MTH633 Group Theory

Question No : 10 of 41

Let H be a subgroup of G and let N be a normal subgroup of G Then $(HN)/N =$ ______

page 1009

Answer (Please select your correct option)

- $H/(H \cup N)$
- $H/(H \cap N)$
- $H \cup (H \cap N)$
- $H \cap (H \cap N)$

H/H∩N

Start Time: [illegible]

Question No : 9 of 41

A non-abelian group G may have __________ of some order d dividing $|G|$

page 1035

Answer (Please select your correct option)

- abelian subgroup
- subset
- no subgroup
- none of these

MTH633 Group Theory

Question No : 8 of 11

A __________ contained in no larger *p*-subgroup.

page 1064

Answer (Please select your correct option)

- *p* - subgroup
- cyclic subgroups
- improper subgroups
- trivial subgroups

MTH633 Group Theory

Question No : 7 of 41

Each orbit in G under conjugation by G is a ____________ in G.

page 1082

Answer (Please select your correct option)

- conjugate class
- subgroup
- trivial group
- none of these

Start Time

Question No : 6 of 41

For a prime number p, every group G of order p^2 is __________.

page 1091

Answer (Please select your correct option)

- abelian
- non-abelian
- infinite
- none of these

Question No : 5 of 41

If H and N are subgroups of a group G, then the their join H V N is the smallest subgroup of G containing both H and N

page 1004

Answer (Please select your correct option)

- True
- False

Let G be a group acting on a set X, and let $x \in X$. Then the set $Gx = \{ax \mid a \in G\}$

is called the orbit of x in G

page 976

Answer (Please select your correct option)

True

False

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MTH633 Group Theory

Question No : 3 of 41

Sometimes a set X is used to study G via a group action of G on X

page 953

Answer (Please select your correct option)

Yes

No

MTH633 Group Theory

Question No : 2 of 41 **Marks: 1 (Budgeted Time 1 Min)**

Generally, for any sets A, B, and C, we can view a map * A x B → C as defining a "multiplication," where any element a of A times any element b of B has as value some element c of C

page 946

Answer (Please select your correct option)

True

MTH633 Group Theory

Question No : 1 of 41

A_3 is a normal subgroup of S_3

page 919

Answer (Please select your correct option)

Yes

No

Question No : 20 of 41

Let the complex numbers of magnitude 1 form a subgroup U of C^*, then the cosets of U are ________

page 718

Answer (Please select your correct option)

- Parallel lines
- circles with center at the origin
- Non-parallel lines
- None of these

Question No : 19 of 41

$Ker(h)$ is called the null space of A if it consists of all $v \in \mathbb{R}^n$ such that ________

page 700

Answer (Please select your correct option)

- $Av = a$
- $Av = b$
- $Av = c$
- $Av = 0$ ✓

Let φ be a homomorphism function from group $(G,\circ)$ to group $(G',\cdot)$ Then, $(Ker\,\varphi,\circ)$ is ________ subgroup of $(G,\circ)$

page 675

Answer (Please select your correct option)

- commutative
- factor
- simple
- normal

MTH633 Group Theory

Question No : 17 of 41

If N is ______________ subgroup of a group G, the left cosets of N in G are the same as the right cosets of N in G.

page 652

Answer (Please select your correct option)

- normal
- proper
- improper
- trivial

Group Theory

Let H be a subgroup of G and a is in G, then ________

page 564

Answer (Please select your correct option)

- $|aH| = |bH|$
- $|aH| \neq |bH|$
- $|aH| < |bH|$
- $|aH| > |bH|$

Question No : 15 of 41

$Z_2 \times Z_3$ is cyclic group with generator ---------------

page 496

Answer (Please select your correct option)

(1, 1)

(0, 0)

(1, 0)

(0, 1)

Question No : 14 of 41

$\prod_{i=1}^{n} G_i$ is the ---------------- notation for all the group operations for $G_1, \cdots, G_n$.

page 484

Answer (Please select your correct option)

- additive
- multiplicative
- subtractive
- division

Start Time: 11:07 AM

114:00

14

MTH633 Group Theory

Question No : 13 of 41

S_3 is NOT the group D_3 of symmetries of an equilateral triangle.

page 441

Answer (Please select your correct option)

