Dynamics MTH404
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Lecture 1-1

Introduction to Mechanics- Dynamics

Mechanics: The study of how rigid bodies react to forces acting on them. The study of bodies in equilibrium is called statics. Newton's first Law: $F_R=0$. The study of object in motion is called Dynamics. It is dived into two branches

Kinematics concerned with the geometrical aspect of motion, s, v, a, & time.

Kinematics 1- how for you went (displacement) 2- How fast you went (speed) 3- How quickly you speed up or down (acceleration).

Scalar Quantity: - A quantity that can be specified by its magnitude & has no direction. For example time, distance & speed.

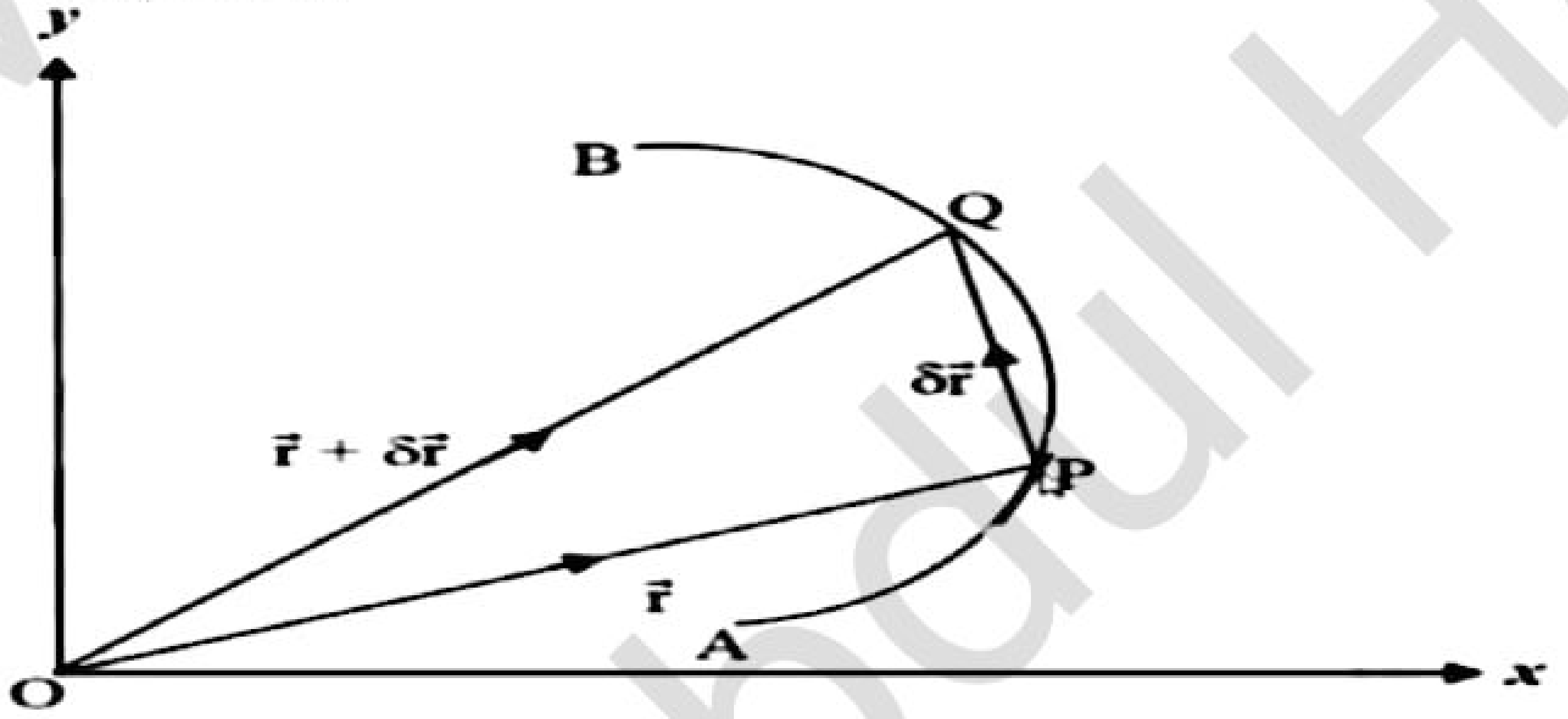
Vector Quantity: - A quantity that has both magnitude & direction for example displacement, velocity & acceleration.

Kinetics concerned with how forces causing the motion. Kinetics Example includes Pressure, Torque, Impulse, Movements & power.

Lecture 2-1

Kinematics

Position Vector: -The Position of particle can be specified by a vector r whose initial point is at the origin of some fixed coordinate system & the terminal point is at the particle. This vector is called Position Vector. If the particle is moving, the vector r changes with time, i.e. Position Vector is a function of time. The curve traced by moving particle is called **the trajectory or path** of particle.



$$\vec{r} = \vec{r}(t)$$

$$x_i + y_j + z_k = x(t)\hat{i} + y(t)\,j + z(t)k$$

$$x = x(t),\ y = y(t),\ z = z(t)$$

These are called parametric equation of path.

Velocity

$$Average\ rate\ of\ change\ of\ displacement = \vec{V} = \frac{dr}{dt} = \frac{\partial r}{\partial t}$$

$$Ins\tan\tan eous\ Velocity = \lim_{\partial t \to 0} \frac{\partial r}{\partial t} = \frac{dr}{dt}$$

Acceleration

$$Average\ rate\ of\ change\ of\ Velocity = \vec{a} = \frac{d\vec{V}}{dt} = \frac{\partial \vec{V}}{\partial t}$$

$$Ins\tan\tan eous\ Velocity = \lim_{\partial t \to 0} \frac{\partial \vec{V}}{\partial t} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(\vec{V}) = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d^2\vec{r}}{dt^2}$$

$$In\ Cartesian\ Coordinate\ system\ if\ \ \vec{r} = x_i + y_j$$

$$Velocity = \frac{d\vec{r}}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j$$

$$Acceleration = \frac{d^2x}{dt^2}i + \frac{d^2y}{dt^2}j$$

Example: -If a car goes from 0 to 100 km/hr in only 3 seconds then what is the acceleration.

$$velocity = 100km/hr = \frac{100 \times 1000}{3600} = 27.8m/s$$

$$\vec{a} = \frac{d\vec{V}}{dt} = \frac{Change\ in\ velocity}{Change\ in\ time} = \frac{27.8}{3} = 9.26m/\sec$$

Lecture 2-2

Examples of velocity & Acceleration

Example 1: - A particle is moving in such a way that its position at any time t is specified by $r = (t^3 + t^2)i + (\cos t + \sin^2 t)\,j + (e^t + \log t)k$ **Find its velocity & Acceleration.**

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$$r=(t^3+t^2)i+(\cos t+\sin^2 t)\,j+(e^t+\log t)k$$

$$velocity=\frac{dr}{dt}=\frac{d}{dt}\Big((t^3+t^2)i+(\cos t+\sin^2 t)\,j+(e^t+\log t)k\Big)$$

$$\frac{dr}{dt}=\left((3t^2+2t)\hat{i}+(-\sin t+2\sin t\cos t)\,j+(e^t+\frac{1}{t})k\right)$$

$$a=\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{d^2r}{dt^2}=\frac{d}{dt}\left((3t^2+2t)\hat{i}+(\sin 2t-\sin t)\,j+(e^t+\frac{1}{t})k\right)$$

$$a=\frac{d^2r}{dt^2}=(6t+2)\hat{i}+(2\cos 2t-\cos t)\widehat{j+}(e^t-\frac{1}{t^2})\widehat{k}$$

Example 2: - If a particle P start from position O at time t=0. We have to find velocity & acceleration if its position vector is given by $\vec{r}=at^2\hat{i}+4at\,j$.

$$\vec{r}=at^2\hat{i}+4at\,j$$

$$velocity=\frac{dr}{dt}=\frac{d}{dt}\left(at^2\hat{i}+4at\,j\right)=2at\hat{i}+4a\,j$$

$$a=\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{d^2r}{dt^2}=\frac{d}{dt}\left(2at\hat{i}+4a\,j\right)=2a\hat{i}$$

Lecture 2-3

Example 3: -If the position of particle moving along the ellipse is given by $\vec{r}=a\cos t\hat{i}+b\sin t\,j \quad a>b$. Find the position of particle where velocity has maximum & minimum magnitude.

$\vec{r} = a\cos t\hat{i} + b\sin t\, j \quad a > b$

$velocity = \vec{V} = \frac{dr}{dt} = \frac{d}{dt}\left(a\cos t\hat{i} + b\sin t\, j\right) = -a\sin t\hat{i} + b\cos t\, j$

$Magnitude\ of\ V = |V| = x\hat{i} + y\, j = \sqrt{x^2 + y^2} = \sqrt{a^2\sin^2 t + b^2\cos^2 t} = \sqrt{a^2\sin^2 t + b^2(1-\sin^2 t)}$

$|V| = \sqrt{a^2\sin^2 t + b^2 - b^2\sin^2 t} = \sqrt{\sin^2 t\left(a^2 - b^2\right) + b^2}$

$V\ will\ \max imum\ when\ \sin^2 t\ is\ \max imum$

$As\ \sin^2 t\ is\ \max imum\ at\ \ \text{-1 } to \text{ 1 } i.e\ 270\ \&\ 90\ \deg rees$

$\vec{r} = a\cos 90\hat{i} + b\sin 90\widehat{j = bj}$

$\vec{r} = a\cos 270\hat{i} + b\sin 270\widehat{j = -bj}$

$when \sin^2 t = 0\, V\ is\ \min imum$

$\sin t = 0\ t = 0{,}180$

$\vec{r} = a\cos 0\hat{i} + b\sin 0\widehat{j} = a\hat{i}$

$\vec{r} = a\cos 180\hat{i} + b\sin 180\widehat{j} = -a\hat{i}$

$so\ Velocity\ will\ \max imum\ at\ \pm bj\ \&\ \min imum\ at\ \pm aj$

Lecture 3-1

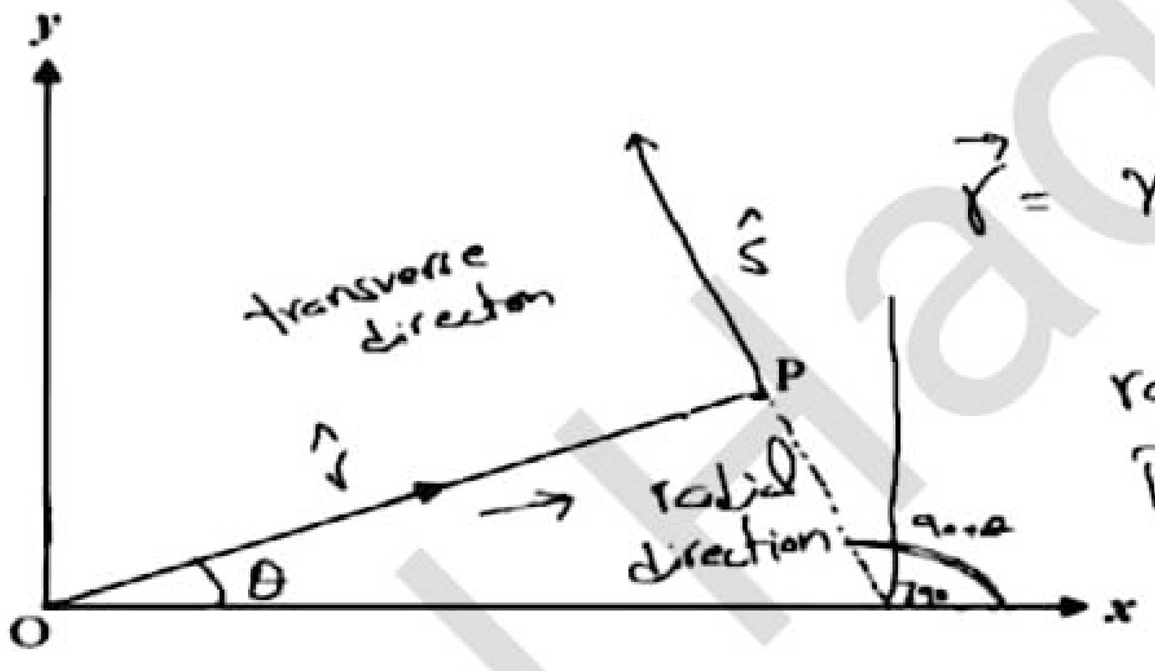
Radial & Transvers Component of Velocity

In polar coordinate, the position of a particle is specified by radius vector r & the polar angle θ which are related to x & y through the relations $x = r\cos\theta$ & $y = r\sin\theta$ provided the two coordinate frames have same origin & the x-axis and the initial line coincide. The direction of radius vector is known as **radial direction** & that perpendicular to it in the direction of increasing θ is called **transverse direction.**

Radial & Transverse Component of Velocity & Acceleration

$x = r\cos\theta$
$y = r\sin\theta$

$\vec{r} = xi + yj$
$= r\cos\theta i + r\sin\theta j$
$\vec{r} = r(\cos\theta i + \sin\theta j)$
$\frac{\vec{r}}{r} = \cos\theta i + \sin\theta j$
$\hat{r} = \cos\theta i + \sin\theta j$



$\hat{r} = \cos\theta i + \sin\theta j$

$\vec{r} = xi + yj$

radius vector r
Polar angle θ

Let $\hat{r}$ & $\hat{s}$ *be units vectors in radial & Transverse direction respectively*

Polar Coordinate $= \hat{r} = \cos\theta i + \sin\theta j$........(*i*)

$$\hat{s} = \cos(90+\theta)i + \sin(90+\theta)j = (\cos 90\cos\theta - \sin 90\sin\theta)i + (\sin 90\cos\theta + \cos 90\sin\theta)$$

$\hat{s} = -\sin\theta i + \cos\theta j$........(*ii*)

Differentiating (*i*) *w.r.t* "*t*"

$$\frac{d\hat{r}}{dt} = \frac{d}{dt}(\cos\theta i + \sin\theta j) = -\sin\theta i\frac{d\theta}{dt} + \cos\theta j\frac{d\theta}{dt} = \frac{d\theta}{dt}(-\sin\theta i + \cos\theta j) = \frac{d\theta}{dt}\hat{s}$$

Differentiating (*ii*) *w.r.t* "*t*"

$$\frac{d\hat{s}}{dt} = \frac{d}{dt}(-\sin\theta i + \cos\theta j) = -\cos\theta i\frac{d\theta}{dt} - \sin\theta j\frac{d\theta}{dt} = -\frac{d\theta}{dt}(\cos\theta i + \sin\theta j) = -\frac{d\theta}{dt}\hat{r}$$

$$\hat{r} = \frac{\vec{r}}{r} \Rightarrow \vec{r} = r\hat{r}$$

$$\text{Now } \vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\hat{r}) = \frac{dr}{dt}.\hat{r} + r\frac{d\hat{r}}{dt} = \frac{dr}{dt}.\hat{r} + r\frac{d\theta}{dt}\hat{s}$$

Thus v_r = *Radial Component of Velocity* $= \frac{dr}{dt} = \dot{r}$

v_θ = *Transverse Component of velocity* $= r\frac{d\theta}{dt} = r\dot{\theta}$

Where . denotes the differentiation w.r.t time "*t*".

Lecture 3-2

Example: - Find the radial & transverse components of velocity of a particle moving along the curves $ax^2 + by^2 = 1$ *at any time t if polar angle* $\theta = ct^2$ **.**

$ax^2 + by^2 = 1$ *at any time t if polar angle* $\theta = ct^2$

Given that $\theta = ct^2$ *Differntiating* w.r.t t we get $\frac{d\theta}{dt} = 2ct$

Also Given that $ax^2 + by^2 = 1$ first we change this into polar form by

x=rcosθ & y=rsin$\theta \Rightarrow ar^2\cos^2\theta + br^2\sin^2\theta = 1 \Rightarrow r^2(a\cos^2\theta + b\sin^2\theta) = 1$

$$r\sqrt{(a\cos^2\theta + b\sin^2\theta)} = 1 \Rightarrow r = (a\cos^2\theta + b\sin^2\theta)^{-\frac{1}{2}}$$

differentiate w.r.t "t" $\frac{dr}{dt} = \frac{d}{dt}(a\cos^2\theta + b\sin^2\theta)^{-\frac{1}{2}}$

$$\frac{dr}{dt} = -\frac{1}{2}(a\cos^2\theta + b\sin^2\theta)^{-\frac{3}{2}}\left(-2a\cos\theta\sin\theta\frac{d\theta}{dt} + 2b\sin\theta\cos\theta\frac{d\theta}{dt}\right)$$

$$\frac{dr}{dt} = \frac{1}{2}(a\cos^2\theta + b\sin^2\theta)^{-\frac{3}{2}}(a-b)2\cos\theta\sin\theta\frac{d\theta}{dt} = \frac{1}{2}(a\cos^2\theta + b\sin^2\theta)^{-\frac{3}{2}}(a-b)\sin 2\theta(2ct)$$

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$$\frac{dr}{dt}=\frac{ct\left(a-b\right)\sin\theta}{\left(a\cos^2\theta+b\sin^2\theta\right)^{\frac{3}{2}}}$$

Radial component of velocity= $\frac{dr}{dt}=\frac{ct\left(a-b\right)\sin\theta}{\left(a\cos^2\theta+b\sin^2\theta\right)^{\frac{3}{2}}}$

Transverse component of velocity=r $\frac{d\theta}{dt}=\frac{2ct}{\left(a\cos^2\theta+b\sin^2\theta\right)^{-\frac{1}{2}}}$

Lecture 4-1

Radial & Transversal Component of Acceleration

Let $\vec{a}$ be the acceleration Then

$$\vec{a}=\frac{d\vec{v}}{dt}=\frac{d}{dt}\left(\frac{d\vec{r}}{dt}.\hat{r}+r\frac{d\theta}{dt}.\hat{s}\right)=\frac{d}{dt}\left(\frac{d\vec{r}}{dt}.\hat{r}\right)+\frac{d}{dt}\left(r\frac{d\theta}{dt}.\hat{s}\right)$$

$$\vec{a}=\left(\frac{d^2\vec{r}}{dt^2}\hat{r}+\frac{d\vec{r}}{dt}\frac{d\hat{r}}{dt}\right)+\left(\frac{dr}{dt}\frac{d\theta}{dt}\hat{s}+r\frac{d}{dt}\left(\hat{s}.\frac{d\theta}{dt}\right)\right)$$

$$\vec{a}=\left(\frac{d^2\vec{r}}{dt^2}\hat{r}+\frac{d\vec{r}}{dt}\frac{d\hat{r}}{dt}\right)+\left(\frac{dr}{dt}\frac{d\theta}{dt}\hat{s}+r\left\{\frac{d\hat{s}}{dt}\frac{d\theta}{dt}+\hat{s}\frac{d^2\theta}{dt^2}\right\}\right)$$

$$\vec{a}=\frac{d^2\vec{r}}{dt^2}\hat{r}+\frac{d\vec{r}}{dt}\frac{d\hat{r}}{dt}+\frac{dr}{dt}\frac{d\theta}{dt}\hat{s}+\frac{d\hat{s}}{dt}\frac{d\theta}{dt}r+r\hat{s}\frac{d^2\theta}{dt^2}$$

$$\vec{a}=\frac{d^2\vec{r}}{dt^2}\hat{r}+\frac{dr}{dt}\frac{d\theta}{dt}\hat{s}+\frac{d\vec{r}}{dt}\frac{d\theta}{dt}\hat{s}-r\left(\frac{d\theta}{dt}\right)^2\hat{r}+r\hat{s}\frac{d^2\theta}{dt^2}$$

$$\vec{a}=\frac{d^2\vec{r}}{dt^2}\hat{r}-r\left(\frac{d\theta}{dt}\right)^2\hat{r}+2\frac{dr}{dt}\frac{d\theta}{dt}\hat{s}+r\frac{d^2\theta}{dt^2}\hat{s}=\left(\frac{d^2\vec{r}}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right)\hat{r}+\left(2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right)+r\frac{d^2\theta}{dt^2}\right)\hat{s}$$

$$a_r=Radial\ component\ of\ acceleration=\frac{d^2\vec{r}}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=\ddot{r}-r(\dot{\theta})^2$$

$$a_\theta=Transverse\ component\ of\ acceleration=2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right)+r\frac{d^2\theta}{dt^2}=2\dot{r}\dot{\theta}-r(\ddot{\theta})^2$$

Lecture 4-2

Example: -A Particle P moves in a plane in such a way that at any time t, its distance from O is $r=at+bt^2$ & the line connecting O & P makes an angle $\theta=ct^{\frac{3}{2}}$ with a fixed line OA. Find the radial & transverse components of velocity & acceleration of particle at t=1.

$$Given\ r = at + bt^2\ \&\ \ \theta = ct^{\frac{3}{2}}$$

$$Differentiating\ w.r.t\ "t"\ \ \frac{dr}{dt} = a + 2bt \quad and \quad \frac{d\theta}{dt} = \frac{3}{2}ct^{\frac{1}{2}}$$

$$Again\ Differentiating\ w.r.t\ "t" \Rightarrow \frac{d^2r}{dt^2} = 2b \quad and \quad \frac{d^2\theta}{dt^2} = \frac{3}{4}ct^{-\frac{1}{2}}$$

$$at \quad t = 1 \qquad r = a + b\ \&\ \theta = c \quad \frac{dr}{dt} = a + 2b \quad and \quad \frac{d\theta}{dt} = \frac{3}{2}c$$

$$\frac{d^2r}{dt^2} = 2b \quad and \quad \frac{d^2\theta}{dt^2} = \frac{3}{4}c$$

$$Radial\ component\ of\ velocity = v_r = \frac{dr}{dt} = a + 2b$$

$$Transverse\ component\ of\ velocity = v_\theta = r\frac{d\theta}{dt} = \frac{3}{2}c\left(a + b\right)$$

$$Radial\ component\ of\ Acceleration = a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 2b - (a+b)\left(\frac{3}{2}c\right)^2 = 2b - \frac{9}{4}c^2(a+b)$$

$$Transverse\ component\ of\ Acceleration = a_\theta = 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + r\left(\frac{d^2\theta}{dt^2}\right) = 2(a+2b)(\frac{3}{2}c) + (a+b)(\frac{3}{4}c) = \frac{3}{4}c(5a+9b)$$

Lecture 4-3

Example: -Find the radial and transverse components of acceleration of a particle moving along the circle $x^2 + y^2 = a^2$ with constant velocity c.

$$x^2 + y^2 = a^2$$

$$first\ we\ change\ this\ \text{into}\ polar\ form\ by\ putting\ x = r\cos\theta\ \&\ y = r\sin\theta$$

$$r^2\cos^2\theta + r^2\sin^2\theta = a^2 \Rightarrow r^2\left(\cos^2\theta + \sin^2\theta\right) = a^2 \Rightarrow r = a$$

$$NowGiven\ that\ \ \frac{d\theta}{dt} = c$$

$$Differentiating\ w.r.t\ "t"\ \ \frac{dr}{dt} = 0 \quad and \quad \frac{d\theta}{dt} = c$$

$$Again\ Differentiating\ w.r.t\ "t" \Rightarrow \frac{d^2r}{dt^2} = 0 \quad and \quad \frac{d^2\theta}{dt^2} = 0$$

$$Radial\ component\ of\ velocity = v_r = \frac{dr}{dt} = 0$$

$$Transverse\ component\ of\ velocity = v_\theta = r\frac{d\theta}{dt} = ac$$

$$Radial\ component\ of\ Acceleration = a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0 - ac^2 = -ac^2$$

$$Transverse\ component\ of\ Acceleration = a_\theta = 2\frac{dr}{dt}\left(\frac{d\theta}{dt}\right) + r\left(\frac{d^2\theta}{dt^2}\right) = 0$$

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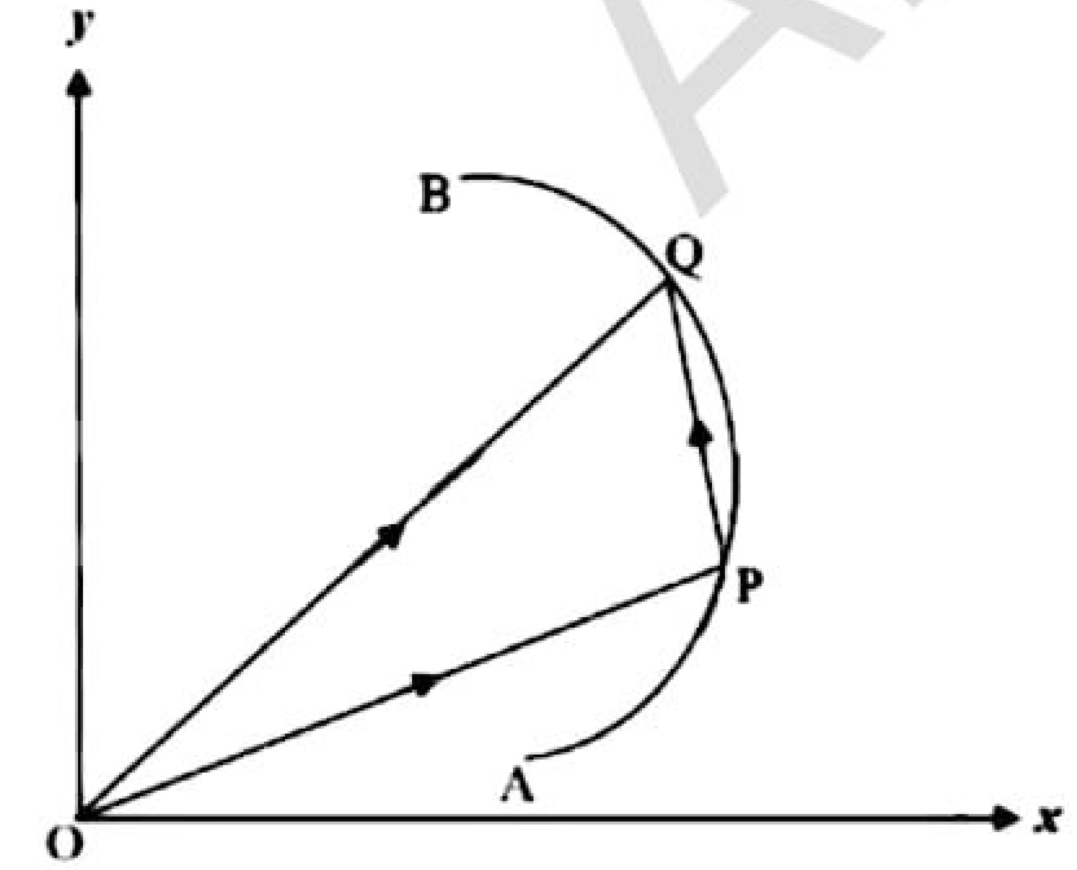
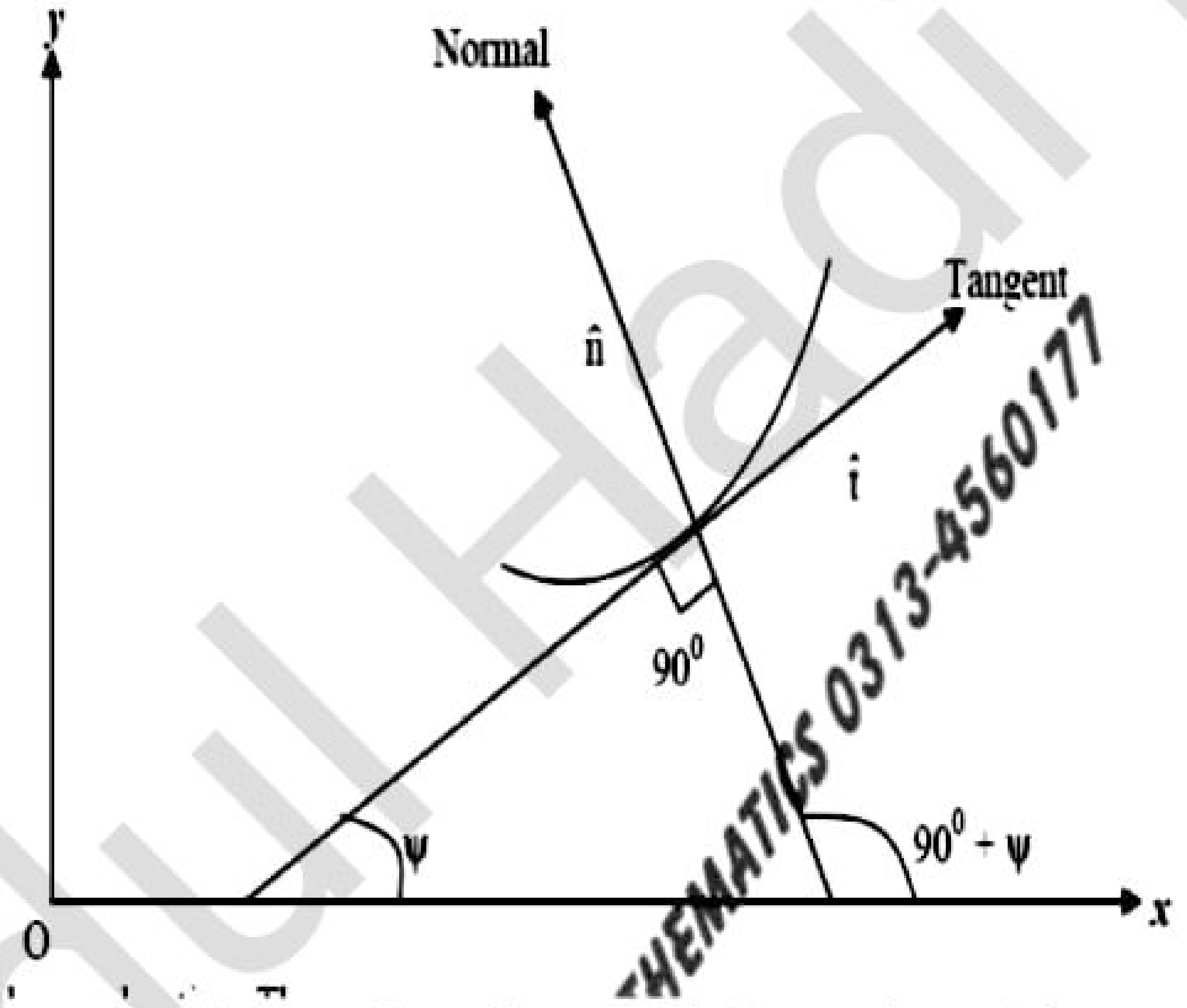
Lecture 5-1

Tangential & Normal Components of Velocity & Acceleration

Tangential and Normal Components of Velocity and Acceleration





Let AB be a part of the trajectory of particle as shown in figure. Let the particle at time **t** be at the point P whose position vector is $\vec r$. After a small time δt, let the particle reach the point Q whose position vector is $\vec r + \delta\vec r$. Then $\overrightarrow{PQ} = \delta\vec r$ and arc PQ= δs. Now

$\vec v = \frac{d\vec r}{dt} = \frac{d\vec r}{ds}\frac{ds}{dt} = v.\frac{d\vec r}{ds}$ *here* $\frac{d\vec r}{ds}$ *is a unit tangential at point P.*

let $\hat t$ *be a unit tangent at point P and n unit vector along normal at the point P.*

Then $\frac{d\vec r}{ds} = \hat t \Rightarrow \vec v = v\hat t + 0.n$

v_t = *tangential Component of velocity* $= v$, v_n = *Normal Component of velocity* $= 0$

Hence the velocity is along the tangent to the path.

Let $\vec a$ *be the acceleration. Then* $\vec a = \frac{d\vec v}{dt} = \frac{d}{dt}(v\hat t) = \frac{dv}{dt}\hat t + v\frac{d\hat t}{dt} = \frac{dv}{dt}\hat t + v\frac{d\hat t}{d\psi}\frac{d\psi}{ds}\frac{ds}{dt} = \frac{dv}{dt}\hat t + v\frac{d\hat t}{d\psi}(Kv)$

Where $\frac{d\psi}{ds} = K$ *is called curvature &* $K = \frac{1}{\rho}$ & $\frac{ds}{dt} = v$

$$\vec a = \frac{dv}{dt}\hat t + \frac{v^2}{\rho}\frac{d\hat t}{d\psi}$$

since $\hat t$ & $\widehat{n}$ *are unit vectors along tangent & normal at P Therefore*

$\hat t = \cos\psi i + \sin\psi j$ & $\widehat{n} = \cos(90+\psi)i + \sin(90+\psi)j = \cos\psi i - \sin\psi j$

$$\frac{d\hat t}{d\psi} = \frac{d}{d\psi}(\cos\psi i + \sin\psi j) = -\sin\psi i + \cos\psi j = \widehat{n}$$

so $\vec a = \frac{dv}{dt}\hat t + \frac{v^2}{\rho}\widehat{n}$

Tangential Component of acceleration $= a_t = \frac{dv}{dt}$

Normal Component of acceleration $= a_n = \frac{v^2}{\rho}$ *where* $\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}$

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Lecture 5-2

Example: -A particle is moving along the parabola $x^2 = 4ay$ with constant speed. Determine tangential & Normal components of its acceleration when it reaches the point whose abscissa is $\sqrt{5}a$.

Given that $x^2 = 4ay$ *Differentiating w.r.t "x"* $2x = 4a\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2a}$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{x}{2a}\right) = \frac{1}{2a}$ *Given that* $x = \sqrt{5}a$ *therefore* $\frac{dy}{dx} = \frac{\sqrt{5}a}{2a} = \frac{\sqrt{5}}{2}$

we know that

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left[1+\left(\frac{\sqrt{5}}{2}\right)^2\right]^{3/2}}{\left|\frac{1}{2a}\right|} = \frac{\left[1+\frac{5}{4}\right]^{3/2}}{\left|\frac{1}{2a}\right|} = 2a\left[\frac{9}{4}\right]^{3/2} = 2a\left(\frac{27}{8}\right) = \frac{27a}{4}$$

Since The Particle is moving with constant speed therefore $\frac{dv}{dt} = 0$

Tangential Component of acceleration $= a_t = \frac{dv}{dt} = 0$

Normal Component of acceleration $= a_n = \frac{v^2}{\rho} = \frac{4v^2}{27a}$

Lecture 5-3

Example: -Find the tangential & Normal component of acceleration of a point describing ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with uniform speed v when the Particle is at (0, b).

