Lecture 23

Maximum & Minimum values of function:-

Absolute Maximum:-

If $f(x_0) \geq f(x)$ for all x in The domain of f, Then $f(x_0)$ is called absolute maximum value or simply maximum value of f.

Absolute Minimum:-

If $f(x_0) \leq f(x)$ for all x in The domain of f Then $f(x_0)$ is called The absolute minimum or simply minimum value of f.

Remark:-

Absolute extremum is The max or min value over The entire domain of f.

Example:-

Find max & min value of $f(x)=2x+1$ over $[0,3)$.

put $x=0$

$f(0)=2(0)+1=1$

put $x=1$

$f(1)=2(1)+1=3$

put $x=2$

$f(2)=2(2)+1=4+1=5$

So minimum (absolute) value is 1 at $x=0$ & function has no ~~max~~ absolute max value.

Extreme value Theorem:-

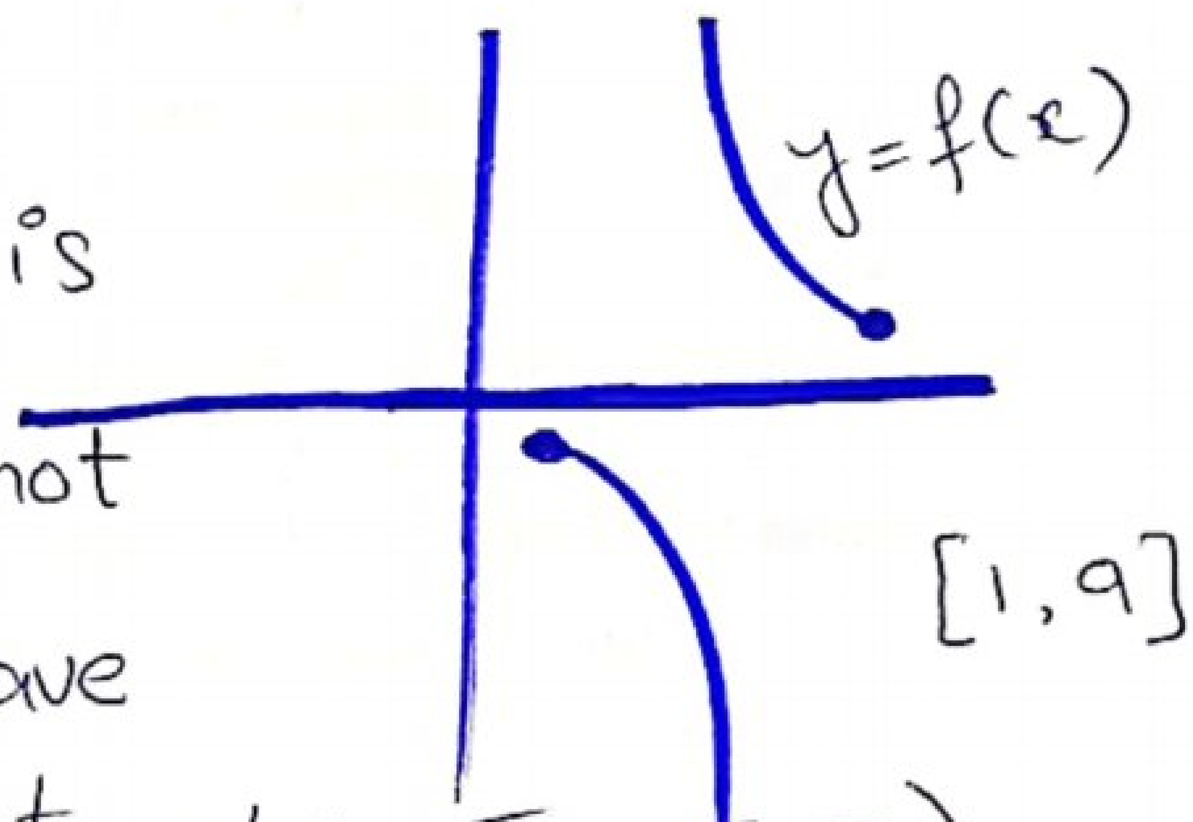
If a function is continuous on a closed interval $[a, b]$, Then f has both a maximum value & a minimum value on $[a, b]$.

Example:-

In previous example $f(x) = 2x+1$, function is continuous but has no maximum value, so it does not obey Extreme value Theorem. bcz interval was not closed.

Example:-

In This graph interval is closed but function is not continuous, so does not have max or min value (doesn't obey The EVT).



