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 1 & 4 \end{pmatrix}$$

Sol:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 3 & 2 & 5 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 1 & 4 & 5 & 6 \end{pmatrix}$$ Ans.

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Example : Multiply the following permutations

$$\begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 9 & 2 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 3 & 9 & 2 \end{pmatrix}$$

Sol:

$$\begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 9 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 8 & 6 & 7 & 3 & 9 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 6 & 7 & 8 & 9 \\ 9 & 7 & 3 & 2 & 8 & 6 \end{pmatrix}$$ Ans

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Example : Let

$$f=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix},\quad g=\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$$

find $f\circ g$, $f^2\circ g$ and $f^3\circ g$.

Sol: $f^2=f\circ f=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$

$$=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$$

$f^3=f^2\circ f=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$

$$=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}$$

Now $f\circ g=\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$

$$=\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}\quad \text{Ans}$$

$f^2\circ g=\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$

$$=\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}\quad \text{Ans}$$

$f^3\circ g=\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$

$$=\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}\quad \underline{\text{Ans}}$$

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Example : Let

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \quad g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Show that $f^4 = g^2$.

Sol:

$$f^2 = f \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$$

$$f^3 = f^2 \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$f^4 = f^3 \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} \quad \text{— (1)}$$

$$g^2 = g \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$g^2 = f^4$ by (1) Hence the proof.

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Example : Let

$$f=\begin{pmatrix}1&2&3&4&5&6\\2&4&6&5&3&1\end{pmatrix}$$

$$g=\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}$$

$$h=\begin{pmatrix}1&2&3&4&5&6\\5&4&1&2&6&3\end{pmatrix}$$

find fog, gof, goh, hog, h^2, g^2.

Solution:

$$fog=\begin{pmatrix}1&2&3&4&5&6\\2&4&6&5&3&1\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5&6\\5&1&3&2&4&6\end{pmatrix}$$ Ans

$$gof=\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\2&4&6&5&3&1\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5&6\\1&3&5&2&4&6\end{pmatrix}$$ Ans

$$goh=\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\5&4&1&2&6&3\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5&6\\3&6&2&5&4&1\end{pmatrix}$$

$$hog=\begin{pmatrix}1&2&3&4&5&6\\5&4&1&2&6&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}$$

$$=\begin{pmatrix}1&2&3&4&5&6\\2&1&6&5&3&4\end{pmatrix}$$

$$h^2=\begin{pmatrix}1&2&3&4&5&6\\5&4&1&2&6&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\5&4&1&2&6&3\end{pmatrix}$$

$$h\circ h=\begin{pmatrix}1&2&3&4&5&6\\6&2&5&4&3&1\end{pmatrix}$$

$$g^2=\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5&6\\6&5&4&1&2&3\end{pmatrix}$$

$$g\circ g=\begin{pmatrix}1&2&3&4&5&6\\3&2&1&6&5&4\end{pmatrix}$$

Ans

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Cyclic Permutation : Let $X = \{x_1, x_2, x_3, \dots, x_n\}$

then a permutation α is called cyclic permutation or cycle of length n if

$$\alpha = \begin{pmatrix} x_1 & x_2 & x_3 & \dots & x_n \\ x_2 & x_3 & x_4 & \dots & x_1 \end{pmatrix}$$

$$= (x_1\ x_2\ x_3 \dots x_n)$$

e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{pmatrix} = (1\ 2\ 3\ 4\ 5)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 1 & 5 \end{pmatrix} = (1\ 2\ 3\ 4)(5)$$
$$= (1\ 2\ 3\ 4)$$

$$X = \{x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n\} \qquad k < n$$

$$\alpha = \begin{pmatrix} x_1 & x_2 & \dots & x_k & x_{k+1} & \dots & x_n \\ x_2 & x_3 & \dots & x_1 & x_{k+1} & \dots & x_n \end{pmatrix}$$

$$= (x_1\ x_2 \dots x_k)$$

Cycle of length k.