Given that $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x^2b^2 + a^2y^2 = a^2b^2$

Differentiating w.r.t "x" $2xb^2 + 2a^2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{xb^2}{a^2y}$

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\frac{d}{dx}\left(\frac{x}{y}\right) = -\frac{b^2}{a^2}\left(\frac{y - x\frac{dy}{dx}}{y^2}\right) = -\frac{b^2}{a^2}\left(\frac{y - x\left(-\frac{xb^2}{a^2y}\right)}{y^2}\right) = -\frac{b^2}{a^2}\left(\frac{\frac{a^2y^2+x^2b^2}{a^2y}}{y^2}\right) = -\frac{b^2}{a^2}\left(\frac{a^2y^2+x^2b^2}{a^2y^3}\right)$$

at $(0,b)\ \frac{dy}{dx} = -\frac{xb^2}{a^2y} = -\frac{(0)b^2}{a^2b} = 0$

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\left(\frac{a^2y^2+x^2b^2}{a^2y^3}\right)=-\frac{b^2}{a^2}\left(\frac{a^2b^2+0^2b^2}{a^2b^3}\right)=-\frac{b^2}{a^2}\left(\frac{a^2b^2}{a^2b^3}\right)=-\frac{b}{a^2}$$

we know that

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}=\frac{\left[1+(0)^2\right]^{3/2}}{\left|-\frac{b}{a^2}\right|}=\frac{a^2}{b}$$

Since *The Particle is moving with uniform speed therefore* $\frac{dv}{dt}=0$

Tangential *Component of acceleration* $=a_t=\frac{dv}{dt}=0$

Normal Component of acceleration $=a_n=\frac{v^2}{\rho}=\frac{v^2}{\frac{a^2}{b}}=\frac{bv^2}{a^2}$

Lecture 5-4

Example: -A particle is moving with uniform speed along the curve $x^2y=a\left(x^2+\frac{a^2}{\sqrt{5}}\right)$. Show that acceleration has maximum value $\frac{10v^2}{9a}$

Given that $x^2y=a\left(x^2+\frac{a^2}{\sqrt{5}}\right)$, $\frac{10v^2}{9a}$

$$y=a\left(1+\frac{a^2}{\sqrt{5}}x^{-2}\right)=a+\frac{a^3}{\sqrt{5}}x^{-2}$$

Differentiating w.r.t "x" $\frac{dy}{dx}=-\frac{2a^3}{\sqrt{5}}x^{-3}$

$$\frac{d^2y}{dx^2}=-\frac{2a^3}{\sqrt{5}}\frac{d}{dx}\left(x^{-3}\right)=\frac{6a^3}{\sqrt{5}}x^{-4}$$

we know that

$$\rho=\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|}=\frac{\left[1+\left(-\frac{2a^3}{\sqrt{5}}x^{-3}\right)^2\right]^{3/2}}{\left|\frac{6a^3}{\sqrt{5}}x^{-4}\right|}=\frac{\left[1+\frac{4a^6}{5}x^{-6}\right]^{3/2}}{\left|\frac{6a^3}{\sqrt{5}}x^{-4}\right|}=\frac{\sqrt{5}x^4}{6a^3}\left[\frac{5x^6+4a^6}{5x^6}\right]^{3/2}=\frac{\left[5x^6+4a^6\right]^{3/2}}{30a^3x^5}$$

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$\text{Since } \textit{The Particle is moving with uniform speed therefore } \frac{dv}{dt} = 0$

$\vec{a} = \frac{v^2}{\rho} n =, \left|\vec{a}\right| = \frac{v^2}{\rho}\left|n\right| = \frac{v^2}{\rho} \because \left|n\right| = 1$

$\left|\vec{a}\right| \textit{ will } \max imum \textit{ when } \rho \textit{ is } \min imum$

$$\frac{d\rho}{dx} = \frac{d}{dx}\left\{\frac{\left[5x^6+4a^6\right]^{3/2}}{30a^3x^5}\right\} = \frac{30a^3x^5\frac{d}{dx}\left[5x^6+4a^6\right]^{3/2} - \left[5x^6+4a^6\right]^{3/2}\frac{d}{dx}30a^3x^5}{\left(30a^3x^5\right)^2}$$

$$\frac{d\rho}{dx} = \frac{30a^3x^5\frac{3}{2}\left[5x^6+4a^6\right]^{1/2}\left[30x^5\right] - \left[5x^6+4a^6\right]^{3/2}\left(150a^3x^4\right)}{\left(30a^3x^5\right)^2}$$

$$\frac{d\rho}{dx} = \frac{900a^3x^{10}\frac{3}{2}\left[5x^6+4a^6\right]^{1/2} - \left[5x^6+4a^6\right]^{3/2}\left(150a^3x^4\right)}{\left(30a^3x^5\right)^2} = \frac{\left(150a^3x^4\right)\left[5x^6+4a^6\right]^{1/2}\left(\frac{3}{2}6x^6 - 5x^6 - 4a^6\right)}{\left(30a^3x^5\right)^2}$$

$$\frac{d\rho}{dx} = \frac{\left[5x^6+4a^6\right]^{1/2}\left(4x^6-4a^6\right)}{6a^3x^6} = \frac{4\left[5x^6+4a^6\right]^{1/2}\left(x^6-a^6\right)}{6a^3x^6} = \frac{2\left[5x^6+4a^6\right]^{1/2}\left(x^2-a^2\right)\left(x^4+x^2a^2+a^4\right)}{3a^3x^6}$$

$\textit{Putting } \frac{d\rho}{dx} = 0, \textit{we get } x = \pm a \text{ since } \frac{d\rho}{dx} < 0 \textit{ before } a \;\&\; \frac{d\rho}{dx} > 0 \textit{ after } x = a$

$\textit{Therefore } \rho \textit{ is } \min imum \textit{ when } x = a$

$$\textit{thus } \rho_{\min} == \frac{\left[5a^6+4a^6\right]^{3/2}}{30a^3a^5} = \frac{\left[9a^6\right]^{3/2}}{30a^8} = \frac{27a^9}{30a^8} = \frac{9}{10}a$$

$$\textit{Maximum value of acceleration} = \frac{v^2}{\rho_{\min}} = \frac{10v^2}{9a}$$

Lecture 6-1

.RECTILINEAR MOTION

The motion of a particle along a straight line is called rectilinear motion. Let the particle start from O along a line. We take line along x-axis. Let after time "t" particle be at a point at a distance 'x' from O. $O \_\_\_\_\_\_\_\_\_\_ \vec{r} \_\_\_\_\_\_\_\_\_\_\_\_ P \quad x-axis$.

$\textit{Let } \vec{r} \textit{ be the position vector of point } P \textit{ w.r.t origin } O. \textit{ then}$

$\vec{r} = \overline{OP} = x\hat{i}$

$\textit{Now } \vec{v} = \frac{d\vec{r}}{dt}\hat{i} \quad \& \vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2x}{dt^2}\hat{i} \textit{ Let } \left|\vec{v}\right| = v \;\&\; \left|\vec{a}\right| = a$

$\textit{Then } v = \frac{dx}{dt} \quad \& \; a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx}.\frac{dx}{dt} = v.\frac{dv}{dx}$

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Lecture 7-1

.RECTILINEAR MOTION

Motion With Constant Acceleration

Let the particle start from O with velocity u at time t=0 with constant acceleration. Let after time "t" particle be at point P at a distance "x" from O. Then

$$a=\frac{dv}{dt}\Rightarrow adt=dv \text{ integrating both side}$$

$$\int adt=\int dv\rightarrow at+A=v \text{ where } A \text{ is constant of acceleration}$$

$$at\ t=0,v=u \qquad a(0)+A=v\ \Rightarrow A=v$$

$$v=u+at \quad \text{Which is 1st equation of motion.}$$

$$As\ v=\frac{dx}{dt}=u+at\Rightarrow dx=udt+atdt \text{ on integrating both side}$$

$$\int dx=\int udt+atdt\Rightarrow x=ut+\frac{1}{2}at^2+B$$

$$put\ t=0,x=0\ B=0\ \ \Rightarrow x=ut+\frac{1}{2}at^2 \text{ which si second equation of motion.}$$

$$As\ a=v.\frac{dv}{dx}\Rightarrow adx=vdv \text{ on integrating both side}$$

$$\int adx=\int vdv\rightarrow ax+C=\frac{1}{2}v^2, put\ t=0,x=0\ \&\ v=u$$

$$C=\frac{1}{2}u^2\Rightarrow ax+\frac{1}{2}u^2=\frac{1}{2}v^2\Rightarrow 2ax+u^2=v^2\Rightarrow 2ax=v^2-u^2 \text{3rd equation of motion}$$

Lecture 7-2

.RECTILINEAR MOTION EXAMPLES

Example: -A particle moving in a straight line starts from rest & is accelerated uniformly to attain a velocity 60 miles per hours in 4 seconds, finds the acceleration of motion & distance travelled by particle in last three seconds.

$$Initial\ velocity=u=0,\quad time=t=4\sec$$

$$Final\ velocity=60miles/h=\frac{60\times1760\times3}{3600}=88ft/\sec$$

$$v=u+at\ ,a=\frac{v-u}{t}=\frac{88-0}{4}=22ft/\sec^2$$

$$Now\ Distance\ covered\ in\ 1st\ second=x_1=ut+\frac{1}{2}at^2=0+\frac{1}{2}(22)(1)^2=11ft$$

$$Now\ Distance\ covered\ in\ four\ second=x_2=ut+\frac{1}{2}at^2=0+\frac{1}{2}(22)(4)^2=176ft$$

$$Distance\ covered\ in\ last\ 3\ seconds=x_2-x_1=176-11=165ft$$

Lecture 7-3

.RECTILINEAR MOTION EXAMPLES

Example: -Two particles start simultaneously from point O & move in straight line one with velocity of 45 mile/h & an acceleration of 2ft/sec^2 & other with a velocity of 90 mile/h & a retardation of 8ft/sec^2. Find the time after which the velocities of particles are same & the distance of O from the point where they meet again.

For 1st particle

$$u = 45 mile/h = \frac{45\times1760\times30}{3600} = 66 ft/\sec$$

$$a = 2 ft/\sec^2 \text{ We know that } v = u + at = 66 + 2t$$

For 2nd particle

$$u = 90 mile/h = \frac{90\times1760\times30}{3600} = 132 ft/\sec$$

$$a = -8 ft/\sec^2 \text{ We know that } v = u + at = 132 - 8t$$

According to given condition $66 + 2t = 132 - 8t$

$10t = 66 \Rightarrow t = 6.6\sec$ *so after* 6.6 sec *velocities of particles will same* .

Let both particle meet after a distance x then

For 1st particle

$$x = ut + \frac{1}{2}at^2 = 66t + \frac{1}{2}2t^2 = 66t + t^2$$

For 2nd particle

$$x = ut + \frac{1}{2}at^2 = 132t + \frac{1}{2}(-8)t^2 = 132t - 4t^2$$

Given condition

$$66t + t^2 = 132t - 4t^2 \Rightarrow 5t^2 = 66t \Rightarrow t = 13.2\sec$$

$$x = 1045.44 ft$$

Lecture 8-1

. MOTION WITH VARIABLE ACCELERATION

Time Dependent Acceleration

$$Acceleration = a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

First condition Time dependent Acceleration

$$\frac{d^2x}{dt^2} = a(t) \Rightarrow \int\frac{d^2x}{dt^2} = \int a(t)dt + A \Rightarrow \frac{dx}{dt} = b(t) + A$$

$$x = \int b(t) + At + B$$

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Example: - Find the distance travelled & the velocity attained by a particle moving in a straight line, at any time t if it starts from rest at t=0 & is subject to an acceleration $t^2+\sin t+e^t$.

$$Acceleration = a = \frac{d^2x}{dt^2} = t^2+\sin t+e^t$$

First condition Time dependent Acceleration

to find velocity we will integrate

$$\int\frac{d^2x}{dt^2} = \int t^2+\sin t+e^t \Rightarrow \frac{dx}{dt} = \frac{t^3}{3}-\cos t+e^t+A$$

$$By\ applying\ initial\ condition\ \frac{dx}{dt}=0, t=0 \quad 0=0-1+1+A$$

$$\Rightarrow A=0, \Rightarrow \frac{dx}{dt} = \frac{t^3}{3}-\cos t+e^t$$

To find distance we will integrate again

$$\int dx = \int\left(\frac{t^3}{3}-\cos t+e^t\right)dt \Rightarrow x = \frac{t^4}{12}-\sin t+e^t+B$$

By applying initial condition $t=0,\ x=0$ *we obtain* $B=-1$

$$x = \frac{t^4}{12}-\sin t+e^t-1$$

Lecture 8-2

. MOTION WITH VARIABLE ACCELERATION

Velocity Dependent Acceleration

$$Acceleration = a = \frac{dv}{dt} = = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

Second condition velocity dependent Acceleration

$$a(v) = v\frac{dv}{dx} \Rightarrow dx = v\frac{dv}{a(v)} \Rightarrow \int dx = \int v\frac{dv}{a(v)} \Rightarrow x = \int\frac{v}{a(v)}dv + C$$

$$\frac{dv}{dt} = a(v) \Rightarrow dt = \frac{dv}{a(v)} \Rightarrow t = \int\frac{dv}{a(v)} + D$$

Example: -A Particle moves in a straight line with an acceleration kv^3. If its initial velocity is u, find the velocity & time spent when the particle has travelled a distance x.

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$$t = 0, initial\ velocity = u,\ final\ velocity = v?$$

$$a = kv^3 \Rightarrow v\frac{dv}{dx} = kv^3 \Rightarrow \frac{dv}{dx} = kv^2 \Rightarrow \frac{dv}{v^2} = kdx \Rightarrow \int \frac{dv}{v^2} = \int kdx$$

$$\Rightarrow -\frac{1}{v} = kx + A \quad By\ applying\ initial\ condition -\frac{1}{u} = k(0) + A, A = -\frac{1}{u}$$

$$-\frac{1}{v} = kx - \frac{1}{u} \Rightarrow \frac{1}{v} = \frac{1}{u} - kx = \frac{1-kux}{u} \Rightarrow v = \frac{u}{1-kux}$$

$$a = kv^3 \Rightarrow \frac{dv}{dt} = kv^3 \Rightarrow \frac{dv}{v^3} = kdt \Rightarrow \int \frac{dv}{v^3} = \int kdt \Rightarrow -\frac{1}{2v^2} = kt + B$$

$$t = 0, v = u, then -\frac{1}{2u^2} = B$$

$$-\frac{1}{2v^2} = kt - \frac{1}{2u^2} \Rightarrow kt = -\frac{1}{2v^2} + \frac{1}{2u^2} \Rightarrow t = \frac{1}{2k}\left(\frac{1}{u^2} - \frac{1}{v^2}\right) = \frac{1}{2k}\left(\frac{1}{u^2} - \left(\frac{1-kux}{u}\right)^2\right)$$

$$t = \frac{1}{2ku^2}(2-kux)$$

Lecture 9-1

. MOTION WITH VARIABLE ACCELERATION

Distance Dependent Acceleration

Acceleration depends on distance than it is called distance dependent acceleration.

$$a(x) = \frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

$$\text{int}\,egrating \int a(x)dx = \int vdv \Rightarrow F(x) = \frac{v^2}{2} + C \Rightarrow 2F(x) - 2C = v^2$$

$$v = \pm\sqrt{2F(x)+D} \Rightarrow \frac{dx}{dt} = \pm\sqrt{2F(x)+D} \Rightarrow \pm\int \frac{dx}{\sqrt{2F(x)+D}} = \int dt$$

$$t = \pm\int \frac{dx}{\sqrt{2F(x)+D}} + E$$

Example: -Discuss the motion of a particle moving in a straight line with an acceleration x^3, where x is the distance of particle from point O on the line, if it starts at t=0, from a point x=c with velocity $\frac{c^2}{\sqrt{2}}$.

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$$t=0, x=c, v=\frac{c^2}{\sqrt{2}} \ \& \ a=x^3$$

$$a=v\frac{dv}{dx}=x^3 \Rightarrow vdv=x^3dx \Rightarrow \int vdv=\int x^3dx \Rightarrow \frac{v^2}{2}=\frac{x^4}{4}+D$$

$$\text{By applying initial conditon } \frac{1}{2}\left(\frac{c^2}{\sqrt{2}}\right)^2=\frac{c^4}{4}+D \Rightarrow D=0$$

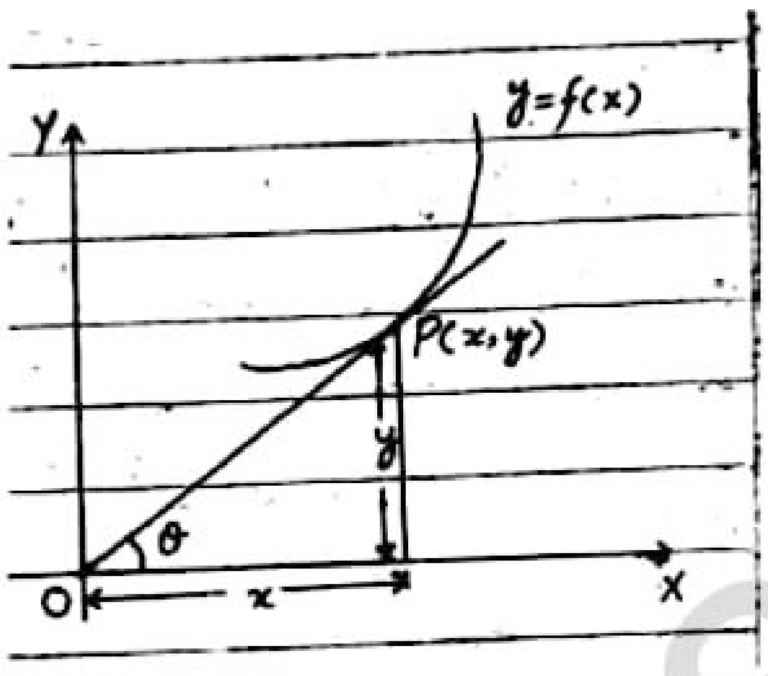
$$\text{so } \frac{v^2}{2}=\frac{x^4}{4} \Rightarrow v=\frac{1}{\sqrt{2}}x^2 \Rightarrow \frac{dx}{dt}=\frac{1}{\sqrt{2}}x^2 \Rightarrow \sqrt{2}x^{-2}dx=dt \Rightarrow \sqrt{2}\int x^{-2}dx=\int dt$$

$$-\frac{\sqrt{2}}{x}+D=t \quad \text{now } t=0, x=c \Rightarrow -\frac{\sqrt{2}}{c}+D=0$$

$$D=\frac{\sqrt{2}}{c}$$

$$t=-\frac{\sqrt{2}}{x}+\frac{\sqrt{2}}{c} \Rightarrow t=\sqrt{2}\left(\frac{1}{c}-\frac{1}{x}\right)$$

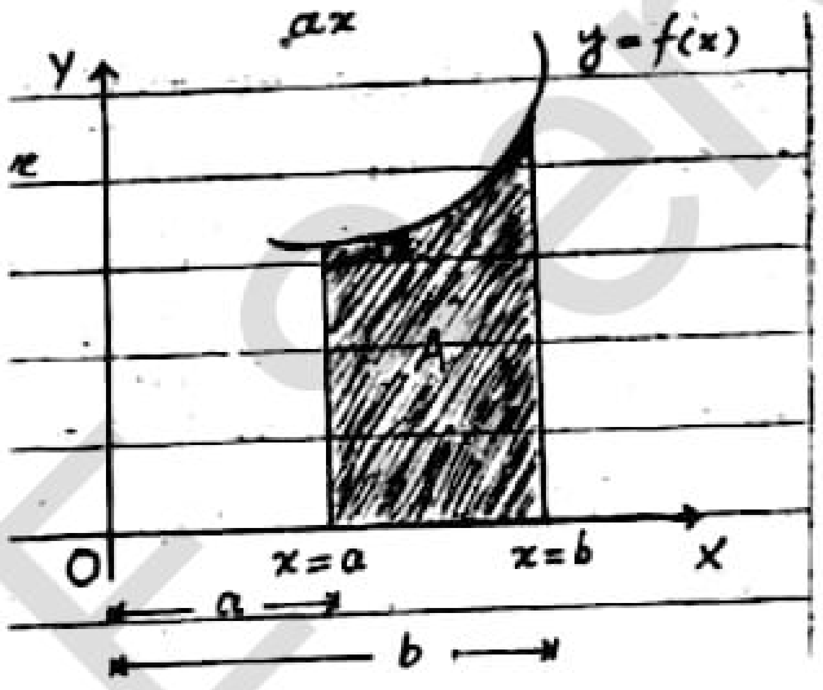
Lecture 10-1



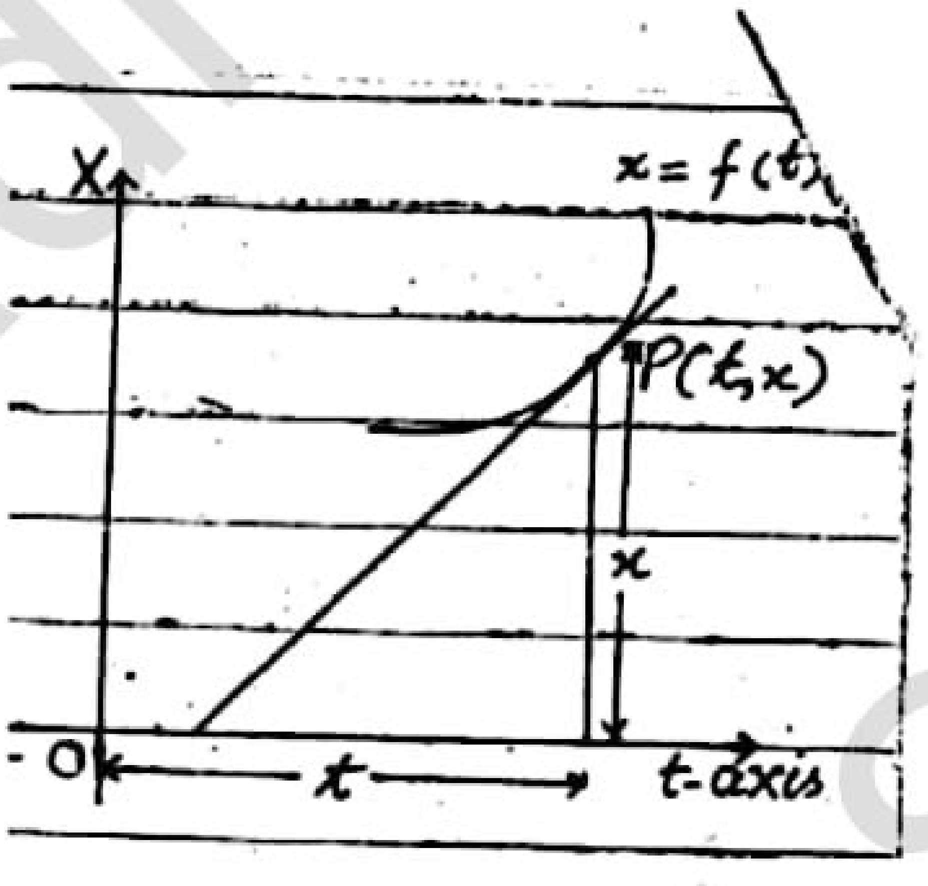
Graphical Method

Considering a particle which is moving along the Cartesian curve y=f(x)
Let at any instant the particle is at point P(x, y). Draw a tangent to the curve at P(x, y) then a slope of tangent line=dy/dx. Let A be the area under the curve y=f(x) & in between the lines x=a & x=b.

then $A=\int_{x=a}^{x=b} ydx$.



Space Time Curve



Considering a particle which is moving along the space time curve x=f(t). Let at any instant the particle is at point P (t, x). Draw a tangent line to this curve at point P. The slope of the tangent line=velocity =dx/dt. Clearly when a particle moves along a space time curve, then at any instant its velocity is given by the slope of tangent line at that point. Let the particle is moving with constant velocity then $\frac{dx}{dt}=c \Rightarrow dx=cdt \Rightarrow \int dx=c\int dt \Rightarrow x=ct+A$ is the first degree equation in ''x'' & ''t'' representing a straight line clearly if a particle moves in a space time plane with constant velocity, its motion will be rectilinear motion.

Velocity Time Curve

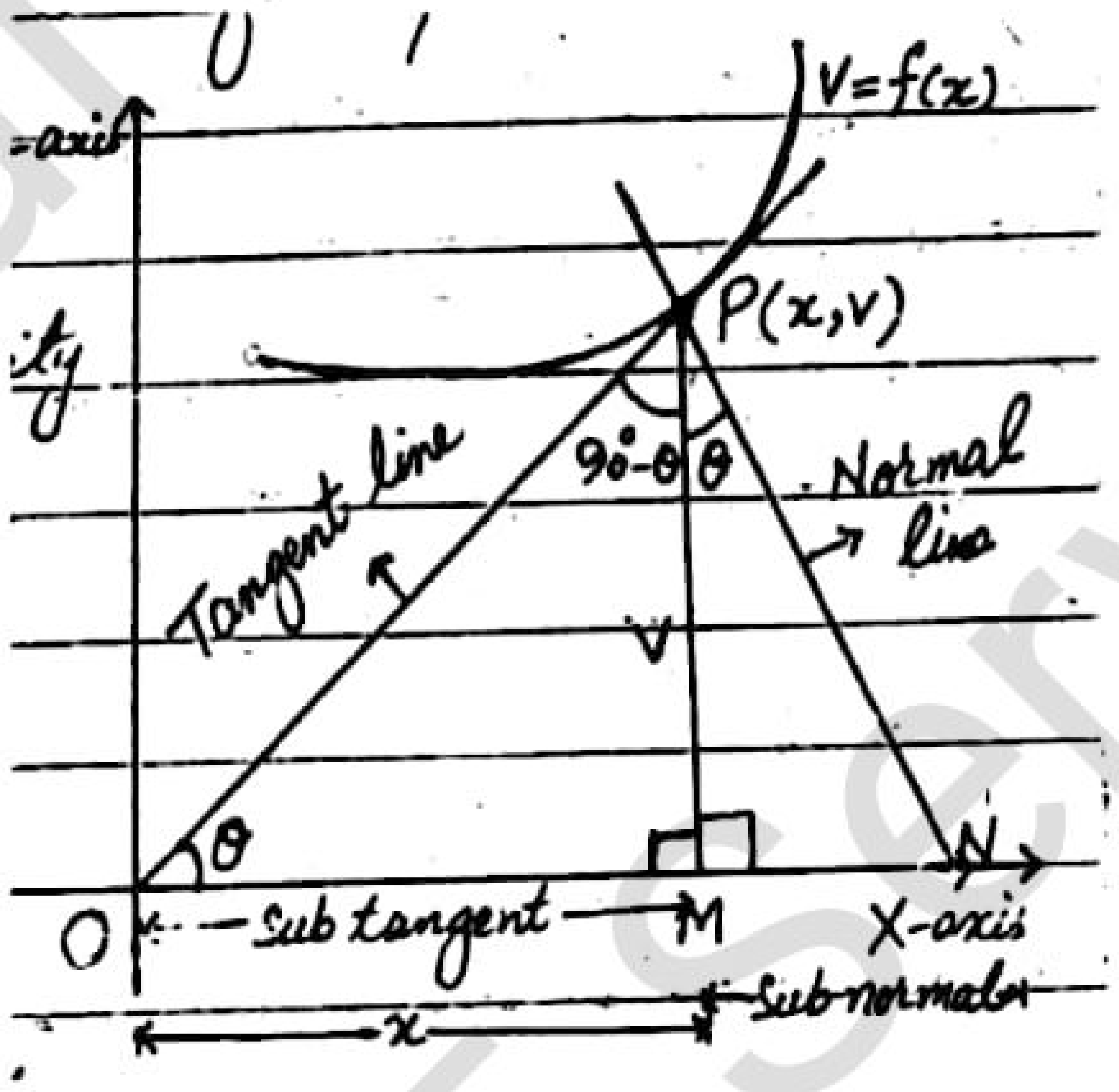
Consider a particle which is moving along the velocity time curve v=f(t). Let at any instant ''t'' the particle is at point P (t, v). Draw a tangent line to this curve through the point P (t, v), then slope of tangent line =acceleration=dv/dt. Clearly when a particle moves along the velocity time curve, then at any point its acceleration is equal to the slope of tangent at that point. Let the particle is moving with constant acceleration then $\frac{dv}{dt} = a(constant) \Rightarrow dv = adt \Rightarrow \int dv = a\int dt \Rightarrow v = at + B$ is the first degree equation in ''v'' & ''t'' representing a straight line. Clearly if a particle moves in a velocity time plane with constant acceleration, its motion will be rectilinear motion. Let ''A'' be the area under the velocity time curve

$v = f(t)$ then A=$\int vdt = \int \frac{dx}{dt}dt = \int dx = |x|_{t_1}^{t_2} = x(t_2) - x(t_1)$ where $x(t_2) - x(t_1)$ is the distance travelled by the particle in the time interval (t_1, t_2).

Lecture 10-2

Show that the length of subnormal at any point of velocity space curve gives the corresponding acceleration of the particle

Considering a particle moving along the velocity space curve v=f(x). Let at any instant the particle is at point P(x, v). Draw tangent & normal to this curve at point P(x, v).



$Slope\ of\ tangent\ line = \frac{dv}{dx} = \tan\theta$

$Now\ from\ right\ angled\ triangle\ PMN\ \ \frac{MN}{PM} = \tan\theta \Rightarrow MN = PM\tan\theta$

$put\ PM = V, \tan\theta = \frac{dv}{dx}, then\ \ MN = v\frac{dv}{dx}$

Length of Subnormal=Acceleration clearly the length of subnormal at any point of velocity space curve gives the corresponding acceleration of the particle

Lecture 10-3

A particle starts from rest from ''O'' with constant acceleration "a". when its velocity acquires a certain value "v" it moves uniformly & then its velocity starts decreasing with a constant retardation 2a till it comes to rest find the distance travelled by the particle, if time taken from rest to rest is "t".

Solution: - Considering a particle that starts moving from rest from the point "O" & move with constant acceleration "a" & attains a velocity "v" in time "t_1", then it moves uniformly for the time "t_2", then its velocity starts decreasing at the rate of "2a" & it comes to rest at point C after further time "t_3". Let t be the total time taken by the particle from rest to rest $t=t_1+t_2+t_3$.

Now the slope of line OA in the velocity time graph is the acceleration & the slope of line BC is the retardation of the particle, Thus we have

$\frac{v}{t_1} = a \,\&\, \frac{v}{t_3} = 2a\ Hence\ t_1 = \frac{v}{a} \,\&\, t_3 = \frac{v}{2a}$

$t = t_1 + t_2 + t_3 \Rightarrow t_2 = t - t_1 - t_3 \Rightarrow t_2 = t - \frac{v}{a} - \frac{v}{2a} = t - \frac{3v}{2a}$

$$The\ distance\ travelled\ in\ time\ "t"\ is\ given\ by$$

$$x = \Delta OAM + Area\ of\ \text{Rectangle}\ ABNM + \Delta BNC$$

$$x = \frac{1}{2}t_1 v + t_2 v + \frac{1}{2}t_3 v = \frac{1}{2}v(t_1 + 2t_2 + t_3) = \frac{1}{2}v\left(\frac{v}{a} + 2\left(t - \frac{3v}{2a}\right) + \frac{v}{2a}\right)$$

$$x = \frac{1}{2}v\left(2t - \frac{3v}{2a}\right)\ \text{is the required distance}$$

Lecture 11-1
SIMPLE HARMONIC MOTION

Periodic motion is the motion in which an object repeats its path in equal intervals of time. For example the emotion of hands of clock is periodic motion; leaves of tree moving to & fro due to wind breeze, these all are the examples of periodic motion. The particle performs the same set of movements again & again in a periodic motion. One such set of movement is called oscillation. An example of oscillatory motion is simple Harmonic Motion. The motion of particle along a straight line with an acceleration whose direction is always towards a fixed point on the line & whose magnitude is proportional to the distance from the fixed point is called SHM. Now consider there is a spring fixed at one end. When there is no force applied to it, it is in equilibrium position. Now if we pull it outwards, there is a force exerted by the string that is directed towards the equilibrium position. If we push the spring inwards, there is a force exerted by the string towards the equilibrium position.

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Let O be a fixed point on the line along the particle is moving. Let the particle be at point P a distance x from O towards its right. Then acceleration of particle is accelertion $= \dfrac{d^2x}{dt^2} = -kx$
where k is the constant of proportionality & negative sign indicates the acceleration directed against the direction in which x is increasing. This motion is taking place in such a way that when particle is moving away from point O, the acceleration is acting against so that as the time progresses, the velocity becomes lesser and lesser.