Theorem:-

If a function f has an extreme value (min/max) on an open interval (a, b), then the extreme value occurs at a critical point of f.

Steps:-

1- Find critical points of f in (a, b).

2- Evaluate f at all critical points & end points.

3- The largest of the values is maximum value.

4- The smallest of the values is minimum value.

Example:- Find maximum & minimum value of $f(x) = 2x^3 - 15x^2 + 36x$ on interval [1, 5].

Step 01 Take derivative to find critical points.

$$f'(x) = 2(3x^2) - 15(2x) + 36(1)$$
$$= 6x^2 - 30x + 36$$

now put $f'(x) = 0$

$$6x^2 - 30x + 36 = 0$$
$$6(x^2 - 5x + 6) = 0$$
$$x^2 - 5x + 6 = 0$$
$$x^2 - (3+2)x + 6 = 0$$
$$x^2 - 3x - 2x + 6 = 0$$
$$x(x-3) - 2(x-3) = 0$$
$$(x-2)(x-3) = 0$$

$x - 2 = 0$ | $x - 3 = 0$

$\boxed{x = 2}$ | $\boxed{x = 3}$

Step 02:-

put $x = 2$ in $f(x)$

$$f(x) = 2x^3 - 15x^2 + 36x$$
$$f(2) = 2(2)^3 - 15(2)^2 + 36(2) \Rightarrow 2(8) - 15(4) + 72$$
$$= 16 - 60 + 72 \Rightarrow \boxed{28}$$

put $x = 3$ in $f(x)$

$$f(x) = 2x^3 - 15x^2 + 36x$$

$$f(3) = 2(3)^3 - 15(3)^2 + 36(3)$$
$$= 2(27) - 15(9) + 108$$
$$= 54 - 135 + 108$$
$$\boxed{f(3) = 27}$$

put $x = 1$

$$f(1) = 2(1)^3 - 15(1)^2 + 36(1)$$
$$= 2 - 15 + 36$$
$$\boxed{f(1) = 23}$$

put $x = 5$

$$f(5) = 2(5)^3 - 15(5)^2 + 36(5)$$
$$= 2(125) - 15(25) + 180$$
$$= 250 - 375 + 180$$
$$\boxed{f(5) = 55}$$

Step 4:-
So $f(x)$ has maximum value at $x = 5$ (i-e) 55.

Step 5 $f(x)$ has minimum value 23 at $x = 3$.

Some Results:-

$$\left.\begin{aligned} \lim_{x \to -\infty} f(x) &= +\infty \\ \lim_{x \to +\infty} f(x) &= +\infty \end{aligned}\right\}$$ $f(x)$ has a ~~maxi~~ minimum but no max value.

$$\left.\begin{aligned} \lim_{x \to -\infty} f(x) &= -\infty \\ \lim_{x \to -\infty} f(x) &= -\infty \end{aligned}\right\}$$ $f(x)$ has a max but no min value.

$$\left.\begin{aligned} \lim_{x \to -\infty} f(x) &= -\infty \\ \lim_{x \to +\infty} f(x) &= +\infty \\ \lim_{x \to -\infty} f(x) &= +\infty \\ \lim_{x \to +\infty} f(x) &= -\infty \end{aligned}\right\}$$ $f(x)$ has neither max nor min value.

Same for $\lim_{x \to a}$ & $\lim_{x \to b}$

Example:-

Find max & min value of $f(x) = x^4 + 2x^3 - 1$ on interval $(-\infty, +\infty)$

$$f'(x) = 4x^3 + 6x^2$$

put $f'(x) = 0$.

$$4x^3 + 6x^2 = 0$$

$$2x^2(2x+3) = 0$$

$$2x^2 = 0 \quad , \quad 2x+3 = 0$$

$$x^2 = 0 \quad , \quad 2x = -3$$

$$\boxed{x = 0} \quad , \quad \boxed{x = \frac{-3}{2}}$$

put $x = 0$ in $f(x)$

$$f(x) = x^4 + 2x^3 - 1$$

$$= (0)^4 + 2(0)^3 - 1$$

$$= 0 + 0 - 1$$

$$f(0) = -1$$

put $x = \frac{-3}{2}$

$$f\left(\frac{-3}{2}\right) = \left(-\frac{3}{2}\right)^4 + 2\left(\frac{-3}{2}\right)^3 - 1$$

$$= \frac{81}{16} - \frac{54}{8} - \frac{1}{1}$$

$$= \frac{81 - 108 - 16}{16}$$

$$f\left(\frac{-3}{2}\right) = -\frac{43}{16}$$

put $x = +\infty$

$$\lim_{x \to +\infty} (x^4 + 2x^3 - 1)$$

$$= +\infty$$

put $x = -\infty$

$$\lim_{x \to -\infty} (x^4 + 2x^3 - 1)$$

$$= +\infty$$

So $f(x)$ has no max value.

