Introduction to Topology

1. What is a Topology?
2. What kind of Math it is?
3. Why use Topology?

Introduction to Topology

Topology as a word is composition of two words

1. Topos(Greek) means "Place".
2. Logy(Latin) means "Study/Analysis".

The study of some thing with respect to its place.

Introduction to Topology

Topology is a kind of Math in which we study Shapes.

Q. Does it mean, Topology is Geometry?

Ans. No.

So what is the difference between Topology and Geometry?

Introduction to Topology

In Geometry we care about the exact measurements such as lengths, angles and areas.

But in Topology we don't care about the measurements.

Introduction to Topology

Geometrically rectangle, triangle, circle and square are different.

But from topological point of view there is no difference between the rectangle, triangle, circle and square.

Introduction to Topology

Instead of measurement, in Topology we care more about

1. Arrangements of shapes
2. Relative positions of shapes
3. Functions/deformations between the shapes.

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So one may say that topology is "Geometry of Position".

In Topology two shapes are considered the same if one shape can be deformed to other shape.

Introduction to Topology

How useful this subject is if we can't tell the difference between shapes?

In fact it's a matter of nature of the study.

Topological study is more qualitative in nature, rather than quantitative.

Introduction to Topology

As an example a circle can be deformed into a square.

So topologically circle and square are of same shape.

But a line and circle are of different shapes.

Introduction to Topology

There are many problem/mathematical models where qualitative study is more effective than the quantitative study.

So topology is a very useful tool in such problems or one may say some times the only effective tool.

Topology

Topology on a Set

Topology on a Set

Power Set:

Let X be a set. Power set of X, denoted as $P(X)$, is the set of all subset of X. Usually we use the word class for a set of sets.

Topology on a Set

Example:

Let $X = \{1, 2, 3\}$

Then

$$P(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}$$

Remarks:

1. ϕ and X are also members of $P(X)$.
2. If is X finite and has n elements then

$$| P(X) | = 2^n$$

Topology on a Set

Let X be a non-empty set and $\Im$ be a subclass of $P(X)$, then $\Im$ is called a topology on X if $\Im$ satisfied the following axioms.

1. $\phi, X \in \Im$.
2. The union of any number of sets in belongs to
3. The intersection of finite number of sets in $\Im$ belongs to $\Im$.

Topology on a Set

To satisfy the 3rd axiom it is enough to show that intersection of any two elements of $\mathfrak{I}$ belongs to $\mathfrak{I}$.

Topology on a Set

Open Set:

The elements of $\mathfrak{I}$ are called open sets or more specifically $\mathfrak{I}$-open sets.

Topology

Topology on a Set

Topology on a Set

Topology on a set can also be defined by stating only 2nd and 3rd axiom.

1st axiom will be followed from 2nd and 3rd.

$$\bigcup\{U \in \Im \mid U \in \phi\} = \phi$$

$$\bigcap\{U \in \Im \mid U \in \phi\} = X$$

Topology

Examples:
Topology on a set

VU

Examples: Topology on a Set

Example 1:
Let $X = \{a, b, c, d\}$
Then

$$P(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, X\}$$

Let us consider the subcollection

$$\Im = \{\phi, \{a\}, \{c\}, \{a,c\}, X\}$$

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Examples: Topology on a Set

$\Im = \{\phi, \{a\}, \{c\}, \{a,c\}, X\}$

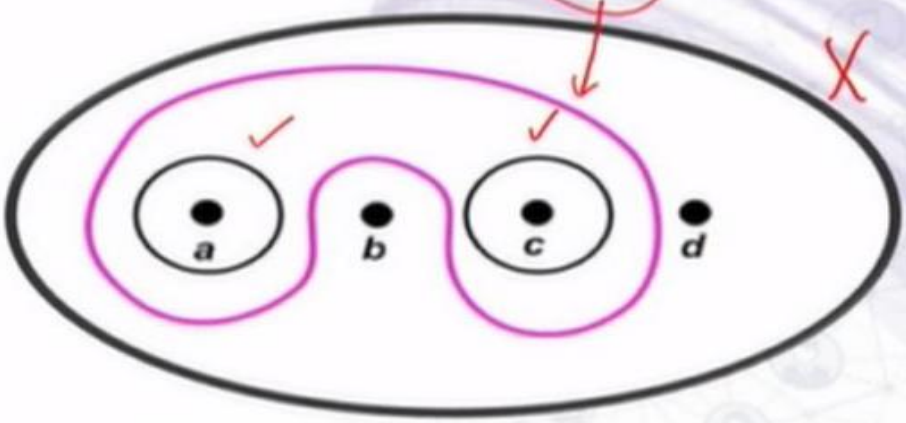
Let us check all three conditions one by one on $\Im$.

1. It is clear that
$$\phi, X \in \Im$$
2. Now if we take union of any number of elements of $\Im$. It belongs to $\Im$.
3. The same is true for intersection.

So $\Im$ is topology on X.

Examples: Topology on a Set

$\Im = \{\phi, \{a\}, \{c\}, \{a,c\}, X\}$



Examples: Topology on a Set

Example 2:

Let $X = \{a, b, c, d\}$

Let us consider

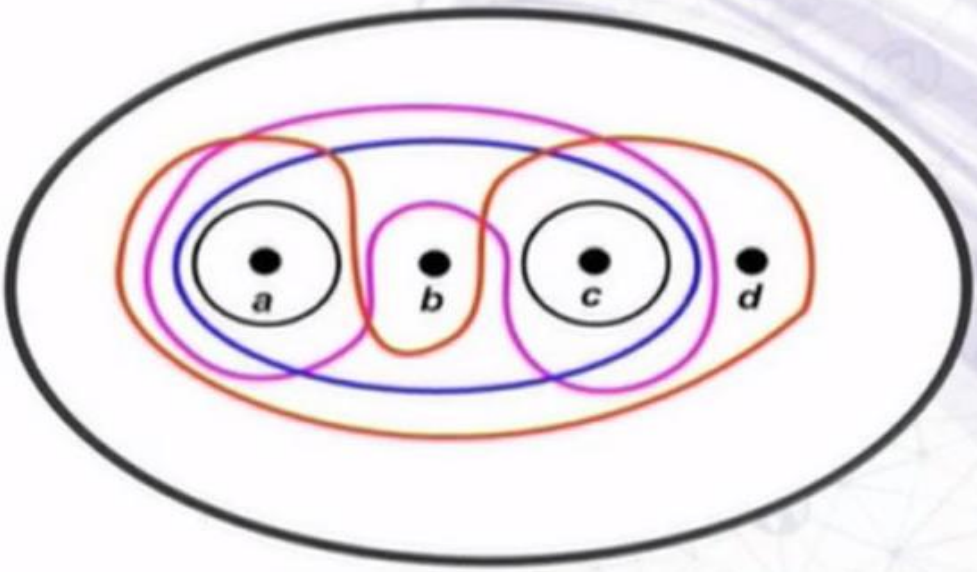
$$\Im' = \{\phi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$$

1. Again in this example it is clear that

$$\phi, X \in \Im'$$

Examples: Topology on a Set

$$\Im' = \{\phi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$$



Examples: Topology on a Set

$X = \{a\}$

$\mathcal{T} = \{\phi, X\}$

Remark 1:
More than one topologies can be defined on a set.

Let us define all the possible topologies on 1-point, 2-point and 3-point sets.

$Y = \{a, b\}$

$\mathcal{T}_1 = \{\phi, \{a\}, Y\}$

$\mathcal{T}_2 = \{\phi, \{b\}, Y\}$

$\mathcal{T}_3 = \{\phi, \{a, b\}, Y\}$

$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\} = Y\}$

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Examples: Topology on a Set

Let us list down all the topologies on a three point set.

$$X = \{a, b, c\}$$

$\Im_1 = \{\phi, X\}.$

$\Im_2 = \{\phi, \{a\}, X\}.$

$\Im_3 = \{\phi, \{b\}, X\}.$

$\Im_4 = \{\phi, \{c\}, X\}.$

$\Im_5 = \{\phi, \{a, b\}, X\}.$

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Examples: Topology on a Set

$$\Im_6 = \{\phi, \{a, c\}, X\}.$$
$$\Im_7 = \{\phi, \{b, c\}, X\}.$$
$$\Im_8 = \{\phi, \{a\}, \{a, b\}, X\}.$$
$$\Im_9 = \{\phi, \{a\}, \{a, c\}, X\}.$$
$$\Im_{10} = \{\phi, \{a\}, \{b, c\}, X\}.$$
$$\Im_{11} = \{\phi, \{b\}, \{a, b\}, X\}.$$
$$\Im_{12} = \{\phi, \{b\}, \{a, c\}, X\}.$$
$$\Im_{13} = \{\phi, \{b\}, \{b, c\}, X\}.$$
$$\Im_{14} = \{\phi, \{c\}, \{a, c\}, X\}.$$

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Examples: Topology on a Set

$$\Im_{15} = \{\phi, \{c\}, \{b, c\}, X\}.$$
$$\Im_{16} = \{\phi, \{c\}, \{a, b\}, X\}.$$
$$\Im_{17} = \{\phi, \{a\}, \{b\}, \{a, b\}, X\}.$$
$$\Im_{18} = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}.$$
$$\Im_{19} = \{\phi, \{b\}, \{c\}, \{b, c\}, X\}.$$
$$\Im_{20} = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}.$$
$$\Im_{21} = \{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}.$$
$$\Im_{22} = \{\phi, \{c\}, \{a, c\}, \{b, c\}, X\}.$$

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Examples: Topology on a Set

$$\Im_{23} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}.$$
$$\Im_{24} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}.$$
$$\Im_{25} = \{\phi, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}.$$
$$\Im_{26} = \{\phi, \{a\}, \{c\}, \{a, c\}, \{a, b\}, X\}.$$
$$\Im_{27} = \{\phi, \{b\}, \{c\}, \{b, c\}, \{a, b\}, X\}.$$
$$\Im_{28} = \{\phi, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}.$$
$$\Im_{29} = P(X).$$

Examples: Topology on a Set

Remark 2:
Not every collection of subsets of a set is eligible to be a topology on the set unless that collection doesn't satisfy all the stated conditions in the definition.

Examples: Topology on a Set

Now let us look into some examples those are not topologies on a set X.

Example 3:
Let $X = \{a, b, c, d\}$ and consider the following collection of subsets of X.

Examples: Topology on a Set

$$\Im_1 = \{\{b\}, \{c\}, \{b, c\}, X\}$$
$$\Im_2 = \{\phi, \{a\}, \{c\}, \{a, c\}\}$$
$$\Im_3 = \{\phi, \{a\}, \{b, c\}, X\}$$
$$\Im_4 = \{\phi, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$$

None of the above $\Im_{i's}$ is a topology on X.

Examples: Topology on a Set

$$\Im_1 = \{\{b\},\{c\},\{b,c\},X\}$$

$$\Im_2 = \{\phi,\{a\},\{c\},\{a,c\}\}$$

$$\Im_3 = \{\phi,\{a\},\{b,c\},X\}$$

$$\Im_4 = \{\phi,\{a,b\},\{b,c\},\{a,b,c\},X\}$$

Reasons:

1. For $\Im_1$:
   $\phi \notin \Im_1$
2. For $\Im_2$:
   $X \notin \Im_2$
3. For $\Im_3$:
   $\{a\} \cup \{b,c\} = \{a,b,c\} \notin \Im_3$
4. For $\Im_4$:
   $\{a,b\} \cap \{b,c\} = \{b\} \notin \Im_4$

Topology

Some More Examples:
Topology on a Set

Examples: Topology on a Set

Example 4:

Let $X = \mathbb{R}$

and $\Im$ be the class consisting $\phi, \mathbb{R}$ and all infinite open intervals of the following the kind

$$I_a = (a, \infty)$$

where a be any real number.

Then $\Im$ is a topology on $\mathbb{R}$.

Examples: Topology on a Set

Let us justify our claim.

It is given that

$$\phi, \mathbb{R} \in \Im$$

For union:

Note that for any subcollection

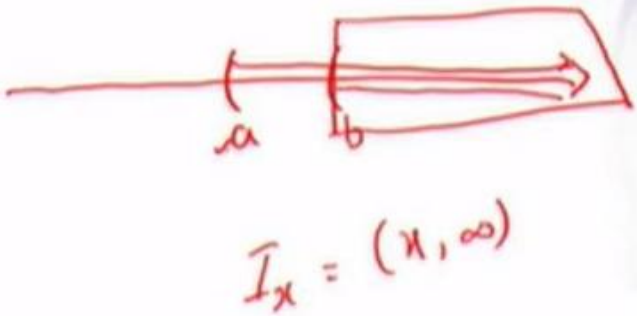
$$\Lambda = \{I_i \mid i \in J\}$$

of $\Im$, where $J \subset \mathbb{R}$.

Now

$$\bigcup_i I_i = \begin{cases} (j, \infty) & \text{if} \quad j = \inf(J) \\ (-\infty, \infty) & \text{if} \quad \text{No inf. exists.} \end{cases}$$

Examples: Topology on a Set



In both cases union belongs to $\Im$.

For intersection:
Let $A = (a, \infty)$ and $B = (b, \infty)$
Consider $a < b$.

$$A \cap B = (b, \infty) \in \Im$$

Examples: Topology on a Set

Example 5:
Let $X = \mathbb{R}$
and consider the class T consisting of $\phi, \mathbb{R}$ along with all infinite open intervals

$$A_q = (-\infty, q)$$

where $q \in \mathbb{Q}$.
This collection is not a topology on real line.

Examples: Topology on a Set

Reason:

The condition of union is not satisfied by the given collection T.

Let us consider the following subclass of T.

$$\aleph = \{(-\infty, q) \mid q \in \mathbb{Q},\ q < \pi\}$$

Since

$$\bigcup_{\substack{q \in \mathbb{Q} \\ q < \pi}} (-\infty, q) = (-\infty, \pi) \notin T$$

Topology

Some Special Topologies

Some Special Topologies

For any set there are always two topologies those are very special.

1. Indiscrete Topology.
2. Discrete Topology:

Some Special Topologies

Indiscrete Topology:
A topology on a set X consists of empty set and the set itself is called "Indiscrete Topology".
It is also called trivial topology.

Some Special Topologies

Remark:
Indiscrete topology is the smallest topology one can define on some set.

Task:
Show that such collection is a topology on any set.

X

$\Im_{ind} = \{\emptyset, X\}$

$\rightarrow \emptyset, X \in \Im_{ind}$

$\rightarrow \emptyset \cup X = X \in \Im_{ind}$

$\rightarrow \emptyset \cap X = \emptyset \in \Im_{ind}$

Some Special Topologies

Discrete Topology:
A topology on a set X consists of all subsets of X is called "Discrete Topology". i.e.

$$\Im = P(X).$$

Remark:
Discrete topology is the largest topology one can define on some set X.

Some Special Topologies

Task:

Let us show that $\Im = P(X)$ is a topology on set X.

✓ $P(X)$ = The set of all subsets of X

→ $\phi, X \in P(X) = \Im$

→ $A_i \subset \Im$

→ $A_j \subset \Im$

$\bigcup_{i \in I} A_i \subset X \Rightarrow \bigcup_{i \in I} A_i \in P(X) = \Im$

$\bigcap_{j=1}^{n} A_j \subset X \Rightarrow \bigcap_{j=1}^{n} A_j = P(X) = \Im$

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Topology

Cofinite Topology

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Cofinite Topology

"Cofinite Topology" is also known as "Finite Complement Topology". For a set X "Cofinite Topology" $\Im_{cof}$ be the collection of all subsets U of X such that U^c is either finite or complete X.

Cofinite Topology

i.e.

$$\Im_{cof} = \{U \subset X \mid U^c \text{ is finite or } X\}$$

or

$$\Im_{cof} = \{U \subset X \mid U^c \text{ is finite or } U = \phi\}$$

Cofinite Topology

Task:
Show that $\Im_{cof} = \{U \subset X \mid U^c \text{ is finite or } X\}$ defines a topology on X.

Proof:

① $\emptyset \in \Im_{cof}$ $\because$ $\emptyset^c = X$

$X \in \Im_{cof}$ $\because$ $X^c = \emptyset$ Finite.

② Let $A_i \in \Im_{cof} \Rightarrow A_i^c$ = finite or X $i \in I$

Now we have to show that $\bigcup_{i \in I} A_i \in \Im_{cof}$??

$\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} (A_i)^c$ = finite or X

finite on X

De Morgan's Law.

Cofinite Topology

③ Let $A_i \in \Im_{cof} \Rightarrow (A_i)^c$ = finite on X, $i = 1, 2, \ldots, n$

✓ $\bigcap_{i=1}^{n} A_i \in \Im_{cof}$??

$\left(\bigcap_{i=1}^{n} A_i\right)^c = \bigcup_{i=1}^{n} (A_i)^c$ = finite on X

Finite union of finite sets is finite.

$\Rightarrow \bigcap_{i=1}^{n} A_i \in \Im_{cof}$.

So $\Im_{cof}$ is a topology.

Cofinite Topology

Remark:

If X is finite then the cofinite topology on X is discrete.

i.e.

$$\Im_{cof} = P(X) \quad \text{if} \quad X \text{ is finite.}$$

Proof: $P(X)$: The set of all subsets of X.

$\Im_{cof} \subset P(X)$ | Now $U \in P(X)$

$\because X$ is finite So U^c = finite.

$\Rightarrow U \in \Im_{cof}$

$\Rightarrow \Im_{cof} = P(X)$ X is fi

Topology

Topological Space

Topological Space

Topological Space:

Let $\Im$ be a topology on X then the pair $(X, \Im)$ is a "Topological Space" or some time called "$\Im$-Topological Space".

Example:

Consider

$$X = \{a, b, c, d\}$$

and

$$\Im = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$$

Then $(X, \Im)$ is a topological space.

Topological Space

Example:

Consider $X = \{a, b, c, d\}$ with

$$\Im' = \{\phi, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}, X\}$$

Then $(X, \Im')$ be another topological space.

Note that in both examples set X is the same but we define two different topological spaces.

Topological Space

Remark:
Correspond to single set one may define multiple topological spaces and all can be different from each other.

Correspond to a three point set one can define 29 different topological spaces.

Topological Space

Discrete Space:
A set X with discrete topology is called "Discrete Topological Space" or simply "Discrete Space".

Indiscrete Space:
A set X with indiscrete topology is called "Indiscrete Topological Space" or simply "Indiscrete Space".

Topology

Arbitrary Intersection

Arbitrary Intersection

Let us recall the 3rd axiom of definition of topology on a set.
"Finite Intersection"
Q: What about arbitrary intersection?
Specifically "Infinite intersection".
Before giving the answer to this question let us look into the next two examples.

Arbitrary Intersection

Example 1:

Consider $\mathbb{R}$ "Real Line" and consider $\Im$ be the set of all open intervals and their unions. This collection is a topology on $\mathbb{R}$.

Consider the following subcollection.

$$A = \left\{ A_n = \left(-\frac{1}{n}, \frac{1}{n} \right) \;\middle|\; n \in \mathbb{N}^* \right\}$$

Now

$$\bigcap_{n \in \mathbb{N}^*} \left(-\frac{1}{n}, \frac{1}{n} \right) = \{0\} \notin \Im$$

Arbitrary Intersection

Example 2:

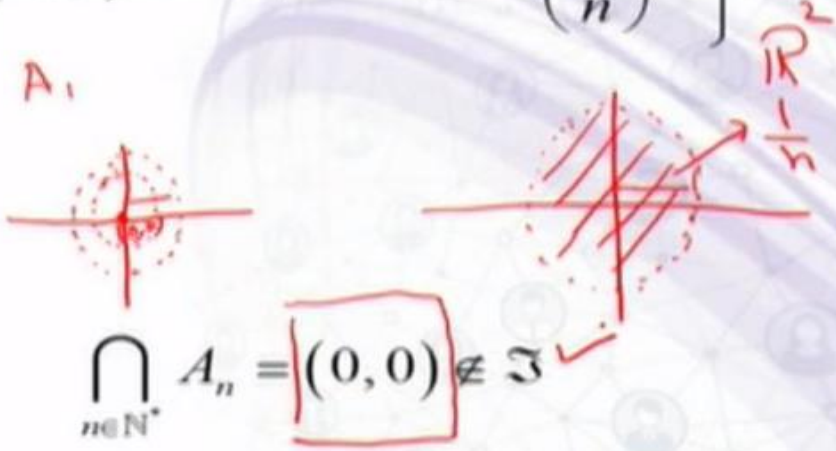
Consider $\mathbb{R}^2$ "Real Plane" and consider $\Im$ the set of all open disks, their arbitrary unions and their finite intersection. This collection is a topology on $\mathbb{R}^2$. Consider the following subcollection of open disks

$$A = \{ A_n \mid n \in \mathbb{N}^* \}$$

where

Arbitrary Intersection

$$A_n = \left\{ (x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < \left(\frac{1}{n}\right)^2 \right\}$$



$$\bigcap_{n \in \mathbb{N}^*} A_n = \{(0,0)\} \notin \Im$$

Topology

Open Sets, Closed Sets

Open Sets, Closed Sets

Open Set:

Let $\Im$ be at a topology on X. Then the elements of $\Im$ are called the open sets, more precisely open subsets of X.