- Yes
- No

MTH633 Group Theory

Question No : 12 of 41

Every subgroup of a cyclic group is ---------------- group.

page 385

Answer (Please select your correct option)

- Cyclic ⇦
- Non- cyclic
- Non-abelian
- Finite

MTH633 Group Theory

Question No : 11 of 41

$\langle Z, + \rangle$ is ------------------ of $\langle R, \cdot \rangle$

page 305

Answer (Please select your correct option)

- subgroup
- not subgroup ⇦
- trivial subgroup
- non-trivial subgroup

QUESTION 1

Let $\phi: G \rightarrow G'$ be a group morphism. Then, ϕ is injective if and only if Ker $\phi = \{e\}$.

Proof :

If Ker(ϕ) = $\{e\}$, then for every $a \in G$, the elements mapped into $\phi(a)$ are precisely the elements of the left coset $a\{e\} = \{a\}$, which shows that ϕ is one to one.

Conversely, suppose ϕ is one to one. Now, we know that $\phi(e)=e'$, the identity element of G'. Since ϕ is one to one, we see that *e* is the only element mapped into *e'* by ϕ, so Ker(ϕ)= $\{e\}$.

QUESTION 2

A homomorphism h: G→G' is injective if and only if Ker h={e}

Proof:

Suppose h is injective, and let $x \in$ Ker h.

Then $h(x)=e'=h(e)$. Hence $x=e$.

Conversely, suppose Ker $h=\{e\}$.

Then $h(x)=h(y)$

$\Rightarrow h(xy^{-1})=h(x)h(y^{-1})$

$=h(x)h(y)^{-1}=e'$

$\Rightarrow xy^{-1} \in \text{Ker } h$

$\Rightarrow xy^{-1}=e$

$\Rightarrow x=y$.

Hence, h is injective.

QUESTION 3

Classify the group $(\mathbb{Z}_4 \times \mathbb{Z}_2) / (\{0\} \times \mathbb{Z}_2)$ according to the fundamental theorem of finitely generated abelian groups.

Solution :

The projection map

$\pi_1 : \mathbb{Z}_4 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ given by $\pi_1(x,y) = x$ is a homomorphism of $\mathbb{Z}_4 \times \mathbb{Z}_2$ onto $\mathbb{Z}_4$ with kernel $\{0\} \times \mathbb{Z}_2$. By fundamental theorem of homomorphism, we know that the given factor group is isomorphic to $\mathbb{Z}_4$.

The projection map

$\pi_1 : \mathbb{Z}_4 \times \mathbb{Z}_2 \rightarrow \mathbb{Z}_4$ given by

$\pi_1(x,y) = x.$

$K = \text{Ker } \pi_1 = \{0\} \times \mathbb{Z}_2$

$= \{(0,0), (0,1)\}.$

$(1,0) + K = \{(1,0), (1,1)\}$

$(2,0) + K = \{(2,0), (2,1)\}$

QUESTION 4

The set of all inner automorphisms of G is a subgroup of Aut(G).

Proof:

(1) Let $i_a, i_b \in \text{Inn}(G)$.

Then $i_a(i_b(x)) = a(i_b(x))a^{-1} = abxb^{-1}a^{-1}$

$= abx(ab)^{-1} = i_{ab} \in \text{Inn}(G)$.

Hence the conjugation by b composed by conjugation by a is conjugation by ab.

(2) The inverse of i_a is conjugation by $a' = a^{-1}$.

$i_a((i_{a'})(x)) = i_a(a'x(a')^{-1}) = aa'xa'^{-1}a^{-1} = aa'x(aa')^{-1} = x$.

Thus Inn (G) is a subgroup.

QUESTION 5

A factor group of a cyclic group is cyclic.

Proof:

Let G be cyclic with generator a, and let N be a normal subgroup of G. We claim the coset aN generates G / N. We must compute all powers of aN. But this amounts to computing, in G, all powers of the representative a and all these powers give all elements in G. Hence the powers of aN certainly give all cosets of N and G / N is cyclic.

