lecture # 8
Trigonometric Function

m Euler's $\Rightarrow$ $e^{iy} = \cos y + i\sin y \to (1)$, $\forall y \in \mathbb{R}$

$e^{-iy} = e^{i(-y)} = \cos(-y) + i\sin(-y)$

$e^{-iy} = \cos y - i\sin y \to (2)$

By (1)+(2)

$e^{iy} + e^{-iy} = (\cos y + i\sin y) + (\cos y - i\sin y)$

$e^{iy} + e^{-iy} = 2\cos y$

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}$$

By (1) - (2)

$e^{iy} - e^{-iy} = (\cos y + i\sin y) - (\cos y - i\sin y)$

$e^{iy} - e^{-iy} = 2i\sin y$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i} \quad \forall y \in \mathbb{R}$$

on the same analogy, we can define for $z \in \mathbb{C}$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \to (3) \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i} \to (4)$$

$$\Rightarrow \sec z = \frac{2}{e^{iz} + e^{-iz}}, \qquad \operatorname{cosec} z = \frac{2i}{e^{iz} - e^{-iz}}$$

By (3) ÷ (4)

$$\underbrace{\frac{\cos z}{\sin z}}_{\cot z} = \frac{\left(e^{iz} + e^{-iz}\right)/2}{\left(e^{iz} - e^{-iz}\right)/2i} = \frac{i\left(e^{iz} + e^{-iz}\right)}{e^{iz} - e^{-iz}}$$

$$\tan z = \frac{1}{\cot z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

* Extended Euler's formula :-

$$e^{iz} = \cos z + i\sin z \; ; \; z \in \mathbb{C}$$

$$R.H.S = \cos z + i \sin z$$

$$= \frac{e^{iz} + e^{-iz}}{2} + \cancel{i}\left(\frac{e^{iz} - e^{-iz}}{2\cancel{i}}\right)$$

$$= \frac{e^{iz} + e^{-iz}}{2} + \frac{e^{iz} - e^{-iz}}{2}$$

$$= \frac{e^{iz} \cancel{+ e^{-iz}} + e^{iz} \cancel{- e^{-iz}}}{}$$

$$= \frac{\cancel{2}e^{iz}}{\cancel{2}} = e^{iz} \; ; \; z \in \mathbb{C}$$

$$e^{iz} = \cos z + i\sin z \; ; \; z \in \mathbb{C}$$

Complex Tringonometric functions:-

prove that:-

(i) $\sin(z+2\pi) = \sin z$

(ii) $\cos(z+2\pi) = \cos z$

(iii) $\tan(\pi + z) = -\tan z$

(iv) $\cos(z+\pi) = -\cos z$

(v) $\sin(z+\pi) = -\sin z, \quad z \in \mathbb{C}$

$\downarrow$

$\sin(z+\pi) = -\sin z.$

$$L.H.S = \sin(z+\pi) = \frac{e^{i(z+\pi)} - e^{-i(z+\pi)}}{2i}$$

$$= \frac{e^{iz+i\pi} - e^{-iz-i\pi}}{2}$$

$$= \frac{e^{iz}\cdot e^{i\pi} - e^{-iz}\cdot e^{-i\pi}}{2i}$$

$$= \frac{e^{iz}(\cos\pi + i\sin\pi) - e^{-iz}(\cos(-\pi) + i\sin(-\pi))}{2i}$$

$$= e^{iz}(-1 + i(0)) - e^{-iz}(-1 + i(0)) \Big/ 2i$$

$$= \frac{-e^{iz} + e^{-iz}}{2i} = -\left(\frac{e^{iz} - e^{-iz}}{2i}\right) = -\sin z$$

Prove that (i) $\sin^2 z + \cos^2 z = 1$

(ii) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$

(iii) $\cos 2z = \cos^2 z - \sin^2 z = 2\cos^2 z - 1 = 1 - 2\sin^2 z$

Middle 1 $= \cos^2 z - \sin^2 z$

$$= (\cos z)^2 - (\sin z)^2$$

$$= \left(\frac{e^{iz} + e^{-iz}}{2}\right)^2 - \left(\frac{e^{iz} - e^{-iz}}{2i}\right)^2$$

$$= \frac{(e^{iz})^2 + (e^{-iz})^2 + 2e^{iz}.e^{-iz}}{4} - \left[\frac{(e^{iz})^2 + (e^{-iz})^2 - 2e^{iz}.e^{-iz}}{4i^2}\right]$$

$$= \frac{e^{2iz} + e^{-2iz} + 2e^{iz-iz}}{4} - \left[\frac{e^{2iz} + e^{-2iz} - 2e^{iz}.e^{-iz}}{4(-1)}\right]$$

$$= \frac{e^{2iz} + e^{-2iz} + 2}{4} + \frac{e^{2iz} + e^{-2iz} - 2}{4}$$

$$= \frac{e^{2iz} \cancel{+e^{-2iz}} \cancel{+2} - e^{2iz} \cancel{+e^{-2iz}} \cancel{-2}}{4}$$

$$= \frac{2e^{2iz} - 2e^{-2iz}}{4}$$

$$= \frac{e^{i2z} + e^{-i(2z)}}{2} = \cos 2z$$

Middle: 2:-

$$2\cos^2 x - 1 = 2(\cos x)^2 - 1$$

$$= 2\left(\frac{e^{ix} + e^{-ix}}{2}\right)^2 - 1$$

$$= \cancel{2}\left[\frac{e^{2ix} + e^{-2ix} + 2e^{ix}e^{-ix}}{\cancel{4}\,2}\right] - 1$$

$$= \frac{e^{2ix} + e^{-2ix} + 2}{2} - 1$$

$$= \frac{e^{2ix} + e^{-2ix} + 2 - 2}{2}$$

$$= \frac{e^{2ix} + e^{-2ix}}{2} = \cos 2x = L.H.S$$

$$R.H.S = 1 - 2\sin^2 x = 1 - 2(\sin x)^2$$

$$= 1 - 2\left[\frac{e^{ix} - e^{-ix}}{2i}\right]^2$$

$$= 1 - \cancel{2}\left[\frac{e^{2ix} + e^{-2ix} - 2e^{ix}e^{-ix}}{4i^2}\right]$$

$$= 1 - \cancel{2}\left[\frac{e^{2ix} + e^{-2ix} - 2}{2}\right]$$

$$= 1 + \frac{e^{2ix} + e^{-2ix} - 2}{2}$$

$$= \frac{\cancel{2} + e^{2ix} + e^{-2ix} - \cancel{2}}{2}$$

$$= \frac{e^{2ix} + e^{-2ix}}{2} = \cos 2x = L.H.S$$

Show that

$\sin(-z) = -\sin z,$

$\tan(-z) = -\tan z$

$\cos(-z) = \cos z \quad z \in \mathbb{C}$

$$L.H.S = \tan(-z) = \frac{\sin(-z)}{\cos(-z)}$$

$$= \frac{e^{i(-z)} - e^{-i(-z)}/2i}{e^{i(-z)} + e^{-i(-z)}/2}$$

$$= \frac{e^{-iz} - e^{+iz}}{i(e^{-iz} + e^{iz})} = -\frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})} = -\tan z$$

Show that $\cos x = 0 \iff x = (2n+1)\frac{\pi}{2}\ ; n \in \mathbb{Z}$

let:-

$P: \cos x = 0$ & $q: x = (2n+1)\frac{\pi}{2} : n \in \mathbb{Z}$

$P \to q$

given $\cos x = 0$ & to prove $x = \frac{(2n+1)\pi}{2}$

$P \Rightarrow \cos x = 0 \Rightarrow \frac{e^{ix} + e^{-ix}}{2} \Rightarrow e^{ix} + e^{-ix} = 0$

$\Rightarrow e^{ix} + \frac{1}{e^{ix}} = 0 \Rightarrow \frac{(e^{ix})^2 + 1}{e^{ix}} = 0$

$e^{2ix} + 1 = 0$

$e^{i(2x)} = -1$

$-1 + i(0) = \cos\pi + i\sin\pi$

$e^{i(2\pi)} = \cos(\pi + n(2\pi)) + i(\sin(\pi + n(2\pi))$

$= \cos((2n+1)\pi) + i\sin[(2n+1)\pi]$

$e^{i(2\pi)} = e^{i(2n+1)\pi}$

$i2x = i(2n+1)\pi + 2ki\pi$

$2x = \pi(2n+1+2k)$

$= \pi(2(n+k)+1)$

$2x = (2n'+1)\pi$

$x = (2n'+1)\pi/2$

Lecture #09

Hyperbolic function:-

Introduction:-

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

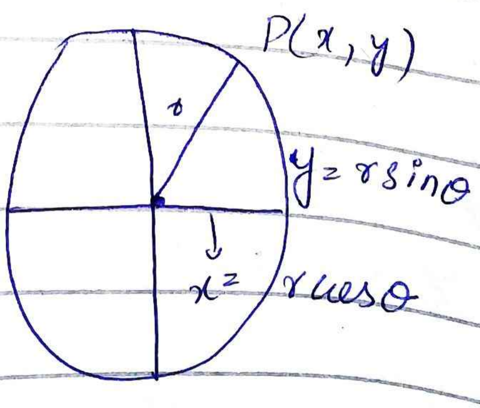
$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x}$$