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Example : **Find the lengths of the following cycles:**

lengths

(i) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix} = (1\ 2\ 3\ 4\ 5\ 6)$ — 6

(ii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 5 & 3 & 6 \end{pmatrix} = (3\ 4\ 5)$ — 3

(iii) $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix} = (1\ 2\ 5)$ — 3

(iv) $\begin{pmatrix} 2 & 3 & 1 & 4 & 5 & 6 \\ 1 & 3 & 4 & 5 & 6 & 2 \end{pmatrix} = (2\ 1\ 4\ 5\ 6)$ — 5

Ans

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Example : **Find the following product and express it as a product of cyclic permutations on mutually disjoint sets.**

$$(14)(235)(35)(45)$$

Sol:

$$= \begin{pmatrix}1&4\\4&1\end{pmatrix}\begin{pmatrix}2&3&5\\3&5&2\end{pmatrix}\begin{pmatrix}3&5\\5&3\end{pmatrix}\begin{pmatrix}4&5\\5&4\end{pmatrix}$$

$$= \begin{pmatrix}1&2&3&4&5\\4&2&8&1&5\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\1&3&5&4&2\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\1&2&5&4&3\end{pmatrix}\begin{pmatrix}1&2&3&4&5\\1&2&3&5&4\end{pmatrix}$$

$$= \begin{pmatrix}1&2&3&4&5\\5&4&3&1&2\end{pmatrix} = \begin{pmatrix}1&5&2&4\\5&2&4&1\end{pmatrix} = (1524)$$ Ans

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Example : **Find the following product and express it as a product of cyclic permutations on mutually disjoint sets.**

(124)(13654)

Sol:

$$(1\,2\,4)(1\,3\,6\,5\,4)$$

$$= \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 6 & 5 & 4 \\ 3 & 6 & 5 & 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 3 & 1 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 6 & 1 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 6 & 3 & 4 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 & 5 & 4 \\ 6 & 5 & 4 & 3 \end{pmatrix}$$

$$= (1\,2)(3\,6\,5\,4) \quad \underline{\text{Ans}}$$

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Example : Express the following permutation as a product of disjoint cycles:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$$

Sol:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 8 & 2 & 6 & 3 & 7 & 4 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 8 \\ 8 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 & 4 \\ 6 & 4 & 3 \end{pmatrix} \begin{pmatrix} 5 & 7 \\ 7 & 5 \end{pmatrix}$$

$$= (1\,8)(3\,6\,4)(5\,7) \quad \text{Ans}$$

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Example : **Express the following permutation as a product of disjoint cycles:**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$$

Sol:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 6 & 4 & 1 & 8 & 2 & 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 \\ 6 & 2 \end{pmatrix} \begin{pmatrix} 5 & 8 & 7 \\ 8 & 7 & 5 \end{pmatrix}$$

$$= (1\,3\,4)(2\,6)(5\,8\,7) \quad \text{Ans}$$

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Theorem : The set S_n of all permutations defined on a set X with n elements is a group under the operation of composition of permutations.

Proof: 1) Closure property:

Let $f, g \in S_n$, then we have to show $fog \in S_n$.

Since $f, g \in S_n$

So $f: X \to X$, $g: X \to X$ are bijective functions

Therefore, the composition of two bijective functions is also bijective.

So $fog: X \to X$ is bijective

$\Rightarrow fog \in S_n$

So the closure property holds.

2) Associative law.

We have to show that

$$(fog)oh = fo(goh)$$

$LHS = (x)[(fog)oh]$

$= ((x)(fog))h = (((x)f)g)h$ — ①

$RHS = (x)[fo(goh)] = ((x)f)(goh)$

$= (((x)f)g)h$ — ②

By ① and ②

$$(fog)oh = fo(goh)$$

3) Identity Permutation: $I(x) \in S_n$

$$foI = Iof = f$$

$(x)(foI) = ((x)f)I = (x)f$ — ⓐ $\quad I(x) = x$

$(x)(Iof) = ((x)I)f = (x)f$ — ⓑ

$$foI = Iof = f$$

Inverse Permutation:

Let $f \in S_n$, then $f: X \to X$ is a bijective mapping. Then $f^{-1}: X \to X$ is also a bijective mapping. $f^{-1} \in S_n$

$$fof^{-1} = f^{-1}of = I$$

$(x)(fof^{-1}) = ((x)f)f^{-1}$ $\quad \because y = f(x)$

$= (y)f^{-1}$ $\quad f^{-1}(y) = x$

$= x = I(x)$ — I $\quad \because I(x) = x$

$(y)(f^{-1}of) = ((y)f^{-1})f = (x)f = y$

$= I(y)$ — II

By I & II $\quad f^{-1}of = fof^{-1} = I$

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Symmetric Group : The group S_n of all permutations defined on a set X with n elements is called symmetric group of degree n.