Let us look into an example for better understanding of the openness.

Open Sets, Closed Sets

Example 1:

Let $X = \{a, b, c, d\}$

Consider

$\Im = \{\phi, \{a\}, \{c\}, \{a, c\}, X\}$

Then

$$\phi$$
$$\{a\}$$
$$\{c\}$$
$$\{a, c\}$$
$$X$$

are open subsets of X.

Open Sets, Closed Sets

Example 2:

Let $X = \{a, b, c, d\}$

Consider

$$\Im = \{\phi, \{c\}, \{a, c\}, \{b, c, d\}, X\}$$

Then

$$\phi$$

$$\{c\}$$

$$\{a, c\}$$

$$\{b, c, d\}$$

$$X$$

are open subsets of X.

Open Sets, Closed Sets

Remark:

- Openness is a relative term.
- ϕ and X are always open.

Open Sets, Closed Sets

Closed Set:

Let $(X, \Im)$ be a topological space. A sub set A of X is closed iff the complement of A in X is open. i.e.

$$X - A \in \Im$$

Let us discuss in example.

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Open Sets, Closed Sets

Example 3:

$$X = \{a, b, c, d\}$$

$$\Im = \{\phi, \{c\}, \{a, c\}, \{b, c, d\}, X\}$$

Now $\{a\}$ is closed since

$$X - \{a\} = \{b, c, d\}$$ open

is open i.e. belongs to $\Im$.

Similarly $\{b, d\}$ is closed since

$$\{b, d\}^c = X - \{b, d\} = \{a, c\}$$

is open.

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Open Sets, Closed Sets

Example 4:

$$X = \{a, b, c, d\}$$

$$\Im = \{\phi, \{c\}, \{a, c\}, \{b, d\}, \{b, c, d\}, X\}$$

$\{a, c\}$ open

$\{a, c\}^c = \{b, d\} \in \Im$

Closed.

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Open Sets, Closed Sets

Example.

X, $\Im_{dis.} = P(X)$

In discrete spaces all subsets of X are open and closed:

Remark:

- Closeness like openness is a relative term.
- ϕ and X are always closed.
- A set can be open and closed at the same time.

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Topology

Properties of Collection of Closed Sets

Properties of Collection of Closed Sets

Let $(X, \mathcal{T})$ be a topological space. Consider the collection

$$\mathcal{A} = \{A \subset X | A^c \in \mathcal{T}\}$$

$\mathcal{A}$ is set of all those subsets of X whose complement is open. Simply "Collection of all closed subsets".

Properties of Collection of Closed Sets

Example:
Let

$$X = \{a, b, c, d\}$$

and

$$\mathcal{T} = \{\emptyset, \{a\}, \{a, d\}, \{a, c, d\}, X\}$$

then

$$\mathcal{A} = \{X, \{b, c, d\}, \{b, c\}, \{b\}, \emptyset\}$$

is the collection of all closed subsets of X.

Properties of Collection of Closed Sets

Example:
Let

$$X = \mathbb{R}$$

and

$$\mathcal{T} = \{(a, \infty) | a \in \mathbb{R}\}$$

Then

$$\mathcal{A} = \{(-\infty, a] | a \in \mathbb{R}\}$$

is the collection of all closed subsets of $\mathbb{R}$ with respect to $\mathcal{T}$.

Properties of Collection of Closed Sets

Theorem:
Let $(X, \mathcal{T})$ be a topological space. Then $\mathcal{A}$, collection of all closed subsets of X, satisfies the following properties.

1. $\emptyset, X \in \mathcal{A}$
2. $\mathcal{A}$ is closed under finite set union.
   i.e. $A_i \in \mathcal{A} \ \ (i \in \{1, 2, \ldots, n\}) \ \Rightarrow \cup_{i=1}^{n} A_i \in \mathcal{A}$
3. $\mathcal{A}$ is closed under arbitrary set intersection.
   i.e. $A_i \in \mathcal{A} \ \ (i \in I \ \Rightarrow \cap_{i \in I} A_i \in \mathcal{A}$

Properties of Collection of Closed Sets

Proof: $\mathcal{A} = \{A \subset X \mid A^c \in \mathcal{T}\}$

1) $\emptyset \in \mathcal{A} \ \because \ \emptyset^c = X \in \mathcal{T}$
   $X \in \mathcal{A} \ \because \ X^c = \emptyset \in \mathcal{T}$

2) Let $A_i \in \mathcal{A}$ i.e $A_i^c \in \mathcal{T}$ $i = 1, 2, \ldots, n$
   $\boxed{\bigcup_{i=1}^{n} A_i \in \mathcal{A}}$ Proved.
   $\left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} (A_i)^c \in \mathcal{T}$
   ∴ The finite intersection of open sets is always open

3) Let $A_i \in \mathcal{A}$ $i \in I$
   $\Rightarrow A_i \in \mathcal{T}$ $\forall\, i \in I$
   $\bigcap_{i \in I}$.

Properties of Collection of Closed Sets

Proof: $\mathcal{A} = \{A \subset X \mid A^c \in \mathcal{T}\}$

1) $\phi \in \mathcal{A} \quad \because \phi^c = X \in \mathcal{T}$ ✓

$X \in \mathcal{A} \quad \because X^c = \phi \in \mathcal{T}$ ✓ ✓

2) Let $A_i \in \mathcal{A}$ i.e $A_i^c \in \mathcal{T}$ $i = 1, 2, \cdots, n$

$\bigcup_{i=1}^{n} A_i \in \mathcal{A}$ ✓ $\quad \left(\bigcup_{i=1}^{n} A_i\right)^c = \bigcap_{i=1}^{n} (A_i)^c \in \mathcal{T}$

Proved.

$\therefore$ The finite intersection of open sets is always open

3) Let $A_i \in \mathcal{A} \Rightarrow A_i \in \mathcal{T}$ $i \in I$

$\bigcap_{i \in I} A_i \in \mathcal{A}$?? $\quad \forall\, i \in I$

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Topology

Intersection of Topologies

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Intersection of Topologies

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Now what can we say about the intersection of $\mathcal{T}_1$ and $\mathcal{T}_2$? Does $\mathcal{T}_1 \cap \mathcal{T}_2$ define a topology on X?
Before answering to this question let us look in to an example.

Intersection of Topologies

Example:
Let

$$X = \{a, b, c\}$$

and

$$\begin{aligned} \mathcal{T}_1 &= \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\} \\ \mathcal{T}_2 &= \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \end{aligned}$$

then

$$\mathcal{T}_1 \cap \mathcal{T}_2 = \{\emptyset, \{a\}, \{a, b\}, X\}.$$

It is clear that $\mathcal{T}_1 \cap \mathcal{T}_2$ is a topology on X.

Intersection of Topologies

Theorem:
Let

$$\{T_i | i \in I\}$$

be any collection of topologies on set X. Then $\cap_{i \in I} T_i$ is also a topology on X.

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Intersection of Topologies

Proof: $\{\overline{S}_i \mid i \in I\}$

Now we have to show that $\bigcap_{i \in I} \overline{S}_i$ is also a top. on X.

1) $\phi \in \bigcap_{i \in I} \overline{S}_i$ $\because \overline{S}_i$ is a top on X $\forall i$
So $\phi \in \overline{S}_i$ $\forall i \in I$

$X \in \bigcap_{i \in I} \overline{S}_i$ $\because \overline{S}_i$ is a top on X, $\forall i$
So $X \in \overline{S}_i$ $\forall i \in I$

2) Let $U_j \in \bigcap_{i \in I} \overline{S}_i$
$\Rightarrow U_j \in \overline{S}_i$ $\forall i$
But $\overline{S}_i$ is a top on X $\forall i$

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Intersection of Topologies

So $\bigcup_{j\in J} U_j \in \overline{S}_i \quad \forall\, i \in I$

$\Rightarrow \bigcup_{j\in J} U_j \in \bigcap_{i\in I} \overline{S}_i$

3) Let $U_j \in \bigcap_{i\in I} \overline{S}_i \qquad j = 1, 2, \ldots, n$

$\Rightarrow U_j \in \overline{S}_i \quad \forall\, i \in I, \ \forall\, j$

But $\overline{S}_i$ is a top on $X \ \forall\, i \in I$

So $\bigcap_{j=1}^{n} U_j \in \overline{S}_i \quad \forall\, i \in I$

$\Rightarrow \bigcap_{j=1}^{n} U_j \in \bigcap_{i\in I} \overline{S}_i$ Proved.

VU

Topology

Union of Topologies

VU

Union of Topologies

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Now what can we say about the union of $\mathcal{T}_1$ and $\mathcal{T}_2$?
As like intersection, can we say that $\mathcal{T}_1 \cup \mathcal{T}_2$ defines a topology on X?
Let us see some examples.

Union of Topologies

Example:
Let

$$X = \{a, b, c\}$$

and

$$\begin{aligned} \mathcal{T}_1 &= \{\emptyset, \{a\}, \{a, b\}, X\} \\ \mathcal{T}_2 &= \{\emptyset, \{b\}, \{a, b\}, X\} \end{aligned}$$

then

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}.$$

In this example $\mathcal{T}_1 \cup \mathcal{T}_2$ is a topology on X.
Does it mean that it is true in general? No, unless we prove it.

Union of Topologies

Example:
Let

$$X = \{a, b, c\}$$

and

$$\begin{aligned} \mathcal{T}_1 &= \{\emptyset, \{a\}, \{c\}, \{a, b\}, \{a, c\}, X\} \\ \mathcal{T}_2 &= \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \end{aligned}$$

then

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}.$$

$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$ is not a topology on X. Since $\{b\} \cup \{c\} = \{b, c\} \notin \mathcal{T}$.

Union of Topologies

Remark:
Union of topologies on a set X may or may not be a topology on X.

Topology

Jsual Topology on $\mathbb{R}$

Usual Topology on $\mathbb{R}$

Real Line:
The real number set, $\mathbb{R}$, one of the most studied/dominated object in Mathematics. Most of the concepts in mathematics are the generalization/abstraction of the properties of the real numbers. To study $\mathbb{R}$ some one needs a usual structure so the rest of the terminologies can be defined according to that.

Usual Topology on $\mathbb{R}$

This usual structure is provided by usual topology. Here we define the usual topology on $\mathbb{R}$ by using order relation.

Open Interval:

Let $a, b \in \mathbb{R}$ with usual order relation "$a < b$" then the open interval from a to b is

$$(a, b) = \{x | a < x < b\} \subset \mathbb{R}$$

Usual Topology on $\mathbb{R}$

Open Set in $\mathbb{R}$:

A subset A of $\mathbb{R}$ is open iff for every $a \in A$ there exists an open interval I_a, containing a, which contained in A. i.e.

$$a \in I_a \subset A.$$

Usual Topology on $\mathbb{R}$:

The set of all open sets of $\mathbb{R}$, $\mathcal{T}_u$,(as defined in above def.) is a topology on $\mathbb{R}$, called usual topology on $\mathbb{R}$.

Usual Topology on $\mathbb{R}$

Task:
Prove that $\mathcal{T}_u$ is a topology on $\mathbb{R}$.

Criteria: ✓
$A \subset \mathbb{R}$
$A \in \mathcal{T}_u$ iff
$\forall\, a \in A \;\exists\, I_a$ s.t $a \in I_a \subset A$

Proof: 1) $\emptyset \in \mathcal{T}_u$ $\because \nexists\, x \in \emptyset$

$X = \mathbb{R} \in \mathcal{T}_u$ $\quad \forall\, x \in \mathbb{R}$, $I_x \subset \mathbb{R}$

2) Let $U_i \in \mathcal{T}_u$ $\quad i \in I$

Let $x \in \bigcup_{i \in I} U_i$

$\Rightarrow x \in U_i$ for some $i \in I$

But U_i is open in $\mathbb{R}$ So $\exists\, I_x \subset U_i \subset \bigcup_{i \in I} U_i$

$\Rightarrow x \in I_x \subset \bigcup_{i \in I} U_i$

VU $\Rightarrow \bigcup_{i \in I} U_i \in \mathcal{T}_u$.

Usual Topology on $\mathbb{R}$

3) Let $U_1, U_2 \in \mathcal{T}_u$

$x \in U_1 \cap U_2 \Rightarrow x \in U_1$ and $x \in U_2$

But U_1 and U_2 both are open So

$\exists\; I_{1x}, I_{2x}$ s.t

$I_{1x} \subset U_1$ and $I_{2x} \subset U_2$

Now $I_{1x} \cap I_{2x} = I_{3x} \subset U_1$

$\subset U_2$

$\Rightarrow x \in I_{3x} \subset (U_1 \cap U_2) \Rightarrow U_1 \cap U_2 \in \mathcal{T}_u$.

($x \in I_{1x}$, $x \in I_{2x}$; I_{3x} open interval)

VU

Topology

Usual Topology on $\mathbb{R}^2$

Usual Topology on $\mathbb{R}^2$

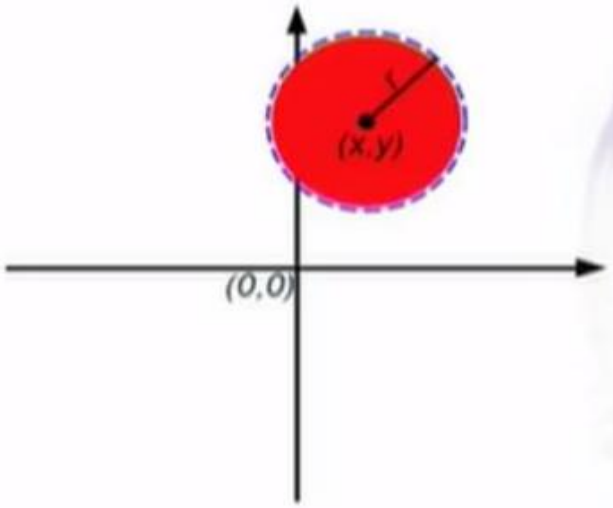


Figure: Open Disk

Real Plane $\mathbb{R}^2$:
The real plane is the cartesian product of two real lines i.e. $\mathbb{R} \times \mathbb{R}$. The real plane is usually written as $\mathbb{R}^2$.

Open Disk:
An open disk $D_{(x,y)}$ in $\mathbb{R}^2$, centered at $(x, y) \in \mathbb{R}^2$, is the set of all those points (x', y') in $\mathbb{R}^2$ whose distance from (x, y) is less than r, where r is a positive real number.

$$D_{(x,y)} = \{(x', y') | \sqrt{(x - x')^2 + (y - y')^2} < r\} \subset \mathbb{R}^2$$

Usual Topology on $\mathbb{R}^2$

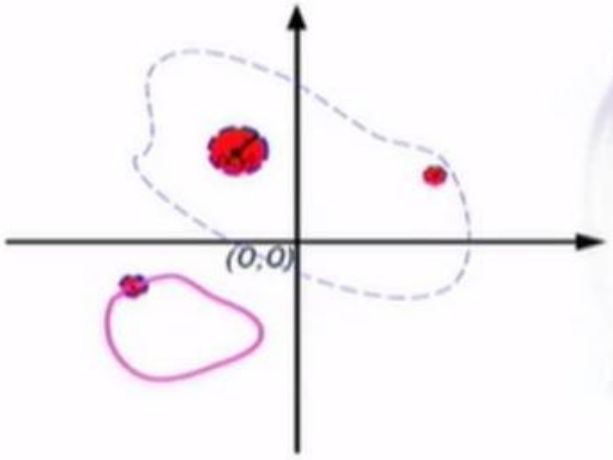


Figure: Open Subset in $\mathbb{R}^2$

Open Set in $\mathbb{R}^2$:
A subset U of $\mathbb{R}^2$ is open in $\mathbb{R}^2$ iff for every $a = (x, y) \in U$ there exists an open disk D_a, which contained in U. i.e.

$$a \in D_a \subset U.$$

Usual Topology on $\mathbb{R}^2$:
The set of all open sets of $\mathbb{R}^2$, $\mathcal{T}_u$,(as defined in above def.) is a topology on $\mathbb{R}^2$, called usual topology on $\mathbb{R}^2$.

Usual Topology on $\mathbb{R}^2$

Task:
Prove that $\mathcal{T}_u$ is a topology on $\mathbb{R}^2$.

Topology

Usual Topology on $\mathbb{R}^n$

pic018.mp4

Usual Topology on $\mathbb{R}^n$

Open n-Ball:
An open ball B_x in $\mathbb{R}^n$, centered at $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, is the set of all those points $y = (y_1, y_2, \cdots, y_n)$ in $\mathbb{R}^n$ whose distance from x is less than r, where r is a positive real number.

$$B_x = \left\{ y; \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} < r \right\} \subset \mathbb{R}^n$$

Usual Topology on $\mathbb{R}^n$

Open Set in $\mathbb{R}^n$:
A subset U of $\mathbb{R}^n$ is open in $\mathbb{R}^n$ iff for every $a \in U$ there exists an open n-ball B_a contained in U. i.e. for every $a \in U \quad \exists \quad B_a$ s.t.

$$a \in B_a \subset U.$$

Usual Topology on $\mathbb{R}^n$:
The set of all open sets of $\mathbb{R}^n$, $\mathcal{T}_u$,(as defined in above def.) is a topology on $\mathbb{R}^n$, called usual topology on $\mathbb{R}^n$.

Usual Topology on $\mathbb{R}^n$

Task:
Prove that $\mathcal{T}_u$ is a topology on $\mathbb{R}^n$.

Topology

Comparison of Topologies

pic019.mp4

Comparison of Topologies

As we have already seen that more than one topologies can be defined on a set X with more than one element. So it is quite natural to ask the following question.

Question:

Is there any relation between two different topologies?

As the topologies consist of collection of subsets of X. So one may put a question on the size of two topologies.

Comparison of Topologies

But this question may not be answered properly unless we will not consider the relation between the elements of two topologies and the elements are the subsets of a set. Naturally we have the relation that is, the set inclusion. And we have the positive answer of our question that is, "yes". There can be class inclusion relation between two topologies.

Comparison of Topologies

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a nonempty set X. Then $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ if and only if $\mathcal{T}_2 \subseteq \mathcal{T}_1$ i.e.

$$\forall U \in \mathcal{T}_2 \Rightarrow U \in \mathcal{T}_1$$

If $\mathcal{T}_1$ is finer than $\mathcal{T}_2$ then we also say that $\mathcal{T}_2$ is weaker than $\mathcal{T}_1$.

Comparison of Topologies

Example:
Let

$$X = \{a, b, c\}$$

and

$$\mathcal{T}_1 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$$
$$\mathcal{T}_2 = \{\emptyset, \{a\}, \{a, b\}, X\}$$

It is clear that

$$\mathcal{T}_2 \subset \mathcal{T}_1$$

Comparison of Topologies

Example:
Let X be any nonempty set. Consider $\mathcal{T}$ be any topology on X and $\mathcal{T}_{InD}$ is indiscrete topology on X. Then it is not very hard to show that $\mathcal{T}_{InD} \subseteq \mathcal{T}$.

$\because$ every $\mathcal{T}$ contains $\emptyset$ and X and $\mathcal{T}_{InD} = \{\emptyset, X\}$.