In similar manner, for $z \in \mathbb{C}$;

$$\cosh z = \frac{e^z + e^{-z}}{2}, \quad \sinh z = \frac{e^z - e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{2}{e^z + e^{-z}}, \quad \operatorname{cosech} = \frac{2}{e^z - e^{-z}}$$

$$\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}; \quad \coth z = \frac{e^z + e^{-z}}{e^z - e^{-z}}$$



2. Relation between Hyperbolic and Circular Function

$\forall \, z \in \mathbb{C}$;

$$\sin(iz) = \frac{e^{i(iz)} - e^{-i(iz)}}{2i}$$

$$= e^{i^2(z)} - e^{-i^2(z)} / 2i$$

$$\sin(iz) = \frac{e^{-z} - e^{-(-1)(z)}}{2i}$$

$$\sin(iz) = \frac{e^{-z} - e^{z}}{2i}$$

$$= i \, \frac{e^{-z} - e^{z}}{2i^2}$$

$$= -\frac{i(e^{z} - e^{-z})}{2(-1)}$$

$$\sin(iz) = i\left(\frac{e^{z} - e^{-z}}{2}\right) = i \sinh z$$

$$\boxed{\sin(iz) = i \sinh z}$$

$$\cos(iz) = \frac{e^{i(iz)} + e^{-i(iz)}}{2}$$

$$= \frac{e^{i^2 z} + e^{-i^2 z}}{2} = \frac{e^{-z} + e^{+z}}{2}$$

$$= e^{z} + e^{-z} / 2$$

$$\boxed{\cos(iz) = \cosh z}$$

$$\tan(iz) = \frac{\sin(iz)}{\cos(iz)} = \frac{i\sinh z}{\cosh z} = i\tanh z$$

$$\boxed{\tan(iz) = i\tanh z}$$

Hyperbolic Identities :-

(i) $\cosh^2 z - \sinh^2 z = 1$

(ii) $\operatorname{sech}^2 z + \tanh^2 z = 1$

(iii) $\coth^2 z = \operatorname{cosech}^2 z = 1$

* $\cosh^2 z - \sinh^2 z = 1$

$$= (\cosh z)^2 - (\sinh z)^2$$

$$= \left(\frac{e^{z}+e^{-z}}{2}\right)^2 - \left(\frac{e^{z}-e^{-z}}{2}\right)^2$$

$$= \frac{e^{2z}+e^{-2z}+2e^{z}\cdot e^{-z}}{4} - \left[\frac{e^{2z}+e^{-2z}-2e^{z}e^{-z}}{4}\right]$$

$$= \frac{e^{2z}+e^{-2z}+2e^{0}}{4} - \left[\frac{e^{2z}+e^{-2z}-2e^{0}}{4}\right]$$

$$= \frac{\cancel{e^{2z}}+\cancel{e^{-2z}}+2-\cancel{e^{2z}}-\cancel{e^{-2z}}+2}{4}$$

$$= \frac{\cancel{4}}{\cancel{4}} = 1$$

* Hyperbolic function (Application-I)

If $z = (x+iy) \in \mathbb{C}$

Then $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$

$\cos z = \cos(x+iy)$

$\because \cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$\cos z = \cos x \cdot \cos(iy) - \sin x \cdot \sin(iy)$

$\because \cos(iy) = \cosh y$

$\because \sin(iy) = i\sinh y$

$\cos z = \cos x \cosh y - \sin x \, i\sinh y$

$= \cos x \cosh y + i(-\sin x \sinh y)$

$|\cos z|^2 = (\cos x \cosh y)^2 + (-\sin x \cdot \sinh y)^2$

$= \cos^2 x \cosh^2 y + \sin^2 x \cdot \sinh^2 y$

$\Rightarrow \because \sin^2 x + \cos^2 x = 1$

$\sin^2 x = (1 - \cos^2 x)$

$\cosh^2 y - \sinh^2 y = 1$

$\cosh^2 y = 1 - \sinh^2 y$

$= \cos^2 x \cdot (1 + \sinh^2 y) + (1 - \cos^2 x) \cdot \sinh^2 y$

$= \cos^2 x + \cancel{\cos^2 x \sinh^2 y} + \sinh^2 y - \cancel{\sinh^2 y \cos^2 x}$

$|\cos z|^2 = \cos^2 x + \sinh^2 y$

(i) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

$L.H.S = |\sin z|^2$

$\sin z = \sin(x+iy)$

$\Rightarrow \because \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin(x+iy) = \sin x \sin(iy) + \cos x \cos(iy)$

$= \sin x\, i \sinh y + \cos x \cosh y$

$= i(\sin x \sinh y) + \cos x \cosh y$

$|\sin z|^2 \Rightarrow (\sin x \sinh y)^2 + (\cos x \cosh y)^2$

$= \sin^2 x \sinh^2 y + \cos^2 x \cosh^2 y$

$\Rightarrow \sin^2 x(1-\cosh^2 y) + \cos^2 x(1-\sinh^2 y)$

$= \sin^2 x - \sin^2 x \cosh^2 x + \cos^2 x - \cos^2 x \sinh^2 y$

$= \sin^2 x(1-\cosh^2 x) + \cos^2 x(1-\sinh^2 y)$

$= (\sin^2 x + \cos^2 x) + (1-\cosh^2 x)$

$= 1 + \sinh^2 y$ ______ .

For any $z = (x+iy) \in \mathbb{C}$; Show that
$\sin(i\bar{z}) = \overline{\sin(iz)} \iff$
$x = n\pi$; $n \in \mathbb{Z}$

Solution :-

let:- $P : \sin(i\bar{z}) = \overline{\sin(iz)}$

$q : z = n\pi i$, $n \in \mathbb{Z}$

(i) $P \to q$: 'P' ~~prove~~ given to prove $z = n\pi i$;

$P \Rightarrow \sin(i\bar{z}) = \overline{\sin(iz)}$

$\sin(i(\overline{x+iy})) = \overline{\sin(i(x+iy))}$

$\sin(i(x-iy)) = \overline{\sin(xi+i^2y)}$

$\sin(ix-i^2y) = \overline{\sin(ix-y)}$

$\sin(ix+y) = \overline{\sin(-y+ix)}$

$\because \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\sin ix\cos y + \cos ix\sin y = \overline{\sin(-y)\cos(ix) + \cos(y)\sin(ix)}$

$i\sinh x\cos y + \cosh x\sin y = \overline{-\sin y\cosh x + \cos y\, i\sinh x}$

$\cosh x\sin y + i\sinh x\cos y = -\sin y\cosh x + i\sinh x\cos y$

$\cosh x\sin y + i\sinh x\cos y = -\sin y\cosh x - i\sinh x\cos y$

$2\cosh x\sin y + 2i\sinh x\cos y = 0 + i0$

$\div$ing by '2'

$\cosh x\sin y + i\sinh x\cos y = 0 + i0$

Equating Real and im. parts.

$\cosh x\sin y = 0$; $\sinh x\cos y = 0$

$\sin y = 0$; $\cosh x \neq 0$

$y = \sin^{-1}(0)$

$y = n\pi$, $n \in \mathbb{Z}$

Therefore

$z = x + iy$

$= 0 + i(n\pi)$

$z = i(n\pi)$

$\sinh x . \cos y = 0$

$\sinh x = 0$

$\frac{e^x - e^{-x}}{2} = 0$

$\Rightarrow e^x - e^{-x} = 0$

$e^x - \frac{1}{e^x} = 0$

$\frac{(e^x)^2 - 1}{e^x}$

$e^{2x} - 1 = 0$

$e^{2x} = 1 = e^0$

$2x = 0$, $\boxed{x = 0}$

Conversly 'q' is given & we prove 'p'

$(q \to p)$

$q \Rightarrow \quad z = in\pi \Rightarrow \bar{z} = -in\pi$

$\sin(i\bar{z}) = \sin i(-in\pi) = \sin(-i^2 n\pi)$

$\sin(i\bar{z}) = \sin n\pi$

$\sin(i\bar{z}) = 0$

$\overline{\sin(iz)} = \overline{\sin i(in\pi)} = \overline{\sin(i^2 n\pi)}$

$= \overline{\sin(-n\pi)}$

$= \bar{0} = 0$

$\therefore \sin(i\bar{z}) = \overline{\sin(iz)}$ if $z = n\pi i$

$(q \to p)$

Periodicity of Hyperbolic function :-

(i) Sinh z and Cosh z is $2\pi i$

(ii) tan hz is πi

P is a period of any function 'F(x)' if $f(x+p) = f(x)$

i) cosh z :-

taking $\cosh(z+2\pi i) = \frac{e^{z+2i} + e^{-(2+2\pi)}}{2}$

$= \frac{e^{z} \cdot e^{2\pi i} + e^{-z} e^{-2\pi i}}{2} =$

$\left\{ \begin{array}{l} e^{i 2\pi} = \cos 2\pi + i\sin(2\pi) \\ = 1 + i(0) = 4 \\ e^{-2\pi i} = 0 \end{array} \right.$