$n = 3$, $S_3 = \{I, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta\}$, $X = \{1, 2, 3\}$

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \alpha\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \alpha^2\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

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Theorem : **Show that (S_3, $\circ$) is a non-abelian group.**

Proof: We know that S_3 is a group of permutations on a set X with three elements.

$$X = \{1, 2, 3\}$$

All possible permutations on X are

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad d_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$d_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \qquad d_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$d_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad d_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

We have to show that $d_1 d_2 \neq d_2 d_1$

$$d_1 d_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \quad \text{---(1)}$$

$$d_2 d_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \quad \text{---(2)}$$

Hence by (1) and (2), $d_1 d_2 \neq d_2 d_1$

Hence $(S_3, \cdot)$ is a non-abelian group.

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Theorem : Find all the subgroups of S_3.

Proof: $X = \{1, 2, 3\}$

All possible permutations on X are

$$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \qquad \alpha\beta = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

$$\beta = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \qquad \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\alpha^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \qquad \alpha^2\beta = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

$$S_3 = \{I, \alpha, \alpha^2, \beta, \alpha\beta, \alpha^2\beta\}$$

where $\alpha^3 = \beta^2 = (\alpha\beta)^2 = (\alpha^2\beta)^2 = I$

Trivial subgroups of S_3 : I, S_3

Non-trivial subgroups of S_3 :

$H_1 = \{I, \alpha, \alpha^2\}$, $H_2 = \{\beta, I\}$

$H_3 = \{\alpha^2\beta, I\}$, $H_4 = \{\alpha\beta, I\}$

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Transposition : A cycle of length 2 is called transposition.

e.g $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = (1\,2)$

$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 5 & 4 \end{pmatrix} = (4\,5)$

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Theorem : **Every cyclic permutation can be expressed as the product of transpositions.**

Proof: Suppose we have a cyclic permutation

$(a_1\ a_2\ a_3 \dots a_n)$ of length n.

We have to show that

$$(a_1\ a_2\ a_3 \dots a_n) = (a_1 a_2)(a_1 a_3) \dots (a_1 a_n) \quad \text{—— (1)}$$

We prove it by using mathematical induction.

Step I:

$n = 2$

$LHS = (a_1\ a_2)$, $RHS = (a_1\ a_2)$

So it is true for $n = 2$.

$n = 3$

$LHS = (a_1\ a_2\ a_3)$

$RHS = (a_1 a_2)(a_1 a_3)$

$$= \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2 & a_1 & a_3 & \dots & a_n \end{pmatrix} \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_3 & a_2 & 1 & \dots & a_n \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_n \\ a_2 & a_3 & a_1 & \dots & a_n \end{pmatrix} = (a_1\ a_2\ a_3) = LHS.$$

So it is also for $n = 3$.

Step II: Let it be true for $n = k$

$$(a_1 a_2 \dots a_k) = (a_1 a_2)(a_1 a_3) \dots (a_1 a_k) \quad \text{—— (2)}$$

Step III: Now we prove it for $n = k+1$

Consider

$$\underbrace{(a_1 a_2)(a_1 a_3) \dots (a_1 a_k)}(a_1 a_{k+1})$$

$$= (a_1 a_2 a_3 \dots a_k)(a_1\ a_{k+1})$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{k-1} & a_k & a_{k+1} \\ a_2 & a_3 & a_4 & \dots & a_k & a_1 & a_{k+1} \end{pmatrix} \begin{pmatrix} a_1 & a_2 & \dots & a_k & a_{k+1} \\ a_{k+1} & a_2 & \dots & a_k & a_1 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & a_2 & a_3 & \dots & a_{k-1} & a_k & a_{k+1} \\ a_2 & a_3 & a_4 & \dots & a_k & a_{k+1} & a_1 \end{pmatrix}$$

$= (a_1\ a_2\ a_3 \dots a_{k+1})$ Hence it is true for $n = k+1$.

$= LHS$.

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Theorem : **Every permutation of degree n can be expressed as the product of transpositions.**

Proof: Since we know that every permutation of degree n can expressed as the product of disjoint cycles.