QUESTION 6

Here the first factor $\mathbb{Z}_4$ of $\mathbb{Z}_4 \times \mathbb{Z}_6$ is left alone. The $\mathbb{Z}_6$ factor, on the other hand, is essentially collapsed by a subgroup of order 3, giving a factor group in the second factor of order 2 that must be isomorphic to $\mathbb{Z}_2$. Thus $(\mathbb{Z}_4 \times \mathbb{Z}_6)/(\langle(0, 2)\rangle)$ is isomorphic to $\mathbb{Z}_4 \times \mathbb{Z}_2$.The trivial subgroup N = {0} of $\mathbb{Z}$ is, of course, a normal subgroup. Compute $\mathbb{Z}/\{0\}$.

Solution:

Since N={0} has only one element, every coset of N has only one element. That is, the cosets are of the form {m} for $m \in \mathbb{Z}$. There is no collapsing at all, and consequently, $\mathbb{Z}/\{0\} \cong \mathbb{Z}$. Each $m \in \mathbb{Z}$ is simply renamed {m} in $\mathbb{Z}/\{0\}$.

QUESTION 7

Let G = H x K be the direct product of groups H and K. Then $\overline{H}$={(h, e)| h ϵ H} is a normal subgroup of G. Also G/$\overline{H}$ is isomorphic to K in a natural way. Similarly, G / $\overline{K}$ ≃ H in a natural way.

Proof

Consider the map π_2: H x K → K given by

$\pi_2(h, k) = k$. The map π_2 is homomorphism since

$\pi_2(h_1h_2, k_1k_2) = k_1k_2 = \pi_2(h_1, k_1)\, \pi_2(h_2, k_2)$.

Because Ker(π_2) = $\overline{H}$, we see that $\overline{H}$ is a normal subgroup of H x K. Because π_2 is onto K, Fundamental Theorem of Homomorphism tells us that (H x K)/ $\overline{H}$ ≃ K.

QUESTION 8

The cyclic group $G=\mathbb{Z}/5\mathbb{Z}$ of congruence classes modulo 5 is simple.

Proof

If H is a subgroup of this group, its order must be a divisor of the order of G which is 5.

Since 5 is prime, its only divisors are 1 and 5, so either H is G, or H is the trivial group.

QUESTION 9

M is a maximal normal subgroup of G if and only if G / M is simple.

Proof :

Let M be a maximal normal subgroup of G. Consider the canonical homomorphism $\gamma: G \to G/M$. Now γ^{-1} of any nontrivial proper normal subgroup of G/M is a proper normal subgroup of G properly containing M. But M is maximal, so this can not happen. Thus G/M is simple.

Conversely, if N is a normal subgroup of G properly containing M, then $\gamma[N]$ is normal in G/M. If also $N \neq G$, then $\gamma[N] \neq G/M$ and $\gamma[N] \neq \{M\}$.

Thus, if G/M is simple so that no such $\gamma[N]$ can exist, no such N can exist, and M is maximal.

QUESTION 10

Show that Z(G) is a normal and an abelian subgroup of G.

Solution :

For each g ∈ G and $z \in Z(G)$ we have $gzg^{-1} = zgg^{-1} = ze = z$, we see at once that Z(G) is a normal subgroup of G. It implies that gz=zg for g ∈ G and z∈Z(G). If G is abelian, then Z(G) = G; in this case, the center is not useful.

QUESTION 11

Let G be a group. The set of all commutators $aba^{-1}b^{-1}$ for $a, b \in G$ generates a subgroup C of G.

Proof:

Let $a, b \in G$. Then,

$(aba^{-1}b^{-1})(aba^{-1}b^{-1})^{-1}$

$=aba^{-1}b^{-1}bab^{-1}a^{-1}$

$=e \in C$

since $e = eee^{-1}e^{-1}$ is a commutator.

QUESTION 12

The Klein 4-group V = {e, a, b, c} is generated.

Proof:

The Klein 4-group V = {e, a, b, c} is generated by {a,b} since ab=c.

It is also generated by {a,c}, {b,c }, and {a,b,c}.

If a group G is generated by a subset S, then every subset of G containing S generates G.

QUESTION 13

The group $\mathbb{Z}_6$ is generated.

Proof:

The group $\mathbb{Z}_6$ is generated by {1} and {5}.It is also generated by {2,3} since 2+3=5, so that any subgroup containing 2 and 3 must contain 5 and must therefore be $\mathbb{Z}_6$. It is also generated by {3,4}, {2,3,4}, {1,3}, and {3,5}. But it is not generated by {2, 4} since

$<2> = \{0, 2, 4\}$ contains 2 and 4.

