5- A theorem was from
Monotonic Function? 5 marks
6-evaluate lim->0 ln sinx/ln x??
7- and Rest two was from the
last 5th chp of about integration
on closed interval.My Yesterday
paper;

Mth621 👇

Mcqs were based on concepts
and 6,7 was from mcqs file of
quizz.
1-Define Removeavle
continuity? 2 marks
2-Deifine chain rule for
composition of functions? 2
marks
3-Define Second mean theorem
of Integration?
4- Find uniform continuity of 2x?
5- A theorem was from
Monotonic Function? 5 marks
6-evaluate lim->0 ln sinx/ln x??
7- and Rest two was from the

Removable discontinuity: Let f be defined on a deleted neighborhood of x_0 and discontinuous (perhaps even undefined) at x_0. We say that f has a removable discontinuity at x_0 if $\lim_{x\to x_0} f(x)$ exists.

In this case, the function

$$g(x) = \begin{cases} f(x) & \text{if } x \in D_f \text{ and } x \neq x_0, \\ \lim_{x\to x_0} f(x) & \text{if } x = x_0, \end{cases}$$

is continuous at x_0.

$$= f(a)\int_a g(x)\,dx + f(b)\Big(\int_a g(x)\,dx - \int_c^a g(x)\,dx\Big)$$

$$= f(a)\int_a^c g(x)\,dx + f(b)\int_c^b g(x)\,dx.$$

5.6 Integration by Substitution

Theorem: Suppose that the transformation $x = \phi(t)$ maps the interval $c \le t \le d$ into the interval $a \le x \le b$, with $\phi(c) = \alpha$ and $\phi(d) = \beta$, and let f be continuous on $[a, b]$.

Let ϕ' be integrable on $[c, d]$. Then

$$\int_\alpha^\beta f(x)\,dx = \int_c^d f(\phi(t))\phi'(t)\,dt. \tag{5.45}$$

Proof: Both integrals in (5.45) exist: the one on the left by the fact that if f is continuous on $[a, b]$, then f is integrable on $[a, b]$, the one on the right by the continuity of $f(\phi(t))$.

The function

$$F(x) = \int_a^x f(y)\,dy$$

is an antiderivative of f on $[a, b]$ and, therefore, also on the closed interval with endpoints α and β.

Hence, by fundamental theorem of calculus,

$$\int_\alpha^\beta f(x)\,dx = F(\beta) - F(\alpha). \tag{5.46}$$

By the chain rule, the function

$$G(t) = F(\phi(t))$$

is an antiderivative of $f(\phi(t))\phi'(t)$ on $[c, d]$. Therefore, we have

$$\int_c^d f(\phi(t))\phi'(t)\,dt = G(d) - G(c) = F(\phi(d)) - F(\phi(c)) = F(\beta) - F(\alpha).$$

Comparing this with (5.46) yields (5.45).

Example: Evaluate the integral

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1 - 2x^2)(1 - x^2)^{-1/2} dx.$$

Solution: We let

$$f(x) = (1 - 2x^2)(1 - x^2)^{-1/2}, \quad -1/\sqrt{2} \le x \le 1/\sqrt{2},$$

and

$$x = \phi(t) = \sin t, \quad -\pi/4 \le t \le \pi/4.$$

Then $\phi'(t) = \cos t$ and

$$I = \int_{-1/\sqrt{2}}^{1/\sqrt{2}} f(x)\,dx = \int_{-\pi/4}^{\pi/4} f(\sin t)\cos t\,dt = \int_{-\pi/4}^{\pi/4} (1 - 2\sin^2 t)(1 - \sin^2 t)^{-1/2} \cos t\,dt. \tag{5.47}$$

4.7 Generalized Mean Value Theorem

Theorem: If f and g are continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then

$$[g(b) - g(a)]f'(c) = [f(b) - f(a)]g'(c)$$

for some c in (a, b).

Proof: The function

$$h(x) = [g(b) - g(a)]f(x) - [f(b) - f(a)]g(x)$$

is continuous on $[a, b]$ and differentiable on (a, b).