$f(x)$ has min value at $x = \frac{-3}{2}$

(i-e) $-\frac{43}{16}$.

Applied minimum & maximum problems:-

Example:-

Find dimension of a rectangle with perimeter 100 ft whose area is as large as possible.

let x = length , y = width , A = area

$A = xy$

put $y = 50 - x$

$A = x(50-x)$

$= 50x - x^2$

$P = 2(x+y)$

$100 = 2(x+y)$

$50 = x+y$

$50 - x = y$

put $50x - x^2 = 0$

$x(50-x) = 0$

$x = 0$ or $50 - x = 0$. So end points are $[0,50]$

Now find critical points

$A'(x) = 0$

$50(1) - 2x = 0$

$50 - 2x = 0$

$50 = 2x$

$\boxed{25 = x}$

put $x = 25$ in

$A = 50x - x^2 \Rightarrow 50(25) - (25)^2 \Rightarrow 1250 - 625 \Rightarrow 625$

put $x = 0$

$A = 50(0) - 0 \Rightarrow 0$

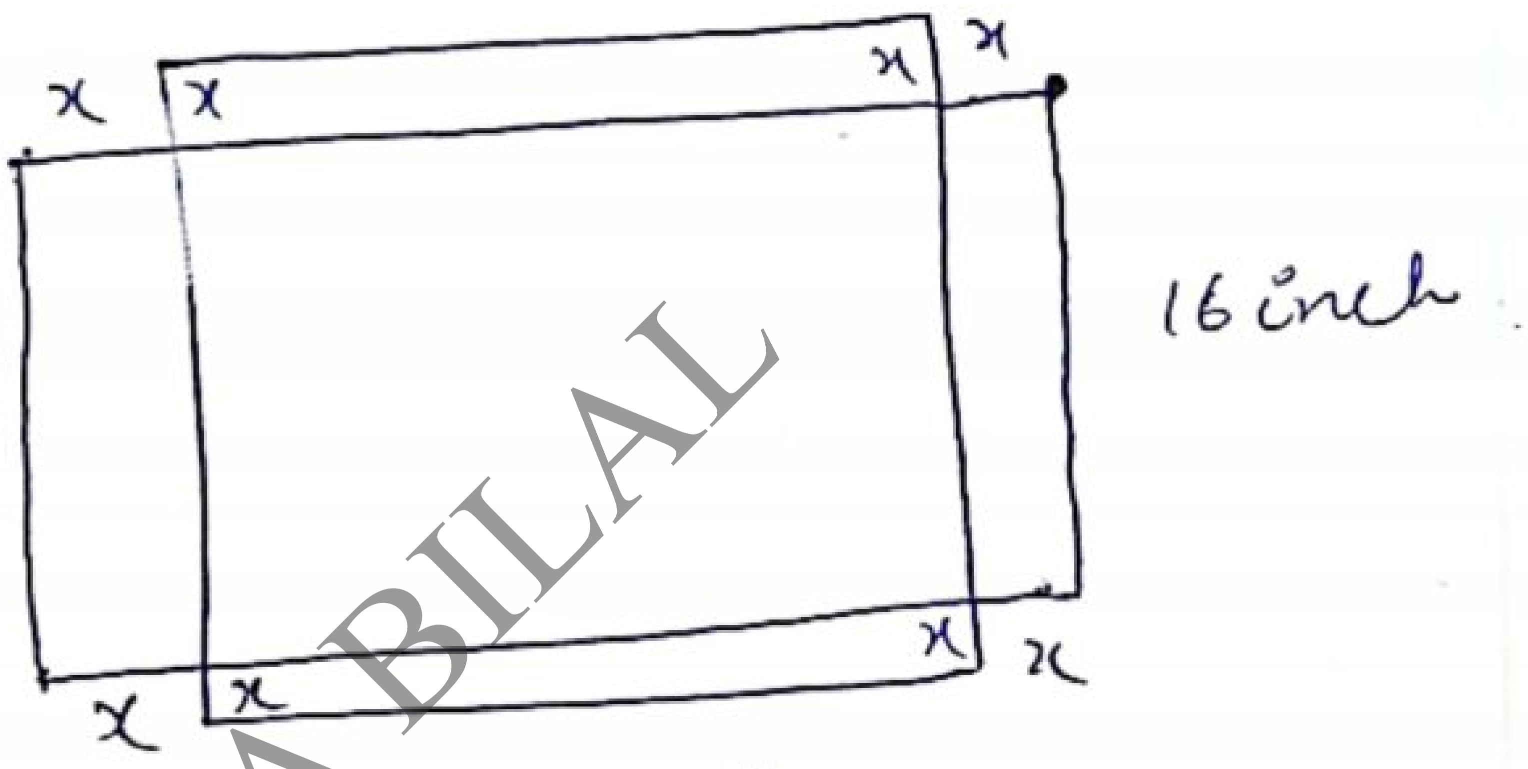
put $x = 50$

$A = 50(50) - (50)^2 \Rightarrow 2500 - 2500 \Rightarrow 0$

So max value of A is 625 at $x = 25$

Example:-

An open box to be made from a 16 inch by 30 inch piece of cardboard by cutting out squares of equal size from the 4 corners & bending up the sides. What size should the squares be to obtain a box with largest possible volume?



Let x = length of sides to be cut

V = Volume of resulting box.

dimensions will be

$(16-2x)$ by $(30-2x)$

As $V = (\text{length})(\text{width})(\text{height})$

$$V = (16-2x)(30-2x)(x)$$

$$= 480x - 92x^2 + 4x^3$$

put $V=0$ to find end points.

$$480x - 92x^2 + 4x^3 = 0$$

$$4x(120 - 23x + x^2) = 0$$

$$4x = 0 \quad , \quad x^2 - 23x + 120 = 0$$

$$\boxed{x=0} \quad , \quad x^2 - (15+8)x + 120 = 0$$

$$x^2 - 15x - 8x + 120 = 0$$