Remark:
Indiscrete topology is the weakest topology on a set X.

Comparison of Topologies

Example:
Let X be any nonempty set. Consider $\mathcal{T}$ be any topology on X and $\mathcal{T}_{dis}$ is discrete topology on X. Then it is not very hard to show that $\mathcal{T} \subseteq \mathcal{T}_{dis}$.

$\because$ every $\mathcal{T} \subseteq P(X) = \mathcal{T}_{dis}$

Remark:
Discrete topology is the finest topology on a set X.

Comparison of Topologies

Comparable Topologies:
Two topologies on a set X are comparable iff one is finer(weaker) than the other.
Remark:
Not all topologies on a set are comparable.
Example:
Let $X = \{a, b, c, d\}$ and consider

$$\begin{aligned} \mathcal{T}_1 &= \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\} \\ \mathcal{T}_2 &= \{\emptyset, \{c\}, \{c, d\}, X\} \end{aligned}$$

$\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable since $\mathcal{T}_2 \not\subseteq \mathcal{T}_1$ and $\mathcal{T}_1 \not\subseteq \mathcal{T}_2$ i.e. no one is finer, no one is weaker in comparison of each other.

Comparison of Topologies

Theorem:
The collection $\mathrm{T} = \{\mathcal{T}_i\}$ of all topologies on X is partially ordered by class inclusion.

Hint:
Show that the following properties hold for all elements of T

1. Reflexivity: $\mathcal{T} \subseteq \mathcal{T}$.
2. Antisymmetry: If $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T} = \mathcal{T}$.
3. Transitivity: If $\mathcal{T} \subseteq \mathcal{T}'$ and $\mathcal{T}' \subseteq \mathcal{T}''$, then $\mathcal{T} \subseteq \mathcal{T}''$.

Topology

Limit Point of a Set

pic020.mp4

Limit Point of a Set

Limit Point:
Let $(X, \mathcal{T})$ be a topological space and A be a subset of X. A point $x \in X$ is a limit point of A iff every open set U_x containing x contains a point of A different from x, i.e.
$\forall\ U_x \in \mathcal{T},\quad x \in U_x$

$$A \cap (U_x \backslash \{x\}) \neq \emptyset$$

Limit point is also called accumulation point or derived point.

Limit Point of a Set

Example:
Let $X = \{a, b, c, d, e\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$$

be a topology of X. Consider $A = \{a, b, c\} \subset X$. Let us check all the elements of X one by one for limit point of A.
For **a**:
a is not a limit point of A. Since there exists an open set $\{a\}$ containing a such that

$$A \cap (\{a\} \backslash \{a\}) = \emptyset$$

Limit Point of a Set

$$\mathcal{T} = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$$

For **b**:
b is a limit point of $A = \{a, b, c\}$. Since all the open sets containing b intersect A other than b.

$$A \cap (\{b, c, d, e\} \backslash \{b\}) = \{c\} \neq \emptyset$$

$$A \cap (X \backslash \{b\}) = \{a, c\} \neq \emptyset$$

For **c**:
c is not a limit point of $A = \{a, b, c\}$. Since there exist an open set $\{c, d\}$ containing c such that

$$A \cap (\{c, d\} \backslash \{c\}) = \emptyset$$

Limit Point of a Set

$$\mathcal{T} = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$$

For **d**:
d is a limit point of $A = \{a, b, c\}$. Since all the open sets containing d intersect A other than d.

$$A \cap (\{c, d\} \backslash \{d\}) = \{c\} \neq \emptyset$$

$$A \cap (\{a, c, d\} \backslash \{d\}) = \{a, c\} \neq \emptyset$$

$$A \cap (\{b, c, d, e\} \backslash \{d\}) = \{b, c\} \neq \emptyset$$

$$A \cap (X \backslash \{d\}) = A \neq \emptyset$$

Limit Point of a Set

$$T = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$$

For **e**:

e is a limit point of $A = \{a, b, c\}$. Since all the open sets containing e intersect A other than e.

$$A \cap (\{b, c, d, e\} \setminus \{e\}) = \{b, c\} \neq \emptyset$$

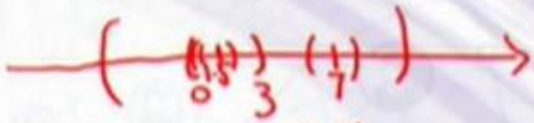
$$A \cap (X \setminus \{e\}) = A \neq \emptyset$$

Limit Point of a Set

Example:

Let $X = \mathbb{R}$ with usual top.

$A = (0, 3)$



7 No, is not a limit point of $A = (0,3)$ — $7 \in (6,8)$, $A \cap [(6,8) \setminus \{7\}] = \emptyset$

1.5 Yes, is a limit point of $A = 0$ — $A \cap [(1.5-\varepsilon, 1.5+\varepsilon) \setminus \{1.5\}] \neq \emptyset$, $\varepsilon \in \mathbb{R}^+$

Limit Point of a Set

3 is a limit point of $A=(0,3)$.

$A \cap [(3-\varepsilon, 3+\varepsilon) \setminus \{3\}] \neq \emptyset$

Example. $X = \mathbb{R}$ with usual top.

$$B = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \dots\} = \{\tfrac{1}{n} \mid n \in \mathbb{N}^*\}$$

0 is a limit point of B $\quad 0 \in (-\varepsilon, \varepsilon) \quad \varepsilon \in \mathbb{R}^+$

VU

Topology

Derived Set: Special Topologies

VU

pic022.mp4

Derived Set: Special Topologies

As we have studied two special topologies on a set X named as "Indiscrete Topology" and "Discrete Topology". Let us figure out the derived set of a subset of X with respect to these special topologies.
Let us first discuss for indiscrete topology.

Derived Set: Special Topologies

Consider an indiscrete space X i.e. X with indiscrete topology

$$\mathcal{T}_{InD} = \{\emptyset, X\}.$$

Note that for any point x in X there is one and only one open set containing x and that is X itself.
So a point $x \in X$ will be limit point of some subset A of X iff

$$A \cap (X \backslash \{x\}) \neq \emptyset.$$

Now if $A = \{y\}$ then any point $x \neq y$ of X satisfies the above said condition. i.e. $\{y\} \cap (X \backslash \{x\}) = \{y\} \neq \emptyset$.
So in this case $A' = A^c$

Derived Set: Special Topologies

Now if A contains two or more points then any point x of X satisfies $A \cap (X \backslash \{x\}) = A \backslash \{x\} \neq \emptyset$.
So in this case $A' = X$
Let us summarize it as our concluding remark.
Remark:
In an indiscrete space X, the derived set A' of any subset A of X is as follows

$$A' = \begin{cases} \emptyset & \text{if } A = \emptyset \\ A^c = X \backslash \{y\} & \text{if } A = \{y\} \\ X & \text{if otherwise} \end{cases}$$

Derived Set: Special Topologies

Now let us discuss for discrete topology.
Consider a discrete space X, i.e. X with discrete topology

$$T_{dis} = P(X).$$

Note that for any point x in X every subset of X containing x is open.
So a point $x \in X$ will be a limit point of some subset A of X iff every subset U of X containing x must intersect with A other than x.
But for every x in a discrete space there is a single ton set containing x i.e. $\{x\}$ which does not intersect with A other than x. So no point of X can be limit point of any subset A of X.

Derived Set: Special Topologies

Following is our concluding remark.

Remark:

In a discrete space X, the derived set A' of any subset A of X is empty.

Topology

Derived Set: Some Results

pic023.mp4

Derived Set: Some Results

In this topic we prove some results related to derived set. Before going towards the theorems let us see when does a point $x \in X$ not belong to derived set of some set A i.e. not be a limit point of A. Recall that $x \in A'$ iff every open set U_x containing x such that $(U_x \setminus \{x\}) \cap A \neq \emptyset$

Derived Set: Some Results

So it means that $x \notin A'$ iff there exist an open set U_x containing x such that $(U_x \setminus \{x\}) \cap A = \emptyset$
or equivalently
iff there exist an open set U_x containing x such that either $U_x \cap A = \emptyset$ or $U_x \cap A = \{x\}$
or equivalently
iff there exist an open set U_x containing x such that

$$U_x \cap A \subset \{x\}$$

Derived Set: Some Results

Theorem: Let $(X, \mathcal{T})$ be a topological space. Let $A, B \subset X$. If $A \subset B$ then $A' \subset B'$.

Proof:

Let $x \in A' \Longrightarrow \forall\ U_x$ (open set containing x)

$$(U_x \backslash \{x\}) \cap A \neq \emptyset$$

Now given $A \subset B$ so

$$\emptyset \neq (U_x \backslash \{x\}) \cap A \supset (U_x \backslash \{x\}) \cap B$$

$\Longrightarrow$

$$x \in B'$$

VUed

Derived Set: Some Results

Theorem: Let $(X, \mathcal{T})$ be a topological space. Let $A, B \subset X$. Then $(A \cup B)' = A' \cup B'$.

Proof:

Let us first show that $A' \cup B' \subset (A \cup B)'$

$$A \subset A \cup B \Longrightarrow A' \subset (A \cup B)'$$
$$B \subset A \cup B \Longrightarrow B' \subset (A \cup B)'$$

So

$$A' \cup B' \subset (A \cup B)'$$

Now we show that $(A \cup B)' \subset A' \cup B'$. For this we take an element from $(A' \cup B')^c$ and show that it belongs to $((A \cup B)')^c$

VU

Derived Set: Some Results

Let $x \notin (A' \cup B')$ this implies that $x \notin A'$ and $x \notin B'$

$\Longrightarrow$

$$\exists\ U_x \quad \text{and} \quad V_x \quad \text{(open sets containing } x\text{)}$$

such that

$$U_x \cap A \subset \{x\} \quad \text{and} \quad V_x \cap B \subset \{x\}$$

Now $U_x \cap V_x$ is an open set containing x and

$$(U_x \cap V_x) \cap (A \cup B) = ((U_x \cap V_x) \cap A) \cup ((U_x \cap V_x) \cap B)$$
$$\subset (U_x \cap A) \cup (V_x \cap B) \subset \{x\} \cup \{x\} = \{x\}$$

$\Longrightarrow$ $x \notin (A \cup B)' \Longrightarrow (A \cup B)' \subset A' \cup B'$

So $(A \cup B)' = A' \cup B'$ is proved.

Topology

Closed Set and Derived Set

ic024.mp4

Closed Set and Derived Set

As we have already studied that a subset A of a topological space X is closed iff A^c is open.
In this topic we characterize the closed set it terms of its limit points. It provides us a criterion to check the closeness of a set.
Theorem: A subset A of a topological space X is closed iff

$$A' \subset A.$$

Closed Set and Derived Set

Example:
Consider

$$X = \{a, b, c, d\}$$

with $\mathcal{T} = \{\emptyset, \{a\}, \{a, c\}, \{a, b, d\}, X\}$
Consider

- $A = \{b, d\}$
- $B = \{d\}$
- $C = \{b, c, d\}$

$A \cap (\overline{\{a, b, d\}} \setminus \{d\}) = \{b\} \neq \emptyset$
$A \cap (X \setminus \{d\}) = \{b\} \neq \emptyset$
$A' = \{b, d\} \subseteq A$

Now A and C are closed since $A' = \{b, d\} = A$ and $C' = \{b, d\} \subset C$ but B is not closed because $B' = \{b\} \not\subset B$.

Closed Set and Derived Set

Example:
Consider

$$X = \{a, b, c, d\}$$

with $\mathcal{T} = \{\emptyset, \{a\}, \{a, c\}, \{a, b, d\}, X\}$
Consider

- $A = \{b, d\}$
- $B = \{d\}$ Not closed.
- $C = \{b, c, d\}$

$B \cap (\{a, b, d\} \setminus \{b\}) = \{d\} \neq b$

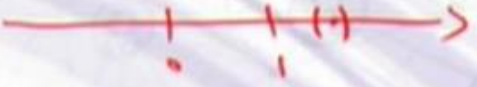
$B' = \{b\} \not\subset B$

Now A and C are closed since $A' = \{b, d\} = A$ and $C' = \{b, d\} \subset C$ but B is not closed because $B' = \{b\} \not\subset B$.

Closed Set and Derived Set

Example:
Consider $\mathbb{R}$ with usual topology. Consider following subsets of $\mathbb{R}$.

- $A = [0, 1)$ $A' = [0, 1] \not\subset A$



Not Closed.

- $B = \{0, 1\}$

$B' = \phi \subset B$

B is closed.

- $C = \{1, 1/2, 1/3, \cdots\}$

No $C' = \{0\} \not\subset C$

C is not closed:

Closed Set and Derived Set

ic025.mp4

Closed Set and Derived Set

Proof: Let A is closed. Now we show that $A' \subset A$.
Let

$$x \notin A \implies x \in A^c$$

$\because$ A is closed. So A^c is open and we have an open set A^c containing x that does not intersect A at all i.e.

$$A^c \cap A = \emptyset \implies (A^c \backslash \{x\}) \cap A = \emptyset$$

So

$$x \notin A' \implies A' \subset A$$

Closed Set and Derived Set

Let us prove the converse.
Let $A' \subset A$. Now we show that A is closed or equivalently A^c is open.
Let

$$x \in A^c \Rightarrow x \notin A \Rightarrow x \notin A' \text{ as given } A' \subset A$$

$\Rightarrow \exists\ U_x$ (open set containing x) s.t. $(U_x \setminus \{x\}) \cap A = \emptyset$
But $x \notin A$ so $U_x \cap A = \emptyset \Longrightarrow U_x \subset A^c$.

So for every x in A^c we have U_x (open set containing x) s.t.

$$U_x \subset A^c.$$

Closed Set and Derived Set

And we can write A^c as union of all these open subsets i.e.

$$A^c = \bigcup_{x \in A^c} U_x$$

This shows that A^c is open and this implies that A is closed.
Proved.

Topology

Criterion for Openness

pic026.mp4

Criterion for Openness

Let $(X, \mathcal{T})$ be a topological space and we have already studied that the elements of $\mathcal{T}$ are called the open sets.
Now the question is "Does there exist any other criterion to check the openness?" the answer is "Yes." Let us state it in following theorem.

Theorem: A subset A of a topological space X is open iff $\forall \quad x \in A \quad \exists\ U_x$ (open set containing x) s.t. $U_x \subset A$.

Criterion for Openness

Example:
Consider

$$X = \{a, b, c, d\}$$

with $\mathcal{T} = \{\emptyset, \{a\}, \{a, c\}, \{b, d\}, \{a, b, d\}, X\}$
Consider

- $A = \{a, b, d\}$
- $B = \{a, b\}$ X
- $C = \{d\}$ X

Now A is open but B and C are not open.

Criterion for Openness

Example:
Consider $\mathbb{R}$ with usual topology.

$A = (0, 1]$ $\quad 1 \in A$ but $\nexists\ U_1$ s.t $U_1 \subseteq A = (0, 1]$

Not an open set

$B = \{1, 4\}$ $\quad 1 \in B$ but $\nexists\ U_1$ s.t $U_1 \subseteq B$

No

Criterion for Openness

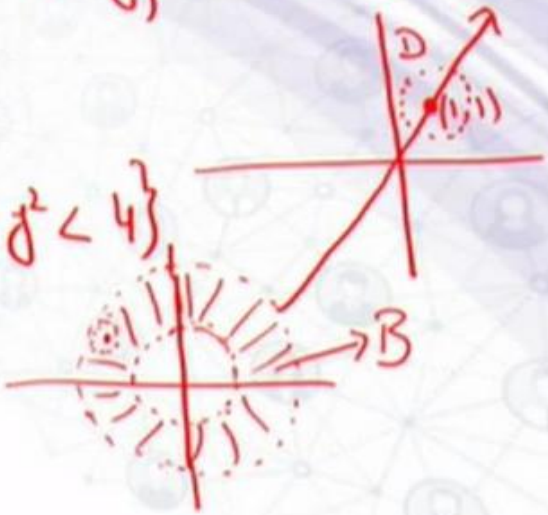
Example:
Consider $\mathbb{R}^2$ with usual topology.

$A = \{(x, y) \in \mathbb{R}^2 \mid x = y\}$ Not open set

$\nexists \; U_{x=(1,1)} \subset A$

$B = \{(x, y) \in \mathbb{R}^2 \mid 1 < x^2 + y^2 < 4\}$

$\cup$



Criterion for Openness

Proof: One way is very obvious. i.e.
Let A is open. It is obvious that $\forall \;\; x \in A \;\; \exists \; U_x$ (open set containing x) s.t. $U_x \subset A$.
$\because$ For every $x \in A$ there exists always the best candidate, that satisfies the condition, is A itself. So
$\forall \;\; x \in A \;\; \exists \; U_x = A$ (open set containing x) s.t. $A \subset A$.
Let us prove the converse.
Let $\forall \;\; x \in A \;\; \exists \; U_x$ (open set containing x) s.t. $U_x \subset A$. So we can write A as union of all U_x i.e.

$$A = \bigcup_{x \in A} U_x$$

shows that A is open. Proved.

Topology

Closure of a Set

pic027.mp4

Closure of a Set

Closure of a Set:
Let $(X, \mathcal{T})$ be a topological space and A be a subset of X. The closure of A, denoted as $\overline{A}$, is the intersection of all closed subsets of X containing A. i.e.

$$\overline{A} = \bigcap_{C \in \mathcal{C}_A} C$$

where $\mathcal{C}_A$ is the class of all closed subsets of X containing A.

Closure of a Set

Example 1:
Let $X = \{a, b, c, d, e\}$ and

$$\mathcal{T} = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$$

be a topology of X. So here we have the closed collection

$$\mathcal{C} = \{X, \{c, d, e\}, \{a, b, e\}, \{e\}, \{a\}, \phi\}$$

Consider $A = \{c\} \subset X$. Then $\mathcal{C}_A = \{X, \{c, d, e\}\}$ is the class of all closed subsets of X containing A. So

$$\overline{A} = \bigcap_{C \in \mathcal{C}_A} C = \{c, d, e\} \bigcap X = \{c, d, e\}$$

Closure of a Set

Example 2:
Let $X = \{a, b, c, d, e\}$ and

$$\mathcal{T} = \{\phi, \{a, b\}, \{c, d\}, \{a, b, c, d\}, \{b, c, d, e\}, X\}$$

So closed collection

$$\mathcal{C} = \{X, \{c, d, e\}, \{a, b, e\}, \{e\}, \{a\}, \phi\}$$

Consider $B = \{a, b\} \subset X$. Then $\mathcal{C}_B = \{X, \{a, b, e\}\}$ is the class of all closed subsets of X containing B. So

$$\overline{B} = \bigcap_{C \in \mathcal{C}_B} C = \{a, b, e\} \bigcap X = \{a, b, e\}$$

Closure of a Set

Remarks:

- $A \subset \overline{A}$

$\bar{A} = \bigcap_{C \in \mathcal{C}_A} C$ $\quad C \subset X$ closed s.t $A \subset C$

$\Rightarrow A \subset \bar{A}$

- $\overline{A}$ is the smallest closed superset of A.

if $\exists\ F \subset X$ closed $A \subset F$

Then $A \subset \bar{A} \subset F$

$A \subset F$ (closed) $\Rightarrow \bar{A} \subset F$ $\quad \bigcap_{C \in \mathcal{C}_A} C$ $\quad \exists\ C = F$

- $\overline{\emptyset} = \emptyset$

$\emptyset \subset \emptyset \Rightarrow \overline{\emptyset} = \emptyset$

- $\overline{X} = X$

$\overline{X} = \bigcap_{C \in \mathcal{C}_X} C$ $\quad \exists$ only one subset of X that is closed and containing X "X"

$X = X$

VU

Topology

Computation of Closure

VU

pic028.mp4

Computation of Closure

For a given topological space X and its subset A, one way to compute the closure of A is to apply the definition in following three steps.

1. List all the closed subset of X.
2. Write $\mathcal{C}_A$, all the closed subset of X containing A.
3. Take intersection of all the elements of $\mathcal{C}_A$.