$acceleration = a = v\dfrac{dv}{dx} \Rightarrow v\dfrac{dv}{dx} = -kx \Rightarrow vdv = -kxdx \Rightarrow \int vdv = -k\int xdx \Rightarrow \dfrac{v^2}{2} = -k\dfrac{x^2}{2} + A$ *where A is constant of acceleration. when* $x = a$ & $v = 0$ *then* $A = k\dfrac{a^2}{2}$

$$\frac{v^2}{2} = -k\frac{x^2}{2} + k\frac{a^2}{2} \Rightarrow v^2 = k(a^2 - x^2) \Rightarrow v = \pm\sqrt{k(a^2 - x^2)}$$

$$v = \frac{dx}{dt} = \sqrt{k(a^2 - x^2)} \Rightarrow \frac{dx}{\sqrt{(a^2 - x^2)}} = \sqrt{k}dt$$

by integrating again $\int \dfrac{dx}{\sqrt{(a^2 - x^2)}} = \int \sqrt{k}dt \Rightarrow \sin^{-1}\left(\dfrac{x}{a}\right) = \sqrt{k}t + B$ *when* $t = 0, x = a$

Where B is constant of integration. $\sin^{-1}\left(\dfrac{a}{a}\right) = \sqrt{k}(0) + B \Rightarrow B = \sin^{-1}(1) = \dfrac{\pi}{2}$

$$\sin^{-1}\left(\frac{x}{a}\right) = \sqrt{k}t + B = \sin^{-1}\left(\frac{x}{a}\right) = \sqrt{k}t + \frac{\pi}{2} \Rightarrow \left(\frac{x}{a}\right) = \sin\left(\sqrt{k}t + \frac{\pi}{2}\right) \Rightarrow x = a\cos\sqrt{k}t$$

Lecture 12-1
NATURE OF SHM

As the displacement of a particle Performing SHM at any time t is $x = a\cos\sqrt{k}t$ Also $-1 \le a\cos\sqrt{k}t \le 1 \Rightarrow -a \le a\cos\sqrt{k}t \le a \Rightarrow -a \le x \le a$. Thus the maximum displacement from a fixed point O is x=a. The fixed point O is called the center of motion. The maximum displacement from the center is called the amplitude of the motion.

Now $x = a\cos\sqrt{k}t = x = a\cos\left(\sqrt{k}t + 2\pi\right) = a\cos\sqrt{k}\left(t + \dfrac{2\pi}{\sqrt{k}}\right)$ which shows that the distance at time t & $t + \dfrac{2\pi}{\sqrt{k}}$ is same.

$x = a\cos\sqrt{k}t$ by differentiating w.r.t "t"

$$\frac{dx}{dt} = v = -a\sqrt{k}\sin\sqrt{k}t = -a\sqrt{k}\sin\left(\sqrt{k}t + 2\pi\right) = -a\sqrt{k}\sin\sqrt{k}\left(t + \frac{2\pi}{\sqrt{k}}\right)$$

It shows that the velocity at time t & $t + \dfrac{2\pi}{\sqrt{k}}$ is same.

Lecture 12-2

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EXAMPLE OF SHM

A particle describes SHM with frequency f. if the greatest velocity is V, find the amplitude & maximum value of the acceleration of the particle. Also show that the velocity v at a distance x from the center of motion is given by $v = 2\pi f\sqrt{a^2 - x^2}$, where a is amplitude .

$Given\ that\ Frequency = f$

$Velocity = V = \sqrt{\lambda\left(a^2 - x^2\right)}\ but\ for\ \max imum\ velocity\ x = 0$

$Given\ that\ Maximum\ velocity = V = \sqrt{\lambda\left(a^2 - 0^2\right)} = \sqrt{\lambda}a$

$V = \sqrt{\lambda}a,\ where\ a\ is\ amplitude$

$As\ T = \frac{2\pi}{\sqrt{\lambda}}\ so\ Frequency = \frac{\sqrt{\lambda}}{2\pi} = f = \frac{\sqrt{\lambda}}{2\pi} \Rightarrow \sqrt{\lambda} = 2\pi f$

$By\ both\ equation\ V = 2\pi af$

$Amplitude = a = \frac{V}{2\pi f}$

$Acceleration = \left|-\lambda x\right| = \lambda x\ for\ Maximum\ Acceleration\ x = a$

$Now\ Maximum\ Acceleration = \lambda a = \left(2\pi f\right)^2 \frac{V}{2\pi f} = 2\pi fV$

$velocity\ at\ dis\tan ce\ x\ is\ given\ by$

$v = \sqrt{\lambda(a^2 - x^2)} = \sqrt{\left(2\pi f\right)^2 (a^2 - x^2)} = 2\pi f\sqrt{a^2 - x^2},\ where\ a\ is\ amplitude$

Lecture 13-1

EXAMPLE OF SHM

A particle describing SHM has velocities 5ft/sec & 4ft/sec, when their distances from the center are 12 ft & 13 ft respectively. Find the time period of motion.

$Time\ period = \frac{2\pi}{\sqrt{\lambda}}$

$We\ know\ that\ v^2 = k(a^2 - x^2)$

$when\ x = 12\,ft\ then\ v = 5\,ft/\sec\ \&\ when\ x = 13\,ft\ then\ v = 4\,ft/\sec$

$25 = \lambda(a^2 - 144)\ \&\quad 16 = \lambda(a^2 - 169)$

$by\ solving\ both\ 25\lambda = 9 \Rightarrow \sqrt{\lambda} = \frac{3}{5}$

$Time\ period = \frac{2\pi}{\sqrt{\lambda}} = Time\ period = \frac{10\pi}{3}$

Lecture 13-2

EXAMPLE OF SHM

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If a point P moves with a velocity v given by $v^2 = n^2\left(ax^2 + abx + c\right)$ show that P executes a SHM. Find the center, the amplitude & the time period of the motion.

Given that $v^2 = n^2\left(ax^2 + 2bx + c\right)$

Diff. w.r.t "x" we get

$$2v\frac{dv}{dx} = n^2\left(2ax + b\right) \Rightarrow v\frac{dv}{dx} = n^2 a\left(x + \frac{b}{a}\right) \because v\frac{dv}{dx} = acceleration$$

$$acceleration = -n^2 a\left[-\left(x + \frac{b}{a}\right)\right] \quad put\ X = -\left(x + \frac{b}{a}\right)$$

$$acceleration = -n^2 aX \Rightarrow acceleration \propto -X$$

Which shows that P executes SHM

$$To\ find\ Center\ put\ X = 0 \Rightarrow X = -\left(x + \frac{b}{a}\right) \Rightarrow 0 = -\left(x + \frac{b}{a}\right), x = -\frac{b}{a}$$

$$To\ find\ Amplitude\ put\ v = 0 \Rightarrow v^2 = n^2\left(ax^2 + 2bx + c\right)$$

$$n^2\left(ax^2 + 2bx + c\right) = 0 \Rightarrow \left(ax^2 + 2bx + c\right) = 0$$

$$x = \frac{-2b \pm \sqrt{4b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

Let O be the origin then

$$OA = \frac{-b + \sqrt{b^2 - ac}}{a} \ \&\ OB = \frac{-b - \sqrt{b^2 - ac}}{a}$$

Let C be the center then

$$OC = -\frac{b}{a}\ then\ Amplitude = CA = OA - OC = \frac{-b + \sqrt{b^2 - ac}}{a} - \frac{b}{a} = \frac{\sqrt{b^2 - ac}}{a}$$

$$Time\ Period = \frac{2\pi}{\sqrt{\lambda}}, \lambda = n^2 a$$

$$Time\ Period = \frac{2\pi}{n\sqrt{a}} =$$

Lecture 14-1

METHOD OF DYNAMICS

A dynamic system is called two dimensional if the orbit of each particle of system is parallel to a plane.

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Equation of Motion: - if in a time t a force is acting on a body whose mass is m & moving with velocity v then Equation of Motion will be

$$\frac{d}{dt}(mv) = F \Rightarrow m\frac{d}{dt}v = F \Rightarrow m.\dot{v} = F \Rightarrow ma = F$$

$$As\ \frac{dr}{dt} = v\ then \quad m\frac{d}{dt}\left(\frac{dr}{dt}\right) = F \Rightarrow m\frac{d^2r}{dt^2} = F \Rightarrow m.\ddot{r} = F$$

Central Force:- If the force F on a particle always passes through a fixed point O, the force is called Central force & O is called center of force. $\frac{d}{dt}(mv) = 0$ By integrating $\Rightarrow mv = C$.

Hence the momentum of the system moving under the influence of No Force is constant throughout the motion which is called principle of conservation of momentum.

Lecture 15-1

Two-Dimension Cartesian Form

Suppose that particle is moving in x y plane then position vector

$$r = xi + yj \ \ \& \ \ F = Xi + Yj$$

$$m\frac{dv}{dt} = F \Rightarrow m.\dot{v} = F \Rightarrow F = m.\ddot{r}$$

$$Xi + Yj = m\left(\ddot{x}i + \ddot{y}j\right) \Rightarrow m\ddot{x} = X \ \& \ m\ddot{y} = Y$$

$$v = \dot{r} = \frac{dr}{dt} = \frac{dx}{dt}i + \frac{dy}{dt}j = \dot{x}i + \dot{y}j$$

$$v^2 = (\dot{x})^2 + (\dot{y})^2$$

Lecture 15-2

Example

A particle moves in such a way that its position vector at time t is $r = (a\cos nt)i + (b\sin nt)$ where a, b, n are constants & a>b>0. Show that the path of particle is an ellipse of semi-major & minor axis a, b respectively, & that the field of force is directed towards the center of ellipse.

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Find also the maximum speed.

$r = xi + yj$

$x = a\cos nt \ \& \ y = b\sin nt$

$r = (a\cos nt)i + (b\sin nt)$

As equation of ellipse is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{a^2\cos^2 nt}{a^2} + \frac{b^2\sin^2 nt}{b^2} = 1 \Rightarrow \cos^2 nt + \sin^2 nt = 1$$

which shows that path of particle ia an ellipse

In cartesian system

$v^2 = (\dot{x})^2 + (\dot{y})^2$

$\dot{x} = -an\sin nt \quad \& \ \dot{y} = bn\cos nt$

$v^2 = (-an\sin nt)^2 + (bn\cos nt)^2 \Rightarrow a^2n^2\sin^2 nt + b^2n^2\cos^2 nt$

$v^2 = a^2n^2\sin^2 nt + b^2n^2(1-\sin^2 nt) = a^2n^2\sin^2 nt + b^2n^2 - b^2n^2\sin^2 nt = n^2(a^2 - b^2)\sin^2 nt + n^2b^2$

when $\sin^2 nt = 1$ *then* v^2 *will* max*imum*

$v_{\max}^2 = a^2n^2 - b^2n^2 + n^2b^2 = a^2n^2$

$v_{\max} = na$

Now we will find force

$F = ma = m\dot{v} = m\ddot{r}$

$\dot{x} = -an\sin nt \ \ then \ \ddot{x} = -an^2\cos nt \ \& \ \dot{y} = bn\cos nt \ \ then \ \ddot{y} = -bn^2\sin nt$

$\ddot{r} = \ddot{x}i + \ddot{y}j \Rightarrow \left(-an^2\cos nt\right)i + \left(-bn^2\sin nt\right) = -n^2\left(a\cos nti + b\sin ntj\right) = -n^2r$

$F = m\ddot{r} = m(-n^2r) = -mn^2r$

Lecture 16-1

Two dimensional Polar form of equation of Motion

Let suppose the motion of particle is restricted to x y. then polar coordinate of particle is (r, θ). The radial & transverse component of acceleration is

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$$a_r = \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right) = \ddot{r} - r(\dot{\theta})$$

$$\text{transverse component of acceleration} = a_\theta = 2\frac{dr}{dt}.\frac{d\theta}{dt} + r\frac{d^2\theta}{dt^2}$$

$$a_\theta = 2.\dot{r}.\dot{\theta} + r\ddot{\theta}$$

if $\hat{r}$ & $\hat{s}$ *are radial & transverse unit vector*

$$F = F_r\hat{r} + F_\theta\hat{s}$$

$$a = a_r\hat{r} + a_\theta\hat{s}$$

$$F = ma \Rightarrow F_r\hat{r} + F_\theta\hat{s} = m\left(a_r\hat{r} + a_\theta\hat{s}\right)$$

$$F_r = ma_r = m(\ddot{r} - r(\dot{\theta})^2)$$

$$F_\theta = ma_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

if the particle is moving along a cirlce of radius "a"

$$r = a, \dot{r} = 0, \ddot{r} = 0$$

then $F_r = -ma(\dot{\theta})^2$ & $F_\theta = ma(\ddot{\theta})$

Lecture 16-2

Example: -A particle of mass m moves on xy-plane under the force $F = -\frac{k}{r^4}\vec{r}$ where r is its distance from the origin O, If it starts on Positive x-axis at a distance **a** from O with speed v_0 in a direction making an angle α with the positive x-direction. Prove that at time t $\ddot{r} = \frac{ma^2v_0^2\sin^2\alpha - k}{mr^3}$.

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$$Radial\ Force = F_r = -\frac{k}{r^4}\bar{r} = -\frac{k}{r^4}r.\hat{r} = -\frac{k}{r^3}\hat{r}, F_\theta = 0$$

$$F_r = m(\ddot{r} - r(\dot{\theta})^2) \quad \& F_\theta = m(2\dot{r}\dot{\theta} - r(\ddot{\theta}))$$

$$m(\ddot{r} - r(\dot{\theta})^2) = -\frac{k}{r^3} \Rightarrow (\ddot{r} - r(\dot{\theta})^2) = -\frac{k}{mr^3}$$

$$m(2\dot{r}\dot{\theta} - r(\ddot{\theta})) = 0 \Rightarrow 2\dot{r}\dot{\theta} - r(\ddot{\theta}) = 0 \quad by\ \text{int}\,egrating \Rightarrow \frac{1}{r}\left(\frac{d}{dt}(r^2\dot{\theta})\right) = 0$$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \Rightarrow r^2\dot{\theta} = c$$

$$v = \dot{v}\hat{r} + r\dot{\theta}\hat{s}\ at \quad r = a\ \&\ v = v_0$$

$$v_0 = (v_0\cos\alpha)\hat{r} + (v_0\sin\alpha)\hat{s}$$

$$(\dot{r})_{r=a} = v_0\cos\alpha \quad \& (r\dot{\theta})_{r=a} = v_0\sin\alpha \Rightarrow a\dot{\theta} = v_0\sin\alpha \Rightarrow \dot{\theta} = \frac{v_0\sin\alpha}{a}$$

$$r^2\dot{\theta} = c \Rightarrow a^2\left(\frac{v_0\sin\alpha}{a}\right) = c \Rightarrow av_0\sin\alpha = c$$

$$r^2\dot{\theta} = av_0\sin\alpha \Rightarrow \dot{\theta} = \frac{av_0\sin\alpha}{r^2}$$

$$(\ddot{r} - r(\dot{\theta})^2) = -\frac{k}{mr^3} \Rightarrow \ddot{r} = r(\dot{\theta})^2 - \frac{k}{mr^3} = r\left(\frac{av_0\sin\alpha}{r^2}\right)^2 - \frac{k}{mr^3}$$

$$\ddot{r} = r\frac{a^2v_0^2\sin^2\alpha}{r^4} - \frac{k}{mr^3}$$

$$\ddot{r} = \frac{ma^2v_0^2\sin^2\alpha - k}{mr^3}$$

Lecture 17-1

If force F acting on particle whose position vector r produces a displacement, then work done will equal to

$$dw = F.dr$$

$$W = \int_{P_1}^{P_2} F.dr = \int_{r_1}^{r_2} F.dr \Rightarrow As\ \ dw = F.dr\ \ so\ \ P = \frac{dw}{dt} = F.\frac{dr}{dt} \Rightarrow \frac{dw}{dt} = F.v\ \ \& K.E = T = \frac{1}{2}mv^2$$

Principle of Energy (Theorem)

The total work done on a particle in moving it along a curve C from P_1 to P_2 is equal to increase T_2-T_1 in the K.E, where T_2 & T_1 are K.E of particles at t_1 & t_2, Corresponding to the positions P_1, P_2.

Proof

Work done by the particle as it moves from P_1 *to* P_2 *is given by*

$$W = \int_{r_1}^{r_2} F.dr \ \Rightarrow F = ma = m\frac{dv}{dt}$$

$$W = \int_{r_1}^{r_2} ma.dr = \int_{r_1}^{r_2} m\frac{dv}{dt}dr = \int_{r_1}^{r_2} m\frac{dr}{dt}dv = \int_{r_1}^{r_2} mvdv = \left|\frac{mv^2}{2}\right|_{r_1}^{r_2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = \tau_2 - \tau_1$$

The rate of increase of K.E of a particle is equal to the power applied to the particle.

$$\tau = \frac{1}{2}mv^2 \Rightarrow \frac{d\tau}{dt} = \frac{1}{2}2mv\frac{dv}{dt} = mva = F.v = P \quad \because \tau = K.E$$

Lecture 18-1

Conservative Force & Principle of Conservation of Energy

The Work done by a conservative force is independent of the path, in other words, the work done by conservative force is the same for any path connecting two points.

$$if\ Position\ Vector = x\hat{i} + yj + zk$$

$$F = -\frac{\partial v}{\partial x}i - \frac{\partial v}{\partial y}j - \frac{\partial v}{\partial z}k = -\nabla v = gradient\ of\ zero$$

Let $f(x, y, z)$ *be a scalar-valued function. then its gradient is*

$$\nabla f(x, y, z) = \left(\frac{\partial f}{\partial x}(x, y, z), \frac{\partial f}{\partial y}(x, y, z), \frac{\partial f}{\partial z}(x, y, z)\right) \ \text{is a vector field.}$$

Which is denoted by $F = \nabla f$. *We can easily calculate the curl of* F *is zero.*

We use the formula for curl F *in terms of its components*

$$Curl\ F = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_2}{\partial z} - \frac{\partial F_1}{\partial x}, \frac{\partial F_1}{\partial x} - \frac{\partial F_2}{\partial y}\right)$$

Since each component of F *is a derivative of* f, *we can rewrite the curl as*

$$curl\nabla f = \left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}, \frac{\partial^2 f}{\partial z\partial x} - \frac{\partial^2 f}{\partial x\partial z}, \frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)$$

if f *is twice continuously differentiable, then its second derivatives are independent of the orde in which derivatives are applied. all the terms cancel in expression for curl* ∇f, *we conclude* $\nabla f = 0$.
The work done on a particle in moving from P1 to P2 under a conservative field of force, is the difference between the P.E of the particle at P1&P2 respectively.

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$$W=\int_{P_1}^{P_2} F.dr = W = \int_{P_1}^{P_2} -\left(\frac{\partial v}{\partial x}i+\frac{\partial v}{\partial y}j+\frac{\partial v}{\partial z}k\right).\left(dx\hat{i}+dy\,j+dz\,k\right)$$

$$W=-\int_{P_1}^{P_2}\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy+\frac{\partial v}{\partial z}dz\right)=-\int_{P_1}^{P_2} dv=-|v|_{P_1}^{P_2}=-\left(v_{P_2}-v_{P_1}\right)=v_1-v_2$$

if τ *is K.E & V is potentail energy then* $E=\tau+V$

Theorem: - In a conservative field of force the total energy of a particle remains constant throughout the motion.

Proof: - $W=v_1-v_2$ & $W=T_2-T_1$ so $v_1-v_2=T_2-T_1 \Rightarrow T_1+v_1=T_2+v_2$

Lecture 19-1

Earth Gravitational Field

Earth pulls all material with a force which is called Earth Gravitational force. If the vertical acceleration is g which is produced by free falling body, is called gravitational acceleration. If m is the mass of body force is equal to g than F=ma implies $W=mg=-mg_j$ then $g=-g_j$.

$v=mgy \Rightarrow$ *y is the height of particle at time t & v is P.E*

Field of force will conservative if we have scalar field (x, y, z)

$$F=-\frac{\partial v}{\partial x}i-\frac{\partial v}{\partial y}j-\frac{\partial v}{\partial z}k \quad \because -\frac{\partial v}{\partial y}j=-mg=W \text{ hence } W \text{ will conservative}$$

Example: -A particle of mass m falls freely in a vertical plane under gravitational field from a height h with speed v_0. Find the speed with which it strikes the ground.

$K.E=\frac{1}{2}mv^2$ & $P.E=mgy$

By principle of conservation of energy

$$\frac{1}{2}mv^2+mgy=C$$

$t=0, y=h, v=v_0$

$\frac{1}{2}mv_0^2+mgh=C$ *by putting the value of C*

$$\frac{1}{2}mv^2+mgy=\frac{1}{2}mv_0^2+mgh \Rightarrow \frac{1}{2}mv^2-\frac{1}{2}mv_0^2=mgh-mgy$$

$$\frac{1}{2}m\left(v^2-v_0^2\right)=mg\left(h-y\right) \Rightarrow v^2-v_0^2=2g\left(h-y\right) \Rightarrow v^2=v_0^2+2g\left(h-y\right)$$

when object strike the Groung $y=0$

$$v^2=v_0^2+2gh \Rightarrow v=\sqrt{v_0^2+2gh}$$

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Lecture 20-1

Principle of Angular Momentum

The moment of linear momentum of an object is called angular momentum. Angular momentum is represented by L. Momentum is the motion contained in a body. Quantity of motion possessed by a body depends upon both of its mass & velocity. So the product of mass & velocity is the measure of the momentum. The angular momentum of an object about an axis of rotation is measured by the product of the linear momentum of the object & the perpendicular distance between the object & the axis of rotation.

Suppose an object of mass m is revolving around a circle of radius r with speed v about an axis passing through center. The linear momentum of object, p=mv

m of object, L=linear momentum × Perpendicular distance from axis of rotation. L=P r which implies that L=mvr. This equation expresses the angular momentum of body. It is a vector quantity. $torque = \tau = r \times F$

Theorem: - The rate of change of angular momentum of particle about a point O is equal to the torque of forces acting on the particle.

$$torque = \tau = r \times F$$
$$L = m(v \times r)$$
$$\frac{dL}{dt} = \frac{d}{dt}\left(m(v \times r)\right)$$
$$\frac{dL}{dt} = \frac{dr}{dt}mv + r\frac{d}{dt}(mv)$$
$$\frac{dL}{dt} = v \times mv + r\frac{d}{dt}(mv) \because cross\ \ product\ is\ zero v \times mv = 0$$
$$\frac{dL}{dt} = r\frac{d}{dt}(mv) \Rightarrow r \times F = \tau$$

Lecture 21-1

Motion of Centre of Mass of a System of Particles

Let O be the origin & $r_1, r_2, ...r_n$ the position vectors of a system S of n Particles of masses $m_1, m_2, ..., m_n$ respectively at time t. Then the center of mass G of the system S is the point whose position vector is

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$$r_G = \frac{\sum_{i=1}^{n} m_i r_i}{\sum_{i=1}^{n} m_i} = \frac{1}{M}\sum_{i=1}^{n} m_i r_i$$

$$linear\ Momentum = \rho = \sum_{i=1}^{n} m_i v_i$$

$$Let\ external\ force = F_i$$

$$Internal\ Force = F_i^{'} = \sum_{j=1}^{n} F_{ij} \qquad \because F_{ii} = 0$$

$$By\ Newton\ 3rd\ law\ of\ motion \quad F_{ij} = -F_{ji}$$

$$\sum_{i=1}^{n} F_i^{'} = \sum_{i=1}^{n}\sum_{j=1}^{n} F_{ij} = \sum_{i.j=1}^{n} F_{ji} = -\sum_{i.j=1}^{n} F_{ij} = -\sum_{i=1}^{n}\sum_{j=1}^{n} F_{ji} = -\sum_{i=1}^{n} F_i^{'}$$

$$\sum_{i=1}^{n} F_i^{'} = -\sum_{i=1}^{n} F_i^{'} \Rightarrow 2\sum_{i=1}^{n} F_i^{'} = 0 \Rightarrow \sum_{i=1}^{n} F_i^{'} = 0$$

This proves that sum of internal forces acting on body is zero.

Lecture 21-1

The Total work done by the forces on a system of particles in moving from the configuration at t₁ to the configuration at time t₂ is equal to increase in K.E.

$$W = \int_{P}^{Q} Fdr = \sum\int_{P_i}^{Q_i} F_i dr_i = \sum_{i=1}^{n}\int_{P_i}^{Q_i} m_i \frac{dv_i}{dt}.dr_i = \sum_{i=1}^{n}\int_{P_i}^{Q_i} m_i \frac{dr_i}{dt}.dv_i = \sum\int_{P_i}^{Q_i} m_i v.dv_i = \sum\left|\frac{mv_i^2}{2}\right|_{P_i}^{Q_i}$$

$$As\ K.E = T = \frac{1}{2}mv^2 \quad W = \sum\left|\frac{mv_i^2}{2}\right|_{P_i}^{Q_i} = T_2 - T_1$$

$$F_i = -\nabla v_i = -\left(\frac{\partial v_i}{\partial x} i + \frac{\partial v_i}{\partial y} j + \frac{\partial v_i}{\partial z} k\right)$$

$$F_i dr_i = -dv_i$$

$$W = -\sum\int_{P_i}^{Q_i} dv_i = -\sum |v_i|_{P_i}^{Q_i} = -(v_2 - v_1) = v_1 - v_2$$

$$T_2 - T_1 = v_1 - v_2$$

$$T_2 + v_2 = T_1 + v_1$$

Lecture 22-1

Example: -A ring of mass m slides on a smooth vertical rod & is attached to a light string which passes over a small pulley distant **a** from the rod, & has mass M>m fastened to its other end.

Show that if the ring is dropped from the point in the rod in the same horizontal plane, it will descend a distance $\frac{2amM}{M^2-m^2}$ before coming to the rest.

The system starts from rest. Suppose that after it is released the system comes to rest again (instantaneously) when the ring has fallen a **distance y** below its initial position and the block has been raised by **distance h.** The PE lost by the ring must equal the PE gained by the block, since neither mass has KE at this instant. Therefore
mgy=Mgh

Now we have to find how y is related to h.
When the ring has fallen a distance y, the length of the string between the peg and the ring is L where

$$L^2 = a^2 + y^2 \qquad here\ h = L - a\ impiesL = h + a$$

$$L^2 = (h+a)^2 \Rightarrow (h+a)^2 = a^2 + y^2 \Rightarrow h^2 + a^2 + 2ah = a^2 + y^2$$

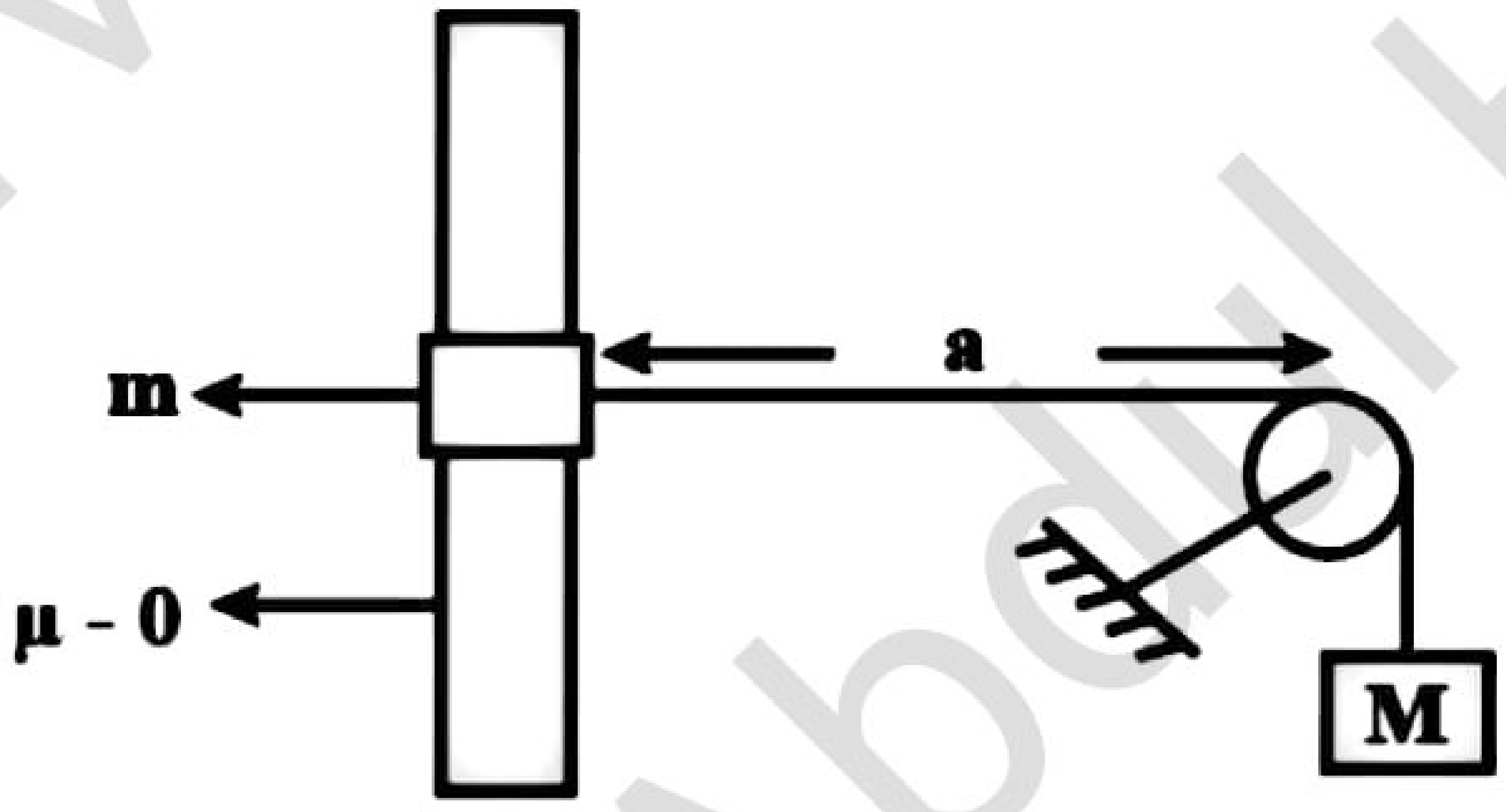
$$y^2 = h^2 + 2ah$$

$$As\ \ mgy = Mgh \Rightarrow my = Mh \Rightarrow \frac{m}{M}y = h$$

$$let\ k = \frac{m}{M}\ then\ h = ky$$

$$y^2 = k^2y^2 + 2aky \Rightarrow y^2 - k^2y^2 = 2aky \Rightarrow y^2\left(1-k^2\right) = 2aky$$

$$y\left(1-k^2\right) = 2ak \Rightarrow y = \frac{2ak}{1-k^2} = \frac{2a\frac{m}{M}}{1-\left(\frac{m}{M}\right)^2} = \frac{\frac{2am}{M}}{\frac{M^2-m^2}{M^2}} = \frac{2amM}{\left(M^2-m^2\right)}$$



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Dynamics MTH (404) Notes For Final Term Preparation

Lecture (23)

Projectile Motion

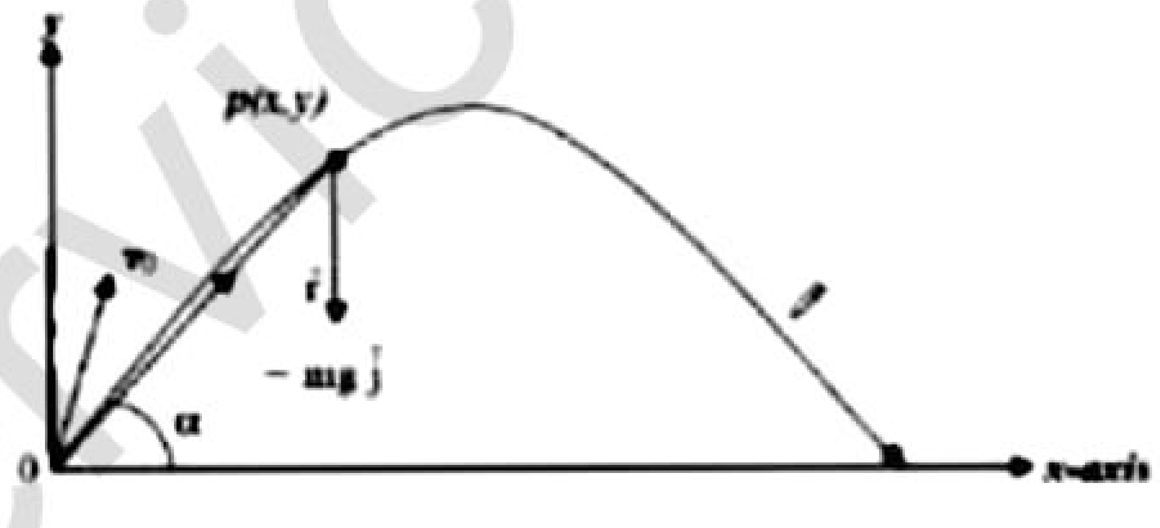
Projectile motion is a form of motion where an object moves in a bilaterally Symmetrical, parabolic path. The path that the object follows is called its trajectory.

Projectile motion only occurs when there is one force applied at the beginning on the

Trajectory, after which the only interference is from gravity.

Trajectory of a Projectile

Let a particle of mass 'm' is projected from a point 'O' with initial velocity 'v_o' making an angle 'α' with horizontal. Take 'O' as origin and horizontal and vertical lines through 'O' as x-axis and y-axis respectively.



$$r = x\hat{i} + y\hat{j}$$

$$\frac{dr}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\vec{v} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j}$$

$$\vec{F} = -mg\,\hat{j}$$

$$F = ma$$

$$-mg\,\hat{j} = m\left(\frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}\right)$$

$$0.i - g\,\hat{j} = \frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}$$

$$\Rightarrow \frac{d^2x}{dt^2} = 0\ ,\quad \frac{d^2y}{dt^2} = g$$

$$\int \frac{d^2x}{dt^2} - \frac{dx}{dt} = A$$

$$\int \frac{d^2y}{dt^2} - \frac{dy}{dt} = -gt + B$$

$$t = 0 \qquad \frac{dx}{dt} = V_\circ \cos\alpha$$

$$\frac{dy}{dt} = V_\circ \sin\alpha$$

$$A = V_\circ \cos\alpha \qquad B = V_\circ \sin\alpha$$

$$\frac{dx}{dt} = (V_\circ \cos\alpha) \qquad \frac{dy}{dt} = -gt + V_\circ \sin\alpha$$

$$x = (V_\circ \cos\alpha) + C$$

$$y = (V_\circ \sin\alpha)t - \frac{t^2}{2}g + D$$

$$t = 0 \quad ,x = 0 \qquad ,y = 0$$

$$c = 0 \qquad ,\quad D = 0$$

$$y = x\tan\alpha - \frac{gx^2}{2V_\circ^2}\sec^2\alpha$$

$$x = V_\circ \cos\alpha t$$

$$y = V_\circ \sin\alpha - \frac{1}{2}gt^2$$

$$t = \frac{x}{V_\circ \cos\alpha}$$

$$y = V_\circ \sin\alpha\left(\frac{x}{V_\circ \cos\alpha}\right)$$

$$-\frac{1}{2}g\left(\frac{x}{V_\circ \cos\alpha}\right)$$

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Definition:

Trajectory is the path described by any projectile.