$$x(x-15) - 8(x-15) = 0$$

$$(x-15)(x-8) = 0$$

$x - 15 = 0$ | $x - 8 = 0$

$x = 15$ | $\boxed{x = 8}$

x can't be 15 bcz it makes ~~length~~ length negative. So end points are $[0, 8]$

Now we'll find critical points.

$$V = 480x - 92x^2 + 4x^3$$

$$\frac{dV}{dx} = 480 - 184x + 12x^2$$

put $\frac{dV}{dx} = 0$

$$480 - 184x + 12x^2 = 0$$

$$4(120 - 46x + 3x^2) = 0$$

$$3x^2 - 46x + 120 = 0$$

$$3x^2 - (36 + 10)x + 120 = 0$$

$$3x^2 - 36x - 10x + 120 = 0$$

$$3x(x - 12) - 10(x - 12) = 0$$

$$(x - 12)(3x - 10) = 0$$

$x - 12 = 0$ | $3x - 10 = 0$

$x = 12$ | $3x = 10$

$\boxed{x = \frac{10}{3}}$

ignore $x = 12$ bcz it is out of $[0, 8]$.

Now use values of x $(0, 8, \frac{10}{3})$ to find maximum value of V.

put $x = 0$

$$V = 480x - 92x^2 + 4x^3$$

$$= 480(0) - 92(0)^2 + 4(0)^3$$

$$V = 0$$

put $x = 8$

$$V = 480x - 92x^2 + 4x^3$$

$$V = 480(8) - 92(8)^2 + 4(8)^3$$

$$V = 3840 - 5888 + 2048$$

$$V = 0$$

put $x = \frac{10}{3}$

$$V = 480\left(\frac{10}{3}\right) - 92\left(\frac{10}{3}\right)^2 + 4\left(\frac{10}{3}\right)^3$$

$$= \frac{4800}{3} - \frac{9200}{9} + \frac{4000}{27}$$

$$= \frac{43200 - 27600 + 4000}{27}$$

$$\boxed{V = \frac{19600}{27}}$$

So function of volume has its maximum value $\frac{19600}{27}$ at $x = \frac{10}{3}$.

Lecture 24

Newton's Method:-
Rolle's Theorem:-
Mean value Theorem:-

⇒ Newton's Method:-

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

(i-e) if $n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

if $n=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

& so on.