Computation of Closure

Example:
Let $X = \mathbb{R}^2$ with usual topology and
$A = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 1\}$.
Then the smallest closed superset of A is
$\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$.
So
$\overline{A} = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$

Computation of Closure

The third way to compute $\overline{A}$ is to use the relation between the $\overline{A}$, A and A' given in the following theorem.

Theorem 1:
Let A be a subset of a topological space X. Then

$$\overline{A} = A \cup A'$$

Theorem 2:
Let A be a subset of a topological space X. Then $x \in \overline{A}$ iff for every open set U_x containing x intersects A.

Computation of Closure

Theorem 1:
Let A be a subset of a topological space X. Then

$$\overline{A} = A \cup A'$$

Proof: Let us first prove that $A \cup A' \subset \overline{A}$

Now. $A \subset \overline{A}$ Now Let $x \in A' \Rightarrow \forall\, U_x$ (open set containing x)

$(U_x \setminus \{x\}) \cap A \neq \emptyset$

$\Rightarrow U_x \cap A \neq \emptyset \Rightarrow x \in \overline{A}$ (By Th. 2)

$\Rightarrow A' \subset \overline{A}$,

$\Rightarrow A \cup A' \subset \overline{A}$

Computation of Closure

Now let us prove the converse. $\bar{A} \subset A \cup A'$

Let $x \in \bar{A}$ if $x \in A$ we have done.

$\Rightarrow \bar{A} \subset A \subset A \cup A'$

If $x \notin A$ But $x \in \bar{A}$ By Th (2)

$\forall U_x, \; U_x \cap A \neq \emptyset$

$(U_x \cap A \setminus \{x\}) \neq \emptyset$

$\Rightarrow x \in A'$

$\Rightarrow x \in A \cup A' \Rightarrow \bar{A} \subset A \cup A'$

$\Rightarrow \bar{A} = A \cup A'$

VU

Topology

Properties of Closure of a Set

VU

pic029.mp4

Properties of Closure of a Set

Let us prove some results related to closure of a set.

Proposition:
Let A be a subset of topological space X. Then

1. $\overline{A}$ is closed subset of X.
2. A is closed iff $\overline{A} = A$
3. $\overline{(\overline{A})} = \overline{A}$

Properties of Closure of a Set

Proof: 1) $\bar{A}$ is a closed subset of X.

By def $\bar{A} = \bigcap_{C \in C_A} C$ — C is closed subset of A.

We know that the intersection of closed sets is closed

$\Rightarrow$ $\bar{A}$ is closed. Proved

$\rightarrow$ 2) A is closed iff $\bar{A} = A$

Let A is closed. $\because A \subset \bar{A}$ and $\bar{A}$ is smallest closed superset of A

$\Rightarrow \bar{A} \subset A$

$\bar{A} = A$

If $\bar{A} = A$ Proved.

closed $\overline{(\bar{A})} = \bar{A}$ $\because$ $\bar{A}$ is closed By (2)

$(\bar{A}) = \bar{A}$

3)

Properties of Closure of a Set

Theorem:
Let X be a topological space and $A, B \subset X$. If $A \subset B$ then $\overline{A} \subset \overline{B}$.

Proof:. $A \subset B \Rightarrow A' \subset B'$

$\Rightarrow A \cup A' \subset B \cup B'$

$\overline{A} \subset \overline{B}$

Properties of Closure of a Set

Theorem:
Let X be a topological space and $A, B \subset X$. Then

$$\overline{A \cup B} = \overline{A} \cup \overline{B}$$

Proof: First we prove that $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$

By previous theorem. $A \subset A \cup B \Rightarrow \overline{A} \subset \overline{A \cup B}$

Similarly $\overline{B} \subset \overline{A \cup B}$

$\Rightarrow \overline{A} \cup \overline{B} \subset \overline{A \cup B}$

Now let us prove that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$

we know that $A \subset \overline{A}$, $B \subset \overline{B}$

$\Rightarrow A \cup B \subset \overline{A} \cup \overline{B}$

Properties of Closure of a Set

$\Rightarrow \quad \overline{A \cup B} \subset \overline{\bar{A} \cup \bar{B}} = \bar{A} \cup \bar{B}$ But $\bar{A}, \bar{B}$ both are closed

$\Rightarrow \overline{A \cup B} \subset \bar{A} \cup \bar{B}$ $\Rightarrow$ $\bar{A} \cup \bar{B}$ is closed

$\Rightarrow \quad \overline{A \cup B} = \bar{A} \cup \bar{B}$

But we know that a set is closed iff Set = Closure of itself.

Proved.

Topology

Dense Set

pic030.mp4

Dense Set

Dense Set:
Let A be a subset of a topological space X. Then A is said to be dense in X iff every point $p \in X$ is either point of A or limit point of A.

or in other words

A is dense in X iff $\overline{A} = X$.

Dense Set

Example 1:
Let $X = \{a, b, c, d, e\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{c, d\}, \{a, c, d\}, \{b, c, d, e\}, X\}$$

be a topology of X.
Consider $A = \{a, b, c\}, B = \{b, e\} \subset X$.
Then A is dense in X since

$$\overline{A} = \{a, b, c, d, e\}$$

But $B = \{b, e\}$ is not dense in X since

$$\overline{B} = \{b, e\} \neq X$$

Dense Set

Example 2:
Let $X = \mathbb{R}$ with usual topology and consider $\mathbb{Q}$
Then

$$\overline{\mathbb{Q}} = \mathbb{R}$$

So $\mathbb{Q}$ is dense in $\mathbb{R}$.

Now consider $\mathbb{N}$
Then

$$\overline{\mathbb{N}} = \mathbb{N} \neq \mathbb{R}$$

So $\mathbb{N}$ is not dense in $\mathbb{R}$.

Dense Set

Remarks:

- X is dense in itself.
- A proper closed subset of X can never dense in X.

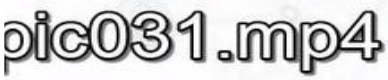
Proof: → $\overline{X} = X$ X is dense in X.

→ $A \subset_{\text{closed}} X$. $\overline{A} = A \neq X$

Proved

Topology

Convergent Sequence



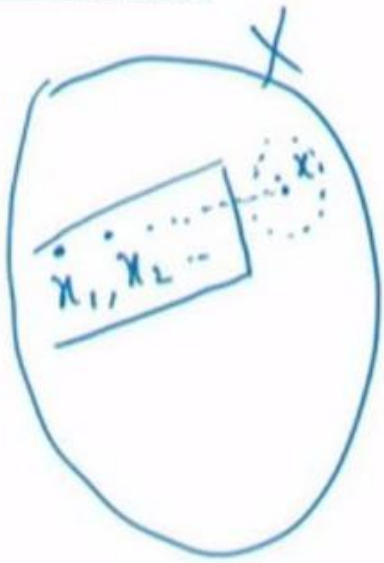
Convergent Sequence

Let $(X, \mathcal{T})$ be a topological space and

$$\{x_1, x_2, x_3, \cdots\} \subset X$$

be a sequence of points in X. The sequence $\{x_n\}$ converges to a point $x \in X$ if for every U_x (open set containing x) in X there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the points $x_n \in U_x$.

Convergent Sequence



Let $(X, \mathcal{T})$ be a topological space and

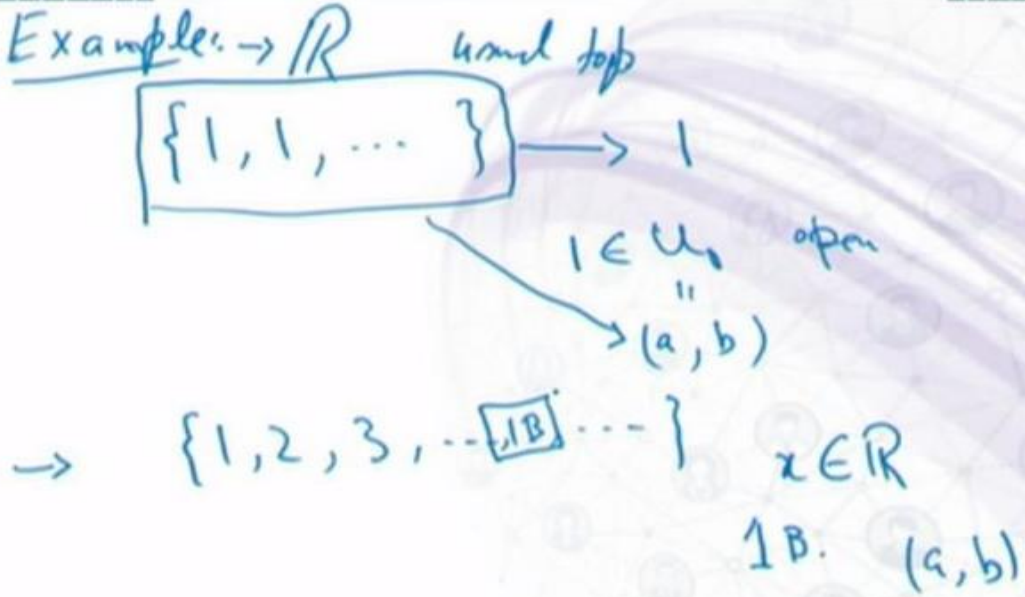
$$\{x_1, x_2, x_3, \cdots\} \subset X$$

be a sequence of points in X. The sequence $\{x_n\}$ converges to a point $x \in X$ if for every U_x (open set containing x) in X there exists $N \in \mathbb{N}$ such that for all $n \geq N$ the points $x_n \in U_x$.

Convergent Sequence

If $\{x_n\}$ converges to x then we say that x is the limit of the sequence $\{x_n\}$ and it is written as $\lim_n x_n = x$ or $x_n \to x$

Convergent Sequence



VU

Topology

Example: Convergent Sequence

VU

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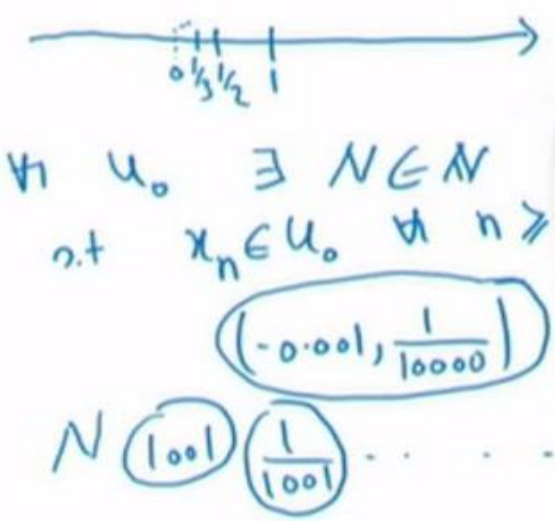
Example: Convergent Sequence



Example:
Consider $\mathbb{R}$ with usual topology and consider the sequence

$$\left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \cdots\right\}$$

$\frac{1}{n} \longrightarrow 0$

Example: Convergent Sequence

Example:
Let X be an indiscrete space and $\{x_n\} \subset X$ be a sequence of points in X. Let y be a point in X. Now note that

- There is only one open set containing y that is X itself.
- X contains all the terms of $\{x_n\}$.

So $x_n \to y$

Remark:
In an indiscrete space X any sequence in X converges to any point in X.

Example: Convergent Sequence

Example:
Let X be a discrete space and $\{x_1, x_2, x_3, \cdots\} \subset X$ be a sequence of points in X. Let y be a point in X. Now note that

- There is a single ton open set containing y that is $\{y\}$.

So x_n can not converge to y unless there exists an $N \in \mathbb{N}$ such that $x_n = y$ for all $n \geq N$.

$\{x_1, x_2, x_3, \cdots, x_N = y, y, \cdots\} \to y$

Example: Convergent Sequence

Example:
Consider $\mathbb{R}$ with usual topology and consider the sequence

$$\{(-1)^n\} = \{-1, 1, -1, \cdots\}$$

$N \in \mathbb{N}$ s.t $x_n \in (a,b)$

$x_n = 1, -1$

Example: Convergent Sequence

Example:
Consider $\mathbb{R}$ with partition topology

$$\mathcal{T}_p = \{[4a-2, 4a+2) | a \in \mathbb{Z}\}$$

and consider the sequence $\{(-1)^n\}$.

$\mathcal{T}_p = \{\ldots [-6,-2), [-2,2), [2,6), [6,10), \ldots\}$

$(-1)^n \longrightarrow x \qquad x \in [-2,2)$

Topology

Interior Point

pic033.mp4

Interior Point

Interior Point:
Let $(X, \mathcal{T})$ be a topological space and A be a subset of X. A point $a \in A$ is an interior point of A iff there exists U_a (open set containing a), such that $U_a \subset A$.

Interior Point

Example 1:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, X\}$$

be a topology of X. Let

$$A = \{a, b, c\}$$

$a \in A$ a is an interior point of A.

$b \in A$ b is " " " " "

$c \in A$ c is not an " " " "

Interior Point

Example 2:
Let $X = R$ with usual topology and $A = [0, 1)$.

$\frac{1}{2} \in (0.3, 0.7) \subset A$
open subset

$\frac{1}{2}$ is an int. pt of A of $\mathbb{R}$

$1 \notin A$
$0 \in A$ $\not\exists\ U_0 \subset A$
$0 \in$

Interior Point

Ex. 3.

$X = \mathbb{R}^2$ with usual top.

$A = \{(x, y) \mid x^2 + y^2 < 3\}$

Topology

Interior of a Set

pic034.mp4

Interior of a Set

Interior of a Set:
Let $(X, \mathcal{T})$ be a topological space and A be a subset of X. Interior of A is the set of all interior points of A.

i.e. $\{a \in A | \exists U_a \in \mathcal{T},\ a \in U_a \subset A\}$

Interior of A is denoted by $int(A)$ or A°.

Interior of a Set

Example 1:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, X\}$$

be a topology of X. Let $A = \{a, b, c\}$

$A^\circ = \{a, b\}$

$B = \{c, d\}$

$B^\circ = \phi$

$C = \{a, b, c, d\} = X$

$X^\circ = X$

Interior of a Set

Example 1:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, X\}$$

be a topology of X. Let $A = \{a, b, c\}$

$A^\circ = \{a, b\}$

$B = \{c, d\}$

$B^\circ = \phi$

$C = \{a, b, c, d\} = X$

$X^\circ = X$

Interior of a Set

Ex: $X = \mathbb{R}^2$

$A = \{(x, y) \mid x^2 + y^2 \leq 1\}$

$A^\circ = \{(x, y) \mid x^2 + y^2 < 1\}$

$B = \{(x, y) \mid x = y\}$

$B^\circ = \emptyset$

VU

Interior of a Set

Remarks:

- $A^\circ \subset A$
- $\emptyset^\circ = \emptyset$
- $X^\circ = X$

VU

Properties of Interior

ic035.mp4

Proposition:

Let A be a subset of topological space X. Then

1. A° is the union of all open subsets of X contained in A.
2. A° is open in X.
3. A° is the largest open subset of X contained in A, i.e. if there exists an open subset B contained in A then $B \subset A^\circ \subset A$
4. A is open iff $A = A^\circ$

Properties of Interior

Proof: ① $A^\circ = \bigcup_i U_i$

Let $\mathcal{U}$ = The class of all open subsets of X contained in A

$U_i \in \mathcal{U}$.

Let $a \in A^\circ$

$\exists\ U_a$ s.t $a \in U_a \subset A$

$\Rightarrow a \in U_a \in \mathcal{U}$.

$\Rightarrow A^\circ \subset \bigcup_i U_i$ — ①

Now Let $x \in \bigcup_i U_i \Rightarrow \exists\ j$ s.t $x \in U_j$ But U_j is open subset of A

$x \in A^\circ$

$\bigcup_i U_i \subset A^\circ$ — ②

$\rightarrow$ ① and ② $\Rightarrow$ $A^\circ = \bigcup_i U_i$

VU

Properties of Interior

② A° is open.

① $A^\circ = \bigcup_i U_i$

open. open

Proved.

③ A° is the largest open subset of X contained in A

i.e if $\exists$ B open s.t $B \subset A$ then $B \subset A^\circ \subset A$

$B \in \mathcal{U}$

$A^\circ = \bigcup_i U_i$ $\exists\ j$ s.t $U_j = B$

$B \subset I$

VU

Properties of Interior

(4) A is open iff $A^\circ = A$

If A is open then By (3)

$A \subset A^\circ \subset A \Rightarrow A^\circ = A$

If $A = A^\circ$ open

$\Rightarrow A$ is open

VU

Topology

Exterior of a Set

VU

ic036.mp4

Exterior of a Set

Exterior Point:
Let $(X, \mathcal{T})$ be a topological space and A be a subset of X. A point $b \in A^c$ is an exterior point of A iff there exists U_b (open set containing b), such that $U_b \subset A^c$.
i.e. Exterior point of A is interior point of A^c.

Exterior of a Set

xterior of a Set:
t $(X, \mathcal{T})$ be a topological ace and A be a subset of . Then exterior of A, noted as $Ext(A)$, is the terior of A^c, i.e.

$$Ext(A) = (A^c)^\circ$$

Exterior of a Set

Example 1:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$$

be a topology of X. Let

$A = \{a, c\}$

$Ext(A) = (A^c)^\circ$

$A^c = \{b, d\}$

$(A^c)^\circ = \{b\}$

$B = \{d\}$

$B^c = \{a, b, c\}$

$Ext(B) = B^c$

VU

Exterior of a Set

Example 2:
Let $X = R$ with usual topology and $A = (-1, 2)$.

- $Ext(A) = (A^c)^\circ = \left((-\infty, -1] \cup [2, \infty)\right)^\circ$
  $= (-\infty, -1) \cup (2, \infty)$

- $Ext(\mathbb{Q}) = (\mathbb{Q}^c)^\circ = (\mathbb{Q}')^\circ = \emptyset$

VU

Topology

Boundary of a Set

pic037.mp4

Boundary of a Set

Boundary Point:
Let $(X, \mathcal{T})$ be a topological space and A be a subset of X. A point $x \in X$ is called a boundary point of A iff for all U_x (open set containing x),

$$U_x \cap A \neq \emptyset$$

and

$$U_x \cap A^c \neq \emptyset.$$

Boundary point is also called frontier point.

Boundary of a Set

Boundary of a Set:
Let (X, T) be a topological space and A be a subset of X. Boundary of A is the set of all boundary points of A.

Boundary of A is denoted as $fr(A)$ or $bd(A)$.

$$bd(A) = (A^{\circ} \cup Ext(A))^{c}$$

VU

Boundary of a Set

Example 1:
Let $X = \{a, b, c, d\}$ and

$$T = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, X\}$$

be a topology of X. Let

$$A = \{a, b, c\}$$

$A^c = \{d\}$

c is a boundary point of A: $\{a,c,d\} \cap A \neq \phi$, $X \cap A \neq \phi$

d is also a boundary point of A: $\{a,c,d\} \cap A \neq \phi$, $X \cap A^c \neq \phi$

"a" $\subset \{a\}$ $\{a\} \cap A^c = \phi$

VU

Boundary of a Set

Example 2:
Consider $X = \mathbb{R}$ with usual topology.

$A = [0, 2)$

"1" No
$(0.9, 1.5) \cap A^c = \emptyset$

"0" Yes.
$c < 0 < d$

$0 \in (c, d)$

$bd(A) = \{0, 2\}$

$B = \mathbb{R}$ $\quad x \in \mathbb{R}$

$bd(\mathbb{R}) = \mathbb{R}$.