QUESTION 14

If N is a normal subgroup of G, then G/N is abelian if and only if $C \leq N$.

Proof :

If N is a normal subgroup of G and G/N is abelian, then

$(a^{-1}N)(b^{-1}N)=(b^{-1}N)(a^{-1}N)$; that is, $aba^{-1}b^{-1}N=N$,

so $aba^{-1}b^{-1} \in N$, and $C \leq N$.

Finally, if $C \leq N$, then

$(aN)(bN)=abN$

$=ab(b^{-1}a^{-1}ba)N$

$= (abb^{-1}a^{-1})baN$

$= baN$

$= (bN)(aN)$.

QUESTION 15

The set Aut(G) of all automorphisms of a group G is a group under composition of mappings, and Inn(G) $\triangleleft$ Aut(G). Moreover, G/Z(G)$\simeq$Inn(G).

Proof:

Clearly, Aut(G) is nonempty. Let $\sigma, \tau \in$ Aut(G). Then for all $x, y \in G$, $\sigma\tau(xy)=\sigma((\tau(x)\tau(y)) = (\sigma\tau(x))(\sigma\tau(y))$.

Hence, $\sigma\tau \in$ Aut(G). Again,

$\sigma(\sigma^{-1}(x)\sigma^{-1}(y))=$
$\sigma\sigma^{-1}(x)\sigma\sigma^{-1}(y)$

$=xy$.

Hence $\sigma^{-1}(x)\sigma^{-1}(y)=$
$\sigma^{-1}(xy)$. Therefore, $\sigma^{-1} \in$ Aut(G). This proves that Aut(G) is a subgroup of the symmetric group S_G and, hence, is itself a group.

QUESTION 16

Let G be a finite abelian group of order n, and let m be a positive integer relative prime to n. Then the mapping $\sigma: x \rightarrow x^m$ is an automorphism of G.

Solution:

$(m,n) = 1 \Rightarrow$ there exist integers u and v such that $mu + nv = 1 \Rightarrow$ for all $x \in G$, $x^{mu+nv} = x^{mu} x^{nv} = x^{um}$ since $o(G)=n$.

Now for all $x \in G$, $x = (x^u)^m$ implies that σ is onto. Further,

$x^m = e \Rightarrow x^{mu} = e \Rightarrow x = e$, showing that σ is 1-1.

That σ is a homomorphism follows from the fact that G is abelian. Hence, σ is an automorphism of G.

QUESTION 17

Let $G = \langle a \mid a^n = e \rangle$ be a finite cyclic group of order n. Then the mapping $\sigma : a \rightarrow a^m$ is an automorphism of G iff (m,n) = 1.Further, if (m,n)=d, then $(a^m)^{n/d} = (a^n)^{m/d} = e$. Thus, the order of a^m divides n/d; that is, n|n/d. Hence, d = 1, and the solution is complete.Let X be a G-set. Then G_x is a subgroup of G for each $x \in X$.

Proof:

Let $x \in X$ and let $g_1, g_2 \in G_x$. Then $g_1x=x$ and $g_2x=x$. Consequently, $(g_1g_2)x=g_1(g_2x)=g_1x=x$, so $g_1g_2 \in G_x$, and G_x is closed under the induced operation of G. Of course $ex=x$, so $e \in G_x$. If $g \in G_x$, then $gx = x$, so $x=ex=(g^{-1}g)x= g^{-1}(gx)=g^{-1}x$, and consequently $g^{-1} \in G_x$. Thus G_x is a subgroup of G.

QUESTION 18

Let X be a G-set. For $x_1, x_2 \in X$, let $x_1 \sim x_2$ if and only if there exists $g \in G$ such that $gx_1 = x_2$. Then $\sim$ is an equivalence relation on X.

Proof:

For each $x \in X$, we have $ex = x$, so $x \sim x$ and $\sim$ is reflexive.