Example:- The equation $x = \cos x$ has a solution between 0 and 1. Approximate it using newton's method. $f(x) = x - \cos x$

$$x_{n+1} = x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)}$$

put $x_1 = 1$

put $n=1$, $x_{1+1} = x_1 - \frac{x_1 - \cos(x_1)}{1 + \sin(x_1)}$

$$x_2 = 1 - \frac{1-\cos(1)}{1+\sin(1)} = 0.7503$$

put ~~x~~ $n=2$

$$x_{2+1} = x_2 - \frac{x_2 - \cos(x_2)}{1+\sin(x_2)}$$

$$x_3 = 0.7503 - \frac{0.7503 - \cos(0.7503)}{1+\sin(0.7503)}$$

$$x_3 = 0.7391$$

You may continue, but approximate solution is 0.7391.

Some difficulties with Newton's Method.

⇒ It does not always work.

⇒ If for some values of n . $f'(x_n)$ Then formula involves division by zero & we are out of business.

⇒ Such a ~~slope~~ case will occur if the Tangent line for some some approximation has slope zero or is parallel to x-axis.

⇒ Sometimes approximation don't converge to a solution.

For example in $x^{\frac{1}{3}} = 0$

$$x_{n+1} = x_n - \frac{(x_n^{\frac{1}{3}})}{\frac{1}{3}(x_n)^{-\frac{2}{3}}} \Rightarrow x_n - 3(x_n)^{\frac{1}{3}} \cdot (x_n)^{\frac{2}{3}} \Rightarrow x_n - 3(x_n)^{\frac{1}{3}+\frac{2}{3}}$$

$$= x_n - 3x_n = -2x_n$$

let $x_1 = 1$ & Then follow approximation, value do not converge

Rolle's Theorem:-

Let f be differentiable on (a, b) & continuous on $[a, b]$. If $f(a) = f(b) = 0$ Then There is at least one point c in (a, b) where $f'(c) = 0$.

Example:- $f(x) = \sin x$ $[0, 2\pi]$

As $\sin x$ is differentiable on $(0, 2\pi)$ & continuous on $[0, 2\pi]$. & $\sin(0) = \sin(2\pi) = 0$

So $f'(c) = \cos x = 0$ at any number in 0 & 2π

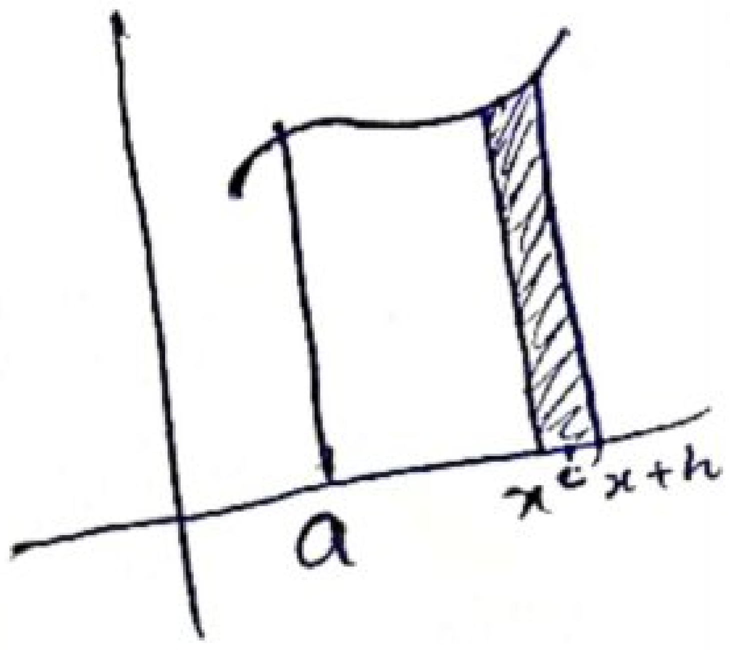
Here The value of c is $\frac{\pi}{2}$ or 90°.

Mean value Theorem:-

Let f be differentiable on (a, b) & continuous on $[a, b]$. Then There is at least one point c in (a, b) where $f'(c) = \frac{f(b) - f(a)}{b - a}$

Lecture 25 — Integration

Area problem



$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \to 0} f(c)$$

As h goes to zero, c approaches x.
$f(c)$ goes to $f(x)$ as c goes to x.

Antiderivative:-

If $F(x)$ is any antiderivative of $f(x)$ on a given interval, then for any value of C, the function $F(x)+C$ is also an antiderivative of $f(x)$ on that interval, moreover every antiderivative of $f(x)$ on the interval is expressible in the form $F(x)+C$, where C is constant.