Furthermore

$$h(a) = h(b) = g(b)f(a) - f(b)g(a).$$

Therefore, Rolle's theorem implies that $h'(c) = 0$ for some c in (a, b).

$$\text{Since} \quad h'(c) = [g(b) - g(a)]f'(c) - [f(b) - f(a)]g'(c).$$

Example: Check the uniform continuity of the function

$$f(x) = 2x.$$

Solution: For the function $f(x)$, we have

$$|f(x) - f(x')| = 2|x - x'| < \varepsilon \quad \text{if} \quad |x - x'| < \varepsilon/2.$$

The function $f(x)$ is uniformly continuous on $(-\infty, \infty)$,

Remark: A function f is *not* uniformly continuous on S if there is an $\varepsilon_0 > 0$ such that if δ is any positive number, there are points x and x' in S such that

$$|x - x'| < \delta$$

but

$$|f(x) - f(x')| \geq \varepsilon_0.$$

Theorem: Suppose that f is monotonic on (a, b) and define

$$\alpha = \inf_{a<x<b} f(x) \quad \text{and} \quad \beta = \sup_{a<x<b} f(x).$$

1. If f is nondecreasing, then $\lim_{x\to a_+} f(x) = \alpha$ and $\lim_{x\to b_-} f(x) = \beta$.

2. If f is nonincreasing, then $\lim_{x\to a_+} f(x) = \beta$ and $\lim_{x\to b_-} f(x) = \alpha$. (Here $a+ = -\infty$ if $a = -\infty$ and $b- = \infty$ if $b = \infty$.)

3. If $a < x_0 < b$, then $\lim_{x\to x_{0+}} f(x)$ and $\lim_{x\to x_{0-}} f(x)$ exist and are finite; moreover,

$$\lim_{x\to x_{0-}} f(x) \le f(x_0) \le \lim_{x\to x_{0+}} f(x)$$

if f is nondecreasing, and

$$\lim_{x\to x_{0-}} f(x) \ge f(x_0) \ge \lim_{x\to x_{0+}} f(x)$$

if f is nonincreasing.

Proof: We first show that $\lim_{x\to a_+} f(x) = \alpha$.

If $M > \alpha$, there is an x_0 in (a, b) such that $f(x_0) < M$. Since f is nondecreasing, $f(x) < M$ if $a < x < x_0$. Therefore, if $\alpha = -\infty$, then $\lim_{x\to a_+} f(x) = -\infty$.

If $\alpha > -\infty$, let $M = \alpha + \varepsilon$, where $\varepsilon > 0$.

Then $\alpha \le f(x) < \alpha + \varepsilon$, so

$$|f(x) - \alpha| < \varepsilon \quad \text{if} \quad a < x < x_0. \tag{3.14}$$

If $a = -\infty$, this implies that $f(-\infty) = \alpha$. If $a > -\infty$, let $\delta = x_0 - a$. Then (3.14) is equivalent to

$$|f(x) - \alpha| < \varepsilon \quad \text{if} \quad a < x < a + \delta,$$

which implies that $f(a+) = \alpha$.

We now show that $\lim_{x\to b_-} f(x) = \beta$.

If $M < \beta$, there is an x_0 in (a, b) such that $f(x_0) > M$.

Since f is nondecreasing, $f(x) > M$ if $x_0 < x < b$. Therefore, if $\beta = \infty$, then $\lim_{x\to b_-} f(x) = \infty$.

If $\beta < \infty$, let $M = \beta - \varepsilon$, where $\varepsilon > 0$. Then $\beta - \varepsilon < f(x) \le \beta$, so

$$|f(x) - \beta| < \varepsilon \quad \text{if} \quad x_0 < x < b. \tag{3.15}$$

If $b = \infty$, this implies that $f(\infty) = \beta$. If $b < \infty$, let $\delta = b - x_0$.

Then (3.15) is equivalent to

$$|f(x) - \beta| < \varepsilon \quad \text{if} \quad b - \delta < x < b,$$

which implies that $f(b-) = \beta$.