We every cyclic Permutati' can be written as the product of transpositions.

Therefore, every permutation of degree n can be expressed as the product of transpositions.

Hence proved.

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Example : Express the following permutation as the product of transpositions.

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

Sol:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix}$$

$$= (1\ 2)\ (3\ 4)$$

Ans

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Example : **Express the following permutation as the product of transpositions.**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix}$$

Sol:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 4 \end{pmatrix}$$

$$= (1\ 5)(2\ 3)$$

Ans

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Example : **Express the following permutation as the product of transpositions.**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 1 & 6 & 4 \end{pmatrix}$$

Sol:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 2 & 1 & 6 & 4 \end{pmatrix}$$

$$= (1\ 3\ 2\ 5\ 6\ 4)$$

$$= (1\ 3)(1\ 2)(1\ 5)(1\ 6)(1\ 4)$$

Ans

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Example : **Express the following permutation as the product of transpositions.**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 1 & 3 & 5 & 8 & 9 & 7 \end{pmatrix}$$

Sol:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 1 & 3 & 5 & 8 & 9 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 & 5 \\ 6 & 5 & 3 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 \\ 8 & 9 & 7 \end{pmatrix}$$

$$= (1\ 2\ 4)(3\ 6\ 5)(7\ 8\ 9)$$

$$= (1\ 2)(1\ 4)(3\ 6)(3\ 5)(7\ 8)(7\ 9)$$ Ans

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Example : **Express the following permutation as the product of transpositions.**

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 6 & 2 & 9 & 8 & 3 & 1 & 4 & 5 & 7 \end{pmatrix}$$

Sol:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 6 & 2 & 9 & 8 & 3 & 1 & 4 & 5 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 10 & 7 \\ 10 & 7 & 1 \end{pmatrix} \begin{pmatrix} 2 & 6 & 3 \\ 6 & 3 & 2 \end{pmatrix} \begin{pmatrix} 4 & 9 & 5 & 8 \\ 9 & 5 & 8 & \end{pmatrix}$$

$$= (1\ 10\ 7)(2\ 6\ 3)(4\ 9\ 5\ 8)$$

$$= (1\ 10)(1\ 7)(2\ 6)(2\ 3)(4\ 9)(4\ 5)(4\ 8)$$

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Even Permutation : A permutation α in S_n is called an even permutation if it can be written as a product of even number of transpositions.

Example :

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 1 & 3 & 5 & 8 & 9 & 7 \end{pmatrix}$$

$$= (1\ 2)(1\ 4)(3\ 6)(3\ 5)(7\ 8)(7\ 9)$$

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Odd Permutation : A permutation α in S_n is called an odd permutation if it can be written as a product of odd number of transpositions.

Example :

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 6 & 2 & 9 & 8 & 3 & 1 & 4 & 5 & 7 \end{pmatrix}$$

$$= (1\ 10)(1\ 7)(2\ 6)(2\ 3)(4\ 9)(4\ 5)(4\ 8)$$

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Example : Is the permutation $(1\,2\,3)(2\,5\,6)(4\,3\,5\,1)$ odd or even ?

Sol: $(1\,2\,3)(2\,5\,6)(4\,3\,5\,1)$

$= (1\,2)(1\,3)(2\,5)(2\,6)(4\,3)(4\,5)(4\,1)$

Since there are 7 transpositions, so the given permutation is an odd permutation.

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Example : **Is the permutation (1 4 7)(3 4 5)(8 7)(8 3 4 5) odd or even ?**

Sol: $(1\ 4\ 7)(3\ 4\ 5)(8\ 7)(8\ 3\ 4\ 5)$

$= (14)(17)(34)(35)(87)(83)(84)(85)$

Since these are 8 transpositions, so the given permutation is an even permutation.

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Example : **Is the permutation (1 2 4)(3 2 5)(6 2 4 5) odd or even ?**

Sol: $(1\ 2\ 4)(3\ 2\ 5)(6\ 2\ 4\ 5)$

$= (1\ 2)(1\ 4)(3\ 2)(3\ 5)(6\ 2)(6\ 4)(6\ 5)$

Since there are 7 transpositions, so the given permutation is an odd permutation.

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Example : Is the permutation $(1\,2\,3\,4\,5)(3\,6\,5)(2\,1\,5)$ odd or even ?