Indefinite integral:-

The process of finding anti derivative is called anti-differentiation or Integration. function in the form $F(x)+C$ are called anti derivative of $f(x)$. **For example** $\frac{1}{3}x^3$, $\frac{1}{3}x^3 - \pi$, $\frac{1}{3}x^3 + C$ are all anti-derivatives of $f(x) = x^2$.

We denote anti-derivative by $\int f(x)\,dx = F(x) + c$

The symbol $\int$ is called integral sign & $f(x)$ is called integrand. It is read as "Indefinite integral of $f(x)$ equals $F(x)$"

C is called constant of integration.

Example:- $\int f(x)\,dx = ?$ when $f(x) = x^2$

$$\int f(x)\,dx = \int x^2\,dx$$

$$= \frac{x^{2+1}}{2+1} + c \Rightarrow \frac{x^3}{3} + c$$

Example:-

$$\int x^2\,dx = \frac{x^3}{3} + c$$

$$\int x^3\,dx = \frac{x^{3+1}}{3+1} + c \Rightarrow \frac{x^4}{4} + c$$

$$\int \frac{1}{x^5}\,dx = \int x^{-5}\,dx \Rightarrow \frac{x^{-5+1}}{-5+1} + c \Rightarrow \frac{x^{-4}}{-4} + c \Rightarrow \frac{-1}{4x^4} + c$$

Derivative Formula	Integration Formula
$\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$	$\int 3x^2\,dx = 3\frac{x^{2+1}}{2+1} \Rightarrow \frac{3x^3}{3} \Rightarrow x^3 + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dt}(\tan t) = \sec^2 t$	$\int \sec^2 t\,dt = \tan t + c$
$\frac{d}{du}(u^{\frac{3}{2}}) = \frac{3}{2}u^{\frac{3}{2}-1} \Rightarrow \frac{3}{2}u^{\frac{1}{2}}$	$\int \frac{3}{2}u^{\frac{1}{2}}\,du = \frac{3}{2}\left(\frac{u^{\frac{1}{2}+1}}{\frac{3}{2}}\right) = u^{\frac{3}{2}} + c$

Properties of Definite Integral:-

$\Rightarrow$ constant can be moved through an integral sign.

$$\int c\, f(x)\,dx = c\int f(x)\,dx$$

$\Rightarrow$ Anti-derivative of a sum is the sum of anti-derivatives. $\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$

$\Rightarrow$ Anti-derivative of a difference is the difference of the anti-derivatives.

$$\int [f(x)-g(x)]\,dx = \int f(x)\,dx - \int g(x)\,dx$$

Example:- Evaluate $\int 4\cos x\,dx$

$$\int 4\cos x\,dx = 4\int \cos x\,dx = 4[\sin x + c]$$

$$= 4\sin x + 4c = 4\sin x + k.$$

Example:- $\int (x^2+x)\,dx$

$$= \int x^2\,dx + \int x\,dx = \frac{x^{2+1}}{3} + \frac{x^{1+1}}{2} + c$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + c$$

Generalized Version:-

$$\int [c_1 f_1(x) + c_2 f_2(x) + \cdots\cdots + c_n f_n(x)]\,dx$$

$$= c_1\int f_1(x)\,dx + c_2\int f_2(x)\,dx + \cdots + c_n\int f_n(x)\,dx$$

Example:- $\int (3x^6 - 2x^2 + 7x + 1)\,dx$

$$= 3\int x^6\,dx - 2\int x^2\,dx + 7\int x\,dx + 1\int \cdot dx$$

$$= 3\frac{x^7}{7} - 2\frac{x^3}{3} + \frac{7x^2}{2} + x + c$$

Example:- $\int \frac{\cos x}{\sin^2 x}\,dx$

$$= \int \frac{\cos x}{\sin x . \sin x}\,dx$$

$$= \int \frac{1}{\sin x} \cdot \frac{\cos x}{\sin x}\,dx$$

$$= \int (\operatorname{cosec} x \cot x)\,dx$$

$$= -\operatorname{cosec} x + c$$

Lecture 26:- Integration by Substitution:-

This is like chain rule of differentiation.
Integration by Substitution is used for solving composit functions.

$$\frac{d}{dx}(G(u)) = \frac{d}{du}(G(u)) \cdot \frac{du}{dx} = f(u)\frac{du}{dx}$$

Take integral

$$\int \left(f(u)\frac{du}{dx}\right) dx = G(u) + C.$$

$$\Rightarrow \int \left(f(u)\frac{du}{dx}\right) dx = \int f(u)\, du$$

Example:- $\int (x^2+1)^{50} \cdot 2x \, dx$

Let $u = (x^2+1)$
Take derivative

$$\frac{du}{dx} = 2x + 0 \Rightarrow 2x$$

$$du = 2x\, dx$$

$$\int u^{50}\, du$$

$$= \frac{u^{50+1}}{51} + C$$

$$= \frac{(x^2+1)^{51}}{51} + C.$$

Don't just add 1 to The origional problem, it'll be incorrect.