Lecture (24)

Properties of projectile

Objects experiencing projectile motion have a constant velocity in the horizontal direction, and a constantly changing velocity in the vertical direction. The trajectory resulting from this combination always has the shape of parabola.

Notice that the trajectory is a parabola.

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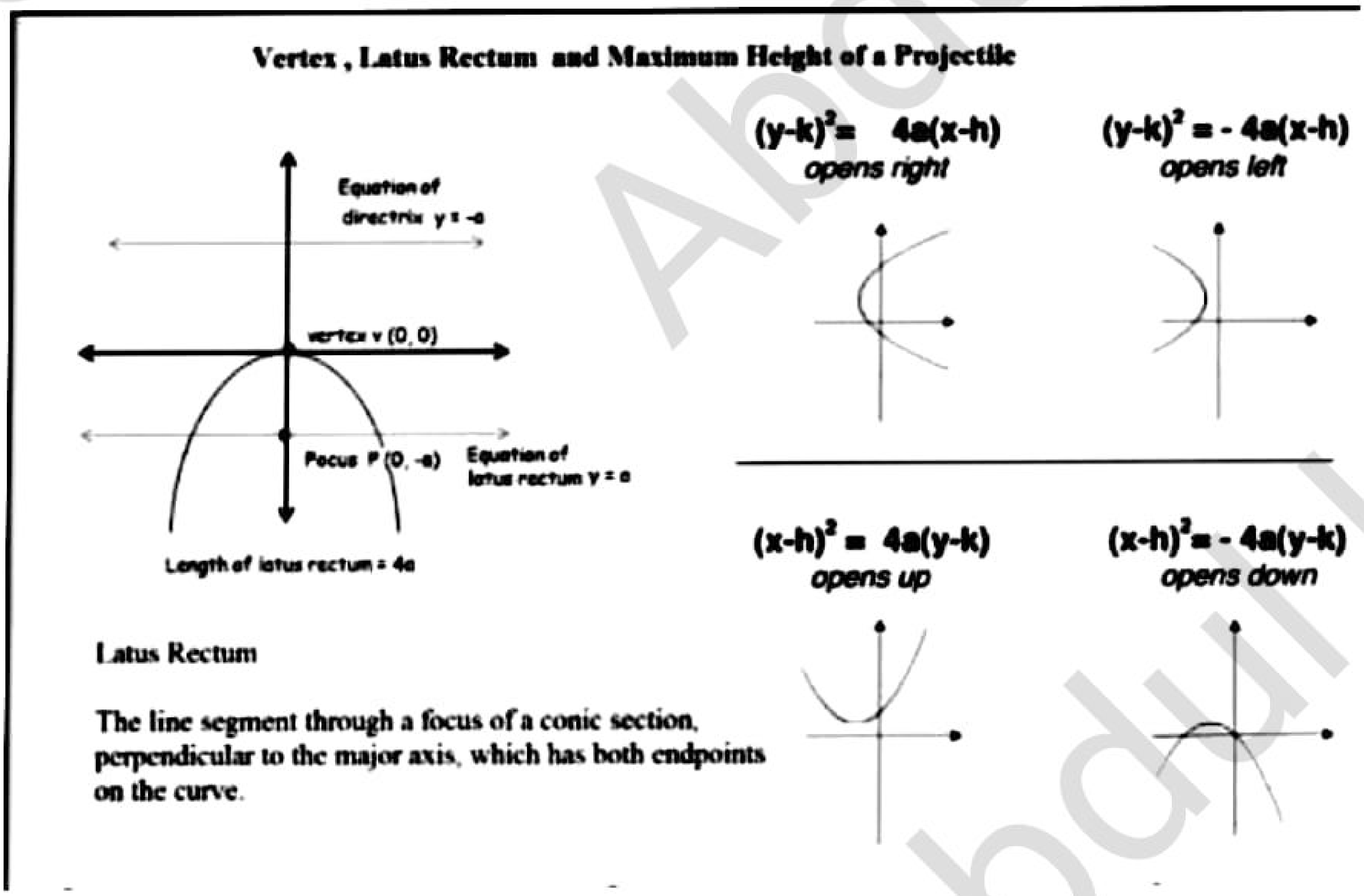

Vertex , Latus Rectum and Maximum Height of a Projectile

Latus Rectum

The line segment through a focus of a conic section, perpendicular to the major axis, which has both endpoints on the curve.

$$(h,k)$$
$$y^2 = 4ax$$
$$(x-4)^2 = 4ay-k$$
$$= -4a(y-k)$$

Latus Rectum

The line through a focus of a conic section,

Perpendicular to the major axis, which has both endpoints on the curve.

Focus of a Parabola

A parabola is set of all points in a plane which arte an equal distance away from a given

Point and agiven line. The point is called the focus of the parabola and the line is called the directrix.

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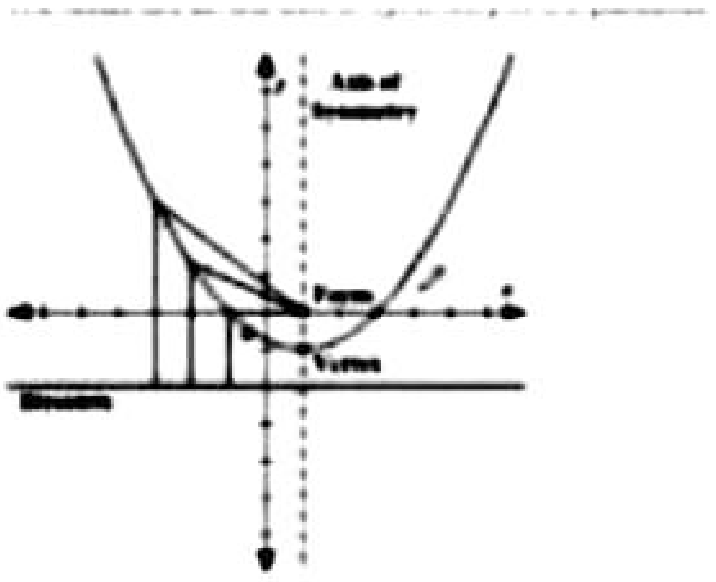
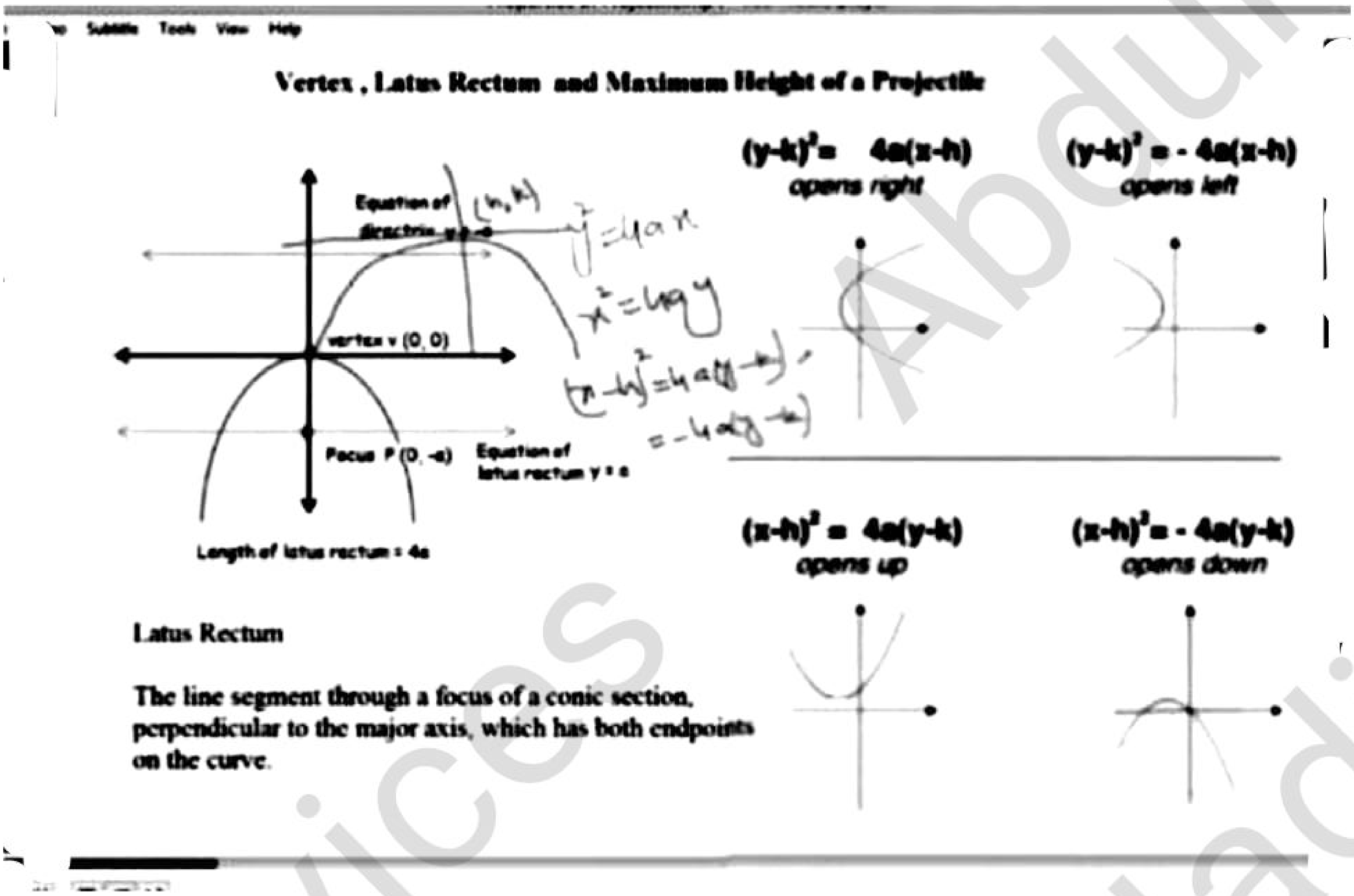
The focus lies on the axis of symmetry of the parabola.





Vertex , Latus Rectum and Maximum Height of a Projectile

$y^2=4ax$

$x^2=4ay$

$(x-h)^2=4a(y-k)$,

$=-4a(y-k)$

$(y-k)^2 = 4a(x-h)$
opens right

$(y-k)^2 = -4a(x-h)$
opens left

$(x-h)^2 = 4a(y-k)$
opens up

$(x-h)^2 = -4a(y-k)$
opens down

Latus Rectum

The line segment through a focus of a conic section, perpendicular to the major axis, which has both endpoints on the curve.

Finding the focus of a parabola given its equation

if you have the equation of a parabola in vertex from $y=a(z-h)^2+k$,

then the vertex is a

is at (h,k) and the focus is $\left(h,k+\frac{1}{4\alpha}\right)$.

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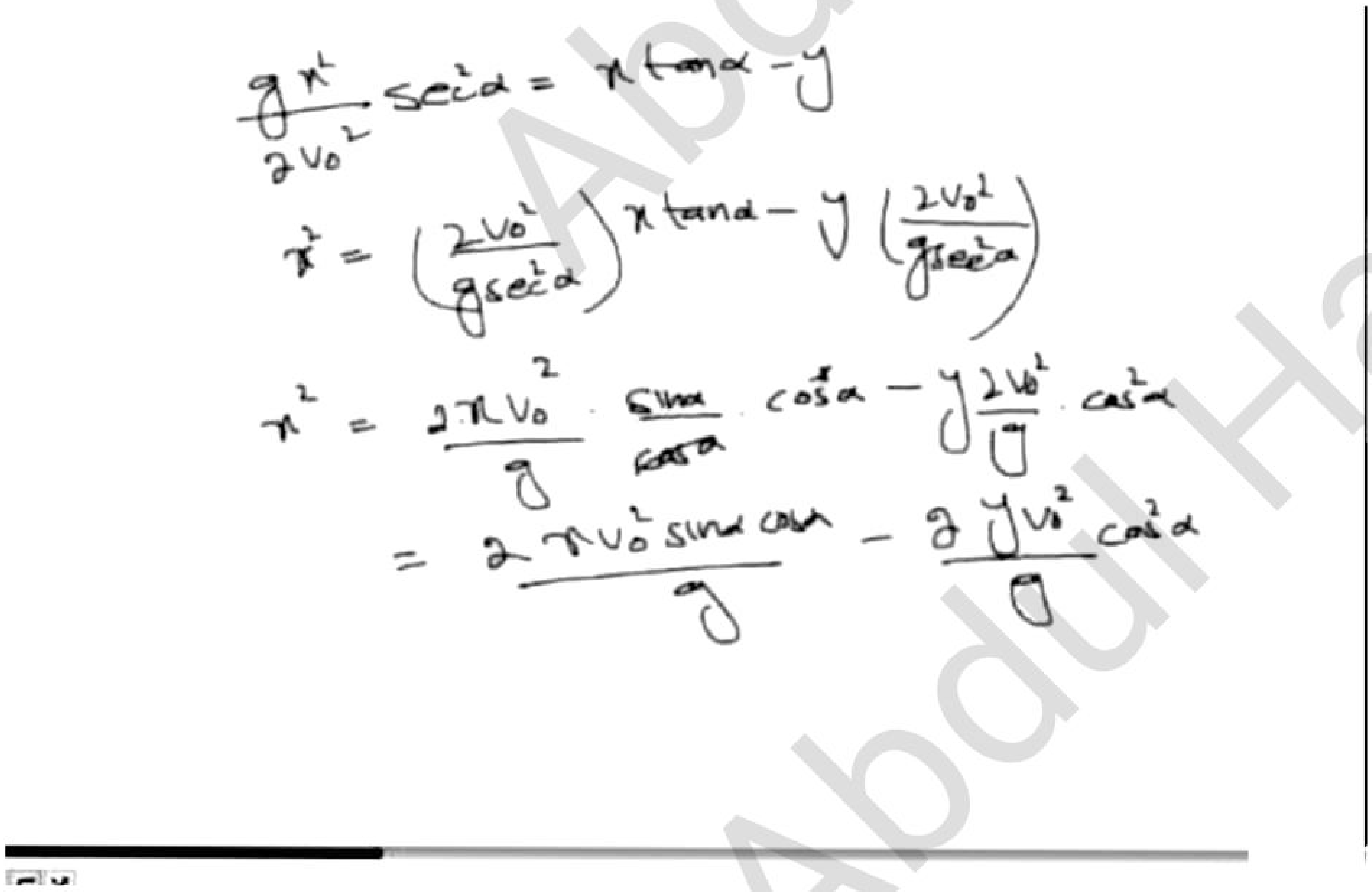
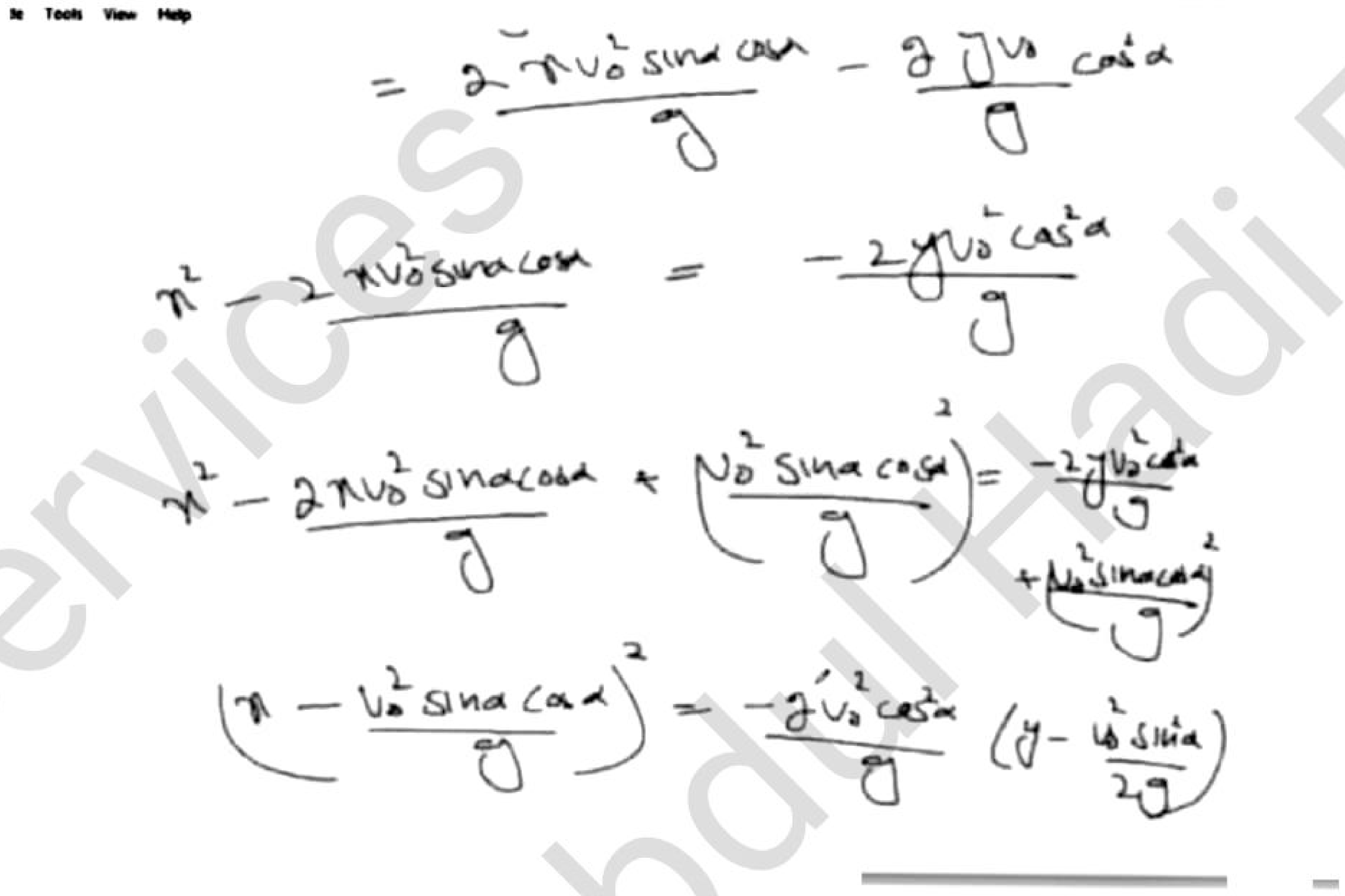


$$\frac{gx^2}{2v_0^2}\sec^2\alpha = x\tan\alpha - y$$

$$x^2 = \left(\frac{2v_0^2}{g\sec^2\alpha}\right)x\tan\alpha - y\left(\frac{2v_0^2}{g\sec^2\alpha}\right)$$

$$x^2 = \frac{2xv_0^2}{g}\cdot\frac{\sin\alpha}{\cos\alpha}\cdot\cos^2\alpha - y\frac{2v_0^2}{g}\cdot\cos^2\alpha$$

$$= \frac{2xv_0^2\sin\alpha\cos\alpha}{g} - \frac{2yv_0^2}{g}\cos^2\alpha$$

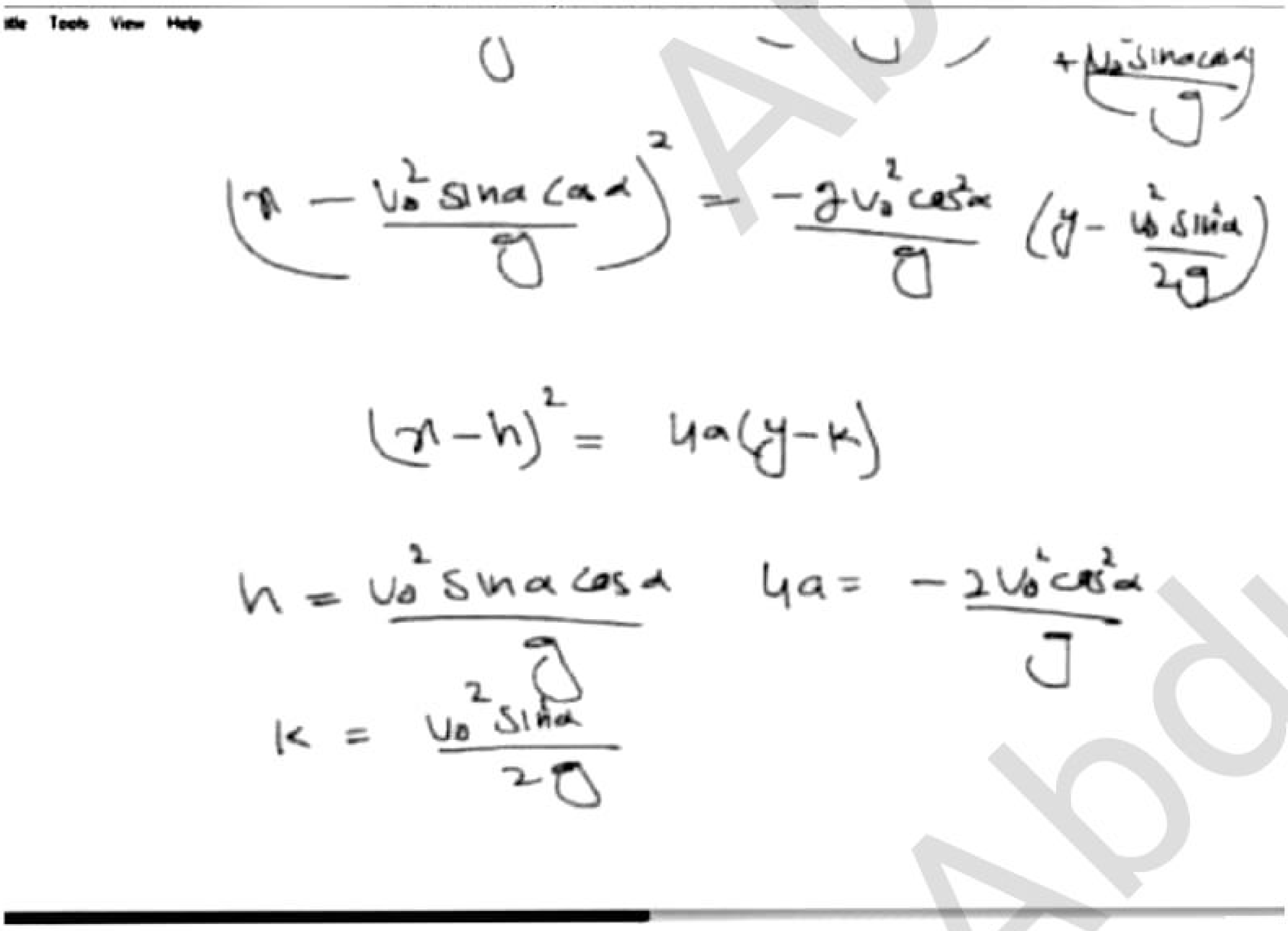


$$= \frac{2xv_0^2\sin\alpha\cos\alpha}{g} - \frac{2yv_0}{g}\cos^2\alpha$$

$$x^2 - \frac{2xv_0^2\sin\alpha\cos\alpha}{g} = -\frac{2yv_0^2\cos^2\alpha}{g}$$

$$x^2 - \frac{2xv_0^2\sin\alpha\cos\alpha}{g} + \left(\frac{v_0^2\sin\alpha\cos\alpha}{g}\right)^2 = -\frac{2yv_0^2\cos^2\alpha}{g} + \left(\frac{v_0^2\sin\alpha\cos\alpha}{g}\right)^2$$

$$\left(x - \frac{v_0^2\sin\alpha\cos\alpha}{g}\right)^2 = -\frac{2v_0^2\cos^2\alpha}{g}\left(y - \frac{v_0^2\sin^2\alpha}{2g}\right)$$



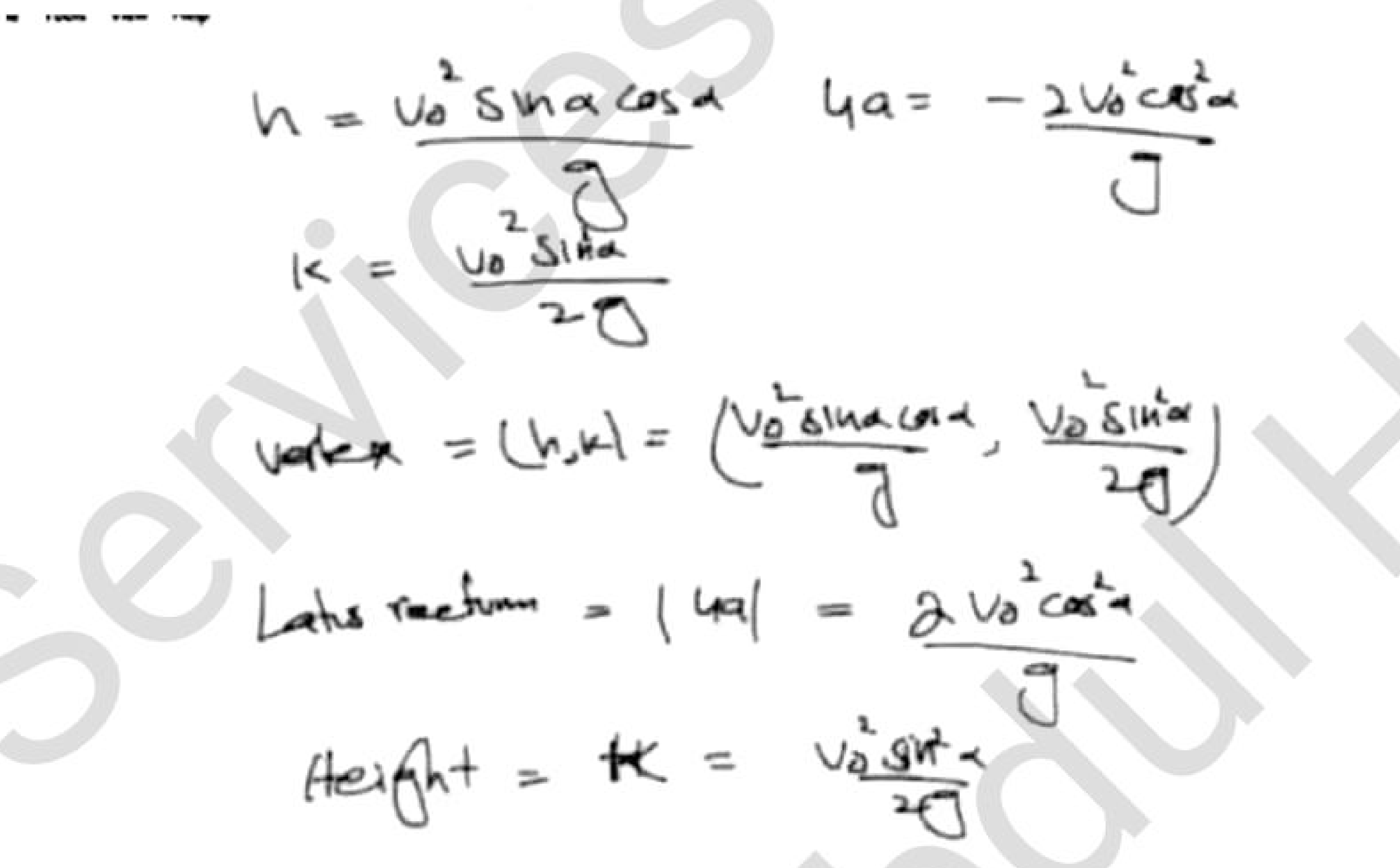
$$+\frac{v_0^2 \sin\alpha\cos\alpha}{g}$$

$$\left(x - \frac{v_0^2 \sin\alpha\cos\alpha}{g}\right)^2 = -\frac{2v_0^2\cos^2\alpha}{g}\left(y - \frac{v_0^2\sin^2\alpha}{2g}\right)$$

$$(x-h)^2 = 4a(y-k)$$

$$h = \frac{v_0^2 \sin\alpha\cos\alpha}{g} \qquad 4a = -\frac{2v_0^2\cos^2\alpha}{g}$$

$$k = \frac{v_0^2\sin^2\alpha}{2g}$$



$$h = \frac{v_0^2 \sin\alpha\cos\alpha}{g} \qquad 4a = -\frac{2v_0^2\cos^2\alpha}{g}$$

$$k = \frac{v_0^2\sin^2\alpha}{2g}$$

$$\text{vertex} = (h,k) = \left(\frac{v_0^2\sin\alpha\cos\alpha}{g}, \frac{v_0^2\sin^2\alpha}{2g}\right)$$

$$\text{Latus rectum} = |4a| = \frac{2v_0^2\cos^2\alpha}{g}$$

$$\text{Height} = k = \frac{v_0^2\sin^2\alpha}{2g}$$

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$$g$$

$$\text{Height} = H = \frac{v_0^2 \sin^2\alpha}{2g}$$

<u>Focus</u>

x-cordinate of Focus = x-cordinate of vertex

$$= \frac{v_0^2 \sin\alpha \cos\alpha}{g}$$

$$= \frac{2}{2g} v_0^2 \sin\alpha \cos\alpha$$

$$= \frac{v_0^2 \sin 2\alpha}{2g}$$

y-cordinate of Focus = $H - \frac{1}{4}$ (Latus rectum)

$$= \frac{v_0^2 \sin^2\alpha}{2g} - \frac{1}{4}\left(\frac{2 v_0^2 \cos^2\alpha}{g}\right)$$

$$= \frac{v_0^2 \sin^2\alpha}{2g} - \frac{v_0^2 \cos^2\alpha}{2g}$$

$$= \frac{-v_0^2}{2g}\left(\cos^2\alpha - \sin^2\alpha\right)$$

$$= \frac{-v_0^2}{2g} \cos 2\alpha$$

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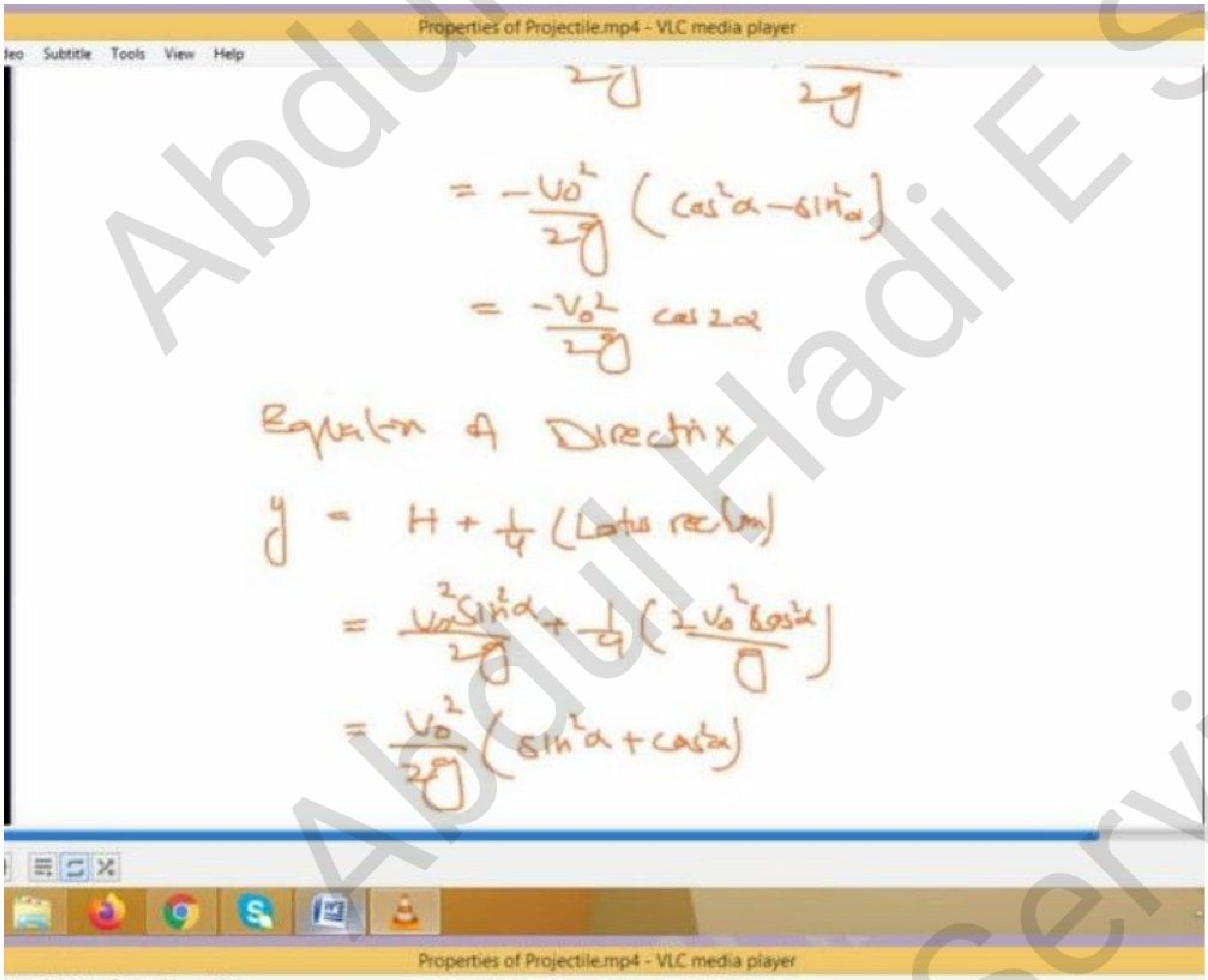


