MTH 404: Measure and Integration

Quiz 1 (Feb 18, 2016)

Maximum Duration: 55 minutes

Note: Each problem carry 10 points. Clearly state the results used.

1. Let $\mathcal{A} = \{A \subseteq \mathbb{R} \mid \text{either } A \text{ or } A^c \text{ is finite}\}$. Determine whether
   (a) $\mathcal{A}$ is an algebra of sets.
   (b) $\mathcal{A}$ is a σ-algebra of sets.

2. Let E be a measurable subset of $\mathbb{R}$ with finite measure and $c > 0$ be a fixed real number. Let (E_n) is a sequence of measurable sets in E with $m(E_n) > c$, for all $n \geq 1$. Let $F = \limsup E_n$.

   (a) Prove that F is a measurable set.
   (b) Prove that $m(F) \geq c$.
   (c) Give an example to show that conclusion (b) can fail if $m(E) = \infty$.

3. Let $(f_n) : \mathbb{R} \to \mathbb{R}$ be a sequence of non-negative integrable functions converging a.e. to an integrable function f. Let $\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$. Prove that for any measurable subset E of $\mathbb{R}$,
$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

4. Let $f : [1, \infty) \to \mathbb{R}$ be a non-negative measurable function defined as $f(x) = 1/x$. Is the function f Lebesgue integrable over $[1, \infty)$? Justify your answer.