$= \frac{e^{z} \cdot (1) + e^{-z}(1)}{2} = \frac{e^{z} + e^{-z}}{2}$

$\cosh z = \frac{e^{z} + e^{-z}}{2}$

If $\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$, then Show that $\cos^2\theta = \pm \sin\alpha$

given that

$\sin(\theta + i\phi) = \cos\alpha + i\sin\alpha$

$\sin\theta \cos i\phi + \cos\theta\, i\sin\phi = \cos\alpha + i\sin\alpha$

$\sin\theta \cosh\theta + \cos\theta\, i\sinh\theta = \cos\alpha + i\sin\alpha$

$\sin\theta \cosh\theta + i\cos\theta \sinh\phi = \cos\alpha + i\sin\alpha$

By equating real and Imp:-

$\sin\theta \cosh\phi = \cos\alpha$; $\cos\theta \sinh\phi$

$\cosh\phi = \frac{\cos\alpha}{\sin\theta}$; $\sinh\phi = \frac{\sin\alpha}{\cos\theta}$

Squaring and Subtracting :-

$$\cosh^2\phi - \sinh^2\phi = \frac{\cos^2\alpha}{\sin^2\theta} - \frac{\sin^2\alpha}{\cos^2\theta}$$

$$1 = \frac{\cos^2\alpha\cos^2\theta - \sin^2\alpha\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\sin^2\theta\cos^2\theta = \cos^2\alpha\cos^2\theta - \sin\alpha\sin^2\theta$$

$$(1-\cos^2\theta)\cos^2\theta = (1-\sin^2\alpha)\cos^2\theta - \sin^2\alpha\,(1-\cos^2\theta)$$

$$\cancel{\cos^2\theta} - \cos^4\theta = \cos^2\theta - \cancel{\sin^2\alpha\cos\theta} - \sin^2\alpha + \cancel{\sin^2\alpha}\cos^2\theta$$

$$-\cos^4\theta = -\sin^2\alpha$$

$$\cos^4\theta = \sin^2\alpha$$

$$\sqrt{\cos^4\theta} = \sqrt{\sin^2\alpha}$$

$$\cos^2\theta = \pm\sin a$$

Lecture # 10:-

Inverse Trigonometric and Hyperbolic Functions

1- one- to- one function:-

Inverse function:-

Function:-

Relation b/w independent & dependent Variables.

$A, B \neq 0$, $R \subseteq A \times B = \{(x, y) \mid x \in A, y \in B\}$

If $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

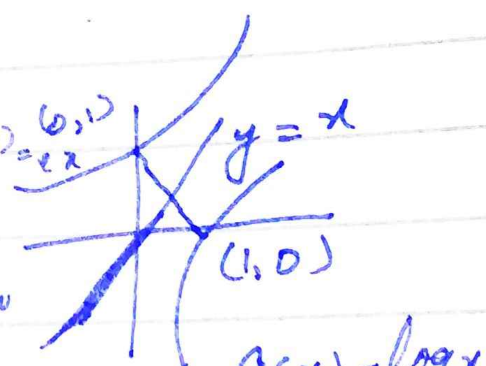
Then 'f' is one-one function.

2. Inverse function:-

If a function is 1-1, then it has an inverse

Inverse of $e^x = \log x =$ reflection of e^x about $y=x$

Inverse of $\log x = e^x =$ reflection of $\log_x$ about y-axis.



$y = f(x) \Rightarrow x = f^{-1}(y)$

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Rang f = Dom^{-1} f

3. Inverse Trigonometric Functions:-

$f(x) = \sin^{-1}x$ inverse function

$y = \sin x$

$y = \sin(x + 2k\pi)$; $k \in \mathbb{Z}$

$\sin^{-1}y = x + 2k\pi \Rightarrow x = \sin^{-1}y - 2k\pi$

$x = \sin^{-1}y + 2(k)\pi$

$\Rightarrow \boxed{x = \sin^{-1}y + 2n\pi}$

$n = -k \in \mathbb{Z} = \{0, \mp 1, \mp 2 \ldots\}$

4. Inverse Trigonometric and hyperbolic Functions for Complex variables:-

For $z \in \mathbb{C}$;

if $w = \sin z$; then we define

$z = \sin^{-1}(x)$, i.e called inverse

Sine function for complex variables.

$\cos^{-1}w$, $\sec^{-1}w$, $\tan^{-1}w$, $\cot^{-1}w$,

$\operatorname{cosec}^{-1}w$; $w \in \mathbb{C}$

Hyperbolic functions :-

$\cosh^{-1}z$, $\sinh^{-1}z$, — $z \in \mathbb{C}$

$\tan^{-1}A + \tan^{-1}B = \tan^{-1}\left(\frac{A+B}{1-AB}\right)$

5. Log ~~Inverse~~ expressions of Inverse Trigonometric functions:-

(i) $\sin^{-1} z = \frac{1}{i} \log(iz + \sqrt{1-z^2})$

(ii) $\cos^{-1} z = \frac{1}{i} \log(z + \sqrt{z^2-1})$

(iii) $\tan^{-1} z = \frac{1}{2i} \log\left(\frac{1+iz}{1-iz}\right)$

(iv) $\cot^{-1} z = \frac{1}{2i} \log\left(\frac{z+i}{z-i}\right)$

(v) $\sec^{-1} z = \frac{1}{i} \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

(vi) $\operatorname{cosec}^{-1} z = \frac{1}{i} \log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$

(v) $\sec^{-1} z = \frac{1}{i} \log\left(\frac{1+\sqrt{1-z^2}}{z}\right)$

Set $w = \sec^{-1} z$

$\sec w = z \quad ; z, w \in \mathbb{R}$

$\frac{1}{\cos w} = z \Rightarrow \cos w = \frac{1}{z}$

$\frac{e^{iw} - e^{-iw}}{2} = \frac{1}{z}$

$e^{iw} - e^{-iw} = \frac{2}{z} \Rightarrow e^{iw} - \frac{1}{e^{iw}} = \frac{2}{z}$.

$$\frac{(e^{iw})^2 - 1}{e^{iw}} = \frac{2}{z}$$

$$z(e^{iw})^2 - z = 2e^{iw}$$

$$z(e^{iw})^2 - 2e^{iw} - z = 0$$

i.e quad in e^{iw}

$$e^{iw} = \frac{+2 \pm \sqrt{(-2)^2 - 4(z)(z)}}{2z}$$

$$= \frac{2 \pm \sqrt{4 - 4z^2}}{2z}$$

$$= \frac{2 \pm 2\sqrt{1-z^2}}{2z}$$

$$= \frac{1 \pm \sqrt{1-z^2}}{z}$$

$$e^{iw} = \frac{1 + \sqrt{1-z^2}}{z}$$

$$iw = \log \frac{1 + \sqrt{1-z^2}}{z}$$

$$w = \frac{1}{i} \log \frac{1 \pm \sqrt{1-z^2}}{z}$$

6- Inverse Trigonometric Function (Application)

Seperate into real and Imaginary parts:-

(i) $Tan^{-1}(x+iy)$, (ii) $Cos^{-1}(\cos\theta + i\sin\theta)$

(iii) $Sin^{-1}(\cos\theta + i\sin\theta)$, (iv) $tan^{-1}(\cos\theta + i\sin\theta)$

(iii) $Sin^{-1}(\cos\theta + i\sin\theta)$

let $\sin^{-1}(\cos\theta + i\sin\theta) = (x+iy) \in \mathbb{R}$

$\cos\theta + i\sin\theta = \sin(x+iy)$

$\because \{ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$\cos\theta + i\sin\theta = \sin x\cos(iy) + \cos x\sin(iy)$

$= \sin x\cosh y + \cos x \, i\sinh y$

Comparing real and Imaginary parts:-

$\cos\theta = \sin x\cosh y \to (i)$, $\sin\theta = \cos x\sinh y \to (ii)$

(1) $\Rightarrow \sin x = \dfrac{\cos\theta}{\cosh y}$, (2) $\Rightarrow \cos x = \dfrac{\sin\theta}{\sinh y}$

$\{ \sin^2 x + \cos^2 x = 1 \to (A)$

Putting $\sin x$ & $\cos x$ in (A)

(A) $\Rightarrow \dfrac{\cos^2\theta}{\cosh^2 y} + \dfrac{\sin^2\theta}{\sinh^2 y} = 1$

$$\Rightarrow \frac{\cos^2\theta \sinh^2 y + \sin^2\theta \cosh^2 y}{\cosh^2 y \cdot \sinh^2 y} = 1.$$

$$\cos^2\theta \sinh^2 y + \sin^2\theta \cosh^2 y = \cosh^2 y \sinh^2 y$$

$$(1-\sin^2\theta)\sinh^2 y + \sin^2\theta(1+\sinh^2 y) = (1+\sinh^2 y)(\sinh^2 y)$$

$$\cancel{\sinh^2 y} - \cancel{\sin^2\theta \sinh^2 y} + \sin^2\theta + \cancel{\sin^2\theta \cdot \sinh^2 y}$$
$$= \cancel{\sinh^2 y} + \sinh^4 y$$

$$\sin^2\theta = \sinh^4 y$$
$$\sqrt{\sin^2\theta} = \sqrt{\sinh^4 y}$$
$$\sinh^2 y = \pm \sin\theta$$

$$\sinh y = \sqrt{\sin\theta}$$
$$\sinh y = \sqrt{\sin\theta}$$
$$y = \sinh^{-1}\sqrt{\sin\theta}$$ ✓ It is also true

$$\therefore \boxed{y = \log \sqrt{\sin\theta} + \sqrt{\sin\theta + 1}}$$

↓ Imaginary part.