$2g$ $2g$

$$= -\frac{V_0^2}{2g}\left(\cos^2\alpha - \sin^2\alpha\right)$$

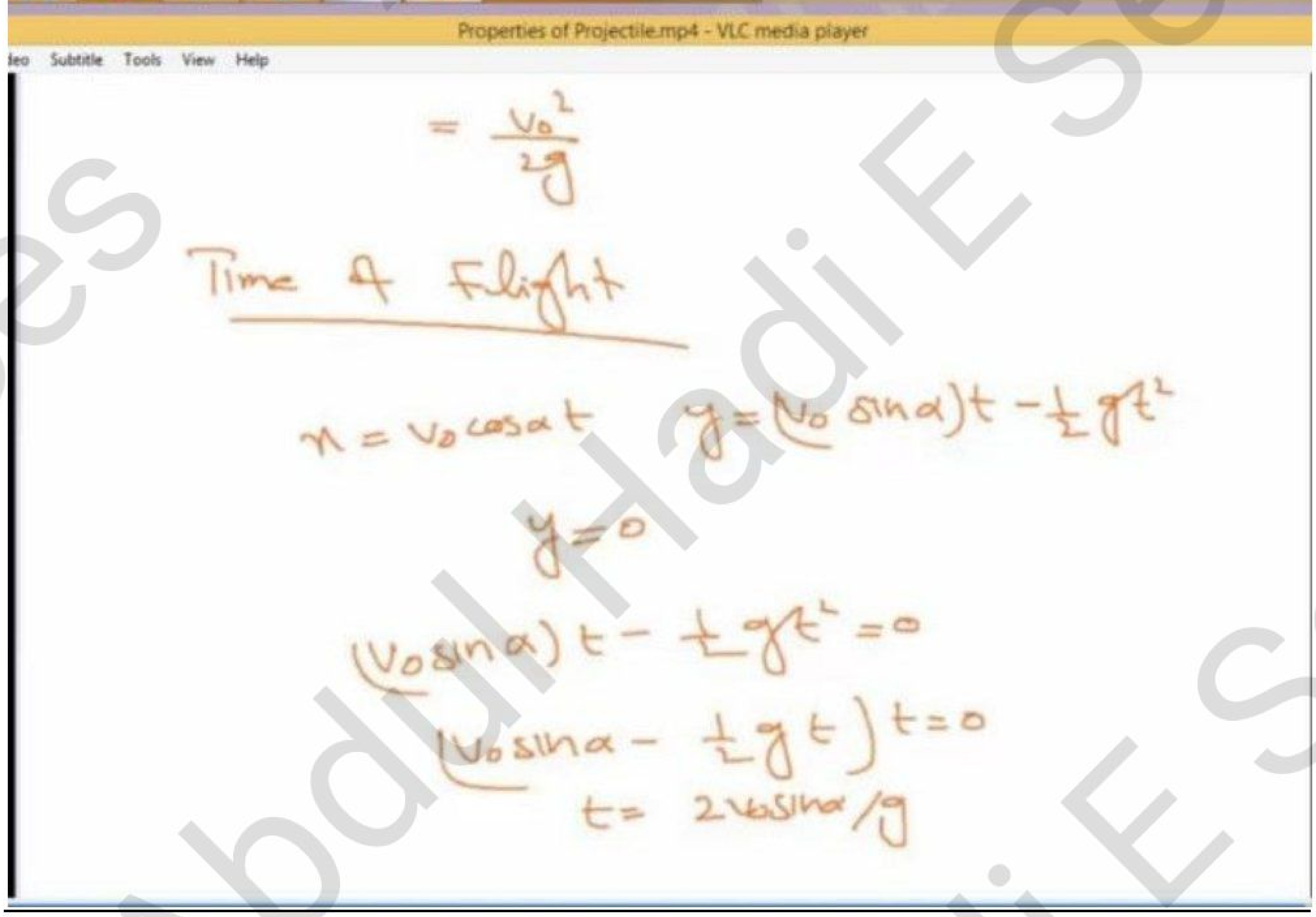
$$= \frac{-V_0^2}{2g}\cos 2\alpha$$

Equation of Directrix

$$y = H + \frac{1}{4}(\text{Latus rectum})$$

$$= \frac{V_0^2\sin^2\alpha}{2g} + \frac{1}{4}\left(\frac{2V_0^2\cos^2\alpha}{g}\right)$$

$$= \frac{V_0^2}{2g}\left(\sin^2\alpha + \cos^2\alpha\right)$$



$$= \frac{V_0^2}{2g}$$

Time of Flight

$$x = V_0\cos\alpha\, t \qquad y = (V_0\sin\alpha)t - \frac{1}{2}gt^2$$

$$y = 0$$

$$(V_0\sin\alpha)t - \frac{1}{2}gt^2 = 0$$

$$\left(V_0\sin\alpha - \frac{1}{2}gt\right)t = 0$$

$$t = 2V_0\sin\alpha/g$$

$$R = (\text{Horizontal velocity}) \times (\text{Time of flight})$$

$$= (v_0 \cos\alpha)\left(\frac{2v_0\sin\alpha}{g}\right)$$

$$= \frac{v_0^2}{g}\boxed{\sin 2\alpha}$$

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$$= \frac{v_0^2}{g}\boxed{\sin 2\alpha}$$

$$\sin 2\alpha = 1$$

$$2\alpha = \sin^{-1}(1)$$

$$2\alpha = 90$$

$$\alpha = 45^\circ$$

$$= \frac{v_0^2}{g}\sin(90)$$

$$R_{max} = \frac{v_0^2}{g}$$

<u>Lecture (25)</u>

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Examples of projectile Motion -1

Question

Determine the maximum possible range for a projectile fired from a cannon having

muzzle velocity v_0 and prove that the height reached in this case is

$$\frac{v_\circ^2}{4g}$$

Solution:

Range = horizontal velocity × Time of flight

$$= v_\circ \cos\alpha \times \frac{2v_\circ \sin\alpha}{g}$$
$$= \frac{v_\circ^2 \sin 2\alpha}{g}$$
$$\sin 2\alpha \ is \ \max$$
$$\sin 2\alpha = 1$$
$$2\alpha = \sin^{-1}(1)$$
$$2\alpha = 90$$
$$\alpha = 45$$
$$Range = \frac{v_\circ^2}{g}$$
$$Range_{\max} = \frac{v_\circ^2}{g}$$
$$v_\circ = 1 mile/\sec = 1760 \times 3/\sec$$
$$= 5280 \ mile/\sec$$

$$R_{\max} = (5280)^2 \qquad g = 9.8m/\sec^2$$
$$= 32$$

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Lecture (26,27,28,29,33) are included

Examples of Projectile Motion –(II),(III),(IV),(V),(VII)

Question

A projectile having horizontal range T , reaches a maximum height H.

Prove

that it must have been launched d with

a) an initial speed equal to

$$\sqrt{\frac{g\left(R^2+16H^2\right)}{8H}}$$

b) at an angle with horizontal given by

$$\sin^{-1}\left(\frac{4H}{\sqrt{R^2\_+16H^2}}\right)$$

Solution:

Let V_o be the velocity of projection and α is the angle of projection.

$$R=\frac{v_\circ^{2}}{g}\sin 2\alpha$$

$$\Rightarrow R^{2}=\frac{4v_\circ^{4}\sin^{2}\alpha\cos^{2}\alpha}{g^{2}}\ldots\ldots\ldots\ldots\ldots(i)$$

Given H is the maximum height of the projectile. $H=\dfrac{V_\circ^{2}\sin^{2}\alpha}{2g}\ldots\ldots(ii)$

Divide (i) *and* (ii) $\qquad \dfrac{R^{2}}{H}=\dfrac{4v_\circ^{2}\sin^{2}\alpha\cos^{2}\alpha}{g^{2}}\times\dfrac{2g}{v_\circ^{2}\sin^{2}\alpha}$

$$\Rightarrow \frac{R^{2}}{H}=\frac{8V_\circ^{2}\cos^{2}\alpha}{g}$$

$$\frac{gR^{2}}{8H}=V_\circ^{2}\cos^{2}\alpha\ldots\ldots\ldots\ldots(iii)$$

From equation (ii) $2gH=V_\circ^{2}\sin^{2}\alpha\ldots\ldots\ldots\ldots\ldots(iv)$

Adding equation (iii)*and*(iv) $\dfrac{gR^{2}}{8H}+2gH=V_\circ^{2}\cos^{2}\alpha+V_\circ^{2}\sin^{2}\alpha$

$$V_\circ^{2}=\frac{\mathrm{g}\mathrm{R}^{2}+16\,\mathrm{H}^{2}}{8H}\ldots\ldots\ldots\ldots(v)$$

Thus, $V_\circ=\sqrt{\dfrac{\mathrm{g}\mathrm{R}^{2}+16\,\mathrm{H}^{2}}{8H}}$

Now, we find the angle $'\alpha'$.

From equation (ii), $\qquad \sin^{2}\alpha=\dfrac{2gR}{V_\circ^{2}}$

$$\Rightarrow \quad \sin^{2}\alpha=\frac{2gH}{\dfrac{g\left(R^{2}+16H^{2}\right)}{8H}}=\frac{2gH\times 8H}{g\left(R^{2}+16H^{2}\right)}$$

$$\Rightarrow \quad \sin^{2}\alpha=\frac{16H^{2}}{R^{2}+16H^{2}}$$

$$\Rightarrow \quad \sin\alpha=\frac{4H}{\sqrt{R^{2}+16H^{2}}}$$

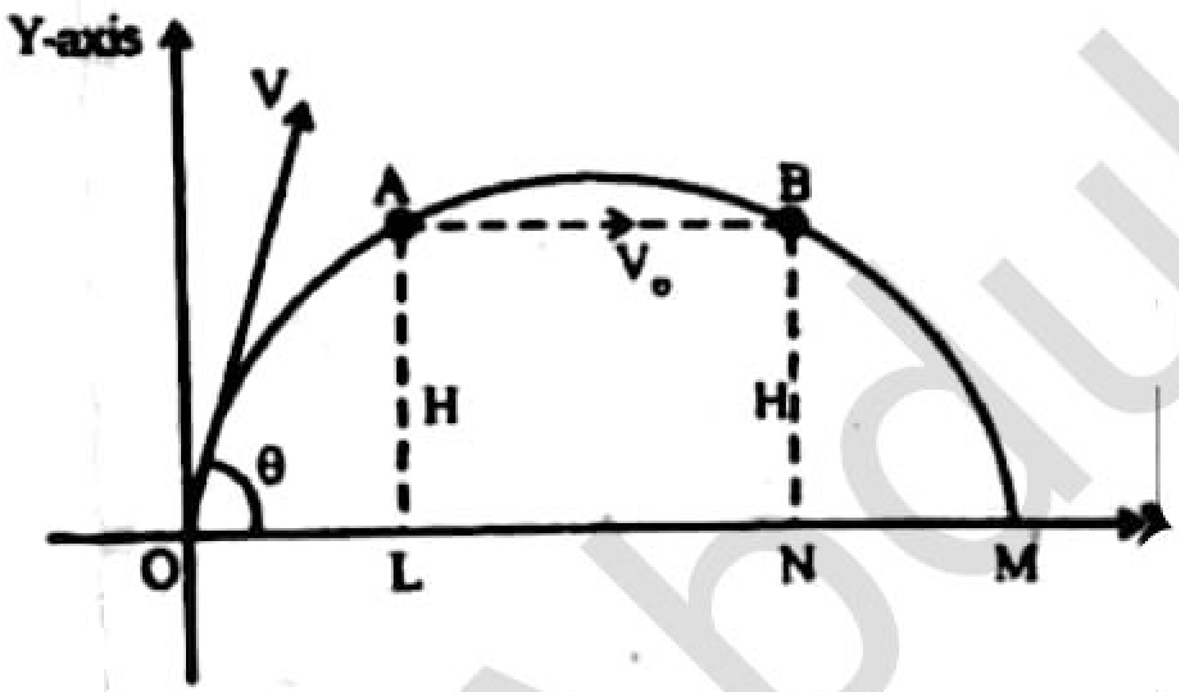
Thus, $\alpha=\sin^{-1}\left(\dfrac{4H}{\sqrt{R^{2}+16H^{2}}}\right)$

Example:

$\therefore \boxed{|AD| = 4\cos\alpha\sqrt{H(H\sin^2\alpha - h)}}$

Q.14. A shell fired with speed V at an elevation θ, hits an airship at height H, which is moving horizontally away from the gun with speed V_o. Show that if $(2V\cos\theta - V_o)(V^2\sin^2\theta - 2gH)^{1/2} = V_o V\sin\theta$, the shell might also have hit the airship if the latter had remained stationary in the position it occupied when the gun was actually fired.

Solution:



Let 'A' be the position of the airship when the gun was actually fired. Let the shell hits the airship at point 'B'. If the airship remains stationary at point A', then the shell also hits the airship at A'.

Given that the shell is fired with speed V at an elevation θ.

Now, $y = (V\sin\theta)t - \frac{1}{2}gt^2$

But y = H Therefore, $H = (V\sin\theta)t - \frac{1}{2}gt^2$

Multiply by '2' $2H = (2V\sin\theta)t - gt^2$

$\Rightarrow \quad gt^2 - (2V\sin\theta)t + 2H = 0$

$$t = \frac{2V\sin\theta \pm \sqrt{4V^2\sin^2\theta - 4(g)(2H)}}{2g}$$

So, $$t = \frac{V\sin\theta \pm \sqrt{V^2\sin^2\theta - 2gH}}{g}$$

Thus, $t_1 = \frac{V\sin\theta - \sqrt{V^2\sin^2\theta - 2gH}}{g}$ is the time of motion of shell from 'O' to 'A'

$t_2 = \frac{V\sin\theta + \sqrt{V^2\sin^2\theta - 2gH}}{g}$ is the time of motion of shell from 'O' to 'B'.

Thus, $t_2 - t_1 = \dfrac{2\sqrt{V^2\text{Sin}^2\theta - 2gH}}{g}$ .... (i)

This the time for shell to move from 'A' to 'B'.

Now, the horizontal distance covered by the shell in time $t_2 - t_1$ with uniform horizontal speed $V\cos\theta$ is

$$|AB| = (V\cos\theta)(t_2 - t_1) \quad (\text{using } S = vt)$$

Or $|AB| = \dfrac{2V\cos\theta\sqrt{V^2\sin^2\theta - 2gH}}{g}$ .... (ii)

Now, the shell will hit the airship at point 'B' if the time taken by the ship to move from 'A' to 'B' is equal to the time for shell to move from O to B.

So, the time for airship to move from A to B is also t_2 given by

$$t_2 = \frac{V\sin\theta + \sqrt{V^2\text{Sin}^2\theta - 2gH}}{g}$$

Now, $|AB| = V_o t_2$ (for ship to move from A to B)

So, $|AB| = \dfrac{V_oV\sin\theta + V_o\sqrt{V^2\text{Sin}^2\theta - 2gH}}{g}$ .... (iii)

From equations (ii) and (iii), we have

$$\frac{2V\cos\theta\sqrt{V^2\sin^2\theta - 2gH}}{g} = \frac{V_oV\sin\theta + V_o\sqrt{V^2\text{Sin}^2\theta - 2gH}}{g}$$

$\Rightarrow 2V\text{Cos}\theta\,(V^2\text{Sin}^2\theta - 2gH)^{1/2} = V_oV\text{Sin}\theta + V_o(V^2\text{Sin}^2\theta - 2gH)^{1/2}$

or $2V\text{Cos}\theta(V^2\text{Sin}^2\theta - 2gH)^{1/2} - V_o(V^2\text{Sin}^2\theta - 2gH)^{1/2} = V_oV\sin\theta$

Thus, $\boxed{(2V\text{Cos}\theta - V_o)(V^2\text{Sin}^2\theta - 2gH)^{1/2} = V_oV\sin\theta}$

Example:

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Q.18. **An aeroplane is flying with uniform speed V_o in an arc of a vertical circle of radius 'a' whose centre is at a height h vertically above a point O of the ground. If a bomb is dropped from the aeroplane when at a height Y and strikes the ground at O, show that Y satisfies the equation**

$$ky^2 + y\,(a^2 - 2hk) + k(h^2 - a^2) = 0, \text{ where } k = h + \frac{ga^2}{2V_o^2}$$

Solution:

Let the bomb is dropped from point O′ (origin). Also let the initial velocity of bomb is V_o and angle of projection is '$-\alpha$' with the horizontal. So, the equation of the path of projectile is

$$y = x\tan(-\alpha) - \frac{gx^2\sec^2(-\alpha)}{2V_o^2}$$

$$\Rightarrow y = -x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V_o^2}$$

Since the bomb hits at O$(x, -y)$, so it will satisfy above equation.

$$-y = -x\tan\alpha - \frac{gx^2\,\text{Sec}^2\alpha}{2V_o^2}$$

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$$\Rightarrow \quad y = x\tan\alpha + \frac{gx^2 \operatorname{Sec}^2\alpha}{2V_o^2} \quad \ldots\ldots (i)$$

For $\Delta O'AC$,

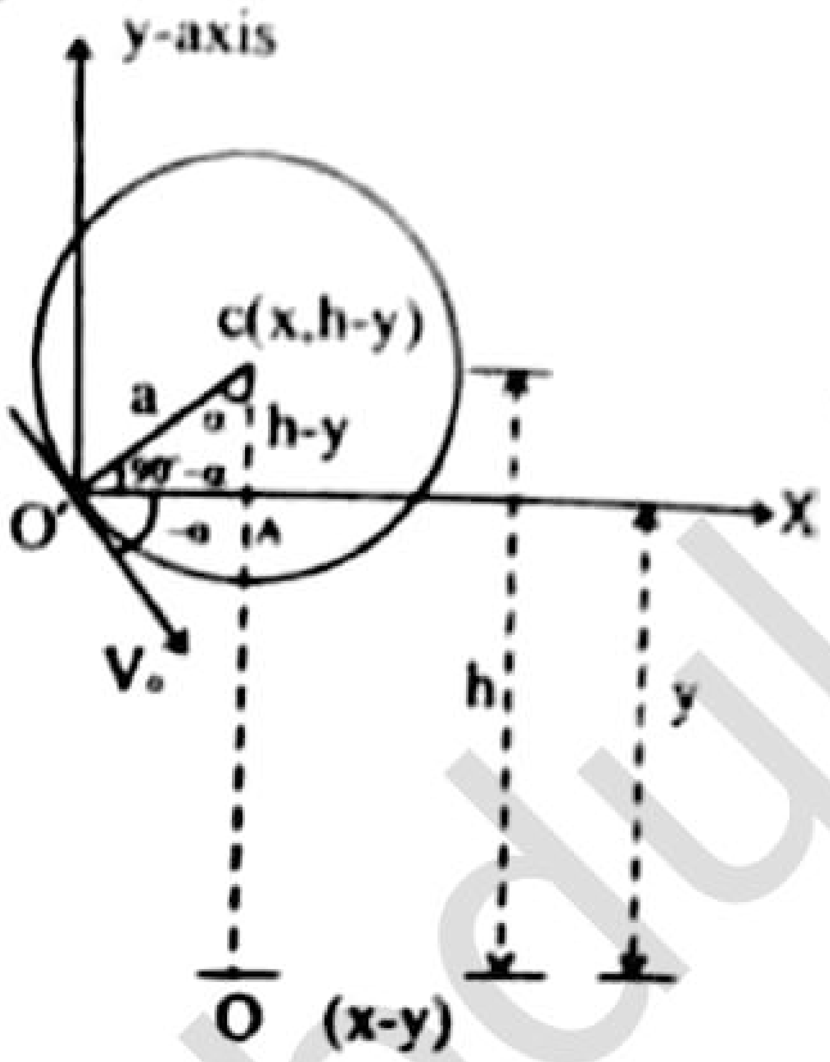
$$|O'A|^2 = |O'C|^2 - |AC|^2$$

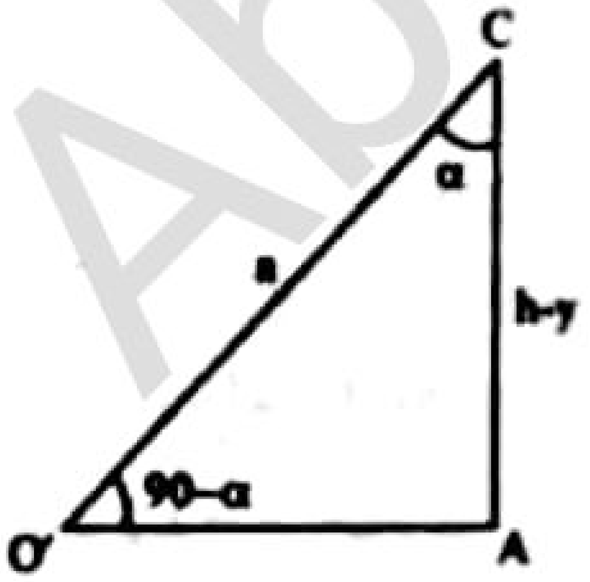
$$\Rightarrow \quad x^2 = a^2 - (h-y)^2$$

$$\text{or} \quad x = \sqrt{a^2 - (h-y)^2}$$

$$\text{Further, } \tan\alpha = \frac{\sqrt{a^2-(h-y)^2}}{h-y}$$

$$\operatorname{Cos}\alpha = \frac{h-y}{a} \Rightarrow \operatorname{Sec}\alpha = \frac{a}{h-y}$$





Putting these values in equation (i)

$$Y = \sqrt{a^2-(h-y)^2}\left(\frac{\sqrt{a^2-(h-y)^2}}{h-y}\right) + \frac{g}{2V_o^2}[a^2-(h-y)^2]\cdot\frac{a^2}{(h-y)^2}$$

Multiply by $(h-y)^2$

$$y(h-y)^2 = [a^2-(h-y)^2](h-y) + \frac{ga^2}{2V_o^2}[a^2-(h-y)^2]$$

$$y(h^2-2hy+y^2) = (a^2-h^2-y^2+2hy)(h-y) + \frac{ga^2}{2V_o^2}[a^2-h^2-y^2+2hy]$$

$$yh^2-2hy^2+y^3 = ha^2-h^3-hy^2+2h^2y-a^2y+h^2y+y^3-2hy^2 + \frac{ga^2(a^2)}{2V_o^2} - \frac{ga^2h^2}{2V_o^2} - \frac{ga^2y^2}{2V_o^2} + \frac{ga^2hy}{V_o^2}$$

$$0 = -\left(h+\frac{ga^2}{2V_o^2}\right)y^2 + \left(-a^2+2h^2+\frac{ga^2h}{V_o^2}\right)y + ha^2 - h^3 + \frac{ga^2(a^2)}{2V_o^2} - \frac{ga^2h^2}{2V_o^2}$$

$$-y^2\left(h+\frac{ga^2}{2V_o^2}\right) + \left[2h\left(h+\frac{ga^2}{2V_o^2}\right)-a^2\right]y + \left[h(a^2-h^2)+\frac{ga^2}{2V_o^2}(a^2-h^2)\right] = 0$$

$$-y^2\left(h+\frac{ga^2}{2V_o^2}\right) + \left\{2h\left(h+\frac{ga^2}{2V_o^2}\right)-a^2\right\}y + \left\{(a^2-h^2)\left(h+\frac{ga^2}{2V_o^2}\right)\right\} = 0$$

$$\text{Put} \qquad h + \frac{ga^2}{2V_o^2} = k$$

So, $-y^2k + (2hk - a^2)y + \{(a^2-h^2)k\} = 0$

Or $+y^2k - (2hk - a^2)y - (a^2 - h^2)k = 0$

Or $y^2k + (a^2 - 2hk)\,y + (h^2 - a^2)k = 0$

Is required.

Lecture (30)

Speed of Projectile Examples

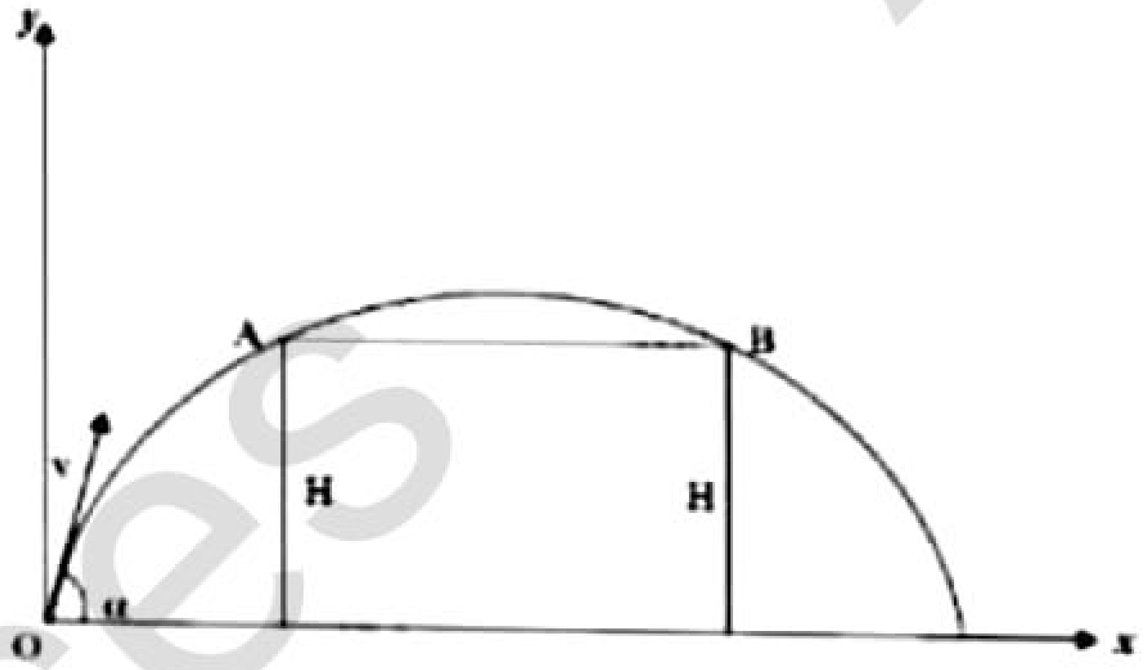
Question:

A shell fired with speed V at an elevation θ, hits an airship at height H... Which is moving

horizontally away from the gun with speed V_o. Show that if

$$(2V\cos\theta - v_\circ)\sqrt{(v^2\sin^2\theta - 2gH)} = v_\circ V\sin\theta$$

The shell might also have hit the air ship if the latter had remained stationary in the position it occupied when the gun was actually fired.



Solution:

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$$t = \frac{2V\sin\theta \pm \sqrt{4V^2 Sin^2\theta - 4(g)(2H)}}{2g}$$

$$So,\quad t = \frac{V\sin\theta \pm \sqrt{V^2 Sin^2\theta - 2gH}}{g}$$

Thus, $t_1 = \frac{V\sin\theta \pm \sqrt{V^2 Sin^2\theta - 2gH}}{g}$ *is the time of motion of shell from 'O' to 'A'*

$t_2 = \frac{V\sin\theta \pm \sqrt{V^2 Sin^2\theta - 2gH}}{g}$ *is the time of motion of shell from* 'O' to *'B'*

Thus, $t_2 - t_1 = \frac{2\sqrt{V^2 Sin^2\theta - 2gH}}{g}$ ....(1)

This the time for shell to move from 'A' to 'B'

Now, the horizontal distance covered by the shell in time $t_2 - t_1$ *with uniform horizontal speed* $V\cos\theta$ *is*

$|AB| = (V\cos\theta)(t_2 - t_1)(u\sin g\ S = vt)$

or $|AB| = \frac{2V\cos\theta\sqrt{V^2 Sin^2\theta - 2gH}}{g}$ ....(2)

Now, the shell will hit the airship at point 'B' if the time taken by the ship to

move from 'A' to 'B' is equal to the time for shell to move from O to B.

So, the time for airship to move from A to B is also t_2 given by

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$$t_2 = \frac{V\sin\theta \pm \sqrt{V^2 Sin^2\theta - 2gH}}{g}$$

Now, $|AB| = V_\circ t_2$ (*for ship to move from A to B*)

So, $$|AB| = \frac{V_\circ \sin\theta + V_\circ\sqrt{V^2 Sin^2\theta - 2gH}}{g} \quad\text{.............}(3)$$

From equations (*ii*) *and* (*iii*), *we have*

$$\frac{2V\cos\theta + V_\circ\sqrt{V^2 Sin^2\theta - 2gH}}{g} = \frac{V_\circ \sin\theta + V_\circ\sqrt{V^2 Sin^2\theta - 2gH}}{g}$$

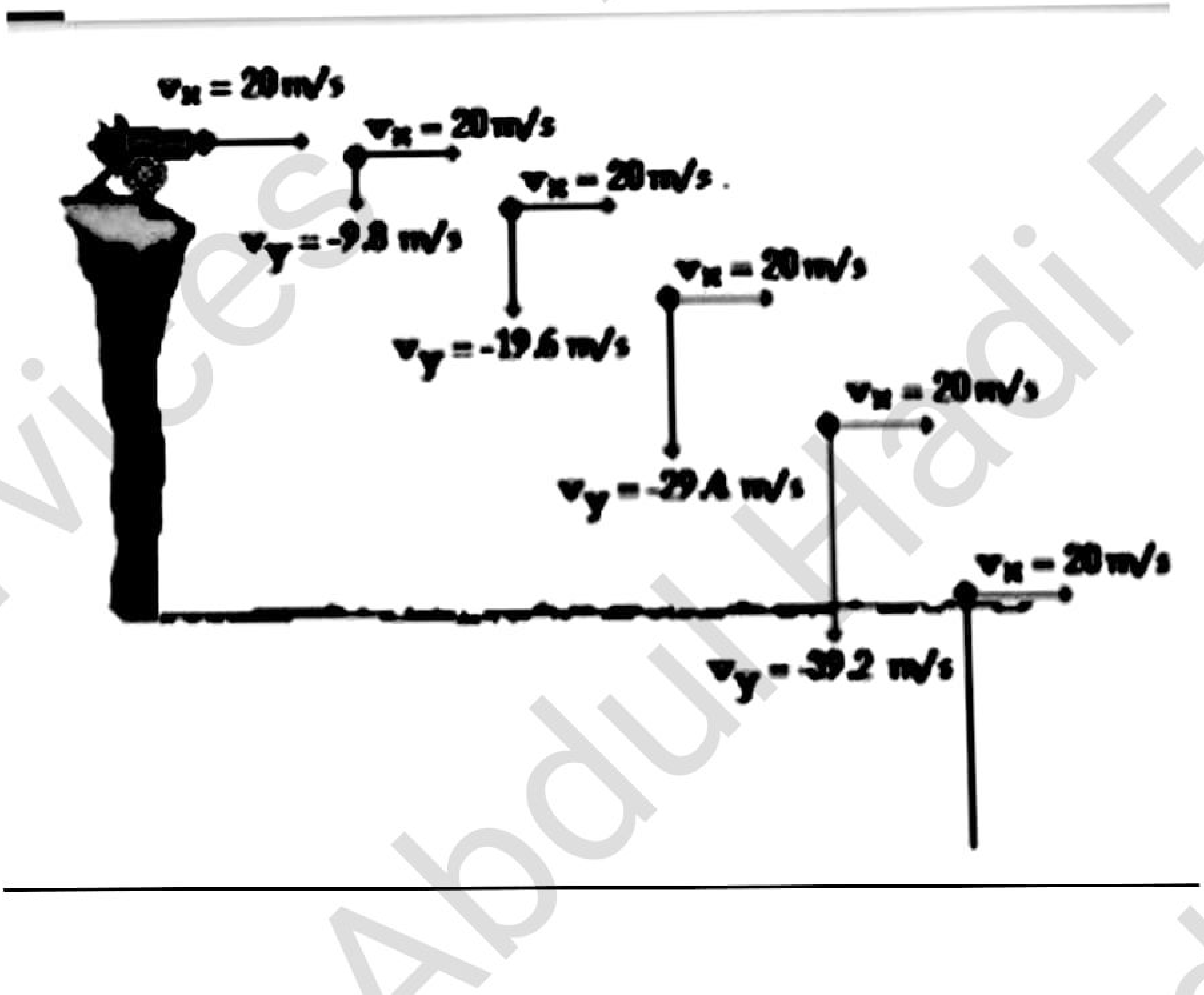
$$\Rightarrow 2VCos\theta\left(V^2 Sin^2\theta - 2gH\right)^{1/2} = V_\circ \sin\theta + V_\circ\left(V^2 Sin^2\theta - 2gH\right)^{1/2}$$

or $$2VCos\theta\left(V^2 Sin^2\theta - 2gH\right)^{1/2} - V_\circ\left(V^2 Sin^2\theta - 2gH\right)^{1/2} = V_\circ V \sin\theta$$

Thus,

$$(2VCos\theta - V_\circ)\left(V^2 Sin^2\theta - 2gH\right)^{1/2} = V_\circ V \sin\theta$$

Example



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The vertical velocity of a projectile changes by 9.8m/s each second, The horizontal motion of a projectile is independent of its vertical motion.

Time	Horizontal Velocity	Vertical Velocity
0 s	73.1 m/s, Right	19.8m/s, up
1 s	73.1 m/s, Right	9.8m/s, up
2 s	73.1 m/s, Right	0m/s
3 s	73.1 m/s, Right	9.8 m/s, down

Example

For example, you throw the ball straight upward, or you kick a ball and give it a speed at

an angle to the horizontal or you just drop things and make them free fall; all these are examples of projectile motion. In projectile motion, gravity is the only force acting on the object.