Sol: $(1\,2\,3\,4\,5)(3\,6\,5)(2\,1\,5)$

$= (1\,2)(1\,3)(1\,4)(1\,5)(3\,6)(3\,5)(2\,1)(2\,5)$

Since there are 8 transpositions, so the permutation is an even permutation.

—

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Theorem : The product of two even or two odd permutations is an even permutation.

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Proof: Let α and β be two even permutations of length n. Suppose α and β can be expressed as m and p transpositions respectively, then $\alpha\beta$ will contain $m+p-2k$ transpositions, where k is a +ve integer.
Since m, p and $2k$ are even integers, so $m+p-2k$ will also be an even integer.

Let f and g be two odd permutations of length n. Suppose that f & g can be expressed as t and u transpositions respectively, then the product fg will contain $t+u-2k$ transpositions. Hence $t+u-2k$ is an even number, so fg has even transpositions.

Hence the proof.

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Q.: The permutation $(1\ 2)(1\ 3)$ is an even permutation.

True ✓

False

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Q : The permutation $(a\ \ b)(a\ \ c)(a\ \ d)$ is an even permutation.

True

False ✓

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Example: Show that the following permutation is even.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 1 & 3 & 5 & 8 & 9 & 7 \end{pmatrix}$$

Sol:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 6 & 1 & 3 & 5 & 8 & 9 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 & 6 & 5 \\ 6 & 5 & 3 \end{pmatrix} \begin{pmatrix} 7 & 8 & 9 \\ 8 & 9 & 7 \end{pmatrix}$$

$$= (1\ 2\ 4)(3\ 6\ 5)(7\ 8\ 9)$$

$$= (1\ 2)(1\ 4)(3\ 6)(3\ 5)(7\ 8)(7\ 9)$$

Since there are 6 transpositions, so it is even.

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Order of Permutation : Let α be a permutation. Then the least positive integer m is called order of α if $\alpha^m = I$, where I is an identity permutation.

Example : $\alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3)$

$$\alpha^2 = \alpha \cdot \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\alpha^3 = \alpha^2 \alpha = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I$$

$\alpha^3 = I$

So the order of $\alpha = 3$ Ans

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Remark : **The order of a transposition is 2.**

Example : $\beta = (1\ \ 2) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$

$$\beta^2 = \beta \cdot \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

$$\beta^2 = I$$

Hence the order of transposition $\beta = 2$ Ans

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Theorem : The order of a cyclic permutation of length m is m.

Proof: let $d = (a_1\ a_2\ a_3\ \cdots\ a_m)$ be a cyclic permutation.

$d : a_1 \to a_2,\ a_2 \to a_3,\ \cdots,\ a_m \to a_1$

$d^2 : a_1 \to a_3,\ a_2 \to a_4,\ \cdots,\ a_m \to a_2$

$d^3 : a_1 \to a_4,\ a_2 \to a_5,\ \cdots,\ a_m \to a_3$

$\vdots$

$d^m : a_1 \to a_1,\ a_2 \to a_2,\ \cdots,\ a_m \to a_m$

$d^m = I$ where I is identity Permutation.

So order of d is m.

Hence the proof.

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Example : Find the order of $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 7 & 9 & 6 & 5 & 8 & 12 & 11 & 10 \end{pmatrix}$

Sol:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 2 & 3 & 4 & 1 & 7 & 9 & 6 & 5 & 8 & 12 & 11 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} \begin{pmatrix} 5 & 7 & 6 & 9 & 8 \\ 7 & 6 & 9 & 8 & 5 \end{pmatrix} \begin{pmatrix} 10 & 12 \\ 12 & 10 \end{pmatrix} \begin{pmatrix} 11 \\ 11 \end{pmatrix}$$

$$= \underbrace{(1\ 2\ 3\ 4)}_{\alpha_1} \underbrace{(5\ 7\ 6\ 9\ 8)}_{\alpha_2} \underbrace{(10\ 12)}_{\alpha_3}$$

length of $\alpha_1 = 4$

length of $\alpha_2 = 5$

length of $\alpha_3 = 2$

$$LCM(2, 5, 4) = 20$$

So the order of $\alpha = 20$ Ans

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Example : Find the order of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$