Now ① $\Rightarrow \cosh y = \dfrac{\cos\theta}{\sin x}$; $\sinh y = \dfrac{\sin\theta}{\cos x}$

& $\cosh^2 y - \sinh^2 y = 1 \rightarrow$ (A)

Putting in eq:-

$$\frac{\cos^2\theta}{\sin^2 x} - \frac{\sin^2\theta}{\cos^2 x} = 1$$

$$\Rightarrow \frac{\cos^2\theta \cos^2 x - \sin^2\theta \sin^2 x}{\cos^2 x \sin^2 x} = 1$$

$$\cos^2\theta \cos^2 x - \sin^2\theta \sin^2 x = \cos^2 x \sin^2 x$$

$$\cos^2 x \cos^2 x - (1-\cos^2\theta)(1-\cos^2 x) = \cos^2 x(1-\cos^2 x)$$

$$\cos^2\theta \cos^2 x - (1-\cos^2\theta - \cos^2\theta + \cos^2\theta \cdot \cos^2\theta)$$

$$= \cos^2\theta - \cos^4 x$$

$$\cancel{\cos^2\theta \cos^2 x} - 1 + \cos^2 x + \cos^2\theta - \cancel{\cos^2\theta\cos^2 x}$$

$$= \cos^2 x - \cos^4 x$$

$$-1 + \cos^2\theta = -\cos^4 x$$

$$+1(1-\cos^2\theta) = +\cos^4 x$$

$$\sin^2\theta = \cos^4 x$$

$$\sqrt{\sin^2\theta} = \sqrt{\cos^4 x}$$

$$\sin\theta = \cos^2\theta$$

$$\cos^2 x = \sin\theta$$

$$\sqrt{\cos^2 x} = \sqrt{\sin\theta}$$

$$\cos x = \pm\sqrt{\sin\theta}$$ Real part.

8. Log expression for inverse Hyperbolic Functions:-

Show that :-

(i) $\sinh^{-1} x = \log(x + \sqrt{x^2+1})$

(ii) $\cosh^{-1} x = \log(x + \sqrt{x^2-1})$

(iii) $\tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

(iv) $\coth^{-1} x = \frac{1}{2} \log\left(\frac{x+1}{x-1}\right)$

(v) $\operatorname{sech}^{-1} x = \log\left(\frac{1+\sqrt{1-x^2}}{x}\right)$

(vi) $\operatorname{cosech}^{-1} x = \log\left(\frac{1+\sqrt{x^2+1}}{x}\right)$

(iii) $\tanh^{-1} x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

let :- $\tanh^{-1} x = w$

$x = \tanh w$

$\left\{ \tanh(w) = \frac{e^{(w)} - e^{-(w)}}{e^{(w)} + e^{-(w)}} \right.$

$\therefore \quad x = \frac{e^{w} - e^{-w}}{e^{w} + e^{-w}}$

$\left\{ \text{if } \frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d} \right.$

componendo & dividendo

$$\frac{z+1}{z-1} = \frac{(e^{w} - e^{-w}) + (e^{w} + e^{-w})}{(e^{w} - e^{-w}) - (e^{w} + e^{-w})}$$

$$\frac{z+1}{z-1} = \frac{\cancel{2}e^{w}}{-\cancel{2}e^{-w}}$$

$$\frac{z+1}{z-1} = \frac{e^{w}}{e^{-w}}$$

$$\frac{z+1}{z-1} = -e^{w+w} \Rightarrow \frac{z+1}{z-1} = -e^{2w}$$

$$(-1)\frac{z+1}{z-1} = e^{2w}$$

$$\frac{z+1}{1-z} = e^{2w}$$

$$e^{2w} = \frac{1+z}{1-z}$$

$$2w = \log\left(\frac{1+z}{1-z}\right)$$

$$w = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)$$

$$\boxed{\tanh^{-1} z = \frac{1}{2}\log\left(\frac{1+z}{1-z}\right)}$$

proved

Practice Questions of lecture '7 to 9'

Lecture # 07:-

Q#01:- Solve the equation:-

$$e^{2ix} = 0 + i0, \text{ where } x \in \mathbb{R}.$$

Here we use the fact:-

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{2ix} = \cos 2x + i\sin 2x = 0 + i0$$

$\cos 2x = 0$, and $i\sin 2x = 0$

$2x = \cos^{-1}(0)$, $2x = i\sin^{-1}(0)$

$2x = (2n+1)\frac{\pi}{2}$, $2x = n\pi$, $n \in \mathbb{Z}$

$x = (2n+1)\frac{\pi}{4}$, $x = \frac{n\pi}{2}$, $n \in \mathbb{Z}$

In particular $x = \frac{\pi}{4}$ and $x = 0, \frac{\pi}{2}$,

$\Rightarrow \nexists\ x \in \mathbb{R}$ which simultaneously can satisfy the vanishing of both real and Imaginary parts of given function.

Q#02:-

Show that the period of e^{2ix} is $2\pi i$ where $x \in \mathbb{R}$.

for any Complex valued function, $p \in \mathbb{C}$ is period $\Leftrightarrow f(z) = f(z+p)$

$$\therefore e^{2ix + 2\pi i} = e^{i(2x + 2\pi)}$$

$$e^{i(2x+2\pi)} = \cos(2x+2\pi) + i\sin(2x+2\pi).$$

$\because$ By Euler's formula

$$= \cos 2x + i \sin 2x \qquad \because \begin{cases} \cos(2\pi + 2x) = \cos 2x \\ \sin(2\pi + 2x) = \sin 2x \end{cases}$$

$$= e^{2ix}$$

$2\pi i$ is a period of e^{2ix}

Q#03:-

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \in \mathbb{C}$, then show that $e^{z_1} e^{z_2} = e^{x_1+x_2}[\cos(y_1+y_2) + i\sin(y_1+y_2)]$.

Solution :-

$$e^{z_1} e^{z_2} = e^{(x_1+iy_1)} e^{(x_2+iy_2)} = e^{x_1} e^{x_2} e^{iy_1} e^{iy_2}$$

$$= e^{x_1+x_2} e^{iy_1+iy_2} = e^{x_1+x_2}(\cos y_1 + i\sin y_1)(\cos y_2 + i\sin y_2)$$

$$= e^{x_1+x_2}(\cos y_1 \cos y_2 + i\cos y_1 \sin y_2 + i\sin y_1 \cos y_2 + i^2 \sin y_1 \sin y_2)$$

$$= e^{x_1+x_2}(\cos y_1 \cos y_2 + i\cos y_1 \sin y_2 + i\sin y_1 \cos y_2 - \sin y_1 \sin y_2)$$

$= e^{x_1+x_2}(\{\cos y_1 \cos y_2 - \sin y_1 \sin y_2\} + i\{\cos y_1 \sin y_2 + \sin y_1 \cos y_2\})$

$= e^{x_1+x_2}(\cos(y_1+y_2) + i\sin(y_1+y_2))$

Hence proved.

Q# 04:-

Show that $|e^{iz}| = e^{-y}$, for $z = (x+iy) \in \mathbb{C}$.

$|e^{iz}| = |e^{i(x+iy)}| = |e^{ix-y}| \Rightarrow i^2 = -1$

$|e^{-y}e^{ix}| = |e^{-y}||e^{ix}|$ As $|z_1 z_2| = |z_1||z_2|$

$\Rightarrow e^{-y}|\cos x + i\sin x| = e^{-y}\sqrt{(\cos^2 x + \sin x)}$

$= e^{-y}(\pm 1)$

$= e^{-y}$

Lecture #08

Q# 05:-

if $e^{2ix} = \cos 2x + i\sin 2x$, then show that $\sin 2x = \dfrac{e^{2ix} - e^{-2ix}}{2i}$

$e^{2ix} = \cos 2x + i\sin 2x \longrightarrow ①$

$e^{-2ix} = \cos(-2x) + i\sin(-2x)$

$= \cos 2x - i\sin 2x$ —— ②

Subtract eq. ① and eq. (2)