Suppose $x_1 \sim x_2$, so $gx_1 = x_2$ for some $g \in G$. Then $g^{-1}x_2 = g^{-1}(gx_1) = (g^{-1}g)x_1 = ex_1 = x_1$, so $x_2 \sim x_1$, and $\sim$ is symmetric. Finally, if $x_1 \sim x_2$ and $x_2 \sim x_3$, then $g_1x_1 = x_2$ and $g_2x_2 = x_3$ for some $g_1, g_2 \in G$. Then $(g_2g_1)x_1 = g_2(g_1x_1) = g_2x_2 = x_3$, so $x_1 \sim x_3$ and $\sim$ is transitive.

QUESTION 19

If N is a normal subgroup of G, and if H is any subgroup of G, then

$H \vee N = HN = NH$. Furthermore, if H is also normal in G, then HN is normal in G.

Proof:

We show that HN is a subgroup of G, from which

$H \vee N = HN$ follows at once. Let $h_1, h_2 \in H$ and $n_1, n_2 \in N$. Since N is a normal subgroup, we have $n_1 h_2 = h_2 n_3$ for some $n_3 \in N$. Then $(h_1 n_1)(h_2 n_2) = h_1(n_1 h_2)n_2 = h_1(h_2 n_3)n_2 = (h_1 h_2)(n_3 n_2) \in HN$, so HN is closed under the induced operation in G. Clearly $e = ee$ is in HN. For $h \in H$ and $n \in N$, we have $(hn)^{-1} = n^{-1}h^{-1} = h^{-1}n_4$ for some $n_4 \in N$, since N is a normal subgroup. Thus $(hn)^{-1} \in HN$, so $HN \leq G$.

A similar argument shows that NH is a subgroup, so $NH = H \vee N = HN$.

Now suppose that H is also normal in G, and let $h \in H$, $n \in N$, and $g \in G$. Then

$ghng^{-1} = (ghg^{-1})(gng^{1}) \in HN$, so HN is indeed normal in G.

QUESTION 20

Let G be a group such that for some fixed integer n >1, $(ab)^n = a^n b^n$ for all a, b∈G. Let $G_n = \{a \in G \mid a^n = e\}$ and $G^n = \{a^n \mid a \in G\}$. Then $G_n \triangleleft G$, $G^n \triangleleft G$, and $G/G_n \simeq G^n$.

Solution:

Let a, b∈G_n and x∈G. Then $(ab^{-1})^n = a^n (b^n)^{-1} = e$, so $ab^{-1} \in G_n$. Also, $(xax^{-1})^n = (xax^{-1})\dots(xax^{-1}) = xa^n x^{-1} = e$ implies $xax^{-1} \in G_n$. Hence, $G_n \triangleleft G$.

Let a, b, x∈G. Then $a^n (b^n)^{-1} = (ab^{-1})^n \in G^n$.

Also, $xa^n x^{-1} = (xax^{-1})\dots(xax^{-1}) = (xax^{-1})^n \in G^n$. Therefore, $G^n \triangleleft G$.

QUESTION 21

Third Isomorphism Theorem

Let H and K be normal subgroups of a group G with $K \leq H$. Then $G/H \simeq (G/K)/(H/K)$.

Proof:

Let $\phi: G \rightarrow (G/K)/(H/K)$ be given by $\phi(a) = (aK)(H/K)$ for $a \in G$. Clearly ϕ is onto $(G/K)/(H/K)$, and for $a, b \in G$,
$\phi(ab) = [(ab)K](H/K)$

$= [(aK)(bK)](H/K)$

$= [(aK)(H/K)][(bK)(H/K)] = \phi(a)\,\phi(b)$, so ϕ is a homomorphism.

The kernel consists of those $x \in G$ such that $\phi(x) = H/K$.

These x are just the elements of H.

Then first isomorphism theorem shows that $G/H \simeq (G/K)/(H/K)$.

QUESTION 22

Let G be a group of order p^n and let X be a finite G-set. Then

$|X| \equiv |X_G|$ (mod p).

Proof:

Recall $|X|=|X_G|+\sum_{i=s+1}^{r}|Gx_i|$.In the notation of above Equation,

we know that

$|Gx_i|$ divides $|G|$.

Consequently p divides $|Gx_i|$ for $s + 1 \leq i \leq r$. Above equation then shows that $|X|-|X_G|$ is divisible by p, so $|X| \equiv |X_G|$ (modp).