Sol:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}}_{\alpha_1} \underbrace{\begin{pmatrix} 4 & 5 & 6 \\ 5 & 6 & 4 \end{pmatrix}}_{\alpha_2} = (1\ 2\ 3)(4\ 5\ 6)$$

length of $\alpha_1 = 3$

length of $\alpha_2 = 3$

Order of $\alpha = LCM(3, 3) = 3$ Ans

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Example : Find the order of $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

Sol: Let $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 4 \\ 4 & 3 \end{pmatrix} = \underbrace{(1\ 2)}_{d_1}\underbrace{(3\ 4)}_{d_2}$

length of $d_1 = 2$

length of $d_2 = 2$

order of $\beta = LCM(2, 2) = 2$ Ans

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Example : Find the order of $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$

Sol :

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 4 & 5 \\ 4 & 5 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} = \underbrace{(1\ 4\ 5)}_{d_1}\ \underbrace{(2\ 3)}_{d_2}$$

Length of $d_1 = 3$

Length of $d_2 = 2$

Order of $f = LCM(2, 3) = 6$ Ans

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Example :

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ and $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$,

Show that $f \circ g \neq g \circ f$

Sol: $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$LHS = f \circ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$ —(1)

$RHS = g \circ f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$ —(2)

By (1) and (2), $f \circ g \neq g \circ f$ Hence proved.

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Example : Show by means of an example that the product of two cyclic permutations need not be a cyclic permutation.

Sol: Let $\alpha = (1\ 2\ 5)$, $\beta = (2\ 1\ 4\ 5\ 6)$ be two cyclic permutations.

Then

$$\alpha\beta = (1\ 2\ 5)(2\ 1\ 4\ 5\ 6)$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 3 & 4 & 1 & 6 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 5 & 6 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 3 & 5 & 4 & 2 \end{pmatrix}$$

This is not a cyclic permutation.

Hence the product of two cyclic permutations is not a cyclic permutation. Hence proved.

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Theorem : Prove that the set A_n of all even permutations in S_n form a subgroup of S_n. Explain why the set B_n of all odd permutations in S_n do not form a subgroup of S_n?

Proof: Let A_n be the set of all even permutations of S_n. We have to show that A_n is a subgroup of S_n.

Clearly $A_n \subseteq S_n$

Let $\alpha, \beta \in A_n$

$\alpha, \beta \in A_n$
$\alpha\beta^{-1} \in A_n$

Since the inverse of even permutation is even permutation.

Therefore, β^{-1} is also an even permutation.

$\Rightarrow \beta^{-1} \in A_n$

Since the product of two even permutations is even permutation.

So, $\alpha\beta^{-1}$ is also an even permutation.

Hence $\alpha\beta^{-1} \in A_n$.

So A_n is a subgroup of S_n.

Let B_n be the set of all odd permutations of S_n.

Let $f, g \in B_n$ be two odd permutations.

Since the inverse of odd permutation is $fg^{-1} \in B_n$

also an odd permutation.

So g^{-1} is an odd permutation. So $g^{-1} \in B_n$

Since the product of two odd permutations is an even permutation.

So $fg^{-1} \notin B_n$. So B_n is not a subgroup of S_n.

Hence proved.

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Example : Given the permutation $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$, verify that $\alpha^6 = I$.

Sol: Given $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 4 & 5 \\ 4 & 5 & 3 \end{pmatrix}$

$= (1\ 2)(3\ 4\ 5)$

$= \alpha_1 \alpha_2$

Length of $\alpha_1 = 2$

Length of $\alpha_2 = 3$

Order of α = LCM (2, 3)

$= 6$

$\alpha^6 = I$ Hence Proved.

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Example : **Write (i) all even permutations in S_3**

(ii) all odd permutations in S_3

Sol: S_3 is the set of all permutation defined on the set $X = \{1, 2, 3\}$.

All the permutations of S_3 are

$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = (1)(2)(3)$

$d_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1)(2\ 3)$

$d_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (1\ 2)(3)$

$d_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = (1\ 2\ 3) = (1\ 2)(1\ 3)$

$d_4 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (1\ 3\ 2) = (1\ 3)(1\ 2)$

$d_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = (1\ 3)(2)$

Since I, d_3, d_4 have even number of transposi-tions, so I, d_3, d_4 are even permutations.

Since d_1, d_2, d_5 have odd number of transpositions, so d_1, d_2, d_5 are the odd permutations.

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