~~$e^{2ix} - e^{-2ix} = 2i\sin 2x$~~

$$e^{2ix} - e^{-2ix} = \cancel{\cos 2x} + i\sin 2x - \cancel{\cos 2x} + i\sin 2x$$

$$e^{2ix} - e^{-2ix} = 2i\sin 2x$$

$$\sin 2x = \frac{e^{2ix} - e^{-2ix}}{2i} \Rightarrow \text{Hence proved.}$$

Q6:- if $e^{2ix} = \cos 2x + i\sin 2x$, then show that $\cos 2x = \frac{e^{2ix} + e^{-2ix}}{2}$

$$e^{2ix} = \cos 2x + i\sin 2x \text{ ---- } ①$$

$$e^{-2ix} = \cos(-2x) - i\sin(-2x)$$

$$= \cos 2x - i\sin 2x$$

Add eq. ① and eq. ②

$$e^{2ix} + e^{-2ix} = \cos 2x + \cancel{i\sin 2x} + \cos 2x - \cancel{i\sin 2x}$$

$$e^{2ix} + e^{-2ix} = 2\cos 2x$$

$$\cos 2x = \frac{e^{2ix} + e^{-2ix}}{2}$$

Hence proved

Q7:- Show that $\left\{\frac{n\pi}{2}\right\}_{n \in z}$ is the solution set of equation $\cos z = 0$

we know that

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \Rightarrow \frac{e^{iz} + e^{-iz}}{2} = 0$$

$$e^{iz} + e^{-iz} = 0 \quad , \quad e^{iz} + \frac{1}{e^{iz}} = 0$$

$$\frac{e^{2iz}+1}{e^{iz}} = 0 \Rightarrow e^{2iz}+1=0$$

$$e^{2iz} = -1 \rightarrow (1)$$

Since :-

$\cos\pi + i\sin\pi = -1$, so we can write eq. (1) as:-

$$e^{2iz} = \cos\pi + i\sin\pi$$

$$e^{2iz} = \cos(\pi+2k\pi) + i s(\pi+2k\pi)$$

$$= e^{(\pi+2k\pi)i} \rightarrow (2)$$

we know that $e^{z_1} = e^{z_2}$ if and only if $z_1 = z_2 + 2im\pi$, $m \in \mathbb{Z}$,

from eq (2)

$$2iz = (\pi+2k\pi)i + 2im\pi$$

$$= (1+2k+2m)\pi i$$

$$2iz = n\pi i \text{ , where } n = 1+2k+2m \text{ is an integers}$$

$$2z = n\pi$$

$$z = \frac{n\pi}{2}$$

Q#08 :- prove that:-

$1+\tan^2 z = \sec^2 z$ for all $z \in \mathbb{C}$.

we know that for any Complex number z,

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz} + e^{-iz}}$$

$$L.H.S = 1 + \tan^2 z$$

$$= 1 + \left(\frac{e^{iz} - e^{iz}}{i(e^{iz} + e^{-iz})}\right)^2$$

$$= 1 + \frac{e^{2iz} - 2 + e^{-2iz}}{i^2(e^{2iz} + 2 + e^{-2iz})}$$

$\because i^2 = -1$

$$= 1 + \frac{e^{2iz} - 2 + e^{-2iz}}{e^{2iz} + 2 + e^{-2iz}}$$

$$= \frac{\cancel{e^{2iz}} + 2 + \cancel{e^{-2iz}} - \cancel{e^{2iz}} + 2 - \cancel{e^{-2iz}}}{e^{2iz} + 2 + e^{-2iz}}$$

$$= \frac{4}{e^{2iz} + 2 + e^{-2iz}} = \left[\frac{2}{e^{iz} + e^{iz}}\right]^2 = \sec^2 z$$

$$L.H.S = R.H.S$$

Lecture # 09

Q# 09:-

Prove that $\sin iy = i\sinh y$.

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

$$= \frac{e^{-y} - e^{y}}{2i} = \frac{e^{-y} - e^{y}}{2i} \times \frac{i}{i}$$

$$= \frac{e^{-y} - e^{y}}{2i\cancel{i}} \times \cancel{i} = \frac{e^{y} - e^{-y}}{2} = i\sinh y$$

Q# 10:- Prove that $\cosh iy = \cos y$.

$$\cosh iy = \frac{e^{iy} + e^{-iy}}{2} = \cos y$$

$$\because \cosh y = \frac{e^{y} + e^{-y}}{2}$$

Q11:- Prove that $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x - \sinh^2 x = \left[\left(\frac{e^{x} + e^{-x}}{2}\right)^2 - \left(\frac{e^{x} - e^{-x}}{2}\right)^2\right]$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{\cancel{e^{2x}} + 2 + \cancel{e^{-2x}} - \cancel{e^{2x}} + 2 - \cancel{e^{-2x}}}{4}$$

$$= \frac{\cancel{4}}{\cancel{4}} = 1 = R.H.S$$

Q# 12:- Show that $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$

$$\coth x = \frac{\cosh x}{\sinh x}$$

$$= \frac{e^x + e^{-x}/2}{e^x - e^{-x}/2}$$

$$= \frac{e^x + e^{-x}}{\cancel{2}} \times \frac{\cancel{2}}{e^x - e^{-x}}$$

$$= \frac{e^x + e^{-x}}{e^x - e^{-x}} \quad \text{prove :-}$$

Q13:- Show that $\operatorname{csch} x = \frac{2}{e^x - e^{-x}}$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$= \frac{1}{e^x - e^{-x}/2} = 1 \times \frac{2}{e^x - e^{-x}}$$

$$= \frac{2}{e^x - e^{-x}} \quad \text{prove}$$

Lecture # 11.

Complex Logarithmic Functions

1- Logarithmic Function in R :-

$$2^3 = 8$$

$$\log_2(8) = 3$$

$$(10)^4 = 10\,000$$

$$\log_{10}(10\,000) = 4$$

i.e.

$$\boxed{y = a^x \iff \log_a y = x}$$

2- Special Cases of log Function in R.

$$e^x = y \Rightarrow \log_e(y) = x$$

$$\ln y = x$$

3. logarithmic function in Complex plane:-

i.e. $z, w \in \mathbb{C}$; Such that $e^w = z$; Then we define natural log of $z \in \mathbb{C}$;

$$\log z = w \rightarrow ①$$

$$\left\{ \begin{array}{l} e^w = z \\ \Rightarrow \log z = w \end{array} \right.$$

let $z = x + iy$, $w = u + iv$

① ⇒

$$\log(x+iy) = u + iv \rightarrow ②$$

$x+iy = e^{u+iv} \Rightarrow x+iy = e^{u}.e^{iv}$

$x+iy = e^{u}[\cos v + i\sin v]$

$x+iy = e^{u}[\cos(v+2k\pi) + i\sin(v+2k\pi)] \rightarrow ②$

Taking Modulus:-

$|x+iy| = |e^{u}| \cdot |\cos(x+2k\pi) + i\sin(v+2k\pi)|$

$\sqrt{x^2+y^2} = e^{u} \cdot \sqrt{\cos^2(v+2k\pi) + \sin^2(v+2k\pi)}$

$\sqrt{x^2+y^2} = e^{u} = z$

$u = \ln|z|$ Real valued function

② $\Rightarrow$ $x+iy = e^{u}\cos(v+2k\pi) + ie^{u}\sin(v+2k\pi)$

$\mathrm{Re}\, z = e^{u}\cos(v+2k\pi) \longrightarrow ©$

$\mathrm{Im}\, z = y = e^{u}\sin(v+2k\pi) \longrightarrow ①$

by ⓓ $\div$ ©

$$\frac{e^{x}\sin(v+2k\pi)}{e^{u}\cos(v+2k\pi)} = \frac{y}{z}$$

$\Rightarrow \tan(v+2k\pi) = y/x$

$v + 2k\pi = \tan^{-1}\left(\frac{y}{x}\right)$

$v = \tan^{-1}\left(\frac{y}{x}\right) - 2k\pi$

$v = \tan^{-1}\left(\frac{y}{x}\right) + 2n\pi$

Now ② $\Rightarrow$

$\log(x+iy) = \ln|z| + i\left[\tan^{-1}\left(\frac{y}{x}\right) + 2n\pi\right]$

Multivalued function

If $n=0$; Then

$$\boxed{\log(x+iy) = \ln|z| + i\,\mathrm{Arg}\,z}$$

Principal logrithm of $z \in \mathbb{C}$

4- Complex Log Function example 01

Find $\log(1-i)$.

given $z = 1-i = 1+i(-1)$

$x = 1, \quad y = -1$

$\because \left\{\log(x+iy) = \ln|z| + i\,\mathrm{Arg}(x+iy)\right\}$

Now $|x+iy| = |1+i(-1)|$

$= \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

$\mathrm{Arg}(1-i) = \mathrm{Tan}^{-1}\left(\frac{-1}{1}\right)$

$= -\mathrm{Tan}^{-1}\left(\frac{1}{1}\right)$ $\because$ T-ray of $\mathrm{Arg}\,z$ lies in '4'

$= -\mathrm{Tan}^{-1}(1) = -45^\circ = 45 \times \frac{\pi}{180}$

$= -\pi/4$

$\therefore \log(1-i) = \ln(\sqrt{2}) + i(-\pi/4)$

$= \ln(2)^{1/2} - i\pi/4$

$\log(1-i) = \frac{1}{2}\log 2 - i\pi/4$

5. Complex log Function (example-2)

$\log x = ?$; $x < 0$

$\because \log z = \log|z| + i \operatorname{Arg} z \rightarrow (1)$

Here $z = x = x + i(0)$; $x < 0$

$|z| = |x + i(0)|$

$|z| = |x| = -x$ if $x < 0$

$\operatorname{Arg} z = \operatorname{Tan}^{-1}\left(\frac{y}{x}\right)$

$= \operatorname{Tan}^{-1}\left(\frac{0}{-x}\right)$

$= \operatorname{Tan}^{-1}(-0)$

$= \pi$

$\therefore (1) \Rightarrow \log z = \log(-x) + i\pi$

6. Complex log Function (example-3)

$\log(1 + \cos\theta + i\sin\theta) = \ln\left(2 \cdot \cos\left(\frac{\theta}{2}\right)\right) + i\left(\frac{\theta}{2}\right)$