$x \in U_x$ open intervals
$U_x \cap \mathbb{R} \neq \emptyset \mid U_x \cap \mathbb{R}' \neq \emptyset$

VU

Boundary of a Set

Example 3:
Consider $X = \mathbb{R}^2$ with usual topology. Let

$$A = \{(x, y) | x^2 + y^2 < 4\}$$

$bd(A) = \{(x, y) \mid x^2 + y^2$

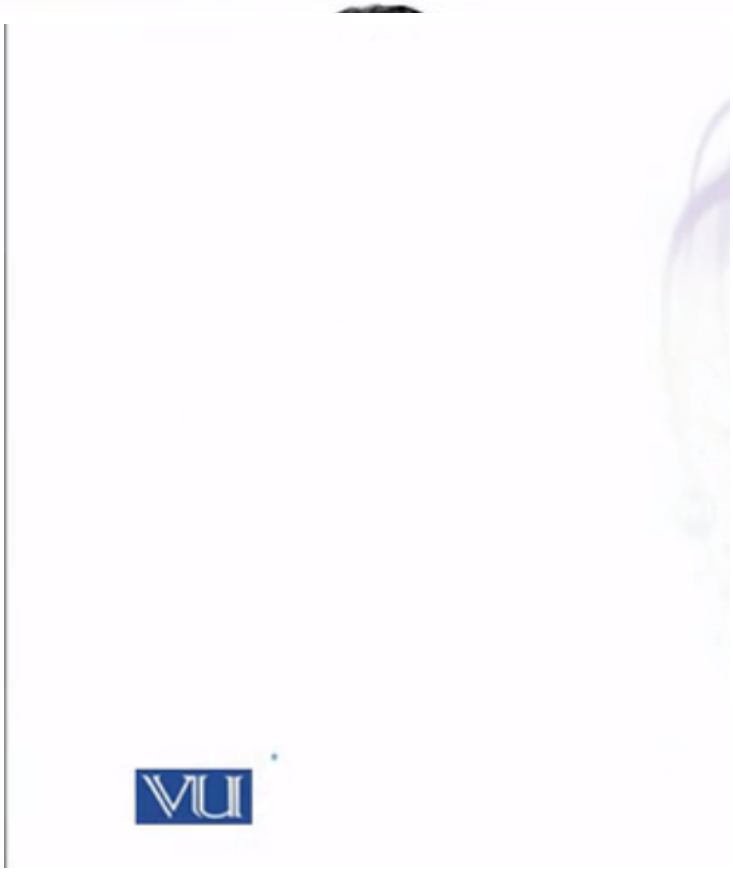
VU

Boundary of a Set

Remarks:

- $bd(A)$ is a closed set.
- Boundary of a set A is empty iff A is both open and closed.

Topology



Partition of a Set

pic038.mp4

Partition of a Set

For a subset A of a topological space X we discussed following three types of sets

- Interior of A.
- Exterior of A.
- Boundary of A.

The set X can be partitioned by these three sets. i.e.

$$X = A^{\circ} \cup Ext(A) \cup bd(A)$$

Partition of a Set

$X = \mathbb{R}$ $\quad A = (0,1]$

$A^{\circ} = (0,1)$

$Ext(A) = (A^{c})^{\circ} = (-\infty, 0) \cup (1, \infty)$

$bd(A) = \{0, 1\}$

$X = \mathbb{R} = \underbrace{(0,1)}_{Int} \cup \underbrace{(-\infty, 0) \cup (1, \infty)}_{Ext} \cup \underbrace{\{0,1\}}_{bd.}$

Partition of a Set

$A \subset X$, $X = \boxed{A^\circ \cup Ext(A) \cup bd(A)}$ — ①

$A^\circ \subset X$, $Ext(A) \subset X$, $bd(A) \subset X$

$\Rightarrow A^\circ \cup Ext(A) \cup bd A \subset X$

Let $x \in X$ $\quad A \subset X$

$\rightarrow x \in A$ or $x \in A^c$

$x \in A^\circ$ $\quad x \notin A^\circ$

$\Rightarrow X \subset$ ① $\quad \Rightarrow \nexists\ U_x$ $\quad x \in U_x \subset A$

$\forall\ U_x$ $\quad U_x \cap A \neq \emptyset$ and $U_x \cap A^c \neq \emptyset$ $\Rightarrow x \in bd A$ Proved.

VU

Partition of a Set

$x \in A^c$

$x \in Ext A = (A^c)^\circ$

$\Rightarrow X \subset Ext A \subset A^\circ \cup Ext(A) \cup bd(A)$

$x \notin (A^c)^\circ$

$\forall\ U_x$

$U_x \not\subset A^c$

$\Rightarrow \forall\ U_x$ opened Contg x

$\boxed{U_x \cap A^c \neq \emptyset}$

$\boxed{U_x \cap A \neq \emptyset}$

$X \subset A^\circ \cup Ext A \cup bd A$

$\rightarrow X = A^\circ \cup Ext A \cup bd A$ $\Rightarrow x \in bd A$

$X \subset bd A$

VU

Topology

Neighborhood System

pic039.mp4

Neighborhood System

Neighborhood of a Point:
Let p be point in a topological space (X, T). A subset N of X is a neighborhood of p iff there exists an open set U_p such that

$$p \in U_p \subset N$$

One may say that N is a neighborhood of p iff p is an interior point of N.

Neighborhood System

Open Neighborhood:
A neighborhood N of a point p is called an open neighborhood iff N is a neighborhood of all of its points. i.e. N is an open set.

Neighborhood System:
The class of all neighborhoods of $p \in X$ is called the neighborhood system of p. It is denoted by $\mathcal{N}_p$.

Neighborhood System

Example:
Let $X = \{a, b, c\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$$

be a topology of X.

$\{a, b\}$ is a neighborhood of b

Is $\{b, c\}$ a neighbourhood of b

No. $\nexists\ U_b$ $\quad b \in U_b \subset \{b, c\}$

Neighborhood System

Example:
Let $X = \mathbb{R}$ with usual topology.

$A = [0, 1)$, $B = (\frac{-1}{2}, \frac{1}{2})$, $C = \{1, 2, 3\}$

- A is not a neighborhood of "0"
  "1" $1 \notin A$
- B is a neighborhood of "0"
- C is not a neighborhood of 1, 2, 3

Neighborhood System

Example:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{a, b, c\}, X\}$$

be a topology of X. Let us list down all the neighborhoods of b.
$\{a, b\}, \{a, b, c\}, \{a, b, d\}, X$
So the neighborhood system of b is

$$\mathcal{N}_b = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, X\}$$

Properties: Neighborhood System

pic040.mp4

Proposition:
Let $\mathcal{N}_p$ is the neighborhood system of p. Then

1. $\mathcal{N}_p$ is non empty.
2. $p \in \bigcap_{N_p \in \mathcal{N}_p} N_p$.
3. $\mathcal{N}_p$ is closed under finite set intersection.
4. Every superset of a member of $\mathcal{N}_p$ belongs to $\mathcal{N}_p$.

Properties: Neighborhood System

Proof: ① Note that X always is the neighborhood of p i.e $X \in \mathcal{N}_p$

$\mathcal{N}_p$ is nonempty.

② $p \in N \quad \forall \; N \in \mathcal{N}_p$

by def. N is neighborhood of p iff $p \in N$ and $\exists \; U_p$ s.t $p \in U_p \subset N$

$\Rightarrow p \in \bigcap_{N \in \mathcal{N}_p} N_p$

Properties: Neighborhood System

③ Let $N, M \in \mathcal{N}_p$

$\Rightarrow \exists \; U_p$ open, V_p open s.t

$p \in U_p \subset N$ and $p \in V_p \subset M$

$\Rightarrow p \in U_p \cap V_p \subset N \cap M$

$N \cap M$ is a neighborhood of p.

$\Rightarrow N \cap M \in \mathcal{N}_p$

Properties: Neighborhood System

(4) $N \in \mathcal{N}_p \Rightarrow \exists\ U_p$ s.t $p \in U_p \subset N$

Let $N \subset M$

$p \in U_p \subset M$

M is a neighborhood of p.

Topology

Subspace Topology

pic041.mp4

Subspace Topology

Subspace Topology:
Let A be a nonempty subset of a topological space $(X, \mathcal{T})$. Then the collection

$$\mathcal{T}_A = \{V = U \cap A | U \in \mathcal{T}\}$$

of subsets of A defines a topology on A.

$\mathcal{T}_A$ is called subspace topology or relative topology on A relative to $(X, \mathcal{T})$.

Subspace Topology

Example 1:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, X\}$$

be a topology of X. Let $A = \{a, b, c\}$

$\mathcal{T}_A = \{\underline{\phi}, \{a\}, \{a, b\}, \{a, c\}, \underline{A}\}$

Subspace Topology

Example 2:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$$

be a topology of X. Let $A = \{a, c\}$

$\mathcal{T}_A = \{\phi, \{a\}, \{a, c\}, A\}$

Subspace Topology

Ex $X = \mathbb{R}$ usual top

$A = \mathbb{Z}$

Let $(-\frac{1}{2}, \frac{1}{2}) \cap \mathbb{Z} = \{0\}$

$\forall\ x \in \mathbb{Z}$ $\exists\ U \in \mathcal{T}_R$.

s.t $\{x\} = U \cap \mathbb{Z}$

$\mathcal{T}_{\mathbb{Z}}$ is discrete top on $\mathbb{Z}$ relative to $\mathcal{T}$

Topology

Proof: Subspace Topology

pic042.mp4

Proof: Subspace Topology

Let A be a nonempty subset of a topological space $(X, \mathcal{T})$. Then show that the collection

$$\mathcal{T}_A = \{V = U \cap A | U \in \mathcal{T}\}$$

of subsets of A defines a topology on A.

Proof: Subspace Topology

Proof: $\mathcal{T}_A = \{V = U \cap A \mid U \in \mathcal{T}\}$

① $\cdot\ \emptyset \in \mathcal{T}_A \quad \because \emptyset \in \mathcal{T} \quad \emptyset = \emptyset \cap A$

$\cdot\ A \in \mathcal{T}_A \quad \because X \in \mathcal{T} \quad A = X \cap A$

② $\to V_i \in \mathcal{T}_A \overset{\cap}{\Longrightarrow} \bigcup_i V_i \in \mathcal{T}_A$

$\Downarrow \exists\ U_i \in \mathcal{T}$ s.t $V_i = U_i \cap A$

Now $\bigcup_i V_i = \bigcup_i (U_i \cap A) = (\bigcup U_i) \cap A$ ($\bigcup U_i \in \mathcal{T}$)

$\Rightarrow \bigcup_i V_i \in \mathcal{T}_A.$

VU

Proof: Subspace Topology

③ Let $V_1, V_2 \in \mathcal{T}_A$

$\Rightarrow \exists\ U_1$ and $U_2 \in \mathcal{T}$

s.t $V_1 = U_1 \cap A$ and $V_2 = U_2 \cap A$

$\Rightarrow V_1 \cap V_2 = (U_1 \cap A) \cap (U_2 \cap A)$

$= (U_1 \cap U_2) \cap A$ ($U_1 \cap U_2 \in \mathcal{T}$)

$\Rightarrow V_1 \cap V_2 \in \mathcal{T}_A$

$\Rightarrow \mathcal{T}_A$ is a top on A.

VU

Topology

Relative Open Sets

opic043.mp4

VU

Relative Open Sets

Relative Open Sets

Let $(A, \mathcal{T}_A)$ be subspace of a topological space $(X, \mathcal{T})$. Then a sub set H of A is open in A or relative open to A iff there exists an open subset G of X such that

$$H = G \cap A$$

One may say that the elements of $\mathcal{T}_A$ are the open subsets of A or called relative open subsets of A.

VU

Relative Open Sets

Example 1:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{a, c, d\}, X\}$$

be a topology of X. Let $A = \{a, b, c\}$

$$\mathcal{T}_A = \{\phi, \{a\}, \{a, b\}, \{a, c\}, A\}$$

$\{a\} = \{a\} \cap A$

$\{b\} \subset A$ No $\quad \{b\} = \square \cap A$

Relative Open Sets

Example 2:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$$

be a topology of X. Let $A = \{a, c\}$

$\{c\} \subset A$

$\{a, c\} \subset A$

$\{c\} = \square \cap A$

$\{a, c\} = \{a, b, c\} \cap A$

Relative Open Sets

Example 3:
Let $X = \mathbb{R}$ with usual topology. Consider $A = (0, 1]$.

$(0, \frac{1}{2}) \subset A$ Yes.
open !!

$(-\frac{1}{2}, \frac{1}{2}) \cap A = (0, \frac{1}{2})$

Is $(0.7, 1]$ open in A? Yes

$(0.7, 1] = (0.7, 3) \cap A$

VU

Relative Open Sets

Example 4:
Let $X = \mathbb{R}$ with usual topology. Consider $A = \mathbb{Z}$.

$\mathcal{T}_{\mathbb{Z}}$ dis.

$\{n\} \subset \mathbb{Z}$
open

VU

Topology

Basis for a Topology

opic044.mp4

Basis for a Topology

Basis for a Topology:

Let $(X, \mathcal{T})$ a topological space. A sub collection $\mathcal{B} \subset \mathcal{T}$ is called a basis for $\mathcal{T}$ if and only if
$\forall \ U \in \mathcal{T}$ and $\forall \ x \in U$
there exists $B \in \mathcal{B}$
s.t. $x \in B \subset U$

Basis for a Topology

Remark:
If $\mathcal{B}$ is a basis for a topology $\mathcal{T}$ on X then every element of $\mathcal{T}$ can be written as union of the elements of $\mathcal{B}$.

Proof:- By def of basis $\forall\ U \in \mathcal{T}$ $\forall\ x \in U$

$\exists\ B_x \in \mathcal{B}$ s.t $x \in B_x \subset U$.

$$U = \bigcup_{x \in U} B_x.$$

VU

Basis for a Topology

Remark:- Let $\mathcal{B}$ be a basis for $\mathcal{T}$ on X

Then every subclass $\mathcal{B}^*$ of open subsets of X of $\mathcal{B}$ is also a basis for $\mathcal{T}$

i.e $\forall\ \mathcal{B} \subset \mathcal{B}^* \subset \mathcal{T}$ $\mathcal{B}^*$ is a basis.

Proof:- $\mathcal{B}$ is a basis so $\forall\ U \in \mathcal{T}$

$\forall\ x \in U$ $\exists\ B \in \mathcal{B}$ s.t $x \in B \subset U$

$\Rightarrow \forall\ U \in \mathcal{T}\ \forall\ x \in U$

$B \in \mathcal{B}^*$ s.t $x \in B \subset$

VU

Topology

Examples: Basis for a Topology

pic045.mp4

VU

Examples: Basis for a Topology

Example 1:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}, X\}$$

be a topology of X. Consider $\mathcal{B} = \{\{a\}, \{b\}, \{a, c, d\}\}$

$\phi = \cup$ {Empty subclass of $\mathcal{B}$}

$\{a\} = \{a\}$ | $\{a, b\} = \{a\} \cup \{b\}$

$\{b\} = \{b\}$ | $\{a, c, d\} = \{a, c, d\}$

$X = \{b\} \cup \{a, c, d\}$.

VU

Examples: Basis for a Topology

Example 2:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, b, c\}, X\}$$

be a topology of X. Consider $\mathcal{B} = \{\{a\}, \{b\}, \{a, b, c\}\}$

$X = \bigcup_{i \in I} B_i$

$U = X \in \mathcal{T}$

$d \in X \not\exists B \in \mathcal{B}$

Examples: Basis for a Topology

Example 3:
Consider $\mathbb{R}$ with usual topology. Consider

$$\mathcal{B} = \{(a, b) \mid a, b \in \mathbb{R}\}$$

$(c,d) \in \mathcal{T}_u \quad (c,d) \in \mathcal{B}$

$(c,d) \cup (e,f) \in \mathcal{T}_u$

$x \in B \subset (c,d) \cup (e,f)$

$B \in \mathcal{B}$

Criteria for a Collection to be Basis

ic046.mp4

Criteria for a Collection to Be Basis

As we discussed for a given topology one can write a collection of open subsets of X that satisfies the condition of basis.

Now the question is "Can a topology be formed by a given collection $\mathcal{B}$ of subsets of X by using set union operation?"

If "Yes" then what should be the conditions on that collection?

Criteria for a Collection to be Basis

One condition is very much clear that is

$$X = \bigcup \{B : B \in \mathcal{B}\}$$

because X is always open in every topology on X.
Is this condition enough for any collection to be a basis?
To understand this let us look an example.

Criteria for a Collection to be Basis

Example:
Let $X = \{a, b, c\}$ and $\mathcal{B} = \{\{a, b\}, \{b, c\}\}$. Let us generate the elements from this collection.

$\tau = \{ \checkmark\emptyset, \{a,b\}, \{b,c\}, \{a,\underset{\checkmark}{b},c\} \}$

$\{a,b\} \cap \{b,c\} = \{b\} \notin \tau$ (τ has X)

τ is not a top. on X.

Criteria for a Collection to be Basis

Theorem:
Let $\mathcal{B}$ be the class of subsets of non-empty set X. Then $\mathcal{B}$ forms a basis for some topology on X if and only if $\mathcal{B}$ satisfies the following conditions:

1. $\forall x \in X \ \exists B \in \mathcal{B}$ s.t $x \in B$
2. For $B_1, B_2 \in \mathcal{B}$ and $\forall \ x \in B_1 \cap B_2 \ \exists \ B_3 \in \mathcal{B}$ s.t $x \in B_3 \subset B_1 \cap B_2$

Criteria for a Collection to be Basis

Example:
Let $X = \{a, b, c\}$ and $\mathcal{B} = \{\{c\}, \{a, c\}, \{b, c\}\}$. It is clear that $\mathcal{B}$ satisfies both conditions stated in the theorem. Let us see generate the elements from this collection.

$$\tau = \{\phi, \{c\}, \{a, c\}, \{b, c\}, \{a, b, c\} = X\}$$

Criteria for a Collection to be Basis

Example:
Let $X = \{a, b, c, d\}$ and $\mathcal{B} = \{\{a, b\}, \{c, d\}, \{d\}\}$. Let us generate the elements from this collection.

$\tau = \{\emptyset, \{d\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c, d\} = X\}$

VU

Topology

Basis for Some Topologies

VU

pic047.mp4

Basis for Some Topologies

In this topic we will discuss the basis of indiscrete topology and discrete topology.

Basis for Indiscrete Topology:

Consider indiscrete space. That is, a nonempty set X with $\mathcal{T} = \{\phi, X\}$.

It is clear that there is only one nonempty open set in this collection that is X so basis of indiscrete space is $\{X\}$.

Basis for Some Topologies

Basis of Discrete Topology:

Example:

Let $X = \{a, b, c\}$, consider the set of all single tons i.e.

$$\mathcal{B} = \{\{a\}, \{b\}, \{c\}\}$$

It is clear that this collection satisfies both conditions of basis. Let us generate the topology from $\mathcal{B}$. Following is the topology

$$\mathcal{T} = \{\{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$$

It is discrete topology

Basis for Some Topologies

Remark:
Consider a discrete space. That is, a nonempty set X with $\mathcal{T} = P(X)$. Then the set of all single tons forms a basis for discrete topology.

i.e $\mathcal{B} = \{\{x\} \mid x \in X\}$

Proof: → $\forall\ x \in X \quad \exists \quad B = \{x\} \in \mathcal{B}$

→ $B_1, B_2 \in \mathcal{B},\ x \in B_1 \cap B_2 \quad \exists \quad B_3 \in \mathcal{B}$.
$x \in B_3 \subset B_1 \cap B_2$ $\quad B_1 \cap B_2 = \emptyset$ □

VU

Topology

Basis for Usual Topology on $\mathbb{R}^n$

VU

pic048.mp4

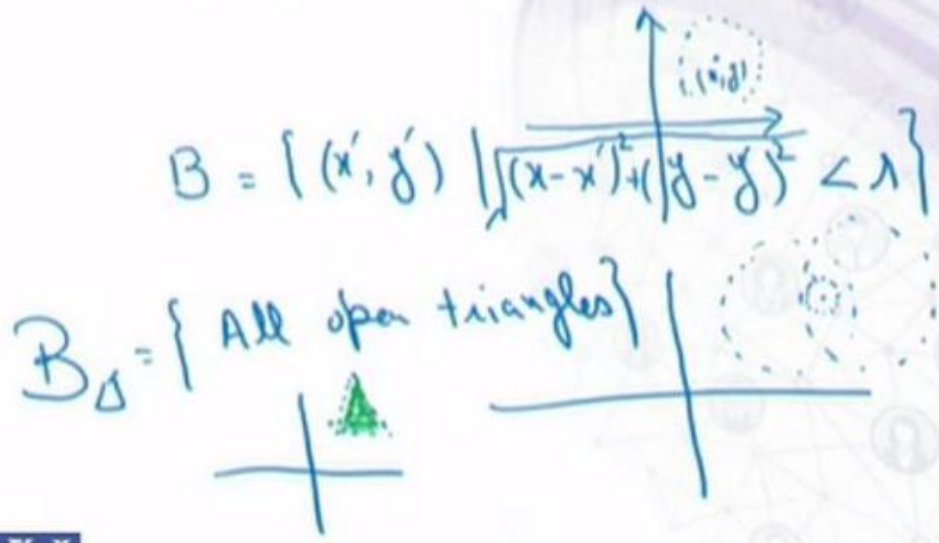
Basis for Usual Topology on $\mathbb{R}^n$

Let us discuss usual basis for the usual topologies $\mathbb{R}^n_u$ on $\mathbb{R}^n$.