Steps for u-substitution:-

1. Make choice for u, say $u = g(x)$.
2. Compute $\frac{du}{dx} = g'(x)$
3. Make substitutions $u = g(x)$ & $du = g'(x)\,dx$.
   (Here no x will be in problem, whole function'll be in the form of u.)
4. Evaluate the resulting integral.
5. Replace u by $g(x)$ so the final answer is in x.

Differentiation formula	Integration formula
$\frac{d}{dx}(x) = 1$	$\int 1\,dx = x + c$
$\frac{d}{dx}\left(\frac{x^{r+1}}{r+1}\right) = x^r$	$\int x^r\,dx = \frac{x^{r+1}}{r+1} + c$
$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\,dx = \sin x + c$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\,dx = -\cos x + c$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + c$
$\frac{d}{dx}(-\cot x) = \operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x\,dx = -\cot x + c$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\int \sec x \tan x \, dx = \sec x + c$$

$$\frac{d}{dx}(-\operatorname{cosec} x) = \cancel{\csc x \tan} \; \csc x \cot x$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

Example:-

$$\int (x-8)^{23} dx$$

let $u = x - 8$

$$\frac{du}{dx} = 1$$

$$du = 1\,dx$$

$$du = dx$$

$$\int u^{23} dx$$

$$= \frac{u^{23+1}}{24} + c$$

$$= \frac{(x-8)^{24}}{24} + c$$

Example:-

$$\int \cos(5x)\,dx$$

let $u = 5x$

$$\frac{du}{dx} = 5$$

$$du = 5\,dx$$

$$\frac{du}{5} = dx$$

$$\int \cos u \cdot \frac{du}{5}$$

$$= \frac{1}{5}\int \cos u \cdot du$$

$$= \frac{1}{5} \sin u + c$$

$$= \frac{1}{5} \sin(5x) + c$$

Example:- $\int \sin^2 x \cos x \, dx$

let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$du = \cos x \, dx$$

$$\int u^2 . du$$

$$= \frac{u^3}{3} + c$$

$$= \frac{(\sin x)^3}{3} + c$$

$$= \frac{\sin^3 x}{3} + c$$

Example:- $\int \frac{\cos\sqrt{x}}{\sqrt{x}} dx$

let $u = \sqrt{x}$

$$\frac{du}{dx} = \frac{1}{2}(x)^{\frac{1}{2}-1}$$

$$\frac{du}{dx} = \frac{1}{2}x^{-\frac{1}{2}}$$

$$\frac{du}{dx} = \frac{1}{2x^{\frac{1}{2}}}$$

$$du = \frac{1}{2\sqrt{x}} dx$$

$$2du = \frac{1}{\sqrt{x}} dx$$

$$\int \cos\sqrt{x} . \frac{1}{\sqrt{x}} dx$$

$$= \int \cos u . 2du$$

$$= 2\int \cos u . du$$

$$= 2 \sin u + c$$

$$= 2 \sin(\sqrt{x}) + c .$$

Example:- $\int t^4\sqrt{3-5t^5}\,dt$

let
$u = 3-5t^5$

$\frac{du}{dt} = 0-5(5t^4)$

$du = -25t^4\,dt$

$\frac{du}{-25} = t^4\,dt$

$\int \sqrt{3-5t^5}\, t^4 dt$

$= \int \sqrt{u} \cdot \frac{du}{-25}$

$= -\frac{1}{25}\int \sqrt{u}\, du$

$= -\frac{1}{25} \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c$

$= -\frac{1}{25} \cdot \frac{u^{3/2}}{3/2} + c$

$= -\frac{1}{25} \times \frac{2}{3} \times u^{3/2} + c$

$= \frac{-2}{75}(3-5t^4)^{3/2} + c$

if $\int t^4 \sqrt[3]{3-5t^4}\, dt$

$= \int \sqrt[3]{3-5t^4}\; t^4 dt$

$= \int \sqrt[3]{u} \cdot \frac{du}{-\cancel{1}25}$

$= -\frac{1}{25}\int \sqrt[3]{u}\, du$

$= -\frac{1}{25} \cdot \frac{u^{\frac{1}{3}+1}}{\frac{1}{3}+1} + c$

$= -\frac{1}{25} \cdot \frac{u^{\frac{4}{3}}}{4/3} + c$

$= -\frac{1}{25} \cdot \frac{3}{4} \cdot u^{\frac{4}{3}} + c$

$= -\frac{1}{25} \cdot \frac{3}{4} \cdot (3-5t^4)^{\frac{4}{3}} + c$