$\log z = \log_n|z| + i \operatorname{Arg} z \rightarrow (1)$

given that :-

$z = (1 + \cos\theta) + i(\sin\theta)$

$|z| = \sqrt{(1 + \cos\theta)^2 + \sin^2\theta}$

$= \sqrt{1 + \cos^2\theta + 2\cos\theta + \sin^2\theta}$

$= \sqrt{1 + 1 + 2\cos\theta}$

$= \sqrt{2 + 2\cos\theta} = \sqrt{2(1 + \cos\theta)}$

$\{ 1 + \cos(\theta) = 2\cos^2\left(\frac{\theta}{2}\right)$

$$|Z| = \sqrt{2 \cdot 2\cos^2(\theta/2)} = 2\cos\theta/_2$$

$$\text{Arg } z = \text{Tan}^{-1}\frac{y}{x} = \text{Tan}^{-1}\left(\frac{\sin\theta}{1+\cos\theta}\right)$$

$$= \text{Tan}^{-1}\left[\frac{\cancel{2}\sin(\theta/2) \cdot \cancel{\cos(\theta/2)}}{\cancel{2}\cos^2\theta/2}\right]$$

$$= \text{Tan}^{-1}\left(\frac{\sin\theta/2}{\cos\theta/2}\right)$$

$$= \text{Tan}^{-1}\tan(\theta/2) = \theta/2$$

Now ① =>

$$\boxed{\log z = \ln[(2\cos\theta/_2)] + i\,[(\theta/2)]}$$

Lecture # 12:-

Complex Powers

1- Exponential and log function in R:-

* e (Euler's constant)

$\forall\ x \in \mathbb{R}$; $e^x \rightarrow$ continuous growth at 100% for 'x' unit of time

=> 1 -> Scaled version of any real no

=> e -> Scaled version of any growth ratio rate.

=> log -> about time

2- Imaginary Growth:-

rotational $e^{i\pi} = -1$

100% growth

3- Complex power (Introduction):-

$$e = 1 \cdot e = \lim_{n \to \infty} \left(1 + \frac{100\%}{n} \right)^n = 2.713 \text{---} = e$$

for any complex number ; we define

$$z^w = e^{w \log z} = e^{w(\ln|z| + i \operatorname{Arg} z)}$$

4- Complex powers (Application - 1)

Evaluate:-

(i) $i^i = e^{i\log i} = e^{i\log(0+1(i))}$

$\log(x+iy) = (\ln|z| + i\,\mathrm{Arg}\,z)$

$= e^{i(\ln\sqrt{(0)^2+(1)^2})} + i\,\mathrm{Tan}^{-1}\left(\frac{1}{0}\right)$

$= e^{i\ln 1 + i\tan^{-1}(\infty)}$

$\Rightarrow \begin{cases} \ln 1 = 0 \\ \mathrm{Tan}^{-1}(\infty) = \pi/2 \end{cases}$

$= e^{i[(0) + i\pi/2]} = e^{i^2(\pi/2)} = e^{-\pi/2}$

5. Complex powers (Application-II)

(ii) a^i ; $a > 0$

$= e^{i\log a} = e^{i\log(a+i(0))}$

$e^{i\log a} = \cos(\log a) + i\sin(\log a)$

$= \cos(\ln a) + i\sin(\ln a)$.

6- Complex powers (Application III)

If $a^{\alpha+i\beta} = (x+iy)^{p+iq}$, $a>0$, prove that:-

(i) $\alpha = \frac{1}{2}p\log_a(x^2+y^2) - q\tan^{-1}\left(\frac{y}{x}\right)\log_a e$

(ii) $\log_a(x^2+y^2) = \alpha^4\left(\frac{\alpha P + \beta q}{P^2+q^2}\right)$

Solve: (i)

$$a^{\alpha+i\beta} = (x+iy)^{P+iq} \qquad \because z^w = e^{w \log z}$$

$$e^{(\alpha+i\beta)\log a} = e^{(P+iq)\log(x+iy)}$$

$$(\alpha+i\beta)\log a = (P+iq)\log(x+iy) + 2k\pi i \; ; k \in \mathbb{Z}$$

for desired result ; put $k=0$

$$(\alpha+i\beta)\log a = (P+iq)\log(x+iy) \longrightarrow (1)$$

$$\log a = \log a + i0 = \ln|a+i0| + i\tan^{-1}\left(\frac{0}{1}\right)$$

$$= \ln\sqrt{a^2+0^2} + i\tan^{-1}(0)$$

$$= \ln a + i(0)$$

$$= \ln a$$

$$\log(x+iy) = \ln|x+iy| + i\tan^{-1}\left(\frac{y}{x}\right)$$

$$= \ln\sqrt{x^2+y^2} + i\tan^{-1}\frac{y}{x}$$

$$= \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

Now (1) $\Rightarrow$

$$(\alpha+i\beta)\ln a = (P+iq)\left[\frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\left(\frac{y}{x}\right)\right]$$

$$\alpha\ln a + i\beta\ln a = \frac{P}{2}\ln(x^2+y^2) + iP\tan^{-1}\left(\frac{y}{x}\right) + i\frac{q}{2}\ln(x^2+y^2) - q\tan^{-1}\left(\frac{y}{x}\right)$$

Comparing real & Imaginary Parts:-

$$\alpha \ln a = \frac{P}{2} \ln(x^2+y^2) - q \operatorname{Tan}^{-1}\left(\frac{y}{x}\right) \rightarrow (A)$$

$$\beta \ln a = p \operatorname{Tan}^{-1}\left(\frac{y}{x}\right) + q_{/2} \ln(x^2+y^2) \rightarrow (B)$$

from eq (A)

$$\alpha = \frac{P}{2} \frac{\ln(x^2+y^2)}{\ln a} - q \operatorname{Tan}^{-1}\frac{y}{x} \cdot \frac{1}{\ln a}$$

$$\left\{ \frac{\ln A}{\ln \beta} = \frac{\log_e A}{\log_e \beta} = \log_\beta A \;\Big|\; \frac{1}{\log a} = \frac{\log_e e}{\log_e a} = \log_a e \right.$$

$$\alpha = \frac{P}{q} \log_a(x^2+y^2) - q \operatorname{Tan}^{-1}\frac{y}{x} \cdot \log_a e$$

eq (B)

$$\beta = P \operatorname{Tan}^{-1}\left(\frac{y}{x}\right) \cdot \frac{1}{\ln a} + \frac{q}{2} \frac{\ln(x^2+y^2)}{\ln a.}$$

$$\beta = P \operatorname{Tan}^{-1}\left(\frac{y}{x}\right) \log_a e + \frac{a}{2} \log_a(x^2+y^2)$$

Now taking:-

$$p\alpha + q\beta = P\left[\frac{P}{2} \log_a(x^2+y^2) - q \operatorname{Tan}^{-1}\left(\frac{y}{x}\right) \log_a e\right] + q\left[P \operatorname{Tan}^{-1}\frac{y}{x} \log_a (x^2+y^2)\right]$$

$$= \frac{P^2}{2} \log_a(x^2+y^2) - Pq \cancel{\operatorname{Tan}^{-1}}\left(\frac{y}{x}\right) \cdot \log_a e + P\cancel{q \operatorname{Tan}^{-1}}\left(\frac{y}{x}\right) \log_a e (x^2+y^2)$$

$= \log_a(x^2+y^2)\left[\frac{p^2}{2}+\frac{q^2}{2}\right]$

$= \log_a(x^2+y^2)\left(\frac{p^2+q^2}{2}\right)$

$p\alpha+q\beta = \log_a(x^2+y^2)\left(\frac{p^2+q^2}{2}\right)$

$\frac{2(p\alpha+q\beta)}{p^2+q^2} = \log_a(x^2+y^2)$

7 Square root Issue:-

What is '$\sqrt{4}$'? ± 2 or '2' -- 1!!