Lecture (31)

Parabola of Safety

For a given launch velocity v and launch angle θ

The parabola of safety or safety parabola in the envelope of the parabolic trajectories of projectiles

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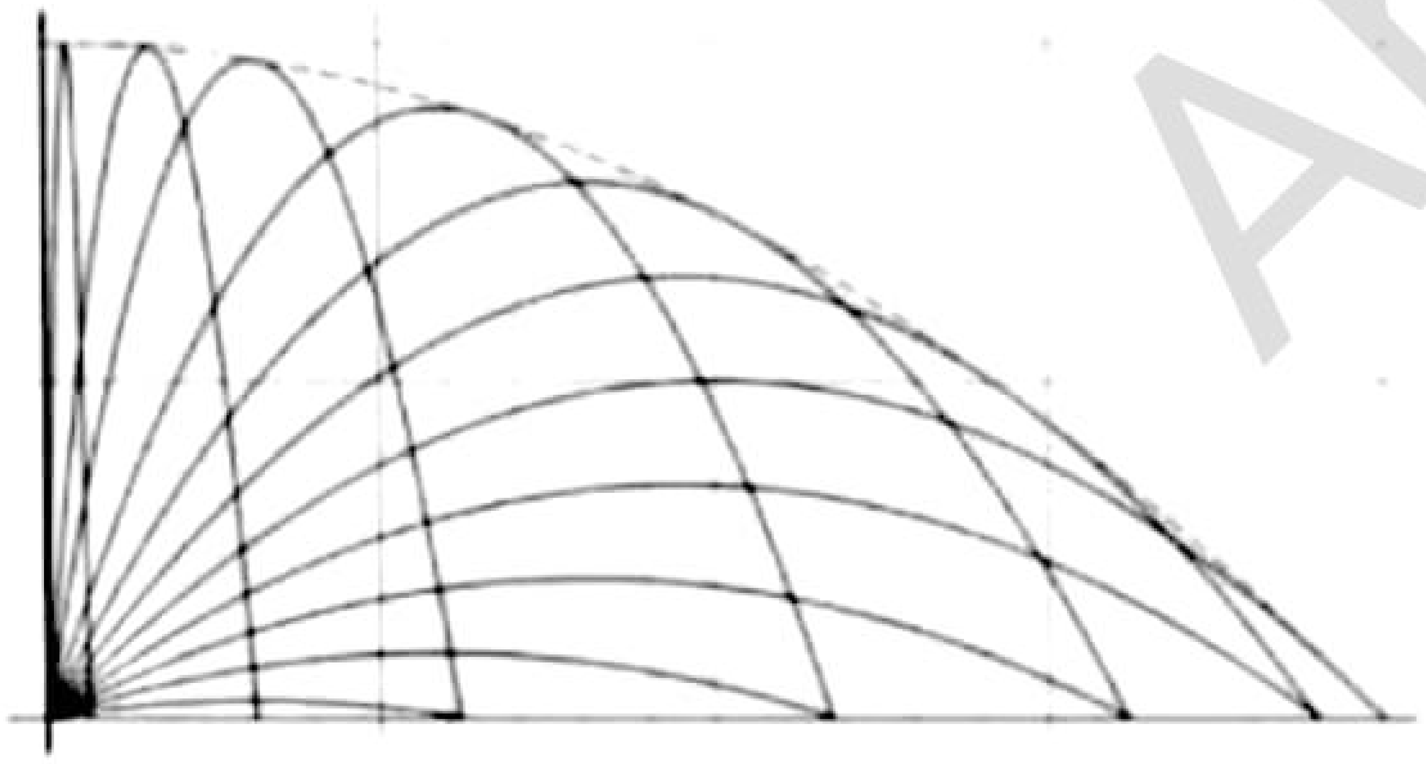
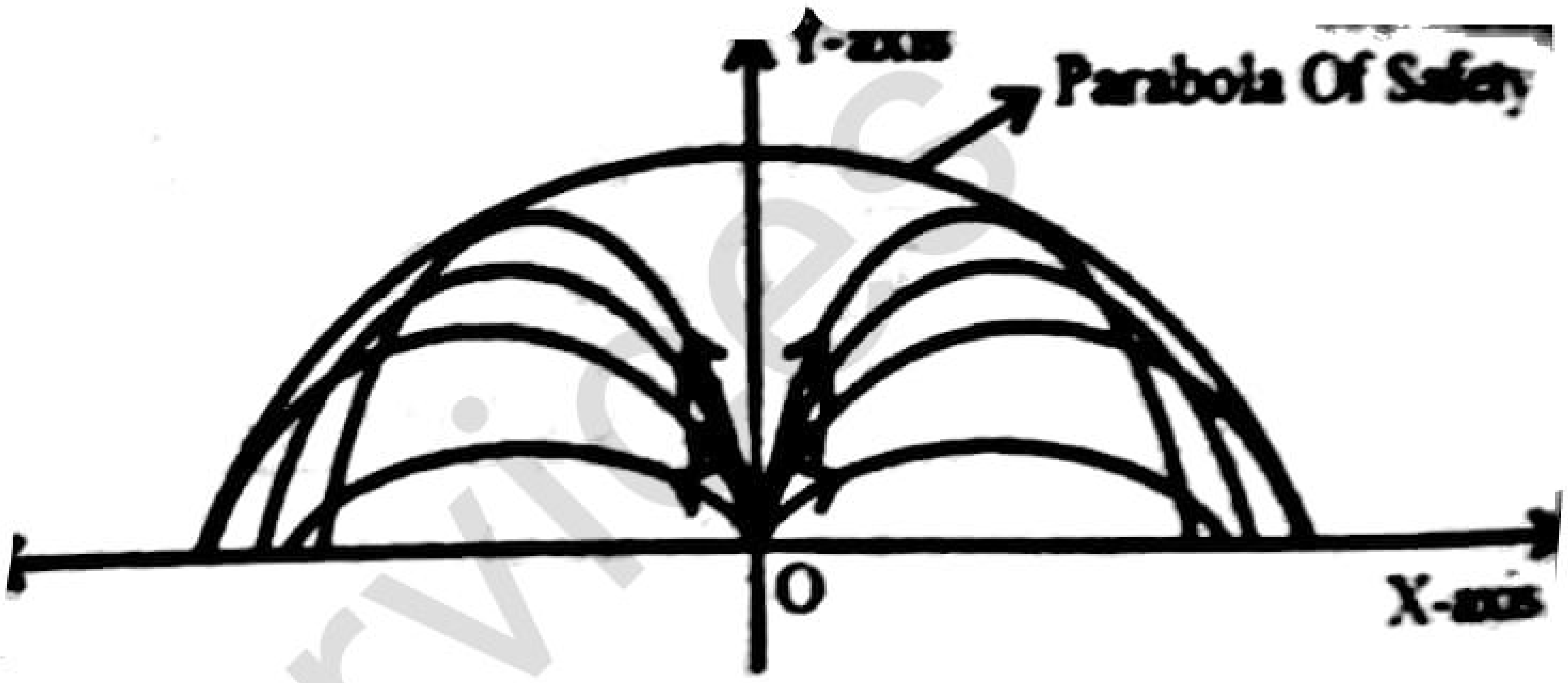


Parabola of Safety:

In classical mechanics and ballistics, the parabola of safety or safety parabola is the envelope of the parabolic trajectories of projectiles shot from a certain point with a given speed at different angles to horizon in a fixed vertical plane.

Define safety of parabola and derive its equation.



The parabola of safety is the boundary curve in a vertical plane which include all possible paths of projectile with the same (a given) initial velocity.

The maximum range of a projectile having initial velocity V_o, on an inclined plane making an angle β with horizontal is

$$R=\frac{V_\circ^2}{g(1+\sin\beta)}$$

$$or \quad R(1+\sin\beta)=\frac{V_\circ^2}{g}$$

$$R+R\sin\beta=\frac{V_\circ^2}{g} \qquad (But\ \ R\ \ Sin\beta=y)$$

$$\Rightarrow R+y=\frac{V_\circ^2}{g}$$

$$or \qquad R=\frac{V_\circ^2}{g}-y \qquad squaring\ \ both\ \ sides,\ \ R^2=\left(\frac{V_\circ^2}{g}-y\right)^2$$

$$\Rightarrow x^2+y^2=\frac{V_\circ^4}{g^2}+y^2-\frac{2V_\circ^2y}{g} \qquad (\because R^2=x^2+y^2)$$

$$So, \qquad x^2=\frac{V_\circ^4}{g^2}-\frac{2V_\circ^2y}{g}$$

$$or \qquad x^2=-\frac{2V_\circ^2y}{g}+\frac{V_\circ^4}{g^2}$$

$$\Rightarrow (x-0)^2=\frac{-2V_\circ^2y}{g}\left(y-\frac{V_\circ^2}{2g}\right)$$

This is the equation of the parabola of safety.

(1) *vertex* $v\left(0,\frac{V_\circ^2}{2g}\right)$

(2) *Length of latus rectum* $=\frac{V_\circ^2}{2g}$

(3) *Focus* $S(0,0)$

Lecture (32)

Range on Inclined Plane

Question:

A particle is projected in a vertical plane with velocity V_o making an angle α with horizontal line. Find the maximum range of projectile and the time of flight on an inclined plane making an angle $\beta(\beta<\alpha)$ with the horizontal line.

Solution:

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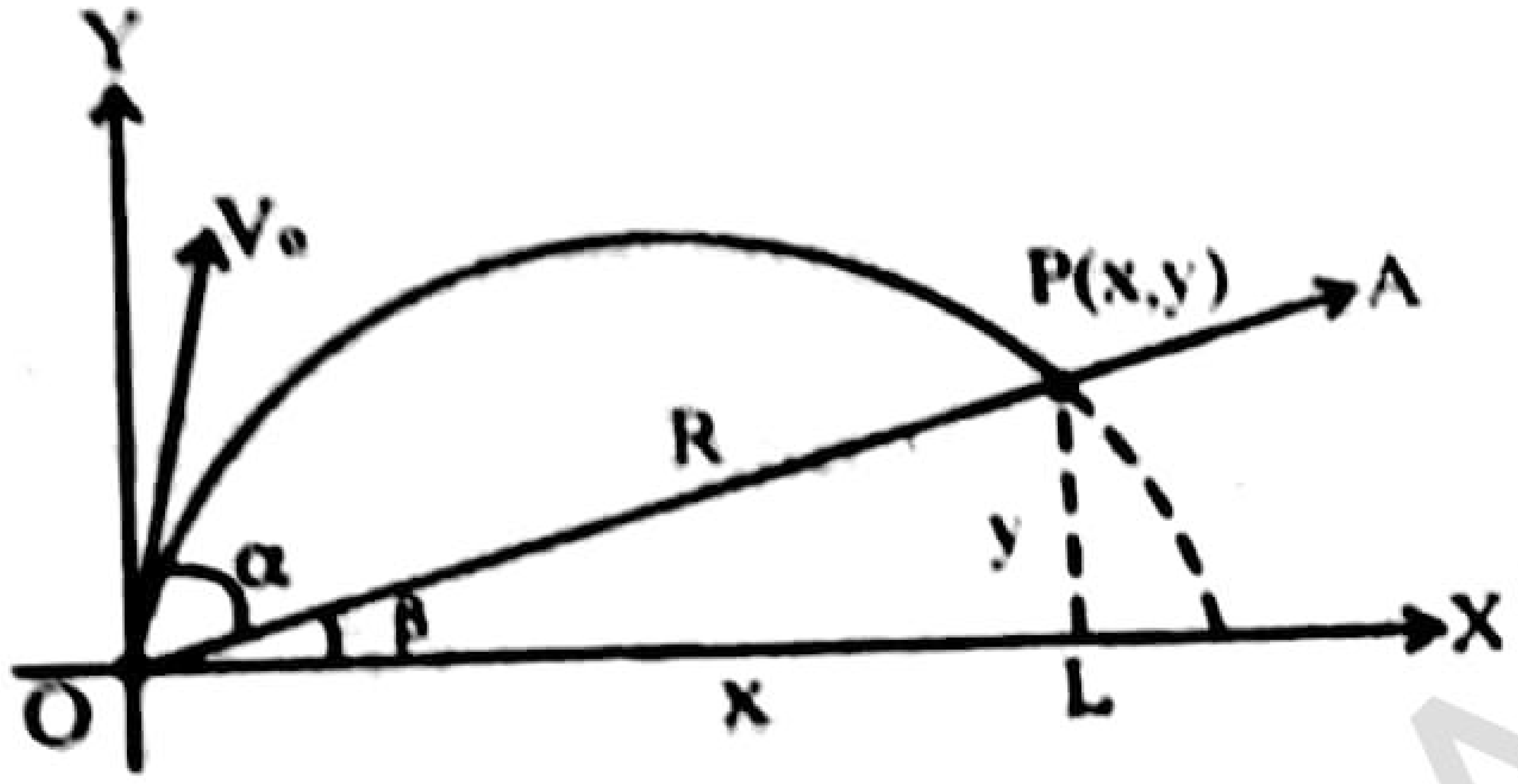
Given that the particle is projected

with velocity V_o making an angle 'α'



with horizontal. During its flight, the

particle hits the inclined plane OA at

point P(x,y) making an angle 'β' with

horizontal.

But $\quad x = R\cos\beta \qquad\qquad y = R\sin\beta$

So, coordinates of P are $(R\cos\beta, R\sin\beta)$.

As p$(R\cos\beta, R\sin\beta)$ lies on the path of parabola whose equation is

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$$y = x\tan\alpha - \frac{gx^2\sec^2\alpha}{2V_o^2}$$

So, we have

$$\mathrm{RSin}\,\beta = RCos\beta\tan\alpha - \frac{gR^2Cos^2\beta Sec^2\alpha}{2V_o^2}$$

$$\Rightarrow Sin\beta = Cos\beta\tan\alpha - \frac{gR^2Cos^2\beta Sec^2\alpha}{2V_o^2}$$

$$\Rightarrow \frac{gR^2Cos^2\beta}{2V_o^2Cos^2\alpha} = Cos\frac{\sin\alpha}{\cos\alpha} - Sin\beta$$

$$\frac{gR^2Cos^2\beta}{2V_o^2Cos^2\alpha} = \frac{\mathrm{Sin}\,\alpha Cos\beta - Cos\alpha Sin\beta}{\mathrm{Cos}\,\alpha}$$

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$$\Rightarrow \quad R = \frac{2V_\circ^2 Cos\alpha}{gCos^2\beta} Sin(\alpha - \beta)$$

$$or \quad R = \frac{V_\circ^2}{gCos^2\beta}[2Sin(\alpha - \beta)Cos\alpha] \ldots\ldots\ldots\ldots(i)$$

$$R = \frac{V_\circ^2}{gCos^2\beta}[2Sin(\alpha - \beta + \alpha) + Sin(\alpha - \beta - \alpha)]$$

Thus $$R = \frac{V_\circ^2}{gCos^2\beta}[Sin(2\alpha - \beta) - Sin\beta]$$

Now, in order to find the time of flight 'T', we use the equation.

$$x = (V_\circ \cos\alpha)t$$

So, $$R\cos\beta = (V_\circ \cos\alpha)T$$

$$\Rightarrow \quad T = \frac{R\cos\beta}{(V_\circ \cos\alpha)}$$

Using equation (i), $$T = \frac{\cos\beta}{(V_\circ \cos\alpha)}\left(\frac{V_\circ^2}{gCos^2\beta}\{2\sin(\alpha - \beta)\cos\alpha\}\right)$$

Thus, $$T = \frac{2V_\circ}{gCos\beta}\sin(\alpha - \beta)$$

Now, we find the maximum range. Take

$$R = \frac{V_\circ^2}{gCos^2\beta}[Sin(2\alpha - \beta) - Sin\beta]$$

R is maximum if $Sin(2\alpha - \beta)$ is maximum i.e., $Sin(2\alpha - \beta) = 1$

$$or \quad Sin(2\alpha - \beta) = Sin\frac{\pi}{2} \Rightarrow 2\alpha - \beta = \frac{\pi}{2}$$

Thus $$R_{max} = \frac{V_\circ^2}{gCos^2\beta}(1 - Sin\beta)$$

$$R_{max} = \frac{V_\circ^2(1 - Sin\beta)}{g(1 + Sin\beta)(1 - Sin\beta)}$$

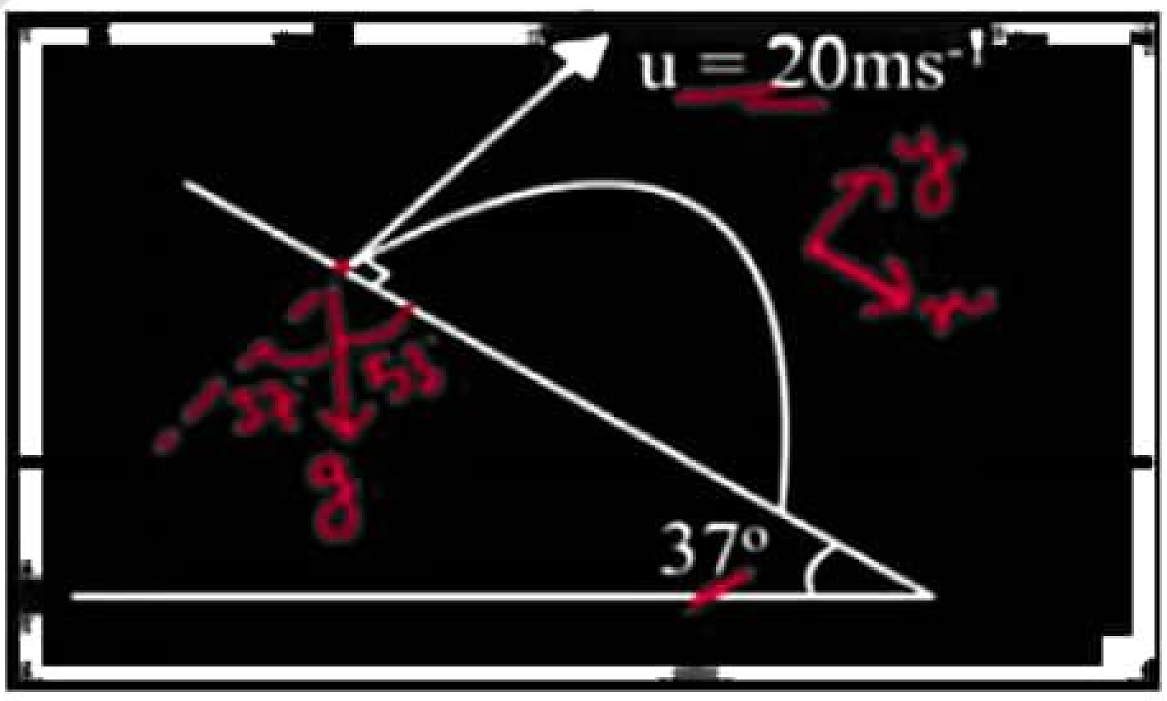
Thus, $$R_{max} = \frac{V_\circ^2}{g(1 + Sin\beta)}$$

This is the maximum range.

Question:

Find the range of the projectile on the inclined plane which is projected perpendicular

to the inclined plane with velocity 20 m/s as shown in figure.



Solution:

$U_x = 0$

$U_y = 20m/s$

$a_x = g\sin 37 = \frac{3}{5}g$

$a_y = g\cos 37 = \frac{-4}{5}g$

$5_y = 4_y t + \frac{1}{2}ayt^2$

$0 = 20t + \frac{1}{2}\times(-g\cos 37)\times t^2$

$20t = \frac{1}{2}\times 10\times\frac{4}{5}\times t^2$

$20t = 4t^2$

$t = 0,5s$

$5_x = U_x t + \frac{1}{2}axt^2$

$= \frac{1}{\cancel{2}}\times\cancel{10}^5\times\frac{3}{\cancel{5}}\times\cancel{20}^5$

$= 7.5m$

Lecture (34)

Introduction to orbital motion

Central Force:

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If a particle is moving in an orbit under the influence of a force whose line of action passes through some fixed point, then such a force is called central force and the fixed point is called its centre. The central force may be attractive or repulsive.

THEOREM: The orbit of a particle moving under a central force is necessarily a plane curve.

Proof:

Let $\vec{F}$ be the central force acting on a particle of mass "m" and the origin "O" be the centre of $\vec{F}$ as shown in the figure.

Let $p(\vec{r})$ be the position of the particle at any time t.

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So, $\vec{OP} = \vec{r}$

Since, $\vec{F}$ and $\vec{r}$ are along the same line,

So, $\vec{r} \times \vec{F} = 0.$ But $\vec{F} = m\vec{a}$

or $\vec{F} = m\dfrac{d\vec{v}}{dt}$ $\left(\because \vec{a} = \dfrac{d\vec{v}}{dt}\right)$

$$\vec{r} \times m\frac{d\vec{v}}{dt} = 0$$

$$\Rightarrow \quad \vec{r} \times \frac{d\vec{v}}{dt} = 0 \ldots\ldots\ldots\ldots (i)$$

Now, $\vec{v} \times \vec{v} = 0$

or $\dfrac{d\vec{r}}{dt} \times \vec{v} = 0 \ldots\ldots\ldots\ldots\ldots (ii)$ $\left(\because \vec{v} = \dfrac{d\vec{r}}{dt}\right)$

Adding equation (i) and (ii), we have

$$r \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v} = 0$$

$\Rightarrow \quad \dfrac{d}{dt}(\vec{r} \times \vec{v}) = 0$ interchanging both *sides*,

$$\vec{r} \times \vec{v} = \vec{h} \ldots\ldots\ldots\ldots\ldots (iii)$$

Where $\vec{h}$ is the constant of integration.

Now, Dot multiply both sides by $\vec{r}$ in equation (iii)

$$\vec{r}.(\vec{r} \times \vec{v}) = \vec{r}.\vec{h}$$

$\Rightarrow \quad 0 = \vec{r}.\vec{h}$ $\left(\because \vec{a}.(\vec{a} \times \vec{b}) = 0\right)$

or $\vec{r}.\vec{h} = 0$

This shows that the position vector of the particle at any time is perpendicular to the fixed constant vector $\vec{h}$ and hence lies in a plane.

Lecture (35)

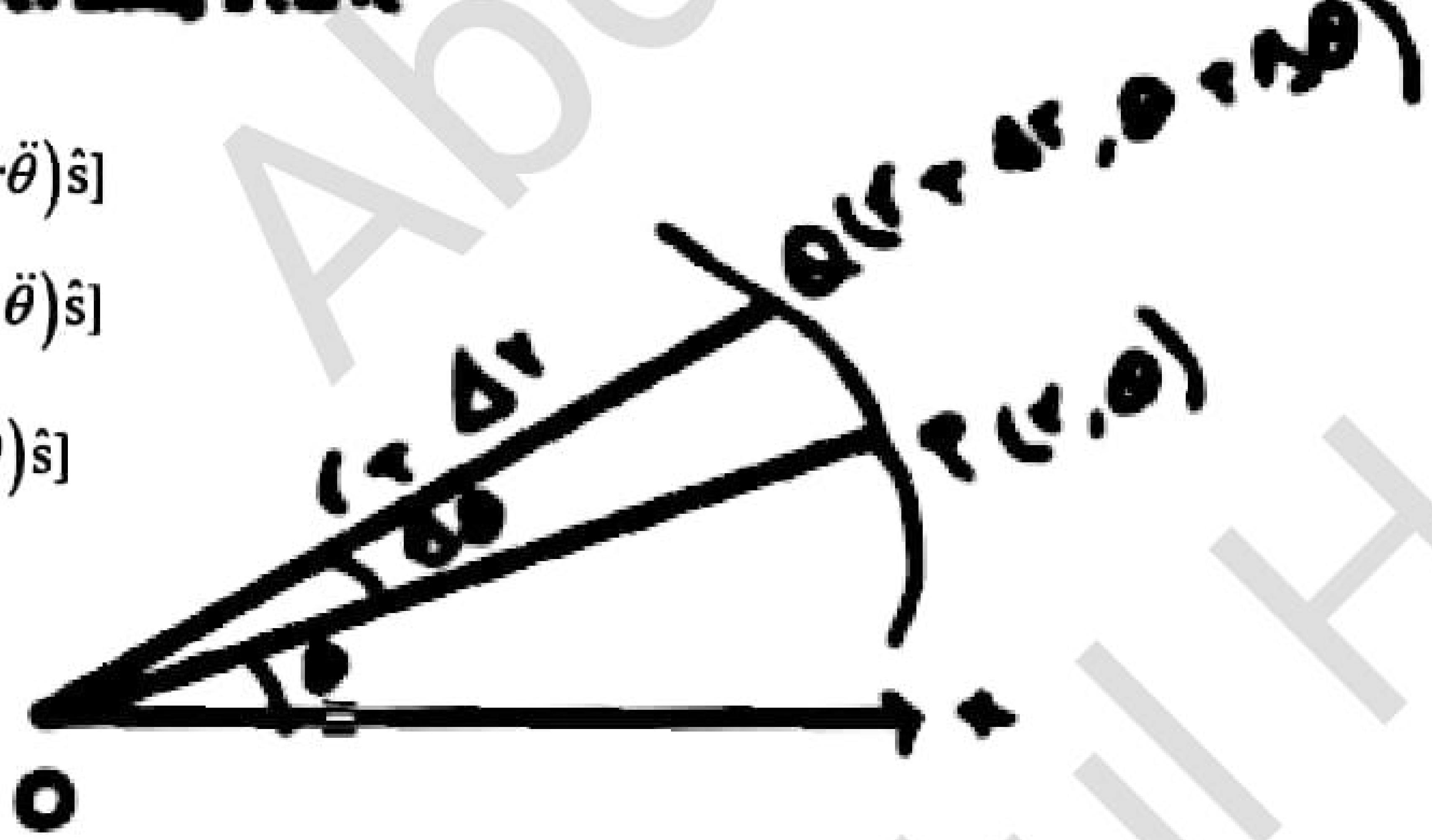
Orbital Motion Theorem 2

Theorem:

When a particle moves under a central force, the areal velocity is constant.

Where areal velocity is the rate at which area is swept out by a particle as it moves along a curve.

moves along a curve



$-F = ma$

$-F = m[\left(\ddot{r} - r\dot{\theta}\right)^2 \hat{r} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\text{s}}]$

$= m[\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \frac{1}{r}\left(2r\dot{r}\dot{\theta} + r^2\ddot{\theta}\right)\hat{\text{s}}]$

$-F = m[\left(\ddot{r} - r\dot{\theta}^2\right)\hat{r} + \frac{1}{r}\frac{d}{dt}\left(\hat{r}\dot{\theta}\right)\hat{\text{s}}]$

$-F_r\hat{r} - F_\theta\hat{s} =$ ……….

$-F_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$

$-F_\theta = \frac{m}{r}\left(\frac{d}{dt}\left(r^2\dot{\theta}\right)\right)$

$F_\theta = 0$

$\frac{m}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$

$\Rightarrow \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$

$\Rightarrow r^2\dot{\theta} = constant = h$

$\Delta A = Area\ of\ OP\theta \cong Area\ of\ OPR$

$= \frac{1}{2}OQ \times PR$

$= \frac{1}{2}(r + \Delta r) r \sin\Delta\theta$

$= \frac{1}{2}\left(r^2 \sin\Delta\theta + r.\Delta r \sin\Delta\theta\right)$

$\frac{\Delta A}{\Delta t} = \frac{1}{2}\left(r^2\frac{\Delta\theta}{\Delta t} + r.\Delta r.\frac{\Delta\theta}{\Delta t}\right)$

to having lim*it as* $\Delta t \to 0$, $\Delta r \to 0$, $\Delta\theta \to 0$

$\frac{\Delta A}{\Delta t} = \frac{1}{2}r^2\dot{\theta} + r.\theta.\dot{\theta}$

$= \frac{1}{2}r^2\dot{\theta}$

$= \frac{1}{2}h$

$2\left(dA/dt\right) = h$

$\Rightarrow$ *h is equal to twice the Areal speed.*

Lecture (36)

Differential Equation of the Orbital in Pedal Form

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As we know the differential equation of the orbit of a particle in polar form is

$$\frac{d^2u}{d\theta}+u=\frac{F}{h^2u^2}\text{............}(i)$$

Also, we have $\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2$ *Differentiating w.r.t.* 'θ',

$$-2p^{-3}.\frac{dp}{d\theta}=2u.\frac{du}{d\theta}+2\left(\frac{du}{d\theta}\right)\left(\frac{d^2u}{d\theta^2}\right)$$

$$\Rightarrow -\frac{1}{p^3}.\frac{dp}{d\theta}=u\frac{du}{d\theta}+\frac{du}{d\theta}.\frac{d^2u}{d\theta^2}$$

or $$-\frac{1}{p^3}.\frac{dp}{d\theta}.\frac{du}{d\theta}=\frac{du}{d\theta}\left(u+\frac{d^2u}{d\theta^2}\right)$$

$$\Rightarrow -\frac{1}{p^3}.\frac{dp}{du}=\frac{d^2u}{d\theta^2}+u \qquad \text{using equation } (i)$$

$$-\frac{1}{p^3}.\frac{dp}{du}=\frac{F}{h^2u^2}$$

$$-\frac{1}{p^3}.\frac{dp}{dr}.\frac{dr}{du}=\frac{F}{h^2u^2}\text{..................}(ii) \quad \text{But} \quad r=\frac{1}{u}\Rightarrow u=\frac{1}{r}$$

$$\Rightarrow \frac{dr}{du}=-\frac{1}{u^2}=-r^2$$

Equation (ii) *becomes* $$-\frac{1}{p^3}.\frac{dp}{dr}.\left(-r^2\right)=\frac{F}{h^2\left(\frac{1}{r^2}\right)}$$

$$\Rightarrow \qquad \frac{h^2}{p^3}.\frac{dp}{dr}=F$$

This is the differential equation of the orbit of a particle in pedal form.

Lecture (37)

Apse and Apsidal Distance

Apse:

An apse is a point on an orbit at which the tangent to the orbit is perpendicular to the radius

vector drawn from the centre of force. An apse is shown in the figure.

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Apsidal Distance:

The distance of an apse from the centre of force is called apsidal distance.

Apse line:

The line joining an apse to the centre of force is called apse line.

Apsidal Angle:

The angle between two consecutive apse line is called an apsidal angle.

The Condition for An Apse:

As we know that the angle ϕ between the tangent to the orbit at a point

$P(r,\theta)$

and radius vector $\vec{r}$ is given by $\tan\varphi = r\frac{d\theta}{dr}, \textit{for an apse}, \varphi = \frac{\pi}{2}$

$$\text{So,} \quad \tan\frac{\pi}{2} = r\frac{d\theta}{dr} \Rightarrow \infty = r\frac{d\theta}{dr}$$

$$\Rightarrow \frac{1}{r}\cdot\frac{dr}{d\theta} = 0$$

$$\text{So,} \qquad \frac{dr}{d\theta} = 0, \ldots\ldots\ldots\ldots\ldots (i) \textit{ is the required codition}$$

$$\text{Also,} \qquad r = \frac{1}{r}$$

$$\Rightarrow \frac{dr}{d\theta} = -\frac{1}{u^2}\cdot\frac{du}{d\theta}$$

$$\textit{Equation } (i) \textit{ becomes} -\frac{1}{u^2}\cdot\frac{du}{d\theta} = 0$$

$$\text{So,} \qquad \frac{du}{d\theta} = 0, \ldots\ldots\ldots\ldots (ii)$$

equation (i) *and* (ii) *are the required conditions.*

Lecture (38)

Classification of Orbits in Term of Energy

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Theorem:

Show that the orbit described by the planet around sun is a conic.

The planet will described an ellipse, a parabola or a hyperbola according as total energy per unit area is negative , zero or positive.

Proof:

We consider the motion of the planet round the sun and the force is governed by Newton's Law of Gravitation. If 'M' and 'm' are the mass of sun and the planet then they attract each other with a force MmG/r^2 where G is constant of gravitation. Take the sun as the pole, the differential equation of the orbit is

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{h^2u^2}\text{.............}(1)$$

$$F=MmG/_{r^2}$$

$$\frac{d^2u}{d\theta^2}+u=\frac{MmG/_{r^2}}{h^2u^2}$$

$$h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right)=MmG/_{r^2}$$

$$h^2\cancel{u^2}\left(\frac{d^2u}{d\theta^2}+u\right)=\mu\times\cancel{u^2} \qquad\qquad MmG=\mu$$

$$\frac{d^2u}{d\theta^2}+u=\frac{\mu}{h^2} \qquad\qquad u=\frac{1}{r}$$

$$u^2=\frac{1}{r^2}$$

$$\frac{d^2u}{d\theta^2}+u-\frac{\mu}{h^2}=0$$

$$\frac{d^2}{d\theta^2}\left(u-\frac{\mu}{h^2}\right)+\left(u-\frac{\mu}{h^2}\right)=0$$

$$\left(D^2+1\right)\left(u-\frac{\mu}{h^2}\right)=0$$

$$m^2+1=0 \qquad \Rightarrow m=\pm 1$$

$$u-\frac{\mu}{h^2}=A\cos\theta+B\sin\theta$$

$$u-\frac{\mu}{h^2}=C\cos\theta_{\cdot}\cos\theta+C\sin\theta_{\cdot}\sin\theta$$

$$u=\frac{\mu}{h^2}+C\cos\left(\theta-\theta_{\cdot}\right)$$

$$\theta_{\cdot}=0$$

$$u=\frac{\mu}{h^2}+C\cos\theta \qquad\qquad u=\frac{1}{r}$$

$$\frac{1}{r}=\frac{\mu}{h^2}\left[1+\frac{C}{\frac{\mu}{h^2}}\cos\theta\right]$$

$$\frac{h^2}{ur}=1+\frac{h^2C}{\mu}\cos\theta\text{......................}(2)$$

$$\frac{h^2/\mu}{r}=1+\frac{h^2c}{\mu}\cos\theta.$$

$$\frac{l}{r}=1+e\cos\theta$$

$$l=h^2/\mu \qquad\qquad e=\frac{h^2c}{\mu}$$

$$u=\frac{\mu}{h^2}+\cos\theta\text{...........................}(A)$$

$$Total\ \ Energy=K.E+\ P.E$$

$$E=T+V$$

$$V=-\int F(r)dr$$

$$=-\int-\frac{\mu}{h^2}dr$$

Lecture (39)

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Kepler's Laws of Planetary Motion

1) Each plant describes an ellipse with sun as focus.

2) The areal speed of the radius vector in any orbit is constant.

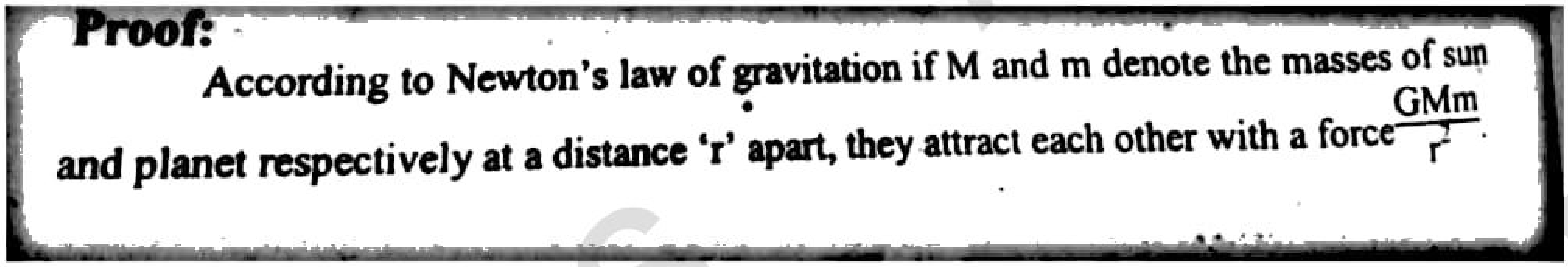
3) The square of time period for describing the whole orbit is proportional to the

cube of the major axis of this orbit.