Basis for Usual on $\mathbb{R}$:
Consider $\mathbb{R}_u$ i.e. $\mathbb{R}$ with usual topology. The usual basis for $\mathbb{R}_u$ is the set of all open intervals.

Basis for Usual Topology on $\mathbb{R}^n$

Basis for Usual on $\mathbb{R}^2$:
Consider $\mathbb{R}^2_u$ i.e. $\mathbb{R}^2$ with usual topology.
The usual basis for $\mathbb{R}^2_u$ is the set of all open disks.



Basis for Usual Topology on $\mathbb{R}^n$

Basis for Usual on $\mathbb{R}^n$:
Consider $\mathbb{R}^n_u$ i.e. $\mathbb{R}^n$ with usual topology.
The usual basis for $\mathbb{R}^n_u$ is the set of all open n-balls.

Topology

Topologies on Real Line

opic049.mp4

Topologies on Real Line

We have already discussed usual topology on $\mathbb{R}$. Now we see more topologies on $\mathbb{R}$ named as

- Lower Limit Topology, $\mathbb{R}_l$
- Upper Limit Topology, $\mathbb{R}_{up}$
- K-topology, $\mathbb{R}_K$

Topologies on Real Line

Lower Limit Topology:
The lower limit topology on $\mathbb{R}$ is denoted by $\mathbb{R}_l$.

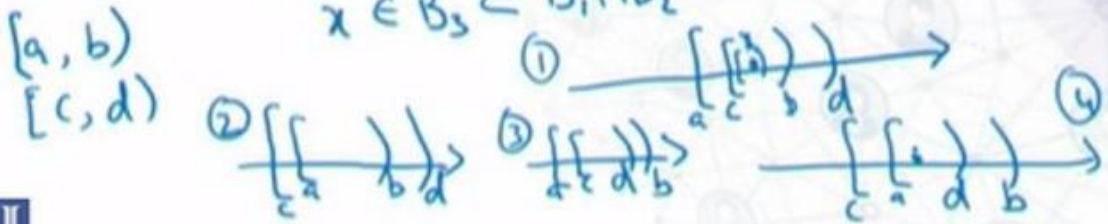
$$\mathcal{B}_l = \{[a, b) | a, b \in \mathbb{R}\}$$

- $x \in \mathbb{R}$ $\quad a \leq x < b$

$x \in [a, b)$

- $B_1, B_2 \in \mathcal{B}_l$ $\quad x \in B_1 \cap B_2$ $\quad \exists\ B_3 \in \mathcal{B}_l$ s.t $x \in B_3 \subset B_1 \cap B_2$

$[a, b)$
$[c, d)$



Topologies on Real Line

K-Topology on $\mathbb{R}$:
Let

$$K = \{1, 1/2, 1/3, \cdots\}$$

be a subset of $\mathbb{R}$ then the K-Topology on real line is denoted as $\mathbb{R}_k$ and its basis are

$$\mathcal{B}_K = \{(a, b) |\ a, b \in \mathbb{R}\} \cup \{(a, b) \backslash K | a, b \in \mathbb{R}\}$$

VU

Topology

Basis for Subspace

VU

pic050.mp4

Basis for Subspace

Recall Subspace Topology.
Topology on a subset A of a topological space $(X, \mathcal{T})$ i.e. the collection of subsets of A defines as

$$\mathcal{T}_A = \{V = U \cap A | U \in \mathcal{T}\}$$

The sub collection

$$\mathcal{B}_A = \{B_A = B \cap A | B \in \mathcal{B}\}$$

forms a basis for $\mathcal{T}_A$, where $\mathcal{B}$ be a basis for $\mathcal{T}$.

VU

Basis for Subspace

Example:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}, X\}$$

be a topology of X. Now $\mathcal{B} = \{\{a\}, \{b\}, \{a, c, d\}\}$ be a basis for $\mathcal{T}$. Let $A = \{a, b, c\}$

$\mathcal{T}_A = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, A\}$

$\mathcal{B}_A = \{\{a\}, \{b\}, \{A, c\}\}$

$\mathcal{T}_A = \{\phi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, A\}$

VU

Basis for Subspace

Theorem:
Let A be a subset of a topological space $(X, \mathcal{T})$. Then the collection

$$\mathcal{B}_A = \{B_A = B \cap A | B \in \mathcal{B}\}$$

of subsets of A forms a basis for $\mathcal{T}_A$, where $\mathcal{B}$ be a basis for $\mathcal{T}$.

Proof: We know that $\mathcal{T}_A = \{V = U \cap A \mid U \in \mathcal{T}\}$

$\rightarrow$ $\forall$ $V \in \mathcal{T}_A$ $\quad V = \bigcup_{i \in I} B_{A_i} = \bigcup_{i \in I} (B \cap A)_i$

$V = U \cap A$ $\quad U \in \mathcal{T}$

But $\quad U = \bigcup_{i \in I} B_i$ $\quad \therefore$ $\mathcal{B}$ is a basis for $\mathcal{T}$

VU

Basis for Subspace

$$V = \left(\bigcup_{i \in I} B_i\right) \cap A$$

$$= \bigcup_{i \in I} (B_i \cap A) \qquad B_i \in \mathcal{B}$$

$\cap$

$\mathcal{B}_A$

So $\mathcal{B}_A$ forms a basis for $\mathcal{T}_A$

VU

Basis: Comparison of Topologies

ic051.mp4

Basis: Comparison of Topologies

Theorem:
Let X be a nonempty set with topologies $\mathcal{T}$ and $\mathcal{T}'$ having basis $\mathcal{B}$ and $\mathcal{B}'$ respectively. Then following are equivalent

1. $\mathcal{T}' \subset \mathcal{T}$
2. For every $B' \in \mathcal{B}'$ and for every $x \in B'$ there exists an element $B \in \mathcal{B}$ such that $x \in B \subset B'$.

Basis: Comparison of Topologies

Proof: 1) $\Rightarrow$ 2)

Let $\mathcal{T}' \subset \mathcal{T}$

Now Let $B' \in \mathcal{B}' \subset \mathcal{T}' \subset \mathcal{T}$

$\Rightarrow$ $B' \in \mathcal{T}$ But $\mathcal{B}$ is a basis for $\mathcal{T}$

So by basis def. $\forall\, x \in B' \;\exists\, B \in \mathcal{B}$ s.t $x \in B \subset B'$

Basis: Comparison of Topologies

2) $\Rightarrow$ 1) $\mathcal{T}' \subset \mathcal{T}$

Let $U' \in \mathcal{T}'$ we have basis for $\mathcal{T}'$ $\mathcal{B}'$

$\forall\, x \in U' \;\exists\; B' \in \mathcal{B}'$ s.t $x \in B' \subset U'$ — (1)

But given that $\forall\, x \in B' \;\exists\; B \in \mathcal{B}$

s.t $x \in B \subset B' \subset U'$

$x \in B \subset U'$

$\Rightarrow$ $U' = \bigcup_{i \in I} B_i$ $\Rightarrow$ $U' \in \mathcal{T}$

$\Rightarrow$ $\mathcal{T}' \subset$

Fineness of $\mathbb{R}_l$

pic052.mp4

The lower limit topology $\mathbb{R}_l$ is finer than the usual topology $\mathbb{R}_u$.
Let us prove this fact.

Fineness of $\mathbb{R}_l$

Proof: Let us recall $\mathcal{B}_l = \{[a,b) \mid a,b \in \mathbb{R}\}$

$\mathcal{B}_u = \{(a,b) \mid a,b \in \mathbb{R}\}$

Now for any $(a,b) \in \mathcal{B}_u$

$\forall \; x \in (a,b)$ $\quad a < x < b$

we can find an element $[x,b) \in \mathcal{B}_l$

$x \in [x,b) \subset (a,b) \Rightarrow R_u \subset R_l$

$R_l \not\subset R_u$ Let $[x,b) \in \mathcal{B}_l$

$\nexists \; (a,b) \in \mathcal{B}_u \;\; x \ll$

s.t $x \in (a,b) \subset [x,b) \Rightarrow R_l \not\subset R_u$.

Topology

Fineness of $\mathbb{R}_K$

pic053.mp4

Fineness of $\mathbb{R}_K$

The K-topology $\mathbb{R}_K$ is finer than the usual topology $\mathbb{R}_u$. Let us prove this fact.

VU

Fineness of $\mathbb{R}_K$

Proof: $\mathcal{B}_u = \{(a,b) \mid a, b \in \mathbb{R}\}$

$\mathcal{B}_K = \{(a,b) \mid a,b \in \mathbb{R}\} \cup \{(a,b) \setminus K \mid a,b \in \mathbb{R}, K = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}\}$

$\mathbb{R}_u \subset \mathbb{R}_K$ $\quad (a,b) \in \mathcal{B}_u$ $\quad x \in (a,b)$ $\exists B \in \mathcal{B}_K$ (= (a,b))

s.t $x \in (a,b) \subseteq B$

$\rightarrow \mathbb{R}_K \not\subset \mathbb{R}_u$ $\quad B = (-1,1) \setminus K \in \mathcal{B}_K$

$0 \in B$ $\quad \nexists$ no open interval (a,b)

s.t $0 \in (a,b) \subset B$.

VU

Topology

Subbasis

opic054.mp4

Subbasis

Let $(X, \mathcal{T})$ a topological space with basis $\mathcal{B}$. A sub collection $\mathcal{S} \subset \mathcal{T}$ is called a subbasis for $\mathcal{T}$ if and only if every element of $\mathcal{B}$ can be written as finite intersection of elements of $\mathcal{S}$.
i.e.

$$\forall \ B \in \mathcal{B}$$

$$\exists \ S_1, S_2, \cdots, S_n \in \mathcal{S}$$

$$B = \bigcap_{i=1}^{n} S_i$$

Subbasis

Remark:
If $\mathcal{S}$ is a subbasis for a topology $\mathcal{T}$ on X then every element of $\mathcal{T}$ can be written as union of finite intersection of elements of $\mathcal{S}$.

Proof: By def of subbasis.

$\forall\ B_j \in \mathcal{B}$ $\quad B_j = \bigcap_{i=1}^{n_j} S_{j_i}$ $\quad S_{j_i} \in \mathcal{S}$

Now by def of basis $\forall\ U \in \mathcal{T}$

$$U = \bigcup_{j \in J} B_j \quad \text{where } B_j \in \mathcal{B}$$

$$= \bigcup_{j \in J} \left(\bigcap_{i=1}^{n_j} S_{j_i} \right)$$

□

Subbasis

Example:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$$

be a topology of X. Now

$$\mathcal{B} = \{\{b\}, \{a, b\}, \{b, c\}, X\}$$

$X = \bigcap$ [Empty class]

The subcollection

$$\mathcal{S} = \{\{a, b\}, \{b, c\}\}$$

Topology

Topology Generated by a Collection

pic055.mp4

Topology Generated by a Collection

As we have already seen that any given class $\mathcal{A}$, of subsets of a nonempty set X, may or may not be a basis for a topology on X.
But $\mathcal{A}$ always able to generate a basis for a topology on X.
Because by definition of subbasis, basis elements can be generated by taking the intersection of finite number of elements of subbasis.

Topology Generated by a Collection

Theorem:
Any class $\mathcal{A}$ of subsets of a nonempty set X is a subbasis for a unique topology $\mathcal{T}$ on X, i.e. finite intersection of members of $\mathcal{A}$ generates a basis $\mathcal{B}$ for a topology $\mathcal{T}$ on X.

Proof: $\mathcal{B} = \{ B \mid B = \bigcap_{i=1}^{n} A_i \quad A_i \in \mathcal{A} \}$

We have to show that the following two cond.

1) $\forall\, x \in X \;\exists\; B \in \mathcal{B}$ s.t $x \in B$ i.e $X = \bigcup B_i$

2) For $B_1, B_2 \in \mathcal{B}$ and $\forall\, x \in B_1 \cap B_2 \;\exists$ $B_3 \in \mathcal{B}$ s.t $x \in B_3 \subset B_1 \cap B_2$.

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Topology Generated by a Collection

Note that $X \in \mathcal{B}$ $\because X = \cap\{\text{Empty Class}\}$

So 1) is proved.

→ Let $B_1, B_2 \in \mathcal{B}$

Then $B_1 = \bigcap_{i=1}^{n} A_i$, $B_2 = \bigcap_{i=1}^{m} A_i'$ $\quad A_i, A_i' \in \mathcal{A}$

Now $B_1 \cap B_2 = \left(\bigcap_{i=1}^{n} A_i\right) \cap \left(\bigcap_{i=1}^{n} A_i'\right)$

$B_3 = B_1 \cap B_2 \in \mathcal{B}$

$\mathcal{A}$ generates a unique topology on X.

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Topology

Example: Topology Generated by a Collection

pic056.mp4

Example: Topology Generated...

Let us see some examples to understand the fact

"Any arbitrary collection of subsets of X always generates a topology on X."

Example: Topology Generated...

Example 1:
Let $X = \{a, b, c, d\}$
and

$$\mathcal{A} = \{\{c\}, \{a, b\}, \{b, c\}\}.$$

Following is the basis $\mathcal{B}$ generated by $\mathcal{A}$.

$$\mathcal{B} = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$$

Now $\mathcal{B}$ will generate $\mathcal{T}$.

$$\mathcal{T} = \{\phi, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$$

Example: Topology Generated...

Example 2:
Let $X = \{a, b, c, d, e\}$
and

$$\mathcal{A} = \{\{c, d\}, \{d, e\}, \{a, b, c\}\}.$$

Following is the basis $\mathcal{B}$ generated by $\mathcal{A}$.

$$\mathcal{B} = \{\phi, \{c\}, \{d\}, \{c, d\}, \{d, e\}, \{a, b, c\}, X\}$$

And following is the topology $\mathcal{T}$ generated by $\mathcal{B}$.

$$\begin{aligned}\mathcal{T} = \{&\phi, \{c\}, \{d\}, \{c, d\}, \{d, e\}, \{a, b, c\},\\ &\{c, d, e\}, \{a, b, c, d\}, X\}\end{aligned}$$

Topology

Example: Topology Generated by a Collection on $\mathbb{R}$

pic057.mp4

Example: Topology Generated...

Example 1:
Consider $\mathbb{R}$
and

$$\mathcal{A} = \{(-\infty, b), (a, \infty) | a, b \in \mathbb{R}\}.$$

Following is the basis $\mathcal{B}$ generated by $\mathcal{A}$.

$$\mathcal{B} = \{\phi, (-\infty, b), (a, \infty), (a, b), \mathbb{R} | a, b \in \mathbb{R}\}$$

Now $\mathcal{B}$ is the basis for usual topology on $\mathbb{R}$.
So $\mathcal{A}$ generates usual topology on $\mathbb{R}$

Example: Topology Generated...

Example 2:
Consider $\mathbb{R}$
and

$$\mathcal{A} = \{[n, n+1] | n \in \mathbb{R}\}.$$

Following is the basis $\mathcal{B}$ generated by $\mathcal{A}$.

$$\mathcal{B} = \{\phi, \{n\}, [n, n+1], \mathbb{R} | n \in \mathbb{R}\}$$

Now $\mathcal{B}$ is the basis for discrete topology on $\mathbb{R}$.
So $\mathcal{A}$ generates discrete topology on $\mathbb{R}$

Topology

ubbasis for Subspace

c058.mp4

Subbasis for Subspace

Recall Subspace Topology.
Topology on a subset A of a topological space $(X, \mathcal{T})$ i.e. the collection of subsets of A defines as

$$\mathcal{T}_A = \{U \cap A | U \in \mathcal{T}\}$$

The collection

$$\mathcal{S}_A = \{S \cap A | S \in \mathcal{S}\}$$

is a subbasis for $\mathcal{T}_A$ on A, where $\mathcal{S}$ be a subbasis for $\mathcal{T}$.

Subbasis for Subspace

Theorem:
Let A be a subset of a topological space $(X, \mathcal{T})$ and $\mathcal{S}$ be a subbasis for $\mathcal{T}$. Then the collection

$$\mathcal{S}_A = \{S \cap A | S \in \mathcal{S}\}$$

is a subbasis for the relative topology $\mathcal{T}_A$ on A.

Proof: Let $V \in \mathcal{T}_A$

$\Rightarrow \exists\ U \in \mathcal{T}$ s.t. $V = U \cap A$

$\because \mathcal{S}$ is a subbasis for $\mathcal{T}$. So

$U = \bigcup_i (S_{i_1} \cap S_{i_2} \cap \cdots \cap S_{i_{n_i}})$ where $S_{ij} \in \mathcal{S}$

Subbasis for Subspace

$$V = \left[\bigcup_i (S_{i_1} \cap S_{i_2} \cap \dots \cap S_{i_{m_i}})\right] \cap A$$

$$= \bigcup_i \left[(S_{i_1} \cap A) \cap (S_{i_2} \cap A) \cap \dots \cap (S_{i_{m_i}} \cap A)\right]$$

S_A

$\Rightarrow S_A$ is a subbasis for $\mathcal{T}_A$. □

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Topology

Smallest Topology
Containing Topologies

VU

pic059.mp4

Smallest Topology...

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a nonempty set X. We have already seen that $\mathcal{T}_1 \cup \mathcal{T}_2$ may or may not be a topology on X. But $\mathcal{T}_1 \cup \mathcal{T}_2$ generates a topology on X.

Moreover this topology is the smallest containing both $\mathcal{T}_1$, $\mathcal{T}_2$.

Smallest Topology...

Example:
Let $X = \{a, b, c, d\}$

$$\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$$

$$\mathcal{T}_2 = \{\emptyset, \{c\}, X\}$$

Now

$$\mathcal{T}_1 \cup \mathcal{T}_2 = \{\emptyset, \{a\}, \{c\}, \{a, b\}, X\}$$

The topology generated by $\mathcal{T}_1 \cup \mathcal{T}_2$ is the following

$\mathcal{T}_3 = \{\emptyset, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, X\}$.

Topology

Largest Topology Contained in Topologies

pic060.mp4

Largest Topology...

et $\mathcal{T}_1$ and $\mathcal{T}_2$ be two
opologies on a nonempty set
.

uestion:
ind the largest topology on
contained in $\mathcal{T}_1$ and $\mathcal{T}_2$?

nswer:
$_1 \cap \mathcal{T}_2$ is the largest
opology on X contained in
$_1$ and $\mathcal{T}_2$.

Largest Topology...

Example:
Let $X = \{a, b, c, d\}$

$$\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$$

$$\mathcal{T}_2 = \{\emptyset, \{b\}, \{a, b\}, X\}$$

The topology

$$\mathcal{T}_1 \cap \mathcal{T}_2 = \{\emptyset, \{a, b\}, X\}$$

is the largest topology contained in $\mathcal{T}_1$ and $\mathcal{T}_2$.

Topology

Local Base

pic061.mp4

Local Base

Local Base at a Point:
Let p be any point of a topological space $(X, \mathcal{T})$. A class $\mathcal{B}_p$ of open sets containing p is called a local base at p if and only if for every $U_p \in \mathcal{T}$ (open set containing p) there exists $B_p \in \mathcal{B}_p$ such that $p \in B_p \subset U_p$.

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Local Base

Remark:
Let $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X and $p \in X$. Then the elements of $\mathcal{B}$ which contain p form a local base at the point p.

Proof: Given $\mathcal{B}$ is a basis for $\mathcal{T}$ on X.
$\forall\ U \in \mathcal{T}\ \ \forall\ x \in U \mid \exists\ B \in \mathcal{B}$ s.t $x \in B \subset U$.