if $\sqrt{4} = 2$

$-\sqrt{4} = -2$

$\pm\sqrt{4} = \pm 2$ $\qquad \sqrt{4} \neq \pm 2 \ \forall x \in \mathbb{R}$

for complex number:-

$\sqrt{4} = \sqrt{4\cdot 1}$

$= \sqrt{4(\cos 0 + i\sin 0)}$

$= \sqrt{4(\cos(0+2k\pi) + i\sin(0+2k\pi))}$

$= (4(\text{cis}(2k\pi)))^{1/2}$

$= (4)^{1/2}[\text{cis}(2k\pi)]^{1/2} = 2$

$= (4)^{1/2}\,\text{cis of } k\pi$

$k=0$

$$\sqrt{4} = 2 \text{cis}(0\pi)$$

$$\sqrt{4} = 2(\cos 0 + i \sin 0)$$

$$\sqrt{4} = \pm 2 .$$

for Complex number

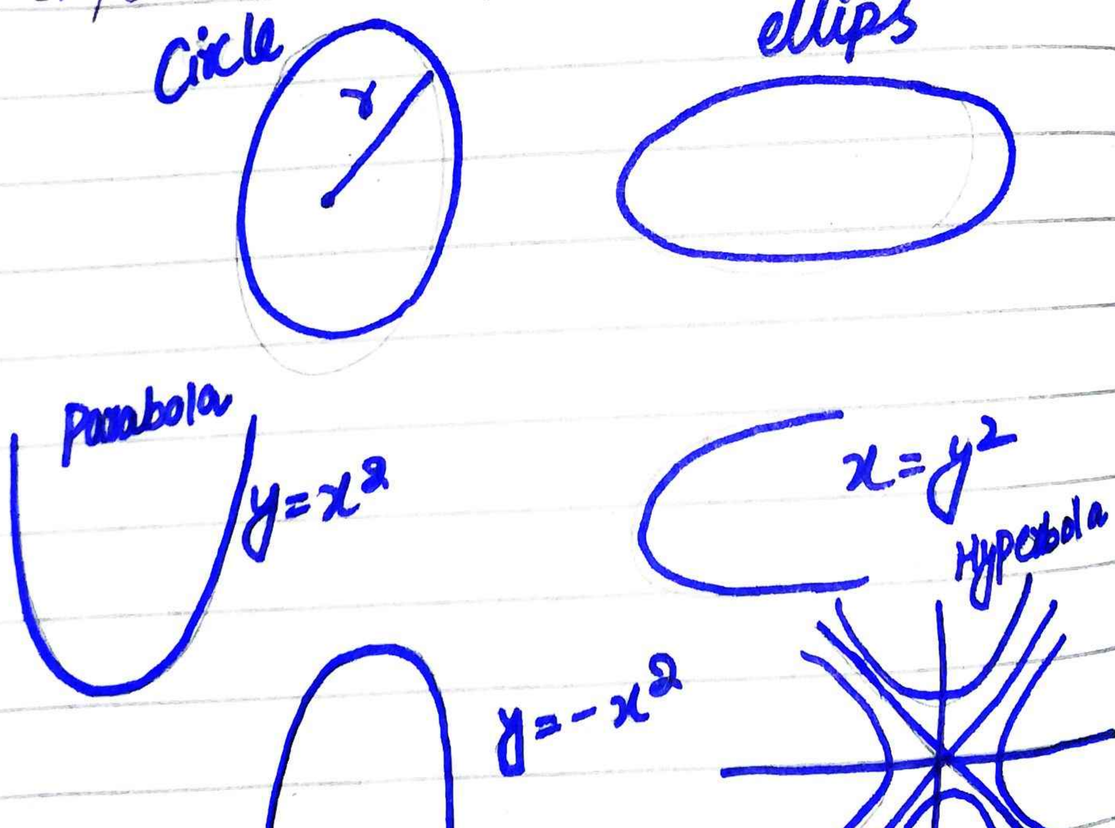
$$\sqrt{4} = \pm 2$$

Lecture # 13

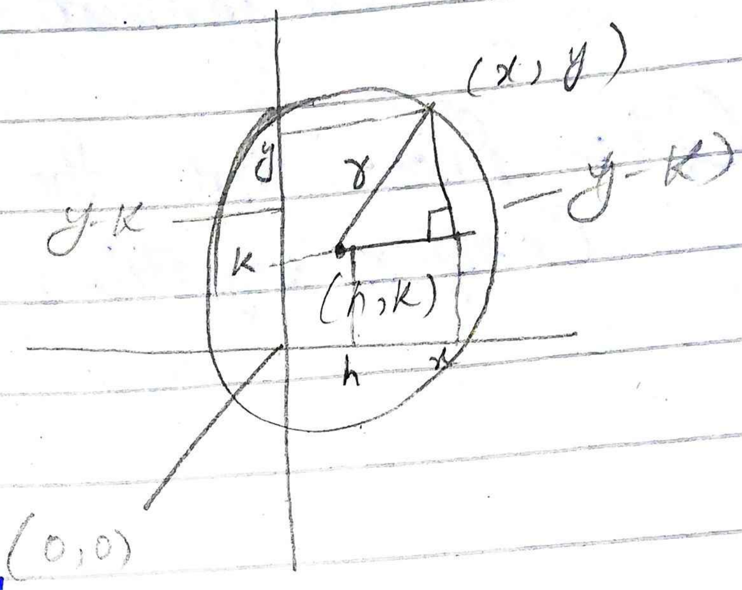
Introduction to Conic Section
Circle and its General Equation

Conic function

- Circle
- ellips
- Parabola
- hyperbola



Circle (Introduction)



general equation of circle:-

$$\boxed{r^2 = (x-h)^2 + (y-k)^2}$$

Area of circle = area = πr^2

Circumference of circle = $2\pi r$

Lecture #14

Parabola and its Porperties

1- Introduction to Parabola:-

Parabola is defined as the set of all points that are equally distance from point and line.

2- Example of Parabola when the vertex is at origin:-

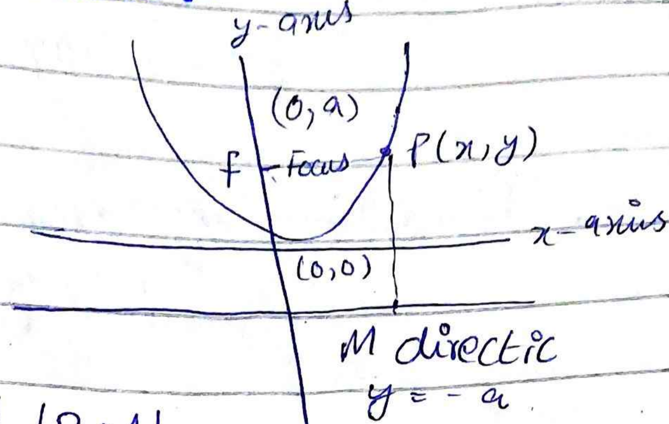
$y^2 = 8x$, $y^2 = 4ax$, $4a = 8$

$a = 2$

A line that passes through the Focus and is perpendicular to the major ~~line~~ axis and has both point on the curve is Called latusretum.

Equations of Parabola :-

vertex is at the origin.



$$|FP| = |PM|$$

$$\sqrt{(x-0)^2 + (y-a)^2} = \sqrt{((y-(-a))^2}$$

$$= (x-0)^2 + (y-a)^2 = (y+a)^2$$

$$\Rightarrow x^2 + y^2 + a^2 + 2ay = y^2 + a^2 + 2ay$$

$$x^2 = 2ay + 2ay$$

$\boxed{x^2 = 4ay}$ parabola equation when opening upward.

$\boxed{x^2 = -4ay}$ Parabola equation when opening diward.

* When parabola opeing right side on x-axis.

$$|FP| = |MP|$$

$$\sqrt{(x-a)^2 + (y-0)^2} = \sqrt{(x-(-a))^2}$$

$$(x-a)^2 + y^2 = (x+a)^2$$

$$\cancel{x^2} + \cancel{a^2} - 2xa + y^2 = \cancel{x^2} + \cancel{a^2} + 2xa$$

$$y^2 = 2xa + 2xa = \boxed{y^2 = 4ax}$$

When vertix at origin

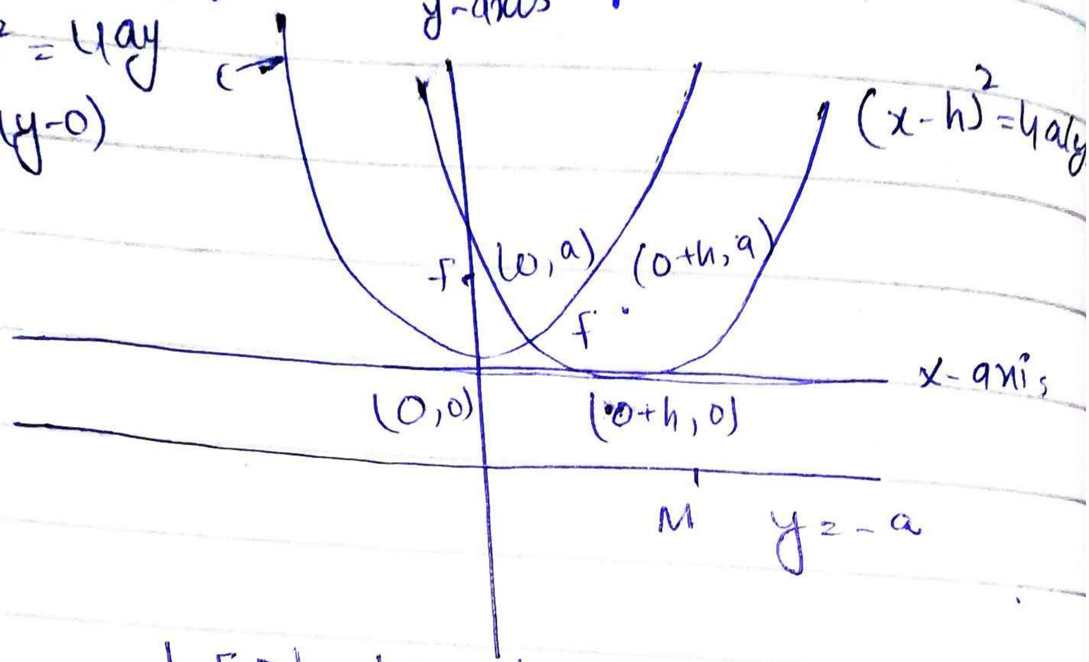
When Parabola opening left side.

$$y^2 = -4ax$$

4- Translation of axes in parabola:-

$x^2 = 4ay$

$(x-0)^2 = 4a(y-0)$



$$|FP| = |PM|$$

$$\sqrt{(x-h)^2 + (y-a)^2} = \sqrt{(y-(-a))^2}$$

$$(x-h)^2 + (y-a)^2 = (y+a)^2$$

$$(x-h)^2 + \cancel{y^2} + \cancel{a^2} - 2ay = \cancel{y^2} + \cancel{a^2} + 2ay$$

$$(x-h)^2 = 2ay + 2ay = 4ay$$

$$\boxed{(x-h)^2 = 4ay}$$

ii) Parabola translate at 'k' unit.