QUESTION 23

Let H be a p-subgroup of a finite group G. Then

$$(N[H]:H) \equiv (G:H) \pmod{p}.$$

Proof:

Let $\mathcal{L}$ be the set of left cosets of H in G, and let H act on $\mathcal{L}$ by left translastion, so that h(xH) = (hx)H. Then $\mathcal{L}$ becomes an H-set. Note that $|\mathcal{L}|=(G:H)$. Let us determine $\mathcal{L}_H$, that is, those left cosets that are fixed under action by all elements of H. Now xH= h(xH) if and only if $H=x^{-1}hxH$, or if and only if $x^{-1}hx \in H$.

Thus xH=h(xH) for all h∈H if and only if $x^{-1}hx$

$=x^{-1}h(x^{-1})^{-1} \in H$ for all h∈H, or if and only if $x^{-1} \in N[H]$, or if and only if x∈N[H]. Thus the left cosets in $\mathcal{L}_H$ are those contained in N[H]. The number of such cosets is (N[H]:H), so $|\mathcal{L}_H| = (N[H]:H)$.

Since H is a p-group, it has order a power of p. Then $|\mathcal{L}| \equiv |\mathcal{L}_H| \pmod{p}$, that is, $(G:H) \equiv (N[H]:H) \pmod{p}$.

QUESTION 24

Second Sylow Theorem

Let P_1 and P_2 be Sylow p-subgroups of a finite group G. Then P_1 and P_2 are conjugate subgroups of G.

Proof:

Here we will let one of the subgroups act on left cosets of the other. Let $\mathcal{L}$ be the collection of left cosets of P_1, and let P_2 act on $\mathcal{L}$ by $z(xP_1)=(zx)P_1$ for $z \in P_2$. Then $\mathcal{L}$ is a P_2-set. We have $|\mathcal{L}_{P_2}| \equiv |\mathcal{L}| \pmod p$, and $|\mathcal{L}| = (G: P_1)$ is not divisible by p, so $|\mathcal{L}_{P_2}| \neq 0$. Let $xP_1 \in \mathcal{L}_{P_2}$.

Then $zxP_1=xP_1$ for all $z \in P_2$, so $x^{-1}zxP_1=P_1$ for all $z \in P_2$. Thus $x^{-1}zx \in P_1$ for all $z \in P_2$, so $x^{-1}P_2x \leq P_1$. Since $|P_1|=|P_2|$, we must have $P_1=x^{-1}P_2x$, so P_1 and P_2 are indeed conjugate subgroups.

QUESTION 25

The center of a finite nontrivial p-group G is nontrivial.

Proof:

We have $|G|=c+n_{c+1}+\ldots+n_r$, where n_i is the number of elements in the ith orbit of G under conjugation by itself.

For G, each n_i divides $|G|$ for $c+1\leq i\leq r$, so p divides each n_i, and p divides $|G|$. Therefore p divides c. Now $e\in Z(G)$, so $c\geq 1$. Therefore $c\geq p$, and there exists some $a\in Z(G)$ where $a\neq e$.

QUESTION 26

For a prime number p, every group G of order p^2 is abelian.

Proof:

If G is not cyclic, then every element except e must be of order p.

Let a be such an element. Then the cyclic subgroup $\langle a \rangle$ of order p does not exhaust G. Also let $b \in G$ with $b \notin \langle a \rangle$. Then $\langle a \rangle \cap \langle b \rangle = \{e\}$, since an element c in $\langle a \rangle \cap \langle b \rangle$ with $c \neq e$ would generate both $\langle a \rangle$ and $\langle b \rangle$, giving $\langle a \rangle = \langle b \rangle$, contrary to construction.

From first Sylow theorem, $\langle a \rangle$ is normal in some subgroup of order p^2 of G, that is, normal in all of G. Likewise $\langle b \rangle$ is normal in G.

Now $\langle a \rangle \vee \langle b \rangle$ is a subgroup of G properly containing $\langle a \rangle$ and of order dividing p^2. Hence $\langle a \rangle \vee \langle b \rangle$ must be all of G.

Thus the hypotheses of last lemma are satisfied, and G is isomorphic to $\langle a \rangle \times \langle b \rangle$ and therefore abelian.