Let $p \in X$. Now consider the following subclass of $\mathcal{B}$.

$$\mathcal{B}_p = \{ B_p \in \mathcal{B} \mid p \in B_p \}$$

Now consider any element $U_p \in \mathcal{T}$ s.t $p \in U_p$
$\exists\ B_p \in \mathcal{B}_p$ s.t $p \in B_p \subset U_p$
$\Rightarrow \mathcal{B}_p$ form a local base at p □

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Local Base

Remark $\mathcal{B} = \bigcup_{p \in X} \mathcal{B}_p$

$\mathcal{B}$ is a basis for $\mathcal{T}$ on X.

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Topology

Examples: Local Base

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pic062.mp4

Examples: Local Base

Example 1:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$$

be a topology of X.

"a" $\mathcal{B}_a = \{\{a\}\}$

"b" $\mathcal{B}_b = \{\{a, b\}\}$

"c" $\mathcal{B}_c = \{\{c, d\}\}$

"d" $\mathcal{B}_d = \{\{a, c, d\}\}$ Not a local base

$\mathcal{B}_a = \{\{a\}, \{a, b\}\}$

$d \in \{a, c, d\}$ & $\{c, d\}$ ✗

$\mathcal{B}_d =$ ·

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Examples: Local Base

Example 1:
Let $X = \{a, b, c, d\}$ and

$$\mathcal{T} = \{\phi, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}, X\}$$

be a topology of X.

"a" $\mathcal{B}_a = \{\{a\}\}$

"b" $\mathcal{B}_b = \{\{a, b\}\}$

"c" $\mathcal{B}_c = \{\{c, d\}\}$

"d" $\mathcal{B}_d = \{c, d\}$

$\mathcal{B}_a = \{\{a\}, \{a, b\}\}$

$d \in \{a, c, d\}$ & $\{c, d\}$ ✗

$\mathcal{B} = \bigcup_{x \in X} \mathcal{B}_x$

$= \{\{a\}, \{a, b\}, \{c, d\}\}$

VU

Examples: Local Base

Example 1:
Let p be a point of a nonempty set X with discrete topology. Then a local base at p is

$$\mathcal{B}_p = \{\{p\}\}$$

Examples: Local Base

Example 3:
Consider $\mathbb{R}$ with usual topology. Let x be any real number then a local base at x is

$$\mathcal{B}_x = \{(x-\epsilon, x+\epsilon) \mid \epsilon \in \mathbb{R}^+\}$$

$x \in (a,b) \subset \mathbb{R}$. $\quad a < x < b$

$a < x-\varepsilon < x < x+\varepsilon < b$

$x \in (x-\varepsilon, x+\varepsilon) \subset (a,b)$

Topology

Continuous Function

pic063.mp4

Continuous Function

In this topic we will discuss about the "Continuous Function" between two topological spaces.

Continuous Function

Def: Let (X, τ_X), (Y, τ_Y) be two top spaces. A function $f: X \to Y$ is cont iff. $\forall\ U \in \tau_Y\ \exists\ V \in \tau_X$ s.t

$$f^{-1}(U) = V$$

i.e the inverse of every open subset of Y is open in X.

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Continuous Function

Remark: If f is cont funct. from (X, τ_X) to (Y, τ_Y). Then we say that

f is a τ_X-τ_Y cont.

f is cont relative to τ_X and τ_Y.

VU

Topology

Examples: Continuous Function

pic064.mp4

VU

Examples: Continuous Function

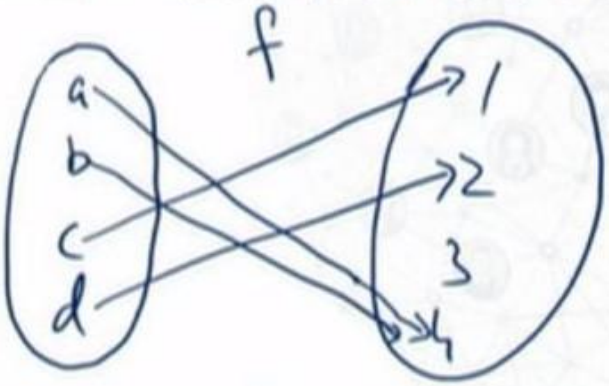
Example 1:

Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$ with topologies

$\rightarrow \mathcal{T}_X = \{\emptyset, \{a, b\}, \{c, d\}, X\}$

$\rightarrow \mathcal{T}_Y = \{\emptyset, \{4\}, \{1, 2\}, \{1, 2, 4\}, Y\}$

$f^{-1}(Y) = X$



$f^{-1}(\emptyset) = \emptyset$

$f^{-1}(\{4\}) = \{a, b\}$

$f^{-1}(\{1, 2\}) = \{c, d\}$

$f^{-1}(\{1, 2, 4\}) = X$

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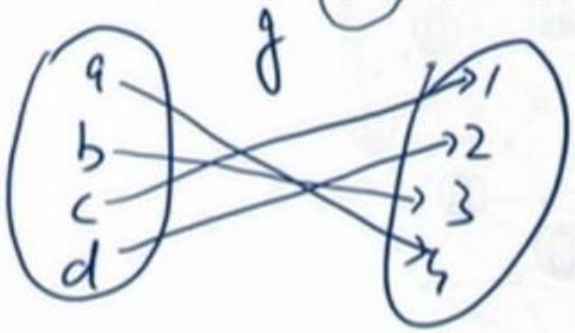
Examples: Continuous Function

Example 2:
Let $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4\}$ with topologies

$$\mathcal{T}_X = \{\emptyset, \{a, b\}, \{c, d\}, X\}$$

$$\mathcal{T}_Y = \{\emptyset, \{4\}, \{1, 2\}, \{1, 2, 4\}, Y\}$$

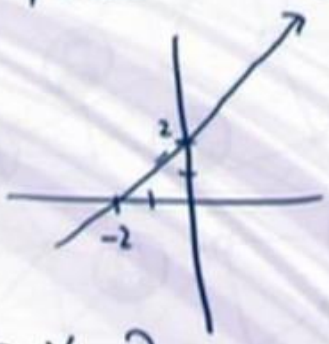


$g^{-1}(\{4\}) = \{a\} \subset X$. Not op

Examples: Continuous Function

Example 3: $X = \mathbb{R}$ with $\mathcal{T}_u$, $Y = \mathbb{R}$ with $\mathcal{T}_u$

$f(x) = x + 2$



$(a, b) \subset Y = \mathbb{R}$

$f^{-1}((a, b)) = (a-2, b-2) \subset X = \mathbb{R}$ op

f is cont function fm X.

Topology

Constant Function

opic065.mp4

Constant Function

Constant Function:
A function whose range is singleton i.e. whose output is the same for every input.

Constant Function

Continuity of Constant Function:
Constant function from any topological space $(X, \mathcal{T}_X)$ to any topological space $(Y, \mathcal{T}_Y)$ is always continuous.

Proof: Let $f: X \to Y$ is constant function
i.e $f(x) = c$ $\forall x \in X$, $c \in Y$
Now take $U \subseteq_{\text{open}} Y$
either $c \in U$ or $c \notin U$
$\Rightarrow f^{-1}(U) = X$ open in X | $f^{-1}(U) = \emptyset$ open in X.
Constant function is cont. fn on $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$

VU

Topology

Functions into Indiscrete Space

VU

pic066.mp4

Functions into Indiscrete Space

In this topic we will discuss the continuity of functions into indiscrete space.

VU

Functions into Indiscrete Space

Theorem:
Let (X, T_X) be any topological space and (Y, T_{Ind}) be an indiscrete space. Then any function $f : X \to Y$ is continuous.

Proof: Given that Y is indiscrete space

i.e $T_Y = \{\phi, Y\}$

Let us consider any function $f: X \to Y$

$\rightarrow f^{-1}(\phi) = \phi$ open in X | $f^{-1}(Y) = X$ open in X.

$\rightarrow f$ is cont.

VU

Topology

Functions from Discrete Space

pic067.mp4

Functions from Discrete Space

Theorem:
Let $(X, \mathcal{T}_{dis})$ be discrete space and $(Y, \mathcal{T}_Y)$ be any topological space. Then any function $f : X \to Y$
is continuous.

Proof: Given that $(X, \mathcal{T}_{dis})$ is a discrete space

$\Rightarrow \quad \mathcal{T}_{dis} = P(X)$

$f : X \to Y$

$U \in \mathcal{T}_Y \qquad f^{-1}(U) = V \subset X$

$\because \quad \mathcal{T}_{dis} = P(X)$

$\Rightarrow \quad V \in \mathcal{T}_{dis} \quad \Rightarrow \quad f^{-1}(U)$ is open in X

$\Rightarrow f$ is cont. function

Topology

Continuity and
Open Sets

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VU

Continuity and Open Sets

Continuity of a function depends on the topology of the domain space and the range space i.e.
"The collection of open subsets of both sets."
The more rich space on domain side provides more room for continuity and more rich space on the range side restricts more a function to be continuous.

VU

Continuity and Open Sets

In our previous three topics we have seen following three things.

1. Constant function $f : X \to Y$ is always continuous.
2. Any function

$$g : X \to \text{Indiscrete Space}$$

is continuous.

3. Any function

$$h : \text{Discrete Space} \to Y$$

is continuous.

Continuity and Open Sets

Let (X, T_X) be strictly finer than (X, T'_X). Then identity function

$$I : (X, T'_X) \to (X, T_X)$$

$I : X \to X$

$I(x) = x \ \forall x$

is not continuous.

Proof: $T'_X \subsetneq T_X \mid \exists\ U \in T_X$ s.t $U \notin T'_X$ ←

Now let $U \in T_X$,

$I(U) = U \notin T'_X$

$\Rightarrow I : (X, T'_X) \longrightarrow (X, T_X)$ is not Cont.

Topology

Examples: Continuity and Open Sets

pic069.mp4

Examples: Continuity...

Let us see some examples to understand the role of open collection in continuity of a function.

Examples: Continuity...

Ex: ① Let $X = \{a, b, c, d\}$

$\mathcal{T}_X = \{\emptyset, \{b\}, X\}$

$Y = \{1, 2, 3, 4\}$, $\mathcal{T}_Y = \{\emptyset, \{2\}, \{1,2\}, \{2,3,4\}, Y\}$

$f: X \longrightarrow Y$ defined as.

$f(a) = 2$, $f(b) = 4$, $f(c) = 2$, $f(d) = 3$

Now f is not cont.

$\because f^{-1}(\{1,2\}) = \{a, c\} \notin \mathcal{T}_X$.

Examples: Continuity...

$\mathcal{T}_X' = \{\emptyset, \{b\}, \{a,c\}, \{a,b,c\}, X\}$

$g: X \longrightarrow Y$ (X with $\mathcal{T}_X'$, Y with $\mathcal{T}_Y$)

$\mathcal{T}_Y = \{\emptyset, \{2\}, \{1,2\}, \{2,3,4\}, Y\}$

$g(a) = 2$, $g(b) = 4$, $g(c) = 2$, $g(d) = 3$

g is cont ($\mathcal{T}_X' - \mathcal{T}_Y$ cont)

Examples: Continuity...

Ex: ② $X = \mathbb{R}_u$, $Y = \mathbb{R}_u$

$f: \mathbb{R}_u \longrightarrow \mathbb{R}_u$

$f(x) = x + 1$

$f(x)$ is cont.

$g: \mathbb{R}_u \longrightarrow \mathbb{R}_l$

$g(x) = x + 1$

g is not cont.

$\because [0,2) \subseteq_{opa} \mathbb{R}_l = Y$

$g^{-1}([0,2)) = [-1,1) \notin \mathbb{R}_u = X.$

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Topology

Continuity and Comparability

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Continuity and Comparability

Proposition 1: Let $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ be a continuous function and $\mathcal{T}'_X$ is finer than $\mathcal{T}_X$. Then $f : (X, \mathcal{T}'_X) \to (Y, \mathcal{T}_Y)$ is continuous.

Proposition 2: Let $g : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is not continuous and $\mathcal{T}''_X$ is weaker than $\mathcal{T}_X$. Then $g : (X, \mathcal{T}''_X) \to (Y, \mathcal{T}_Y)$ is not continuous.

Continuity and Comparability

Proof: (Prop. 1)

$f : (X, \mathcal{T}_X) \xrightarrow[\text{cont.}]{} (Y, \mathcal{T}_Y)$

So $\forall\ U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X$ ✓

Given that $\mathcal{T}_X \subset \mathcal{T}'_X$

$\Rightarrow f^{-1}(U) \in \mathcal{T}'_X$

$\Rightarrow f : (X, \mathcal{T}'_X) \to (Y, \mathcal{T}_Y)$ is cont.

Continuity and Comparability

Proof: (Prop 2) $g:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$

Not Cont.

This means that $\exists\ (U)\in\mathcal{T}_Y$ s.t

$g^{-1}(U)\notin\mathcal{T}_X$

Now $\mathcal{T}_X''\subset\mathcal{T}_X$

$\Rightarrow\quad g^{-1}(U)\notin\mathcal{T}_X''$ ✓

$\Rightarrow$ not cont $\cdot\ g:(X,\mathcal{T}_X'')\to(Y,\mathcal{T}_Y)$ is

Topology

Continuity and Basis

pic071.mp4

Continuity and Basis

Proposition:
A function

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$

is continuous iff the inverse image of each member of a basis $\mathcal{B}_Y$ for $\mathcal{T}_Y$ is an open subset of X.

$\forall B_Y \in \mathcal{B}_Y \Rightarrow f^{-1}(B_Y) \in \mathcal{T}_X$

Continuity and Basis

Proof: Given $\forall B_Y \in \mathcal{B}_Y \Rightarrow f^{-1}(B_Y) \in \mathcal{T}_X$ ←

Let $U \in \mathcal{T}_Y \overset{??}{\Rightarrow} f^{-1}(U) \in \mathcal{T}_X$

Now $\because \mathcal{B}_Y$ is a basis for $\mathcal{T}_Y$ so

$\forall U \in \mathcal{T}_Y \quad U = \bigcup_i B_{Y_i} \quad B_{Y_i} \in \mathcal{B}_Y$

$$f^{-1}(U) = f^{-1}\left(\bigcup_i B_{Y_i}\right) = \bigcup_i \left(f^{-1}(B_{Y_i})\right)$$

Now $f^{-1}(B_{Y_i}) \in \mathcal{T}_X \ \forall i$ and $\mathcal{T}_X$ is a top.

$f^{-1}(U) \in \mathcal{T}_X$

$\Rightarrow$ f is cont

Continuity and Basis

Let $f: X \to Y$ is cont

$\Rightarrow$ $\forall\ U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X$

But we know that $\mathcal{B}_Y \subset \mathcal{T}_Y$

$\Rightarrow$ $\forall\ B_Y \in \mathcal{B}_Y \subset \mathcal{T}_Y$

$\Rightarrow f^{-1}(B_Y) \in \mathcal{T}_X$

□.

VU

Topology

Continuity and Subbasis

VU

opic072.mp4

Continuity and Subbasis

Theorem:
A function

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$

is continuous iff the inverse image of each member of a subbasis $\mathcal{S}_Y$ for $\mathcal{T}_Y$ is an open subset of X.

$\forall\, S_Y \in \mathcal{S}_Y \Rightarrow f^{-1}(S_Y) \in \mathcal{T}_X$.

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Continuity and Subbasis

Proof: Given $\forall\, S_Y \in \mathcal{S}_Y \Rightarrow f^{-1}(S_Y) \in \mathcal{T}_X$

Let $U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X$

$\because \mathcal{S}_Y$ is a subbasis for $\mathcal{T}_Y$

$\Rightarrow \forall\, U \in \mathcal{T}_Y$, $U = \bigcup_i \left(\bigcap_{j=1}^{n_i} S_{Y_{ij}} \right)$ where $S_{Y_{ij}} \in \mathcal{S}_Y$

$$f^{-1}(U) = f^{-1}\left(\bigcup_i \left(\bigcap_{j=1}^{n_i} S_{Y_{ij}} \right) \right)$$

$$= \bigcup_i \left(\bigcap_{j=1}^{n_i} f^{-1}(S_{Y_{ij}}) \right) \in \mathcal{T}_X$$

$\Rightarrow f^{-1}(U)$ is open in $X \Rightarrow f$ is cont.

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Continuity and Subbasis

Let $f: X \to Y$ is cont

$\Rightarrow \forall\ u \in \tau_Y \Rightarrow f^{-1}(u) \in \tau_X$

$S_Y \subset \tau_Y$

$\Rightarrow \forall\ S_Y \in S_Y \quad f^{-1}(S_Y) \in \tau_X.$

Topology

Absolute Value Function

pic073.mp4

Absolute Value Function

In this topic we will discuss the continuity of Absolute Value Function.

$$f: \mathbb{R} \to \mathbb{R}$$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Absolute Value Function

In this topic we will discuss the continuity of Absolute Value Function.

$$f: \mathbb{R} \to \mathbb{R}$$

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Topology

Composition of
Continuous Function

pic074.mp4

Composition of Continuous...

In this topic we will prove the fact
"Composition of two continuous functions is a continuous function."

$f: X \to Y$, $g: Y \to Z$

$h = g \circ f(x) : g(f(x))$

$h: X \to Z$.

Composition of Continuous...

Proposition: Let $f: X \to Y$, $g: Y \to Z$ are two cont. function. Then $g \circ f: X \to Z$ is cont.

Proof: Let $U \in \mathcal{T}_Z$

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) \in \mathcal{T}_X$$

($g^{-1}(U) \in \mathcal{T}_Y$, $f^{-1}(g^{-1}(U)) \in \mathcal{T}_X$)

$\Rightarrow (g \circ f)^{-1}(U)$ is open in X

$\Rightarrow g \circ f$ is a cont fun. $X \to$

VU

Topology

Continuous Function: Another Definition

VU

pic075.mp4

Continuous Function...

Proposition:
A function

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$

is continuous iff the inverse image of every **closed** subset of Y under f is **closed** in X.

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Continuous Function...

Proof:
$\to f$ is a cont function from $X \to Y$
$\forall\ U \in \mathcal{T}_Y \Rightarrow f^{-1}(U) \in \mathcal{T}_X$
Let $A \subseteq_{\text{closed}} Y \Rightarrow A^c \subseteq_{\text{open}} Y \ (A^c \in \mathcal{T}_Y)$
$\because$ f is cont
$\Rightarrow f^{-1}(A^c)$ is open in X.
But $f^{-1}(A^c) = \left(\left(f^{-1}(A)\right)^c\right) \to$ open in X
$\Rightarrow \left(f^{-1}(A)\right)^c$ is closed in X
$\Rightarrow f^{-1}(A)$ is closed in X

VU

Continuous Function...

(Given) We have $\forall\ A \underset{\text{closed}}{\subset} Y \Rightarrow f^{-1}(A) \underset{\text{closed}}{\subset} X$

Let $U \underset{\text{open}}{\subset} Y \Rightarrow U^c \underset{\text{closed}}{\subset} Y$

But given $f^{-1}(U^c) \underset{\text{closed}}{\subset} X$

$f^{-1}(U^c) = (f^{-1}(U))^c$

$f^{-1}(U) \underset{\text{open}}{\subset} X$ or $f^{-1}(U) \in \tau_X$

VU

Topology

Restriction of Function

VU

opic076.mp4

Restriction of Function

Def.
Let

$$f : X \to Y$$

be a function and $A \subset X$.
The restriction of f to A is a function

$$f_A : A \to Y$$

defined as

$$f_A(a) = f(a) \quad \forall \quad a \in A$$

Restriction of Function

Proposition: Let $f: X \to Y$ be a cont function. Then $f_A: A \to Y$ is cont. wrt $A \subset X$.

Proof: For any $B \subset Y$ $f_A^{-1}(B) = A \cap f^{-1}(B)$

Let $U \in \tau_Y$

$\Rightarrow f_A^{-1}(U) = A \cap f^{-1}(U)$

Given f is cont so $f^{-1}(U) \in \tau_X$

$\Rightarrow f_A^{-1}(U) \in \tau_A$. □

Topology

Continuous Function and Arbitrary Closeness

pic077.mp4

Continuous Function...