Translation of axis in parabola:-

$(y-k)^2 = 4a(x-h)$

5. Example '1' of Translation of Axes:-

$2(x-3)^2 = 10(y+1) \rightarrow (i)$

Equation:-

$(x-h)^2 = 4a(y-k) \quad \rightarrow (ii)$

Divide by '2' eq (i)

$\frac{\cancel{2}(x-3)^2}{\cancel{2}} = \frac{\cancel{10}^{5}(y+1)}{\cancel{2}}$

$(x-3)^2 = (y+1) \rightarrow (iii)$

By comparing (ii) and (iii)

$(x-3)^3 = 5(y-(-1))$

vertex $(h, k) = (3, -1)$

focus $(3, -1+1.25) = (3, .25)$

$a(.2,5) = y = -2.25 \Rightarrow$ directix

Practice Question of lecture 10 to 12.

Question # 01:-

Prove that $\text{Cosec}^{-1} z = \frac{1}{i} \log\left(\frac{i+\sqrt{z^2-1}}{z}\right)$, $z \in \mathbb{C}$

let $w = \text{Cosec}^{-1} z$

$$\text{Cosec}\, w = z$$

$$\frac{1}{\sin w} = z$$

$$\sin w = \frac{1}{z} \quad \text{---} \; (i)$$

As we know that $\sin y = \frac{e^{iy} - e^{-iy}}{2i}$,

So eq. (1) can be written as:

$$\frac{e^{iw} - e^{-iw}}{2i} = \frac{1}{z}$$

$$e^{iw} - e^{-iw} = \frac{2i}{z}$$

$$e^{iw} - \frac{1}{e^{iw}} = \frac{2i}{z}$$

$$\frac{(e^{iw})^2 - 1}{e^{iw}} = \frac{2i}{z}$$

$$z(e^{iw})^2 - z = 2ie^{iw}$$

$$z(e^{iw})^2 - 2ie^{iw} - z = 0 \rightarrow (2)$$

which is quadratic in e^{iw}, so apply quadratic formula to eq. (2), we get;

$$e^{iw} = \frac{-(-2) \pm \sqrt{(-2i)^2 - 4(z)(-z)}}{2z}$$

$$= \frac{2 \pm \sqrt{-4 + 4z^2}}{2z}$$

$$= \frac{2 \pm 2\sqrt{z^2-1}}{2z} = \cancel{2}\left(\frac{1 \pm \sqrt{z^2-1}}{\cancel{2}z}\right)$$

$$= \frac{1 \pm \sqrt{z^2-1}}{z}$$

if we consider only positive value and ignore negative sign, then

$e^{iw} = \frac{1 \pm \sqrt{z^2-1}}{z}$ taking log on both sides, we get

$$\log e^{iw} = \log\left(\frac{1 + \sqrt{z^2-1}}{z}\right)$$

$$iw = \log\left(\frac{1 + \sqrt{z^2-1}}{z}\right)$$

$$w = \frac{1}{i}\log\left(\frac{1 + \sqrt{z^2-1}}{z}\right) \Rightarrow \operatorname{Cosec}^{-1} z = \frac{1}{i}\log\left(\frac{1 + \sqrt{z^2-1}}{z}\right)$$ proved.

Question #02:-

Separate into real and imaginary parts of $\tan^{-1}(x+iy)$.

Solution:-

let $\tan^{-1}(x+iy) = (a+ib)$ --- (A)

So real part is a and imaginary part is β. We want to find a and b. first we will show that $\tan^{-1}(x-iy) = a-ib$

from (A), we have

$$x+iy = \tan(a+ib) \text{ --- (B)}$$

{as we know that ; $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$

$\tan i\beta = i\tanh\beta$

So from eq (B)

$$x+iy = \frac{\tan a + \tan(ib)}{1-\tan a\tan(ib)} = \frac{\tan a + i\tan b}{1 - i\tan a\tan b}$$

Taking Conjugates of both sides we get

$$x-iy = \frac{\tan a + i\tan b}{1 - i\tan a\tan b}$$

$$x-iy = \frac{\tan a - i\tan b}{1+\tan a\tan\beta} = \tan(a-ib)$$

which can be written as

$\tan^{-1}(x-iy) = (a-ib)$ —— (C)

Adding eq (A) and (C)

$\tan^{-1}(x+iy)+\tan^{-1}(x-iy) = a+\cancel{ib}+a-\cancel{ib}$

$\tan^{-1}\dfrac{(x+iy)+(x-iy)}{1-(x+iy)(x-iy)} = 2a$

$\tan^{-1}\dfrac{x+\cancel{iy}+x-\cancel{iy}}{1-(x^2+y^2)} = 2a$

$\tan^{-1}\dfrac{2x}{1-x^2-y^2} = 2a$

Hence,

$a = \dfrac{1}{2}\tan^{-1}\dfrac{2x}{1-x^2-y^2} \longrightarrow D$

To find b, we subtract (A) and (C);

$\tan^{-1}(x+iy)-\tan^{-1}(x-iy) = \cancel{a}+ib-\cancel{a}+ib$

$\tan^{-1}\dfrac{(x+iy)-(x-iy)}{1+(x+iy)(x-iy)} = 2ib$

$\tan^{-1}\dfrac{2yi}{1+x^2+y^2} = 2ib$

~~$2yi$ $2b=$~~

$\dfrac{2y\cancel{i}}{1+x^2+y^2} = \tan(2ib) = i\tanh 2b$

$2b = \tanh^{-1}\dfrac{2y}{1+x^2+y^2}$

$b = \dfrac{1}{2}\tanh^{-1}\dfrac{2y}{1+x^2+y^2} \longrightarrow (E)$

Question # 03:-

for any complex number z, prove that $\sinh^{-1} z = \log(z + \sqrt{z^2+1})$.

let $w = \sinh^{-1} z$. Then $z = \sinh w$

$$z = \frac{e^{w} + e^{-w}}{2}.$$

$$\boxed{= \frac{e^{2w} - 1}{2e^{w}}}$$

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سمجھ نہیں آ رہی ؟؟؟؟

= Hence

$$2e^{w} z = e^{2w} - 1$$

$$e^{2w} - 2e^{w} z - 1 = 0$$

Which is a quadratic equation in e^{w}.

Therefore;

$$e^{w} = \frac{2z \pm \sqrt{4z^2+4}}{2}$$

$$e^{w} = z \pm \sqrt{z^2+1}$$

Now taking logarithm on both sides;

$$\log e^{w} = \log\left(z \pm \sqrt{z^2+1}\right)$$

$$w = \log\left(z \pm \sqrt{z^2+1}\right)$$

$$\sinh^{-1} z = \log\left(z \pm \sqrt{z^2+1}\right)$$

Hence prove. Which is required result.

Question # 04:-

Find $\log z$ if

(i) $z = 2i$

Here $z = 2i = 0 + 2i$

$|z| = \sqrt{(0)^2 + 2^2} = \sqrt{4} = 2$

$Arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{0}\right) = \tan^{-1}(\infty)$

$= \frac{\pi}{2}$,

Therefore $\log(2i) = \ln|z| + i\,Arg(z)$

$= \ln 2 + \frac{\pi}{2} i$

(ii) $z = -i$

$z = 0 + (-1)i$

$|z| = \sqrt{(0)^2 + (-1)^2} = 1$

$Arg\, z = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{0}\right) = \tan^{-1}\infty = \frac{\pi}{2}$. So

Since $z = -i$ lies below the x-axis,

$Arg(z) = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$.

Therefore $Log(-i) = \ln|z| + i\,Arg(z)$

$= \ln 1 - \frac{\pi}{2} i = -\frac{\pi}{2} i$

(iii) $z = x = x + 0i$

$|z| = \sqrt{x^2 + 0^2} = x$

$Arg(z) = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{x}\right) = \tan^{-1}(0) = 0$

Therefore:-

$\log(x) = \ln|z| + i\,Arg(z) = \ln x + 0$

$= \ln x$

(iv) $z = 1+\sqrt{3}i$

$|z| = \sqrt{(1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$

$Arg\, z = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

Therefore $\log(1+\sqrt{3}i) = \ln|z| + i\,Arg(z)$

$= \ln 2 + \frac{\pi}{3} i$ Ans.

Lecture # 15

Introduction to Ellipse

Radius is different in Ellipse in x and y direction.

* radius in x direction is Semi major axis.
* radius in y direction is Semiminnor axis.

Major equation :-

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example:-

$$\frac{(y-1)^2}{4} + \frac{(x+2)^2}{9} = 1$$

$a=3, \; b = 2$

$a > b$

② Ellipse foci:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; , \; a > b$$