1ST LAW OFKEPLER'S

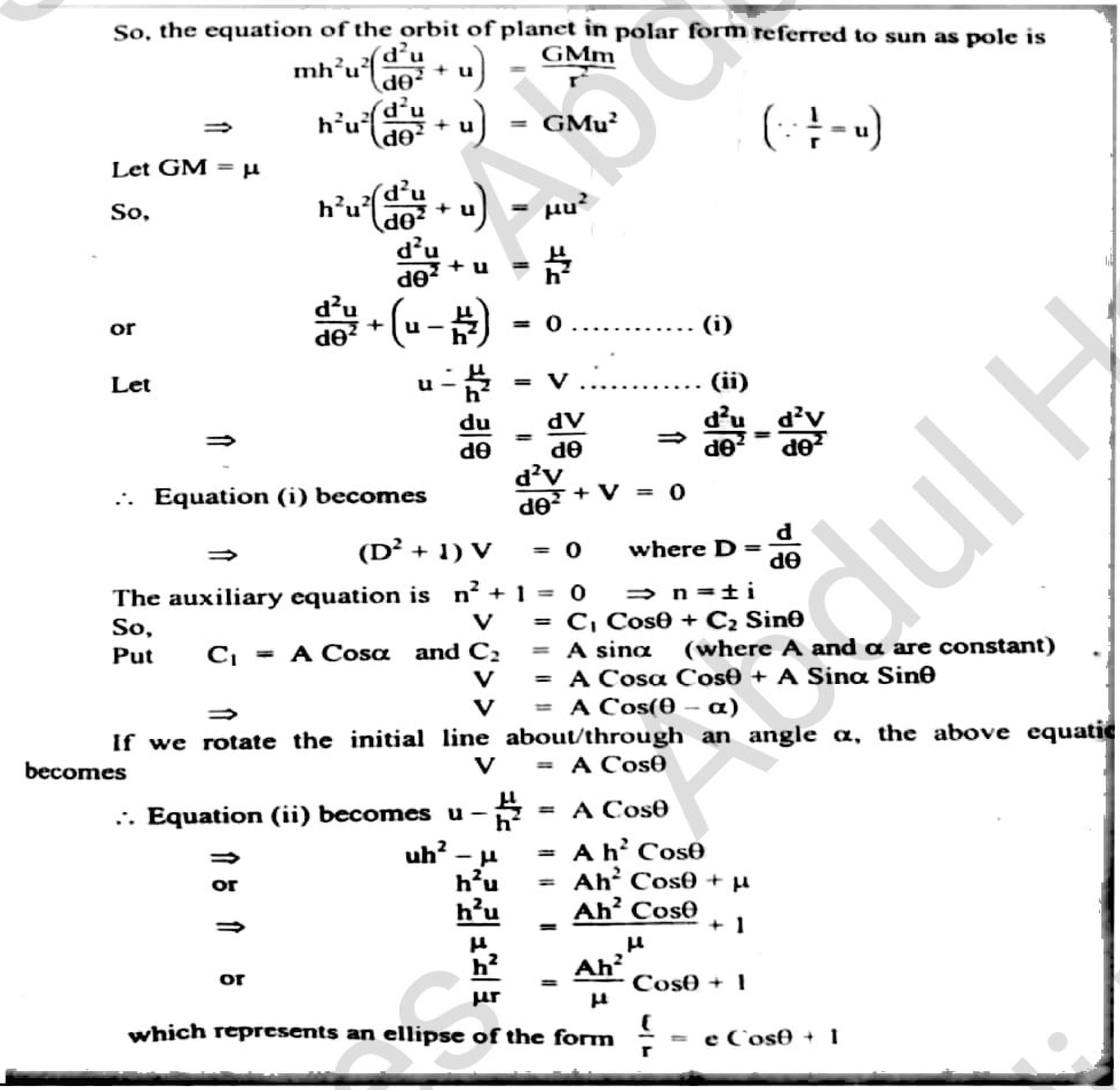
Each planet describes an ellipse with sun as focus.

Proof:



Proof:

According to Newton's law of gravitation if M and m denote the masses of sun and planet respectively at a distance 'r' apart, they attract each other with a force $\frac{GMm}{r^2}$.



So, the equation of the orbit of planet in polar form referred to sun as pole is

$$mh^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = \frac{GMm}{r^2}$$

$$\Rightarrow \quad h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = GMu^2 \qquad \left(\because \frac{1}{r}=u\right)$$

Let $GM = \mu$

So, $$h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = \mu u^2$$

$$\frac{d^2u}{d\theta^2}+u = \frac{\mu}{h^2}$$

or $$\frac{d^2u}{d\theta^2}+\left(u-\frac{\mu}{h^2}\right) = 0 \;\ldots\ldots\ldots\ldots \text{(i)}$$

Let $$u-\frac{\mu}{h^2} = V \;\ldots\ldots\ldots\ldots \text{(ii)}$$

$$\Rightarrow \quad \frac{du}{d\theta} = \frac{dV}{d\theta} \quad \Rightarrow \frac{d^2u}{d\theta^2} = \frac{d^2V}{d\theta^2}$$

∴ Equation (i) becomes $$\frac{d^2V}{d\theta^2}+V = 0$$

$$\Rightarrow \quad (D^2+1)V = 0 \quad \text{where } D = \frac{d}{d\theta}$$

The auxiliary equation is $n^2+1=0 \quad \Rightarrow n = \pm i$

So, $$V = C_1 \text{Cos}\theta + C_2 \text{Sin}\theta$$

Put $C_1 = A\,\text{Cos}\alpha$ and $C_2 = A\,\text{sin}\alpha$ (where A and α are constant)

$$V = A\,\text{Cos}\alpha\,\text{Cos}\theta + A\,\text{Sin}\alpha\,\text{Sin}\theta$$

$$\Rightarrow \quad V = A\,\text{Cos}(\theta-\alpha)$$

If we rotate the initial line about/through an angle α, the above equati becomes

$$V = A\,\text{Cos}\theta$$

∴ Equation (ii) becomes $$u-\frac{\mu}{h^2} = A\,\text{Cos}\theta$$

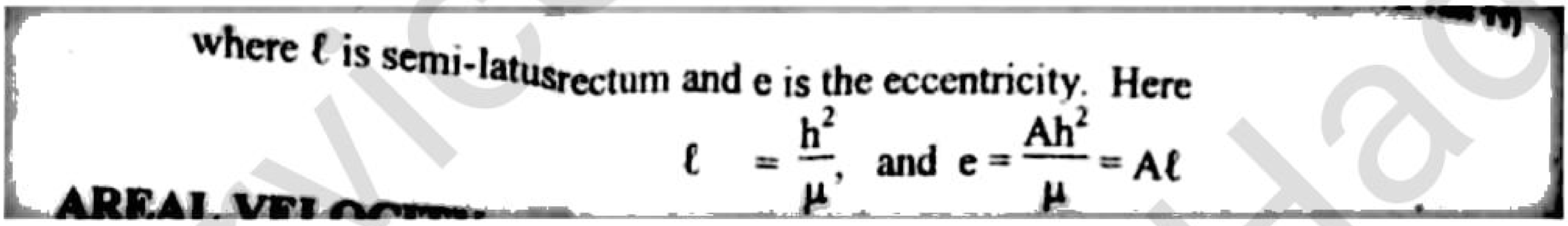
$$\Rightarrow \quad uh^2-\mu = A\,h^2\,\text{Cos}\theta$$

or $$h^2u = Ah^2\,\text{Cos}\theta+\mu$$

$$\Rightarrow \quad \frac{h^2u}{\mu} = \frac{Ah^2\,\text{Cos}\theta}{\mu}+1$$

or $$\frac{h^2}{\mu r} = \frac{Ah^2}{\mu}\text{Cos}\theta+1$$

which represents an ellipse of the form $\frac{\ell}{r} = e\,\text{Cos}\theta+1$



where ℓ is semi-latusrectum and e is the eccentricity. Here

$$\ell = \frac{h^2}{\mu}, \text{ and } e = \frac{Ah^2}{\mu} = A\ell$$

AREAL VELOC

Areal Velocity:

The area swept out by the radius vector per unit time of a particle moving in a central orbit, is called the areal velocity of the particle.

2nd LAW OF KEPLER'S:

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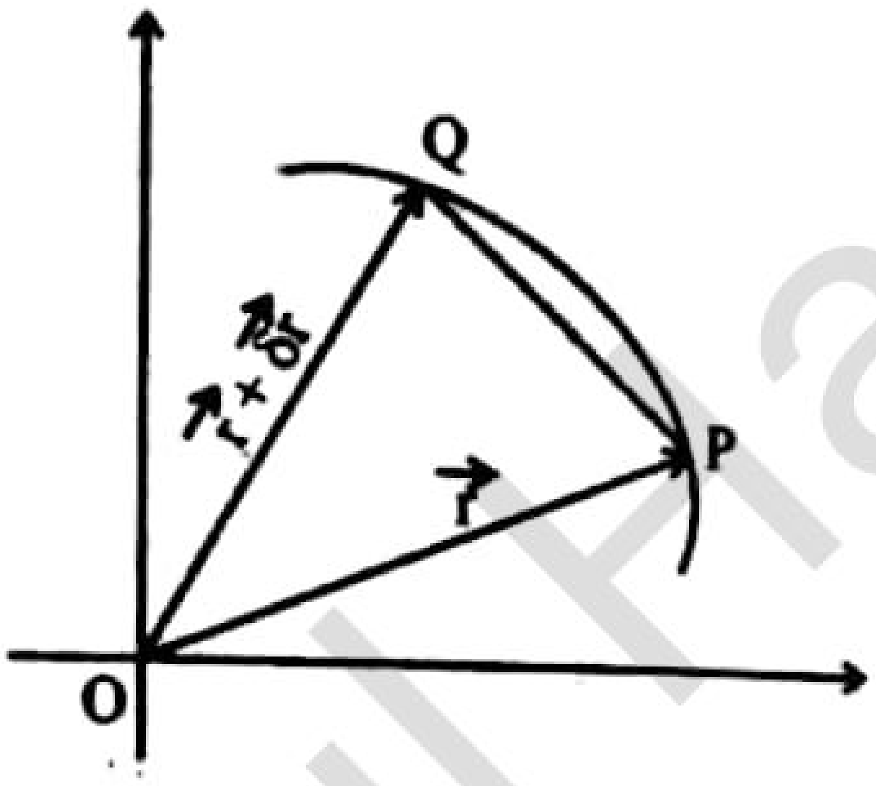
The areal speed of the radius vector in any orbit is constant.

The areal speed of the radius vector in any orbit is constant.

Proof:

Let the sun is located at origin 'O' and 'P' be the position of the planet at any time 't' and $\vec{OP} = \vec{r}$

Let after a very small interval of time δt planet is at point $\vec{Q}$ and $\vec{OQ} = \vec{r} + \delta\vec{r}$



$$\vec{PQ} = \vec{r} + \delta\vec{r} - \vec{r}$$

$$\Rightarrow \quad \vec{PQ} = \delta\vec{r}$$

Let δA be the sectorial area OPQ.

$$\text{Now,} \quad \delta A = \frac{1}{2}|\vec{OP} \times \vec{OQ}|$$

$$\delta A = \frac{1}{2}|\vec{r} \times \delta\vec{r}|$$ Dividing both sides by δt ≠ 0, we get

$$\frac{\delta A}{\delta t} = \frac{1}{2}\left|\vec{r} \times \frac{\delta\vec{r}}{\delta t}\right|$$ Taking limit $\delta t \to 0$,

$$\lim_{\delta t \to 0}\frac{\delta A}{\delta t} = \lim_{\delta t \to 0}\frac{1}{2}\left|\vec{r} \times \frac{\delta\vec{r}}{\delta t}\right|$$

$$\frac{dA}{dt} = \frac{1}{2}\left|\vec{r} \times \frac{\delta\vec{r}}{\delta t}\right|$$

or $$\frac{dA}{dt} = \frac{1}{2}|\vec{r} \times \vec{v}| \qquad \left(\because \frac{d\vec{r}}{dt} = \vec{v}\right)$$

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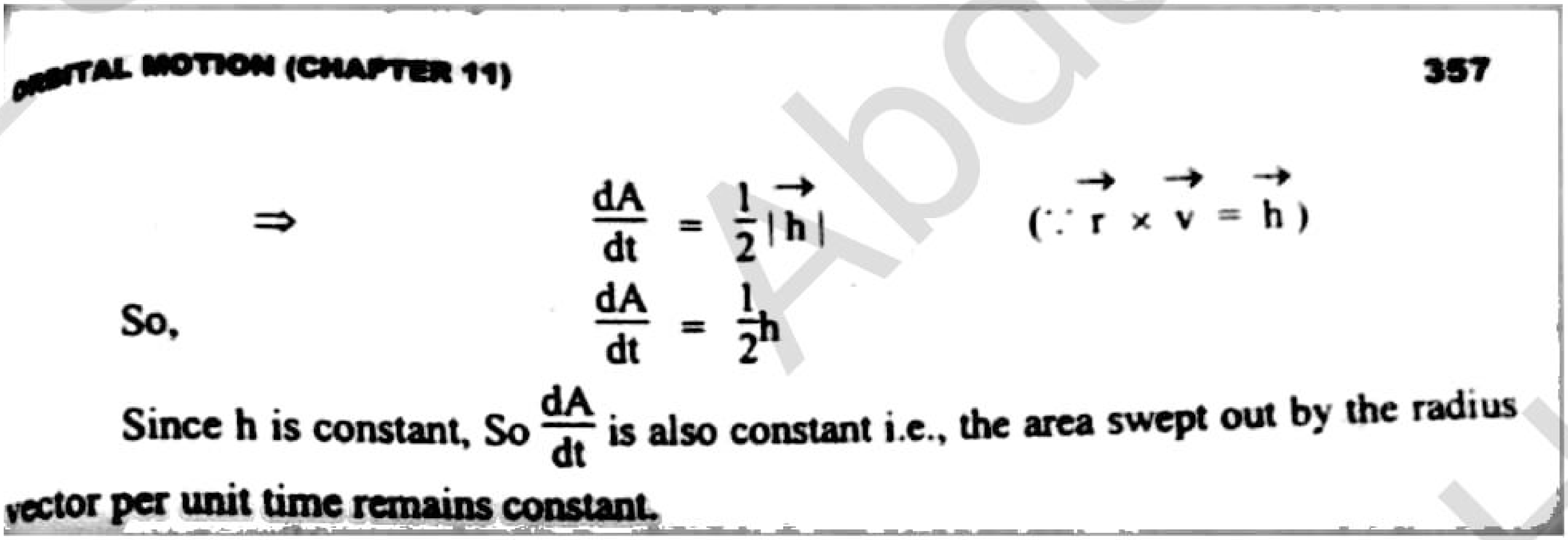

$$\Rightarrow \qquad \frac{dA}{dt} = \frac{1}{2}|\vec{h}| \qquad (\because \vec{r} \times \vec{v} = \vec{h})$$

So,

$$\frac{dA}{dt} = \frac{1}{2}h$$

Since h is constant, So $\frac{dA}{dt}$ is also constant i.e., the area swept out by the radius vector per unit time remains constant.

3RD LAW OF KEPLER'S:

Square of the time period for describing whole orbit is proportional to the cube of major axis of this orbit.

Proof:

vector per unit time remains constant.

3rd LAW OF KEPLER'S

Square of the time period for describing whole orbit is proportional to the cube of major axis of this orbit.

Proof: If a and b are semi axis of the elliptic orbit described by a planet the time for one revolution $= T = \dfrac{\pi ab}{\text{areal velocity}}$

$$= n\,\frac{\pi ab}{\frac{h}{2}}$$

$$= \frac{2\pi ab}{h}$$

Now $h^2 = \mu l$, $\quad h = \sqrt{\mu l}$, But $l = \dfrac{b^2}{a}$

$$h = \sqrt{\frac{\mu b^2}{a}} = \frac{\sqrt{\mu}}{\sqrt{a}}b$$

$$\therefore \quad T = \frac{2\pi ab}{\frac{\sqrt{\mu}}{\sqrt{a}}\,b} = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$$T^2 = \frac{4\pi^2 a^3}{\mu}$$

$$= \frac{\pi^2}{2\mu}.8a^3$$

$$= \frac{\pi^2}{2\mu}(2a)^3$$

$$\Rightarrow \quad T^2 \propto (2a)^3$$

Example:

vector per unit time remains constant.

3rd LAW OF KEPLER'S

Square of the time period for describing whole orbit i proportional to the cube of major axis of this orbit.

Proof: If a and b are semi axis of the elliptic orbit described by a planet the time fo

$$\text{one revolution} = T = \frac{\pi ab}{\text{ared velocity}} \qquad = n \frac{\pi ab}{\frac{h}{2}}$$

$$= \frac{2\pi ab}{h}$$

$$\text{Now } h^2 = \mu l, \qquad h = \sqrt{\mu l}, \text{ But } l = \frac{b^2}{a}$$

$$h = \sqrt{\frac{\mu b^2}{a}} = \frac{\sqrt{\mu}}{\sqrt{a}} b$$

$$\therefore \quad T = \frac{2\pi ab}{\frac{\sqrt{\mu}}{\sqrt{a}} b} = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$$T^2 = \frac{4\pi^2 a^3}{\mu}$$

$$= \frac{\pi^2}{2\mu} . 8a^3$$

$$= \frac{\pi^2}{2\mu} (2a)^3$$

$$\Rightarrow \quad T^2 \propto (2a)^3$$

Example:

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Example 1: Show that the law of force towards the pole, of a particle describing the curve $r^n = a^n \operatorname{Cos} n\theta$ is given by $f = \dfrac{(n+1)h^2 a^{2n}}{r^{2n+3}}$.

Solution:

As we know $\quad f = h^2u^2\left(\dfrac{d^2u}{d\theta^2} + u\right)$ .... (i)

Now, $\quad r^n = a^n \operatorname{Cos} n\theta$

$\Rightarrow \quad \dfrac{1}{u^n} = a^n \operatorname{Cos} n\theta \qquad \left(\because r = \dfrac{1}{u}\right)$

$\Rightarrow \quad u^n = \dfrac{1}{a^n} \operatorname{Sec} n\theta$

Differentiating both sides w.r.t. 'θ', $\quad nu^{n-1}\dfrac{du}{d\theta} = \dfrac{1}{a^n}(\operatorname{Sec} n\theta \tan n\theta)(n)$

$\Rightarrow \quad \dfrac{du}{d\theta} = \dfrac{1}{a^n u^{n-1}}(\operatorname{Sec} n\theta \tan n\theta)$

or $\quad \dfrac{du}{d\theta} = \dfrac{u}{a^n u^n}(\operatorname{Sec} n\theta \tan n\theta)$

But $\quad u^n = \dfrac{1}{a^n} \operatorname{Sec} n\theta$ .... (A)

So, $\quad \dfrac{du}{d\theta} = \dfrac{1}{a^n\left(\dfrac{\operatorname{Sec} n\theta}{a^n}\right)}(\operatorname{Sec} n\theta \tan n\theta)$

$\Rightarrow \quad \dfrac{du}{d\theta} = u \tan n\theta$ .... (ii)

Differentiating again w.r.t. 'θ', $\quad \dfrac{d^2u}{d\theta^2} = \dfrac{du}{d\theta}\tan n\theta + u \operatorname{Sec}^2 n\theta\,(n)$

Using equation (ii), $\quad \dfrac{d^2u}{d\theta^2} = u \tan n\theta \,.\, \tan n\theta + un \operatorname{Sec}^2 n\theta$

$\Rightarrow \quad \dfrac{d^2u}{d\theta^2} = u(\tan^2 n\theta + n \operatorname{Sec}^2 n\theta)$ .... (iii)

Put (iii) in equation (i),

$$f = h^2u^2[u(\tan^2 n\theta + n \operatorname{Sec}^2 n\theta) + u]$$

$\Rightarrow \quad f = h^2u^3(\tan^2 n\theta + n \operatorname{Sec}^2 n\theta + 1)$

$\Rightarrow \quad f = h^2u^3(\operatorname{Sec}^2 n\theta + n \operatorname{Sec}^2 n\theta)$

$\Rightarrow \quad f = h^2u^3(n+1)\operatorname{Sec}^2 n\theta$

From equation (A), $\operatorname{Sec} n\theta = a^n u^n$

So, $\quad f = h^2u^3(n+1)a^{2n}u^{2n}$

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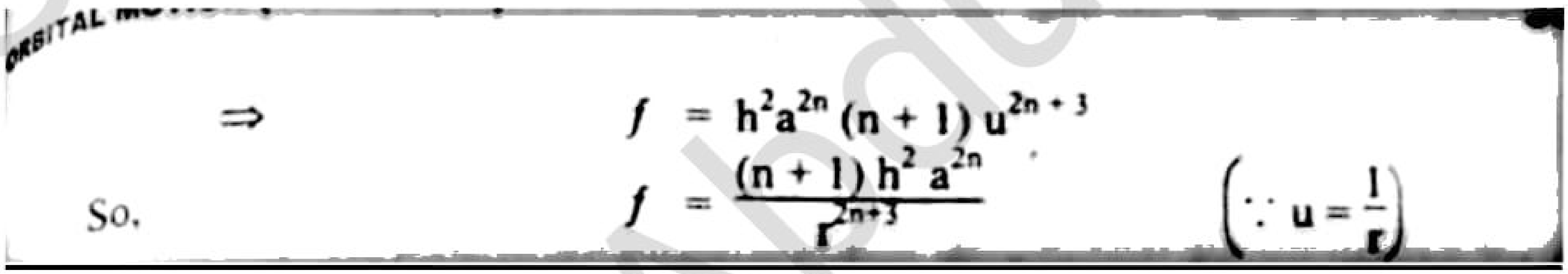

$$\Rightarrow \qquad f = h^2a^{2n}(n+1)\,u^{2n+3}$$

So, $$f = \frac{(n+1)\,h^2\,a^{2n}}{r^{2n+3}} \qquad \left(\because u = \frac{1}{r}\right)$$

Lecture (40, 41) include

(Example of Orbital Motion), (Example of Orbital Motion II)

Example:

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Q.4. A particle of unit mass describes an ellipse under the action of central force Mr. Show that the normal component of the acceleration at any instant is $\frac{abM^{3/2}}{V}$, where V is the velocity at that instant and a, b are semi-axis of ellipse.

Solution:

Since, the particle of unit mass describes an ellipse under the action of central force Mr. So the equation of its orbit is

$$h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = Mr$$

$$\Rightarrow \quad h^2u^2\left(\frac{d^2u}{d\theta^2}+u\right) = \frac{M}{u} \qquad \left(\because r=\frac{1}{u}\right)$$

$$\Rightarrow \quad h^2\left(\frac{d^2u}{d\theta^2}+u\right) = \frac{M}{u^3} \qquad \text{Multiplying both sides by } 2\frac{du}{d\theta}.$$

$$h^2\left(2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}+2u\frac{du}{d\theta}\right) = 2Mu^{-3}\frac{du}{d\theta} \qquad \text{Integrating both sides,}$$

$$h^2\left[\left(\frac{du}{d\theta}\right)^2+u\right] = 2M\frac{u^{-2}}{-2}+C$$

$$\Rightarrow \quad h^2\left[\left(\frac{du}{d\theta}\right)^2+u\right] = -\frac{M}{u^2}+C$$

$$\text{But} \quad \left(\frac{du}{d\theta}\right)^2+u = \frac{1}{p^2}$$

So,

$$\frac{h^2}{p^2} = \frac{-M}{u^2}+C$$

or

$$\frac{h^2}{p^2} = C-Mr^2 \text{ .... (i)}$$

Now, pedal equation of ellipse is $\frac{a^2b^2}{p^2} = a^2+b^2-r^2$ .... (ii)

Comparing equation (i) with equation (ii), we have

$$\frac{h^2}{a^2b^2} = \frac{C}{a^2+b^2} = \frac{M}{1}$$

$$\Rightarrow \quad h^2 = Ma^2b^2 \text{ .... (iii)}$$

Now,

$$Vp = h$$

$$\Rightarrow \quad V^2p^2 = h^2$$

$$\Rightarrow \quad V^2p^2 = Ma^2b^2 \qquad \text{[using equation (iii)]}$$

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$$\Rightarrow \qquad p^2 = \frac{Ma^2b^2}{V^2}$$

Now, for ellipse, we have

$$\rho = \frac{a^2b^2}{p^2}$$

The normal component of acceleration is

$$a_n = \frac{V^2}{\rho} = \frac{V^2}{\frac{a^2b^2}{p^3}}$$

So, $$a_n = \frac{V^2p^3}{a^2b^2} \text{ .... (iv)}$$

Now, $$p^2 = \frac{Ma^2b^2}{V^2} \qquad \text{(Taking square root)}$$

$$\Rightarrow \qquad p = \frac{M^{1/2}ab}{V} \qquad \text{(Taking cube)}$$

$$\Rightarrow \qquad p^3 = \frac{M^{3/2}a^3b^3}{V^3}$$

Put this value in equation (iv), $a_n = \frac{V^2}{a^2b^2} \cdot \frac{M^{3/2}a^3b^3}{V^3}$.

Hence, $a_n = \frac{abM^{3/2}}{V}$ as required.

Example:

Q.10. A planet is describing an ellipse about the sun as focus. Show that its velocity away from the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it then is $\frac{2\pi ae}{T(1-e^2)}$, where 2a the major axis, e the eccentricity and T is the periodic time.

Solution:

Since the orbit of the planet is an ellipse with force center at one of the foci, so its equation is

$$r = \frac{\ell}{1 + e\cos\theta} \qquad \text{or } \frac{\ell}{r} = 1 + e\cos\theta$$

$$\Rightarrow \qquad r = \ell(1 + e\cos\theta)^{-1}$$

$$\Rightarrow \qquad \frac{dr}{d\theta} = -\ell(1 + e\cos\theta)^{-2}(-e\sin\theta)$$

$$\frac{dr}{d\theta} = \frac{e\ell\sin\theta}{(1 + e\cos\theta)^2}$$

Now,

$$\frac{dr}{dt} = \frac{dr}{d\theta}\cdot\frac{d\theta}{dt}$$

$$\Rightarrow \qquad \frac{dr}{dt} = \frac{e\ell\sin\theta}{(1 + e\cos\theta)^2}\cdot\dot{\theta} \quad \left(\because \frac{d\theta}{dt} = \dot{\theta}\right) \quad \text{But } 1 + e\cos\theta = \frac{\ell}{r}$$

So, $$\frac{dr}{dt} = \frac{e\ell \sin\theta}{\frac{\ell^2}{r^2}}\dot{\theta}$$

$$\frac{dr}{dt} = \frac{e \sin\theta}{\ell}.(r^2\dot{\theta}) \quad \text{But} \quad r^2\dot{\theta} = h$$

So, $$\frac{dr}{dt} = \frac{eh \sin\theta}{\ell} \qquad \text{But} \quad h^2 = \mu\ell \quad \Rightarrow h = \sqrt{\mu\ell}$$

So, $$\frac{dr}{dt} = \frac{e \sin\theta}{\ell}.\sqrt{\mu\ell}$$

$\Rightarrow$ $$\frac{dr}{dt} = e \sin\theta\sqrt{\frac{\mu}{\ell}}$$

where e, μ and ℓ are constants.So, $\frac{dr}{dt}$ is maximum if Sin θ is maximum.

i.e., $$\sin\theta = 1$$

$\Rightarrow$ $$\theta = \frac{\pi}{2}$$

So, $$\left(\frac{dr}{dt}\right)_{max.} = e\frac{\sqrt{\mu}}{\sqrt{\ell}} \quad .... (i)$$

But, by Kepler's law, $$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}} \Rightarrow \sqrt{\mu} = \frac{2\pi a^{3/2}}{T} \quad .... (ii)$$

Also, $$\ell = \frac{b^2}{a} \quad \Rightarrow \ell = \frac{a^2(1-e^2)}{a}$$

$$\ell = a(1-e^2) \quad .... (iii)$$

Put (ii), (iii) values in equation (i), we get

$$\left(\frac{dr}{dt}\right)_{max.} = e.\frac{2\pi a^{3/2}}{T}.\frac{1}{\sqrt{a(1-e^2)}}$$

$\Rightarrow$ $$\dot{r} = \frac{2\pi ae}{T\sqrt{1-e^2}} \text{ as required.}$$

Lecture (42)

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Collision of Particles

Direct Collision:

Two bodies are said to be Impinge (collide) directly when the direction of each is along

the common normal at the point of contact.

Oblique Collision:

When the direction of motion either or both is not along the common normal.

This impact is said to be oblique.

Elastic Collision:

In an Elastic collision K.E is the same before and after collision.

Inelastic Collision:

If the K.E. before and after collision is not same then the collision is said to be inelastic collision.