Recall Closure point.

$$\overline{A} = A \cup A'$$

A point that is Arbitrarily Close to a set A.
Continuity preserve arbitrary closeness.

Continuous Function...

$\bar{A} = A \cup A'$

Theorem: A function

$$f : (X, T_X) \to (Y, T_Y)$$

is continuous iff for every subset A of X

$$f(\bar{A}) = \overline{f(A)}$$

Proof: Let $f: X \to Y$ is cont. Let $A \subset X$

Now we know that $f(A) \subset \overline{f(A)}$

So $A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)})$ ——— (1)

But $f(A)$ is closed in Y

VU

Continuous Function...

$\Rightarrow$ $f^{-1}(\overline{f(A)})$ is closed in X $\because f$ is cont.

Now (1) $\Rightarrow$ $A \subset f^{-1}(\overline{f(A)})$

i.e. $A \subset \bar{A} \subset f^{-1}(\overline{f(A)})$

$\Rightarrow$ $f(\bar{A}) \subset f(f^{-1}(\overline{f(A)})) = \overline{f(A)}$

$\Rightarrow$ $f(\bar{A}) \subset \overline{f(A)}$

VU

Continuous Function...

For the other side.

Let $f(\bar{A}) \subset \overline{f(A)}$

Let $B \subset Y$ (closed)

Let $f^{-1}(B) = A$ we have to show that A is closed in X.

i.e. $\bar{A} = A$

We know that $f(\bar{A}) \subset \overline{f(A)}$

$\Rightarrow f(\overline{f^{-1}(B)}) \subset \overline{f(f^{-1}(B))} = \bar{B} = B$

Continuous Function...

$\Rightarrow f(\bar{A}) \subset B$

$\Rightarrow f^{-1}(f(A)) \subset f^{-1}(B)$

$\Rightarrow \bar{A} \subset f^{-1}(B) = A$

$\bar{A} \subset A$, $A \subset \bar{A}$

$\Rightarrow \bar{A} = A$

$\Rightarrow A$ is closed in X

$\Rightarrow f$ is Con

Continuity at a Point

pic078.mp4

We have discussed continuity in global sense i.e. continuity on a set.
Now let us discuss locally i.e. continuity at a point.

Continuity at a Point

Def: Let $f:(X,\mathcal{T}_X)\to(Y,\mathcal{T}_Y)$ be a function. f is cont. at a point $x\in X$ iff.

for every $U_{f(x)}$ (open set containing $f(x)$)

There exists V_x (open set of X containing x)

s.t. $V_x\subset f^{-1}(U_{f(x)})$ ✓

or $f(V_x)\subset U_{f(x)}$ ✓

VU

Continuity at a Point

For a given $x\in X$

1) Find $f(x)$

2) Collect all the open subsets of Y containing $f(x)$. i.e

$$\mathcal{U}_{f(x)}=\{U_{f(x)} \mid f(x)\in U_{f(x)}\}\subset\mathcal{T}_Y$$

3) For each element of $\mathcal{U}_{f(x)}$ ($\forall_i U_{f(x)}$) check is $\exists$ some V_x s.t $V_x\subset f^{-1}(U_{f(x)})$

VU

Continuity at a Point

If your answer is Yes for all $U_{f(x)}$
Then you can say f is cont. at x.
But if $\exists$ some $U_{f(x)}$ s.t $\nexists V_x$ s.t $V_x \subset f^{-1}(U_{f(x)})$ Then you say that f is not cont at x.

VU

Topology

Examples: Continuity at a Point

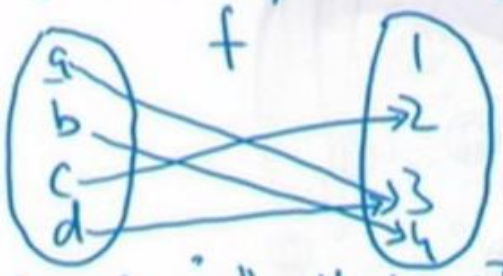
VU

ic079.mp4

Examples: Continuity at a Point

Ex. ① $X=\{a,b,c,d\}$ with $\mathcal{T}_X=\{\phi,\{c\},\{b\},\{a,b\},\{b,c,d\},X\}$

and $Y=\{1,2,3,4\}$ with $\mathcal{T}_Y=\{\phi,\{2\},\{1,2\},\{2,3,4\},Y\}$



f is cont at "a". $\forall\ U_3\in\mathcal{T}_Y\ \exists\ V_a\in\mathcal{T}_X$ s.t $V_a\subset f^{-1}(U_3)$

$U_3=\{\{2,3,4\},Y\}$ $\quad f^{-1}(\{2,3,4\})=X$ open in X

• f is not cont at "c" $U_2=\{\{2\},\{1,2\},\{2,3,4\},Y\}$

$f^{-1}(\{2\})=\{c\}\notin V_c$ s.t $V_c\subset\{c\}$

VU

Topology

Continuity at every Point and Continuous Function

VU

opic080.mp4

Continuity at every Point...

Theorem:
A function $f : X \longrightarrow Y$ is continuous if and only if it is continuous at every point of X.

Continuity at every Point...

Proof: Let $f: X \to Y$ is Cont.

Let $x \in X$

Now take $U_{f(x)} \in \mathcal{T}_Y$ (open subset of Y containing $f(x)$)

$\because$ f is continuous. So

$$f^{-1}(U_{f(x)}) \in \mathcal{T}_X$$

So $V_x = f^{-1}(U_{f(x)}) \subset f^{-1}(U_{f(x)})$

$\rightarrow$ f is continuous at x.

Continuity at every Point...

For the other side.

$f: X \to Y$ is cont at every $x \in X$.

i.e $\forall\ x \in X$ and $\forall\ U_{f(x)} \in \mathcal{T}_Y$

$\exists\ V_x \in \mathcal{T}_X$ s.t $V_x \subset f^{-1}(U_{f(x)})$

$\therefore$ V_x exists for every point of $U_{f(x)}$

So $f^{-1}(U_{f(x)})$ can be written as the union of such V_x.

Continuity at every Point...

i.e $$f^{-1}(U_{f(x)}) = \bigcup \{ V_p \mid p \in f^{-1}(U_{f(x)}) \}$$

$\Rightarrow$ $f^{-1}(U_{f(x)})$ is open in X.

$\Rightarrow$ f is a cont. function from $(X, \mathcal{T}_X)$ to $(Y, \mathcal{T}_Y)$ □

Topology

Topology Induced by Function

pic081.mp4

Topology Induced by Function

Definition:
Let $f : X \longrightarrow Y$ be a function and $\mathcal{T}_Y$ be a topology on Y. The class of subsets of X

$$\mathcal{T}_{X_f} = \{U = f^{-1}(V) | V \in \mathcal{T}_Y\}$$

forms a topology on X. This topology is called "Topology on X induced by f".

Topology Induced by Function

Task:
Show that $\mathcal{T}_{X_f}$ is a topology on X and also show that

$$f : (X, \mathcal{T}_{X_f}) \longrightarrow (Y, \mathcal{T}_Y)$$

is continuous.

Proofs
- $\emptyset \in \mathcal{T}_{X_f}$ $\because$ $\emptyset = f^{-1}(\emptyset)$ $\emptyset \in \mathcal{T}_Y$
- $X \in \mathcal{T}_{X_f}$ $\because$ $X = f^{-1}(Y)$ $Y \in \mathcal{T}_Y$
- Let $U_i \in \mathcal{T}_{X_f}$ $i \in I$
  $U_i = f^{-1}(V_i)$ $(V_i \in \mathcal{T}_Y)$ $\forall\, i \in I$.

VU

Topology Induced by Function

$$U_1 \cap U_2 = f^{-1}(V_1) \cap f^{-1}(V_2)$$
$$= f^{-1}(V_1 \cap V_2)$$

$V_1 \cap V_2 \in \mathcal{T}_Y$

$\Rightarrow U_1 \cap U_2 \in \mathcal{T}_{X_f}$

VU

Topology Induced by Function

$f : (X, \tau_{X_f}) \longrightarrow (Y, \tau_Y)$

f is cont iff $\forall\ V \in \tau_Y$

$\Rightarrow f^{-1}(V) \in \tau_{X_f}$

τ_{X_f} So f is cont.

VU

Topology

Examples: Topology Induced by Function

VU

pic082.mp4

Examples: Topology Induced...

Let us see some examples of topologies induced by functions.

VU

Examples: Topology Induced...

Ex: (1) Let $X = \{a, b, c, d\}$, $Y = \{1, 2, 3\}$

$\rightarrow \tau_Y = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, Y\}$ ←

$\rightarrow f: (X) \rightarrow (Y)$ defined as.

$f(a) = 1$, $f(b) = 2$, $f(c) = 2$, $f(d) = 3$

$\rightarrow \tau_{X_f} = \{\emptyset, \{b, c\}, \{a, b, c\}, \{b, c, d\}, X\}$

VU

Examples: Topology Induced...

Ex:2 $g: X \to Y$ — X, Y are the same as in Ex (1)

$g(a) = 1$, $g(b) = 1$, $g(c) = 3$, $g(d) = 2$

$\rightarrow \overline{\mathcal{S}}_X = \{\emptyset, \{d\}, \{a, b, d\}, \{c, d\}, X\}$

Q: What is $\overline{\mathcal{S}}_X$ on X s.t $f: X \to Y$, $g: X \to Y$ are both cont.:

Examples: Topology Induced...

Ans The top with the subbasis.

$\rightarrow \mathcal{S}_{X_{f,g}} = \{f^{-1}(V) \mid V \in \overline{\mathcal{S}}_Y\} \cup \{g^{-1}(V) \mid V \in \overline{\mathcal{S}}_Y\}$

The top generated by $\mathcal{S}_{X_{f,g}}$ (subbasis) is induced top induced by f and g.

Topology

Topology on Real Line and Linear Function

pic083.mp4

VU

Topology on Real Line and Linear...

In this topic we will show that the topology on real line induced by the set of all linear functions from real line to real line with lower limit topology is discrete.

VU

Topology on Real Line and Linear...

$f: X \to Y$, $g: X \to Y$

$\mathcal{T}_Y$ is a top on Y

Then $\mathcal{T}_{X_{f,g}}$ has basis

$\mathcal{B}_{X_{f,g}} = \{f^{-1}(V) \mid V \in \mathcal{T}_Y\} \cup \{g^{-1}(V) \mid V \in \mathcal{T}_Y\}$

VU

Topology on Real Line and Linear...

Let $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$

$\mathcal{T}_l$ $\mathcal{T}_l$

$f(x) = x - b$, $g(x) = b - x$ $b \in \mathbb{R}$

$A \subset \mathbb{R}$ $A \in \mathcal{T}_l$

$A = [0, 1)$

$f^{-1}([0,1)) = [b, b+1)$, $g^{-1}([0,1)) = [b-1, b)$

VU

Topology on Real Line and Linear...

$$f^{-1}([0,1)) \cap g^{-1}([0,1)) = [p, p+1) \cap (p-1, p]$$

$$= \{p\}$$

$\{p\} \in \mathcal{B}_x$

$\Rightarrow$ The topology on $\mathbb{R}$ induced by set of all Linear functions is Discrete.

VU

Topology

Open Mapping

VU

pic084.mp4

Open Mapping

Def:
A map

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$

is called an open map iff the image of each open subset of X is open in Y.

$\forall\ U \in \mathcal{T}_X\ ,\ f(U) \in \mathcal{T}_Y$

Topology

Open Mapping and Basis

opic085.mp4

Open Mapping and Basis

Proposition:
A map

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$

is an open map iff the image of each element of a basis $\mathcal{B}_X$ for $\mathcal{T}_X$ is an open subset of Y.

VU

Open Mapping and Basis

Proof: Let f is open.

$\Rightarrow$ The image of every element of $\mathcal{T}_X$ is an element of $\mathcal{T}_Y$

Now $\mathcal{B}_X \subset \mathcal{T}_X \Rightarrow f(B_x) \in \mathcal{T}_Y$

$B_x \in \mathcal{B}_X$.

VU

Open Mapping and Basis

For other side.

Let $U \in \mathcal{T}_X$

$\mathcal{B}_X$ is a basis of $\mathcal{T}_X$

So $U = \bigcup_i B_i \quad B_i \in \mathcal{B}_X$

Now $f(U) = f(\bigcup_i B_i) = \bigcup_i f(B_i)$

Now $f(B_i)$ is open in Y

Open Mapping and Basis

$\Rightarrow \bigcup_i f(B_i)$ is open in Y

So $f(U)$ is open in Y

$\Rightarrow f$ is an open map. $\square$.

Closed Mapping

ic086.mp4

Closed Mapping

Def:
A map

$$f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$$

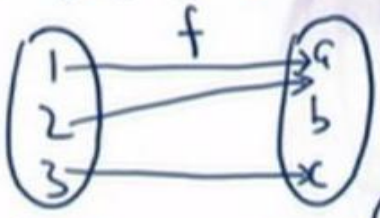
is called a closed map iff the image of each closed subset of X is closed in Y.

$\forall\ U \underset{\text{closed}}{\subset} X \Rightarrow f(U) \underset{\text{closed}}{\subset} Y$

Closed Mapping

Ex: $X = \{1, 2, 3\}$, $\tau_X = \{\phi, \{1\}, \{1, 2\}, X\}$

$Y = \{a, b, c\}$, $\tau_Y = \{\phi, \{a, b\}, \{b\}, Y\}$



$C_X = \{X, \{2, 3\}, \{3\}, \phi\}$

$C_Y = \{Y, \{c\}, \{a, c\}, \phi\}$

$f(\phi) = \phi$ ✓

$f(\{3\}) = \{c\}$ ✓ | $f(\{2, 3\}) = \{a, c\}$ ✓ | $f(X) = \{a, c\}$ ✓

VU

Topology

Projection Mapping

VU

pic087.mp4

Projection Mapping

Projection Map:
Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. The maps

$$\begin{aligned} \pi_1 : (X \times Y, \mathcal{T}_{X\times Y}) &\to (X, \mathcal{T}_X) \\ (x, y) &\mapsto x \end{aligned}$$

$$\begin{aligned} \pi_2 : (X \times Y, \mathcal{T}_{X\times Y}) &\to (Y, \mathcal{T}_Y) \\ (x, y) &\mapsto y \end{aligned}$$

are called projection maps, where $\mathcal{T}_{X\times Y}$ is product topology on $X \times Y$. π_1 is first projection and π_2 is second projection.

VU

Projection Mapping

Fact:
Projection map may or may not be closed.
Justification:
Consider the first projection

$$\pi_1 : (\mathbb{R} \times \mathbb{R}, \mathcal{T}_{\mathbb{R}\times\mathbb{R}}) \to (\mathbb{R}, \mathcal{T}_u)$$

The set $A = \{(x, y) | \ y = \tan x \ , \ \frac{-\pi}{2} < x < \frac{\pi}{2}\}$ is closed in $\mathbb{R} \times \mathbb{R}$ but

$$\pi_1(A) = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \subset \mathbb{R}.$$

Not closed

VU

Continuity and Open Mapping

pic088.mp4

Continuity and Open Mapping

We have studied

- Continuous Mapping
- Open Mapping
- Closed Mapping

Questions:

1. Is a continuous map an open map?
2. Is an open map a continuous map?

Continuity and Open Mapping

Example 1:
Let $X = \{a, b, c, d\}$ with $\mathcal{T}_X = \{\emptyset, \{b, c\}, \{a, b, c\}, X\}$
and
$Y = \{1, 2, 3\}$ with $\mathcal{T}_Y = \{\emptyset, \{1\}, \{1, 2\}, Y\}$.

Consider the following map

$$f : X \to Y$$

Continuous.

$f(\{b,c\}) = \{2\}$ ∩ Not open in Y

defined as $f(x) = 2$ for all $x \in X$.

Continuity and Open Mapping

Example 2:
Let $X = \mathbb{R}$ with usual topology and $Y = \mathbb{R}$ with lower limit topology.

Consider the identity map

$[a, b) \subset \mathbb{R}_l$

$$f : X \to Y$$
$$x \mapsto x$$

$f^{-1}[a,b) = [a,b) \subset \mathbb{R}_u$

f is not cont.

$(a,b) \in \mathcal{B}_u$ $\quad f((a,b)) = (a,b) \in \mathcal{T}_l$

Continuity and Open Mapping

Answers:

1. Continuous map may or may not be open map.
2. Open map may or may not be continuous map.

VU

Topology

Openness and Restriction Mapping

VU

pic089.mp4

Openness and Restriction...

Recall the restriction mapping and the fact about the continuity of restriction mapping.
Now the question is

"Can we prove the same kind of result for restriction of an open mapping?".

Openness and Restriction...

Example: Let $X = \{a, b, c, d\}$ with $\mathcal{T}_X = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, X\}$ and $Y = \{1, 2, 3\}$ with $\mathcal{T}_Y = \{\emptyset, \{1\}, \{1, 2\}, Y\}$.

Consider the map $f : X \to Y$ defined as

$$f(a) = 1,\ f(b) = 2,\ f(c) = 3,\ f(d) = 2$$

Let $A = \{b, c, d\} \subset X$ with relative topology $\mathcal{T}_A = \{\emptyset, \{b\}, \{b, c\}, A\}$. Now the restriction map $f_A : A \to Y$ is defined as

$$f_A(b) = 2,\ f_A(c) = 3,\ f_A(d) = 2$$

Openness and Restriction...

Now f is an open map but f_A is not an open map. Because $\{b\}$ is an open subset of A but

$$f_A(\{b\}) = \{2\} \underset{NotOpen}{\subset} Y$$

Remark:

Restriction of an open map may or may not be an open map.

Topology

Homeomorphism and Homeomorphic Spaces

pic090.mp4

Homeomorphism...

Homeomorphism:
A map $f : X \to Y$ is called a homeomorphism between two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ iff

1. f is bijective.
2. f is continuous.
3. $f^{-1} : Y \to X$ is continuous.

Homeomorphism...

Homeomorphic Spaces:
Two topological spaces $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ are called homeomorphic iff there exists a homeomorphism $f : X \to Y$. We also say such spaces topologically equivalent spaces.

Topology

Examples:
Homeomorphism

MTH634_Topic091.mp4

Examples: Homeomorphism

Example 1: Consider

$X = \{a, b, c, d\}$ with $T_X = \{\emptyset, \{a, b\}, \{c, d\}, X\}$

$Y = \{1, 2, 3, 4\}$ with $T_Y = \{\emptyset, \{2, 3\}, \{1, 4\}, Y\}$

and the map $f : X \to Y$ defined as

$$f(a) = 2,\ f(b) = 3,\ f(c) = 4,\ f(d) = 1$$

$f^{-1}(\{2,3\}) = \{a,b\} \subseteq_{\text{open}} X$ | $f^{-1}(Y) = X \subseteq_{\text{open}} X.$

$f^{-1}(\{1,4\}) = \{c,d\} \subseteq_{\text{open}} X$

Examples: Homeomorphism

Example 1: Consider
$X = \{a, b, c, d\}$ with $T_X = \{\emptyset, \{a, b\}, \{c, d\}, X\}$
$Y = \{1, 2, 3, 4\}$ with $T_Y = \{\emptyset, \{2, 3\}, \{1, 4\}, Y\}$
and the map $f : X \to Y$ defined as

$$f(a) = 2,\ f(b) = 3,\ f(c) = 4,\ f(d) = 1$$

Now $f^{-1} : Y \to X$

$f^{-1}(1) = d$, $f^{-1}(2) = a$, $f^{-1}(3) = b$, $f^{-1}(4) = c$

f^{-1} is cont.

bij. + cont + f^{-1}: cont.

Examples: Homeomorphism

Example 2: Consider
$X = [0, 1]$, $Y = [0, 2]$ as subspaces of $(\mathbb{R}, T_u)$
and the map $f : X \to Y$ defined as

$$f(x) = 2x \quad \forall x \in X$$

- f is bijection.
- f is cont.
- $f^{-1} : Y \to X$, $f^{-1}(y) = \frac{y}{2}$

$X = [0, 1]$ ⟺ $Y = [0, 2]$

So f^{-1} is cont.