Newton's Law Experimental View or Newton's Law of Restitution

Case a:

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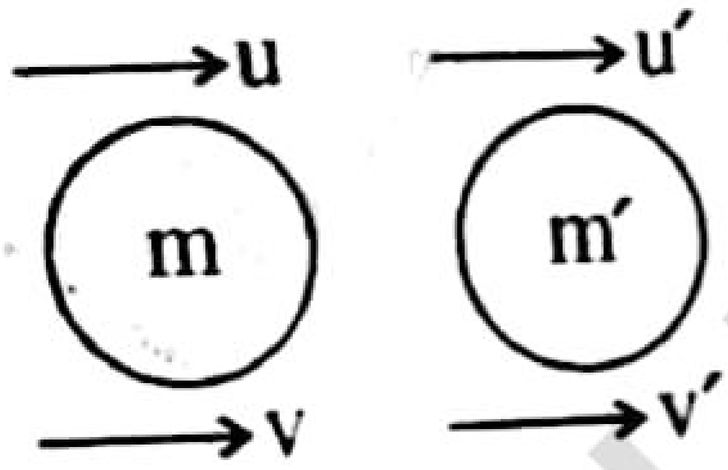
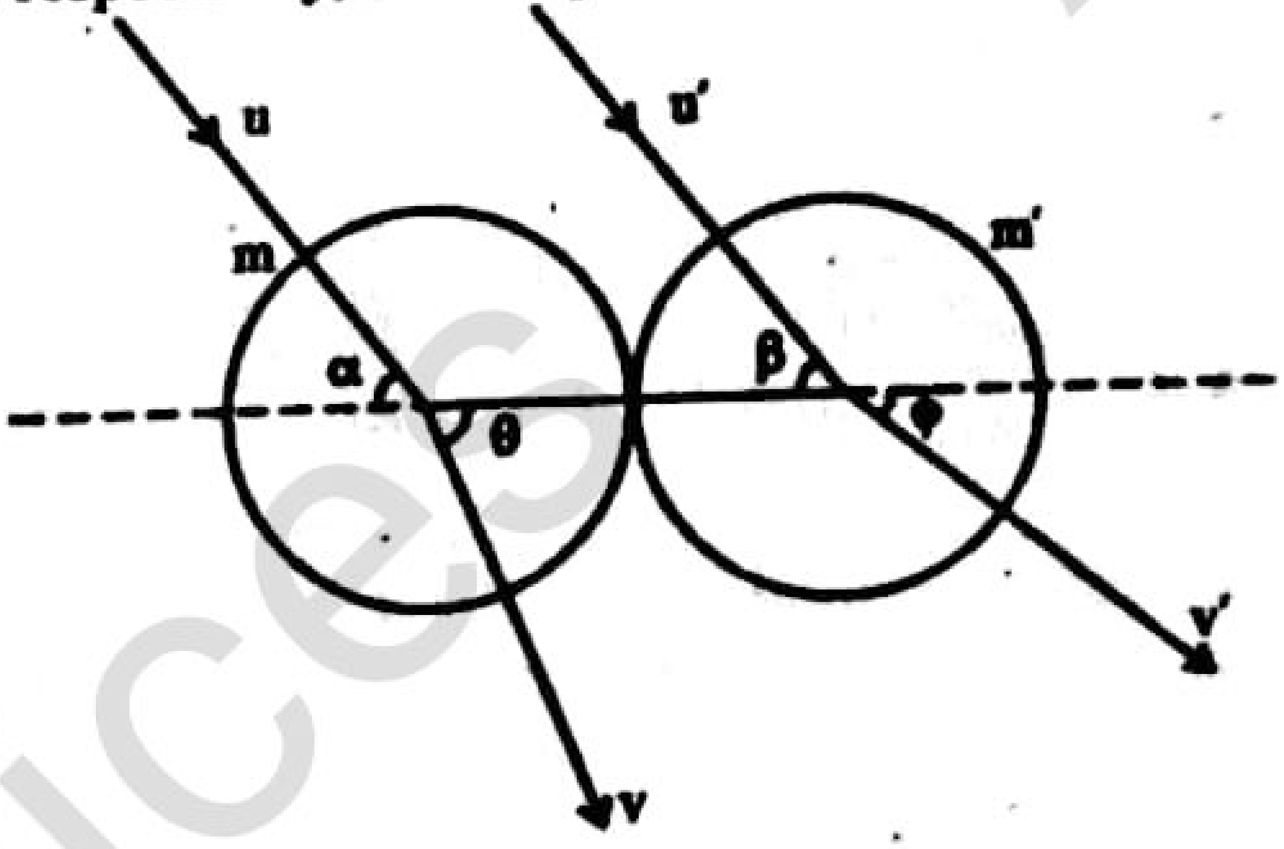
Newton's Law Experimental View or Newton's Law of Restitution

CASE a: When two bodies impinge directly, their relative velocity after impact is in a constant ratio to their relative velocity before impact and is in the opposite direction. Thus if the velocities of two impinging bodies be u, u′ before impact and v, v′ after impact, them $\frac{v-v'}{u-u'}$ is negative and is equal to a constant quantity which is independent of the masses m and m′ of the bodies. This constant quantity is denoted by "e" is called the modulus or coefficient of Elasticity or restitution or resilience of the bodies



$$\frac{v-v'}{u-u'} = -e \qquad \Rightarrow v-v' = -e\,(u-u')$$

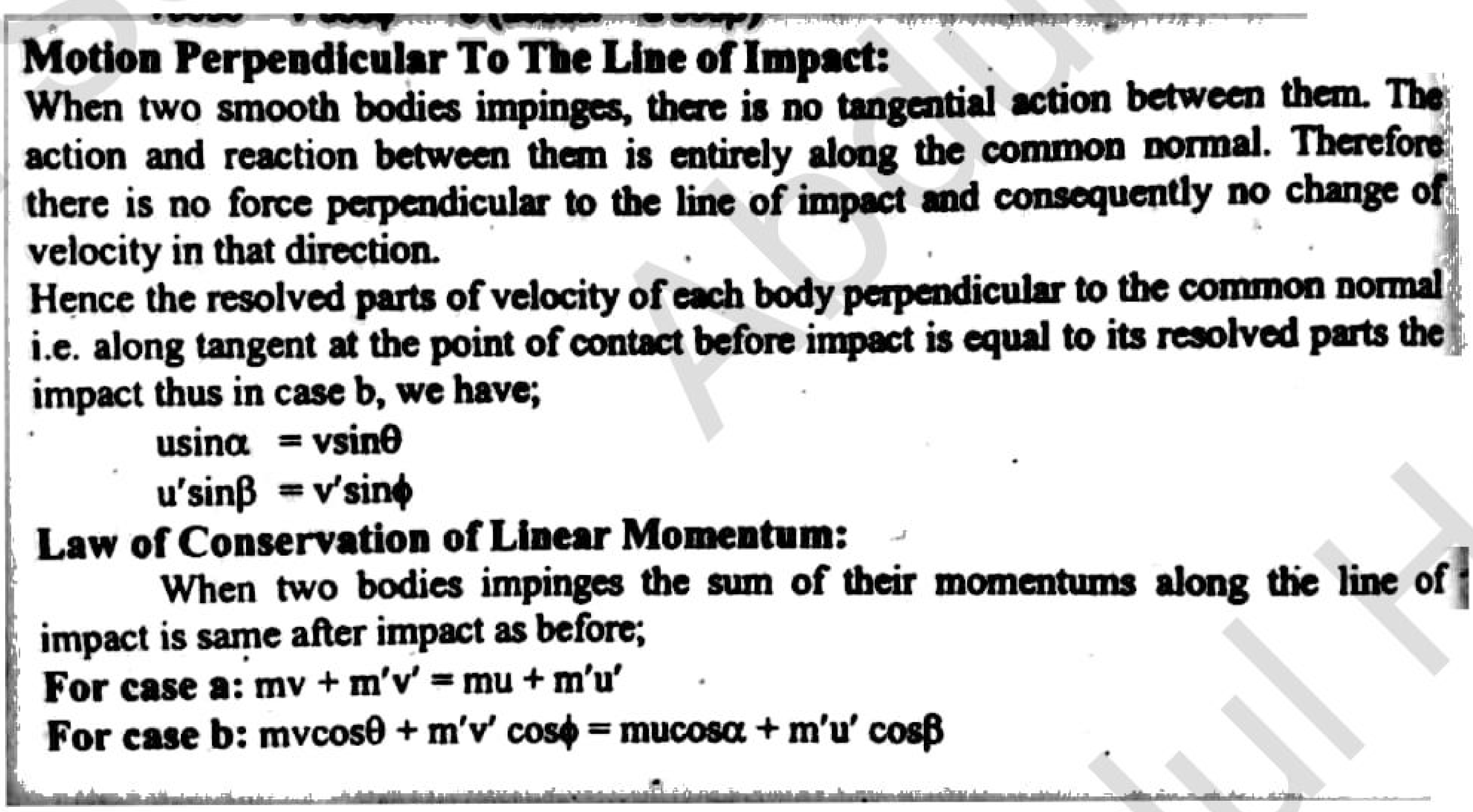
CASE b: When two bodies impinges obliquely their relative velocity resolved along their common normal after impact is in constant ratio to their relative velocity before impact resolved in the same direction and is in the opposite direction.
Thus if two bodies of masses m, m′ be moving with velocities u, u′ in direction inclined at angle α,β respectively to their common normal and v, v′ be their velocities in direction θ and ϕ respectively, after impact.



$$\frac{v\cos\theta - v'\cos\phi}{u\cos\alpha - u'\cos\beta} = -e$$

$$v\cos\theta - v'\cos\phi = -e\,(u\cos\alpha - u'\cos\beta)$$

Motion Perpendicular to The Line of Impact:



Motion Perpendicular To The Line of Impact:

When two smooth bodies impinges, there is no tangential action between them. The action and reaction between them is entirely along the common normal. Therefore there is no force perpendicular to the line of impact and consequently no change of velocity in that direction.

Hence the resolved parts of velocity of each body perpendicular to the common normal i.e. along tangent at the point of contact before impact is equal to its resolved parts the impact thus in case b, we have;

$$u\sin\alpha = v\sin\theta$$

$$u'\sin\beta = v'\sin\phi$$

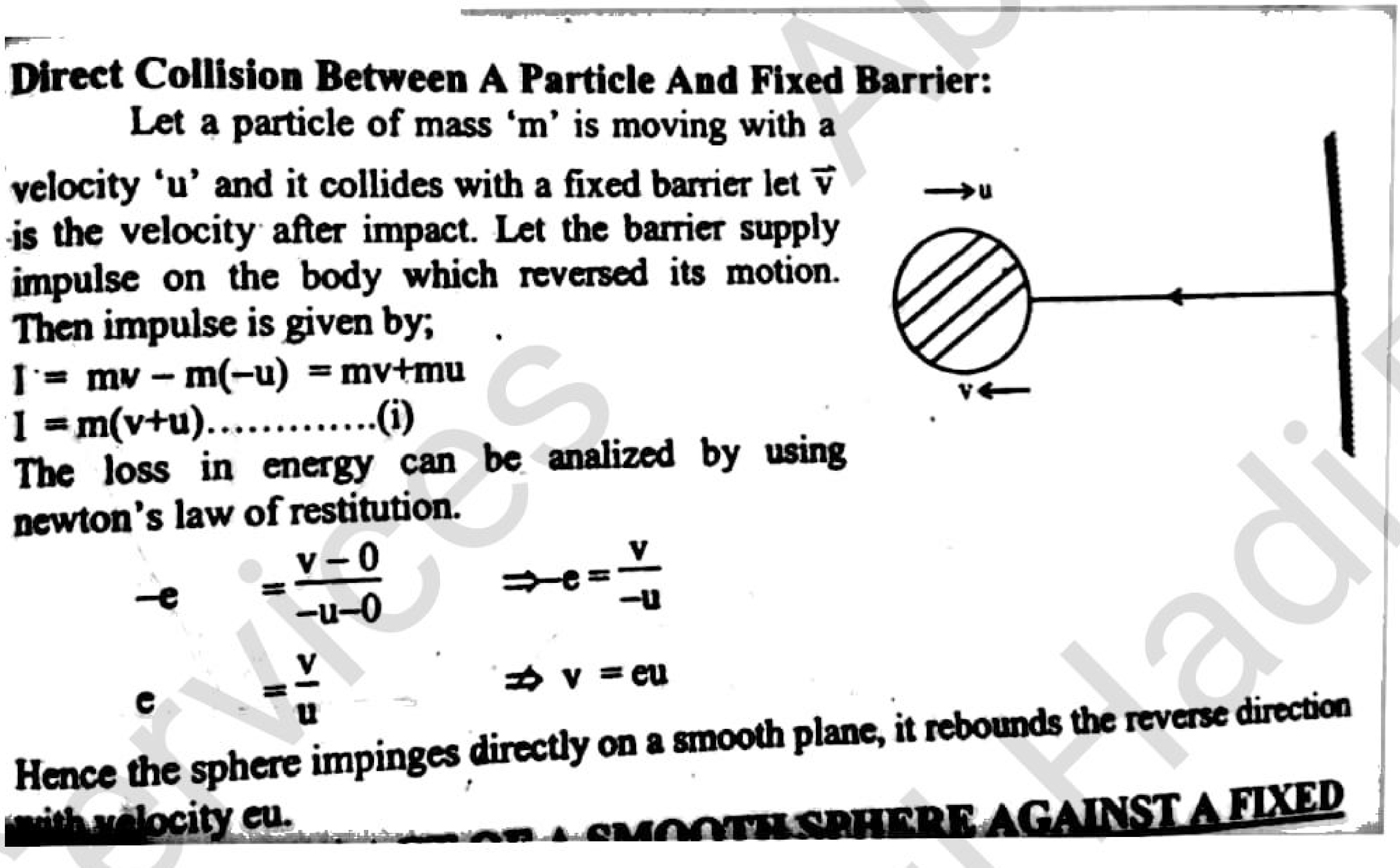
Law of Conservation of Linear Momentum:

When two bodies impinges the sum of their momentums along the line of impact is same after impact as before;

For case a: $mv + m'v' = mu + m'u'$

For case b: $mv\cos\theta + m'v'\cos\phi = mu\cos\alpha + m'u'\cos\beta$

Direct Collision Between A Particle And Fixed Barrier:



Direct Collision Between A Particle And Fixed Barrier:

Let a particle of mass 'm' is moving with a velocity 'u' and it collides with a fixed barrier let $\vec{v}$ is the velocity after impact. Let the barrier supply impulse on the body which reversed its motion. Then impulse is given by;

$$I = mv - m(-u) = mv + mu$$

$$I = m(v+u) \quad \text{............(i)}$$

The loss in energy can be analized by using newton's law of restitution.

$$-e = \frac{v-0}{-u-0} \Rightarrow -e = \frac{v}{-u}$$

$$e = \frac{v}{u} \Rightarrow v = eu$$

Hence the sphere impinges directly on a smooth plane, it rebounds the reverse direction with velocity eu.

OF A SMOOTH SPHERE AGAINST A FIXED

Lecture (43)

Damped Harmonic Oscillator

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The Harmonic Oscillator:

A mathematical model describing problems of oscillations in the form of rectilinear simple

Harmonic motion is usually called for Harmonic oscillator e.g. simple pendulum

Damped Harmonic Oscillator:

When some sort of velocity dependent retarding (or damping) force, due to resistance of the

medium or some other possible cause is ordinary present when the damping force is taken into account. The harmonic oscillator is termed as the Damped Harmonic Oscillator.

Damped Oscillation under the Damping Force Proportional (to the 1st power of) Velocity:

Suppose a particle of mass m is moving along the x-axis having fixed origin O. Let P be the

position of the particle at time t from O. The forces acting on the particle are

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(i) *a restorative force* $-mw^2 x\underline{i}$ *and*

(ii) *a damping force* $-mk\dot{x}\underline{i}$, *where* k *is* $+ve$ *constant.*

The equation of motion is $m\ddot{x}\underline{i} = -mw^2 x\underline{i} - mk\dot{x}\underline{i}$

or $\quad \ddot{x} + k\dot{x} + w^2 x = 0 \ldots\ldots\ldots\ldots (i)$

set $\quad D = \frac{d}{dt} \quad \therefore (1)$ *becomes*

$$\left(D^2 + kD + w^2\right)x = 0$$

if $k = 0$ *and* $w \neq 0$, *this gives S.H.M. if* $k \neq 0$ *then the auxiliary equation is*

Hence $\quad p = \frac{-k \pm \sqrt{k^2 - 4w^2}}{2} = \frac{-k}{2} \pm \sqrt{\frac{k^2}{4} - w^2}$

Now we discuss the three cases for the solution of different equation according as the $\quad disc = \frac{k^2}{4} - w^2 \leq 0$

Case I:

when $\frac{k^2}{4} - w^2 < 0$

$$\frac{k^2}{4} - w^2 = -n^2$$

$$p = -\frac{k}{2} \pm n$$

Therefore the general solution of different equation is

$$x = e^{\frac{-k}{2}t}\left[C_1 \cos nt + C_2 \sin nt\right]$$

put $C_1 = a\cos\alpha$, *where* C_1, C_2, a, α *are constant we get*

$$x = ae^{\frac{-k}{2}t}\left[\sin\alpha \cos nt + \cos\alpha \sin nt\right]$$

$$x = ae^{\frac{-k}{2}t}\sin\left(nt + \alpha\right) \ldots\ldots\ldots\ldots (ii) \qquad as\ t \to \infty, x \to 0$$

Case II:

when $\frac{k^2}{4} - w^2 > 0$ *Let* $\frac{k^2}{4} - w^2 = \lambda^2$

$p = -\frac{k}{2} \pm \lambda$

Therefore the equation is $x = Ae^{\left(-\frac{k}{2}+\lambda\right)t} + Be^{\left(-\frac{k}{2}+\lambda\right)t}$

$$as\ t \to \infty, x \to 0$$

Case III:

$$when\ \frac{k^2}{4} - w^2 = 0 \qquad then\ p = -\frac{k}{2}, -\frac{k}{2} \qquad Let\ \frac{k}{2} = w$$

$$\therefore \qquad p = -w, -w$$

$\therefore$ *The solution of different equation is*

$x = (A + Bt)e^{-wt}$ *as* $t \to \infty, x \to 0$

Lecture (44)

Damped Forced Oscillations

Damped Forced Oscillations:

In order to overcome the damping effect of a medium an applied force called driving force is generally applied, In such a case the system is called damped forced oscillator.

The applied force is usually a periodic force, In addition to the restorative force and damping force. Let we apply a force mFcospt, (F and p being constants) acting in the direction of increasing x

Then equation of motion is $m\ddot{x} = -mw^2x - mk\dot{x} + mF\cos pt$ dividing by m

$$\ddot{x} + k\dot{x} + w^2x = F\cos pt$$

$$\Rightarrow \qquad (D^2 + kD + w^2)x = F\cos pt \ \ldots\ldots\ldots(i)$$

The solution of this Differentiate equation consists of two parts the complementary function and the particular integral solution. First we find the complimentary solution of $(D^2 + kD + w^2)x = 0$

In the above three cases we see that the complimentary function decay with time. For this reason this part of the solution is called the transcent solution. The particular integral on the other hand generally does not die out, therefore the ultimate motion of the system depends upon that part of the solution which is called the steady state solution:

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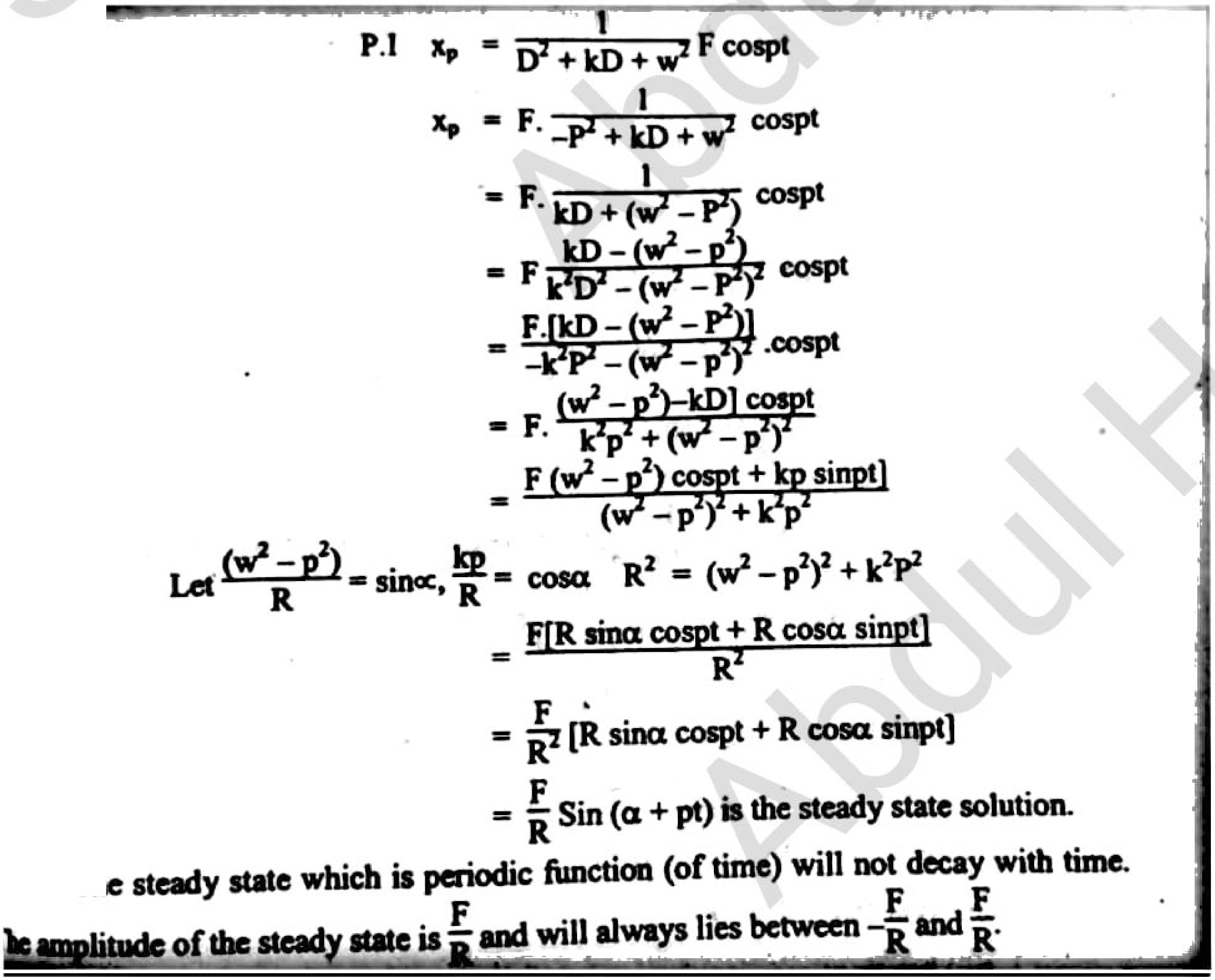

P.1 $x_p = \frac{1}{D^2 + kD + w^2} F \cos pt$

$$x_p = F.\frac{1}{-P^2 + kD + w^2} \cos pt$$

$$= F.\frac{1}{kD + (w^2 - P^2)} \cos pt$$

$$= F\frac{kD - (w^2 - p^2)}{k^2D^2 - (w^2 - P^2)^2} \cos pt$$

$$= \frac{F.[kD - (w^2 - P^2)]}{-k^2P^2 - (w^2 - p^2)^2} .\cos pt$$

$$= F.\frac{(w^2 - p^2) - kD] \cos pt}{k^2p^2 + (w^2 - p^2)^2}$$

$$= \frac{F (w^2 - p^2) \cos pt + kp \sin pt]}{(w^2 - p^2)^2 + k^2p^2}$$

Let $\frac{(w^2 - p^2)}{R} = \sin\alpha, \frac{kp}{R} = \cos\alpha \quad R^2 = (w^2 - p^2)^2 + k^2P^2$

$$= \frac{F[R \sin\alpha \cos pt + R \cos\alpha \sin pt]}{R^2}$$

$$= \frac{F}{R^2} [R \sin\alpha \cos pt + R \cos\alpha \sin pt]$$

$= \frac{F}{R}$ Sin $(\alpha + pt)$ is the steady state solution.

e steady state which is periodic function (of time) will not decay with time.

he amplitude of the steady state is $\frac{F}{R}$ and will always lies between $-\frac{F}{R}$ and $\frac{F}{R}$.

Lecture (45)

Vertical Motion with air Resistance

Case I: Vertically downward Motion:

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CHAPTER 10

DAMPED AND FORCED OSCILLATOR RESISTED MOTION

AIR RESISTANCE PROPORTION TO THE SQUARE OF THE VELOCITY

Case 1: Vertically downward Motion:

To discuss the motion of a particle of mass m released from rest at an initial height above the ground. The resistance of air is assumed to be proportional to the square of velocity.

Let x be the distance fallen through in time t, with air retardation $m\lambda v^2$ where v be the velocity at time t and λ is +ve constant. Then equation of motion is $mv\dfrac{dv}{dx} = mg - m\lambda v^2$ divide by m both side

$$v\frac{dv}{dx} = g - \lambda v^2$$

$$v\frac{dv}{dx} = \lambda\left(\frac{g}{\lambda} - v^2\right)$$

Let $\dfrac{g}{\lambda} = k^2$ $\Rightarrow$ $v\dfrac{dv}{dx} = \lambda(k^2 - v^2)$ .... (i)

Separate the variable and integrating

$$\int\frac{vdv}{k^2-v^2} = \lambda\int dx$$

$$-\frac{1}{2}\int\frac{-2vdv}{k^2-v^2} = \lambda x + C$$

$$-\frac{1}{2}\,ln\,(k^2 - v^2) = \lambda x + C,$$ Where C is constant of integration

Initially when t = 0, $x = 0$, $v = 0$

Putting these values in (i) we get $C = -\dfrac{1}{2}\,lnk^2$

Hence $-\dfrac{1}{2}\,ln\,(k^2 - v^2) = \lambda x - \dfrac{1}{2}\,lnk^2$

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$$\Rightarrow \quad \frac{1}{2}\ln k^2 - \frac{1}{2}\ln(k^2 - v^2) = \lambda x$$

$$\Rightarrow \quad -\frac{1}{2}\{\ln(k^2 - v^2) - \ln k^2\} = \lambda x$$

$$\Rightarrow \quad \ln\left(\frac{k^2 - v^2}{k^2}\right) = -2\lambda x$$

$$\Rightarrow \quad \frac{k^2}{k^2} - \frac{v^2}{k^2} = e^{-2\lambda x} \quad \Rightarrow \quad 1 - \frac{v^2}{k^2} = e^{-2\lambda x}$$

$$\Rightarrow \quad 1 - e^{-2\lambda x} = \frac{v^2}{k^2}$$

$$\Rightarrow \quad v^2 = k^2(1 - e^{-2\lambda x}) \ldots\ldots \text{(ii)}$$

This equation gives us velocity in terms of x.

Again use the expression for acceleration.

$\frac{dv}{dt} = \lambda\,(k^2 - v^2)$again separate the variables and integrating

$$\int\frac{dv}{k^2 - v^2} = \lambda\int dt\ .$$

$$\frac{1}{k}\tan h^{-1}\frac{v}{k} = \lambda t + D$$

Initially when $t = 0$, $v = 0$

$\frac{1}{k}\tan h^{-1}(0) = \lambda(0) + D \quad \Rightarrow \quad D = 0$ put in above

$\frac{1}{k}\tan h^{-1}\frac{v}{k} = \lambda t \quad \Rightarrow \quad \tan h^{-1}\frac{v}{k} = \lambda kt$ But $k^2 = \frac{g}{\lambda} \quad \Rightarrow \quad \lambda = \frac{g}{k^2}$

$$\tan h^{-1}\frac{v}{k} = \frac{g}{k^2}kt$$

$$\Rightarrow \quad \tan h^{-1}\frac{v}{k} = \frac{g}{k}t$$

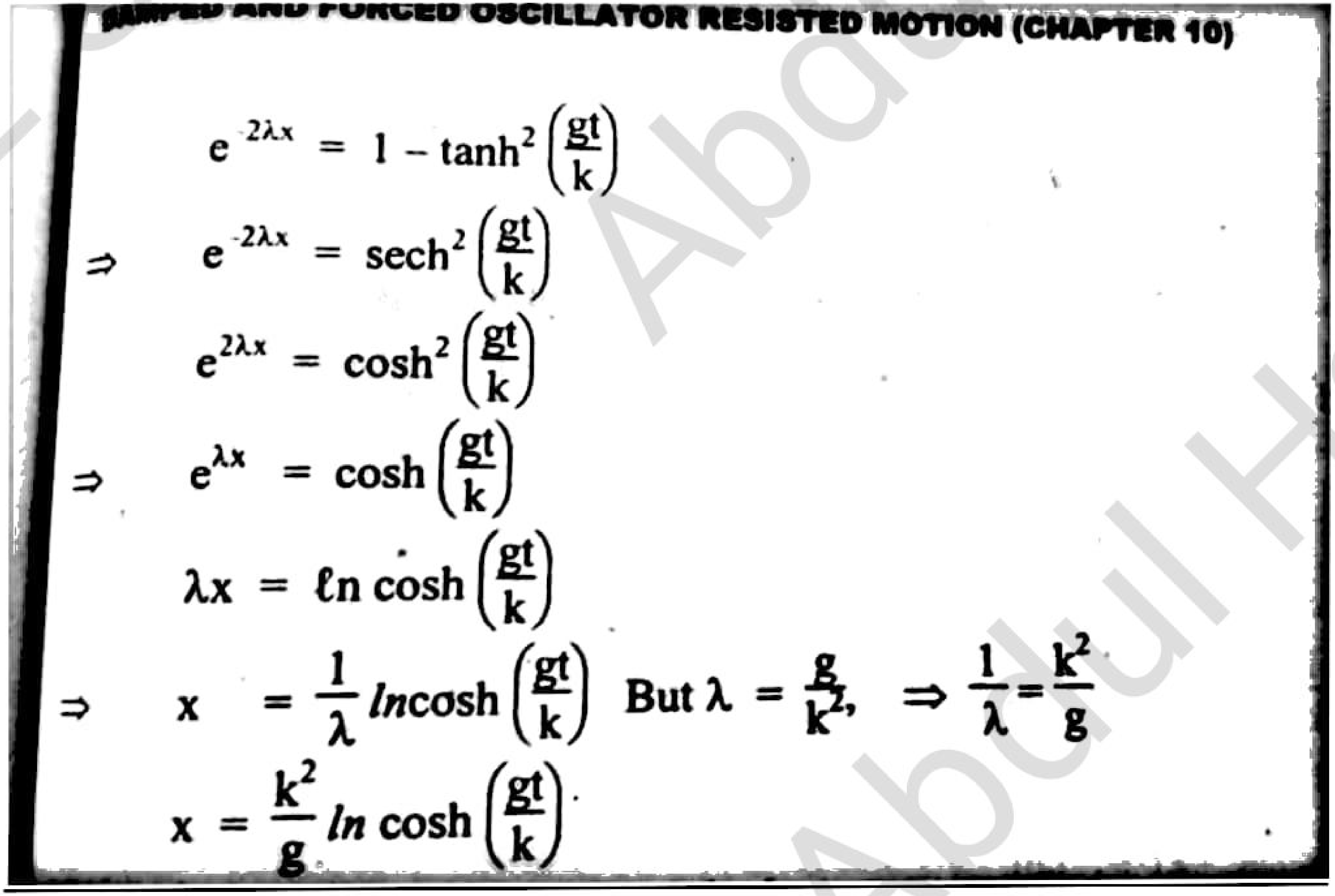
$\frac{v}{k} = \tan h\frac{gt}{k} \quad \Rightarrow \quad v = k\tanh\left(\frac{gt}{k}\right)$ this gives 'v' in terms't'

$$v^2 = k^2\tanh^2\left(\frac{gt}{k}\right) \ldots\ldots\ldots\ldots\ldots \text{(iii)}$$

Now we find the relation between x and t from (ii) and (iii) by comparing.

$$k^2\tanh^2\left(\frac{gt}{k}\right) = k^2(1 - e^{2\lambda x})$$

$$\Rightarrow \quad \tanh^2\left(\frac{gt}{k}\right) = 1 - e^{2\lambda x}$$



$$e^{-2\lambda x} = 1 - \tanh^2\left(\frac{gt}{k}\right)$$

$$\Rightarrow \quad e^{-2\lambda x} = \operatorname{sech}^2\left(\frac{gt}{k}\right)$$

$$e^{2\lambda x} = \cosh^2\left(\frac{gt}{k}\right)$$

$$\Rightarrow \quad e^{\lambda x} = \cosh\left(\frac{gt}{k}\right)$$

$$\lambda x = \ell n \cosh\left(\frac{gt}{k}\right)$$

$$\Rightarrow \quad x = \frac{1}{\lambda} ln\cosh\left(\frac{gt}{k}\right) \quad \text{But } \lambda = \frac{g}{k^2}, \quad \Rightarrow \frac{1}{\lambda} = \frac{k^2}{g}$$

$$x = \frac{k^2}{g} ln \cosh\left(\frac{gt}{k}\right)$$

Case II: Vertically Upward Motion

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Case II: Vertically upward Motion:

Now we discuss the motions of a particle of mass m projected vertically upward with an initial velocity v_o and air resistance proportional to the square of the velocity.

Let x be the height attained by the particle in time t. the equation of motion is $mv\frac{dv}{dx} = -mg - m\lambda v^2$ Divide by m

$$v\frac{dv}{dx} = -g - \lambda v^2 = -\lambda\left(\frac{g}{\lambda} + v^2\right) \text{ Let } \frac{g}{\lambda} = k^2$$

$$v\frac{dv}{dx} = -\lambda(k^2 + v^2) \text{ ...... (i)}$$ Separate the variables and integrating

$$\int\frac{vdv}{k^2 + v^2} = -\int \lambda\, dx$$ Multiply both side by 2

$$\int\frac{2vdv}{k^2 + v^2} = -2\lambda\int dx$$

$$ln(k^2 + v^2) = -2\lambda x + A$$ where A is constant of integration.

Initially when $t = 0, x = 0,$ $v = v_o$ we have

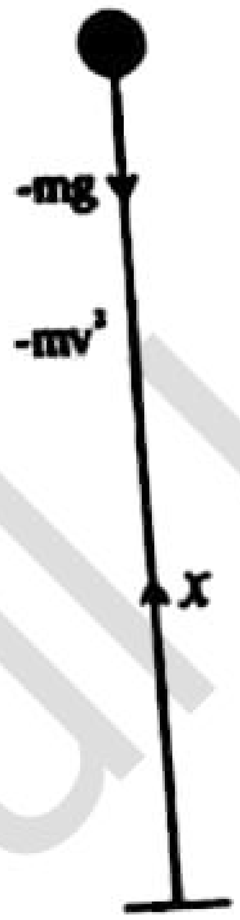
$$A = ln(k^2 + v_o^2)$$ put in above

$$ln(k^2 + v^2) = -2\lambda x + ln(k^2 + v_o^2)$$

$$ln(k^2 + v^2) - ln(k^2 + v_o^2) = -2\lambda x$$

$$ln\left(\frac{k^2 + v^2}{k^2 + v_o^2}\right) = -2\lambda x$$

$$\Rightarrow \quad x = \frac{1}{2\lambda}\, ln\left(\frac{k^2 + v_o^2}{k^2 + v^2}\right) \text{ But } \lambda = \frac{g}{k^2} \text{ Or } \frac{1}{\lambda} = \frac{k^2}{g}$$



$$x = \frac{k^2}{2g} h \left(\frac{k^2 + v_o^2}{k^2 + v^2}\right) \text{ .... (ii)}$$

Again (i) can be written as

$\frac{dv}{dt} = -\lambda (k^2 + v^2)$ Separate the variables and integrating

$$\int \frac{dv}{k^2 + v^2} = -\lambda \int dt$$

$$\frac{1}{k} \tan^{-1} \frac{v}{k} = -\lambda t + B$$

At $t = 0$, $v = v_o$,

$$B = \frac{1}{k} \tan^{-1} \frac{v_o}{k}$$

Above become

$$\frac{1}{k} \tan^{-1} \frac{v}{k} = -\lambda t + \frac{1}{k} \tan^{-1} \frac{v_o}{k}$$

$$\frac{1}{k} \tan^{-1} \frac{v_o}{k} - \frac{1}{k} \tan^{-1} \frac{v}{k} = \lambda t$$

$\tan^{-1} \frac{v_o}{k} - \tan^{-1} \frac{v}{k} = \lambda tk$ But $\lambda = \frac{g}{k^2}$

$$\tan^{-1} \frac{v_o}{k} - \tan^{-1} \frac{v}{k} = tk \frac{g}{k^2}$$

$$\tan^{-1} \frac{v_o}{k} - \tan^{-1} \frac{v}{k} = \frac{gt}{k} \text{ .... (iii)}$$

$$\tan^{-1} \frac{v}{k} = \tan^{-1} \frac{v_o}{k} - \frac{gt}{k}$$

$$\frac{v}{k} = \tan\left(\tan^{-1} \frac{v_o}{k} - \frac{gt}{k}\right)$$

$$v = k \tan\left(\tan^{-1} \frac{v_o}{k} - \frac{gt}{k}\right)$$

$\frac{dx}{dt} = k \tan\left(\tan^{-1} \frac{v_o}{k} - \frac{gt}{k}\right)$ Separate the variable and integrati

$$\int dx = k \int \tan\left(\tan^{-1} \frac{v_o}{k} - \frac{gt}{k}\right) dt$$

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$$x = -k\, ln \cos\left(\tan^{-1}\frac{v_0}{k} - \frac{gt}{k}\right)\left(\frac{-k}{g}\right) + C$$

$$x = \frac{k^2}{g}\, ln \cos\left(\tan^{-1}\frac{v_0}{k} - \frac{gt}{k}\right) + C$$

at $t = 0$, $x = 0$, so $C = -\frac{k^2}{g}\, ln \cos\left(\tan^{-1}\frac{v_0}{k}\right)$

above become

$$x = \frac{k^2}{g}\, ln \cos\left(\tan^{-1}\frac{v_0}{k} - \frac{gt}{k}\right) - \frac{k^2}{g}\, ln \cos\left(\tan^{-1}\frac{v_0}{k}\right)$$

$$= \frac{k^2}{g}\, ln\left[\frac{\cos\left(\tan^{-1}\frac{v_0}{k} - \frac{gt}{k}\right)}{\cos\left(\tan^{-1}\frac{v_0}{k}\right)}\right] \text{.............(iv)}$$

Equation (ii), (iii) and (iv) determine the motion.

Best of luck & Pray for me

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