$= \frac{-3}{100}(3-5t^4)^{\frac{4}{3}} + c$

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Lecture 27:- Sigma Notation:-

⇒ used to write lengthy sums in compact Form.

⇒ Sigma (Σ) is an upper case letter in Greek.

⇒ Σ is called Sigma / Summation.

Examples:-

i) $\sum_{k=1}^{5} k^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

ii) $\sum_{k=4}^{8} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3$

iii) $\sum_{k=1}^{5} 2k = 2(1) + 2(2) + 2(3) + 2(4) + 2(5)$

$= 2 + 4 + 6 + 8 + 10$

iv) $\sum_{k=0}^{5} (-1)^k (2k+1) = (-1)^0(2(0)+1) + (-1)^1[2(1)+1] + (-1)^2[2(2)+1] +$
$(-1)^3(2(3)+1) + (-1)^4[2(4)+1] + (-1)^5[2(5)+1]$

$= 1 - 3 + 5 - 7 + 9 - 11$

In $\sum_{k=0}^{5} (2k)$ The number on The top (5) is called upper limit of summation & The number on The bottom (0) is called lower limit. k is called index of The Summation.

any letter can be used as index instead k.

Example:- $\sum_{i=1}^{4} \frac{1}{i}$, $\sum_{n=1}^{4} \frac{1}{n}$, $\sum_{j=1}^{4} \frac{1}{j}$ all are same.

(i-e) $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

- If upper & lower limits are same then the summation reduces to just one term.

$$\sum_{k=2}^{2} k^3 = 2^3$$

- If in the summation, expression on right of summation does not involve the index, then

$$\sum_{i=1}^{5} 2 = 2+2+2+2+2$$

$$\sum_{k=3}^{6} x^3 = x^3+x^3+x^3+x^3$$

- A sum can be written in more than one notations if we change the limits.

$$\sum_{k=1}^{5} 2k = 2+4+6+8+10$$

$$\sum_{k=0}^{4} (2k+2) = 2+4+6+8+10$$

$$\sum_{k=2}^{6} (2k-2) = 2+4+6+8+10$$

Changing the index of summation:-

Example:- Express $\sum_{k=3}^{7} 5^{k-2}$ in sigma notation so that lower limit is 0 rather than 3.

Define a new index j by

Take $k = 3$

$k - 3 = 0$

$j = 0$

$\left(\text{say } j = k - 3 \right)$

Put $k = 3$
$j = 3 - 3 = 0$

Put $k = 7$
$j = 7 - 3 = 4$

Then $j + 3 = k$

put in

$$\sum_{k=3}^{7} 5^{k-2} = \sum_{j=0}^{4} 5^{(j+3)-2}$$

$$= \sum_{j=0}^{4} 5^{j+1}$$

- To represent a general sum, we'll use letters with subscripts.

Example:- $a_1 + a_2 + a_3$ represent general sum with three terms. can also be written as

$$\sum_{k=1}^{3} a_k .$$

- general sum with n terms can be written as

$$\sum_{k=1}^{n} b_k = b_1 + b_2 + b_3 + \cdots + b_n .$$

Properties of Sigma Notation:-

i) $\displaystyle\sum_{k=1}^{n} c\, a_k = c \sum_{k=1}^{n} a_k$

ii) $\displaystyle\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

iii) $\displaystyle\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

Theorems:-

$$\sum_{k=1}^{n} k = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$\sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=1}^{n} k^3 = 1^3+2^3+3^3+\cdots+n^3 = \left[\frac{n(n+1)}{2}\right]^3$$

Example:- $\sum_{k=1}^{30} k(k+1)$

$$= \sum_{k=1}^{30} (k^2+k) \Rightarrow \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k$$

$$= \frac{30(30+1)[2(30)+1]}{6} + \frac{30(30+1)}{2}$$

$$= \frac{(30)(31)(61)}{6} + \frac{(30)(31)}{2}$$

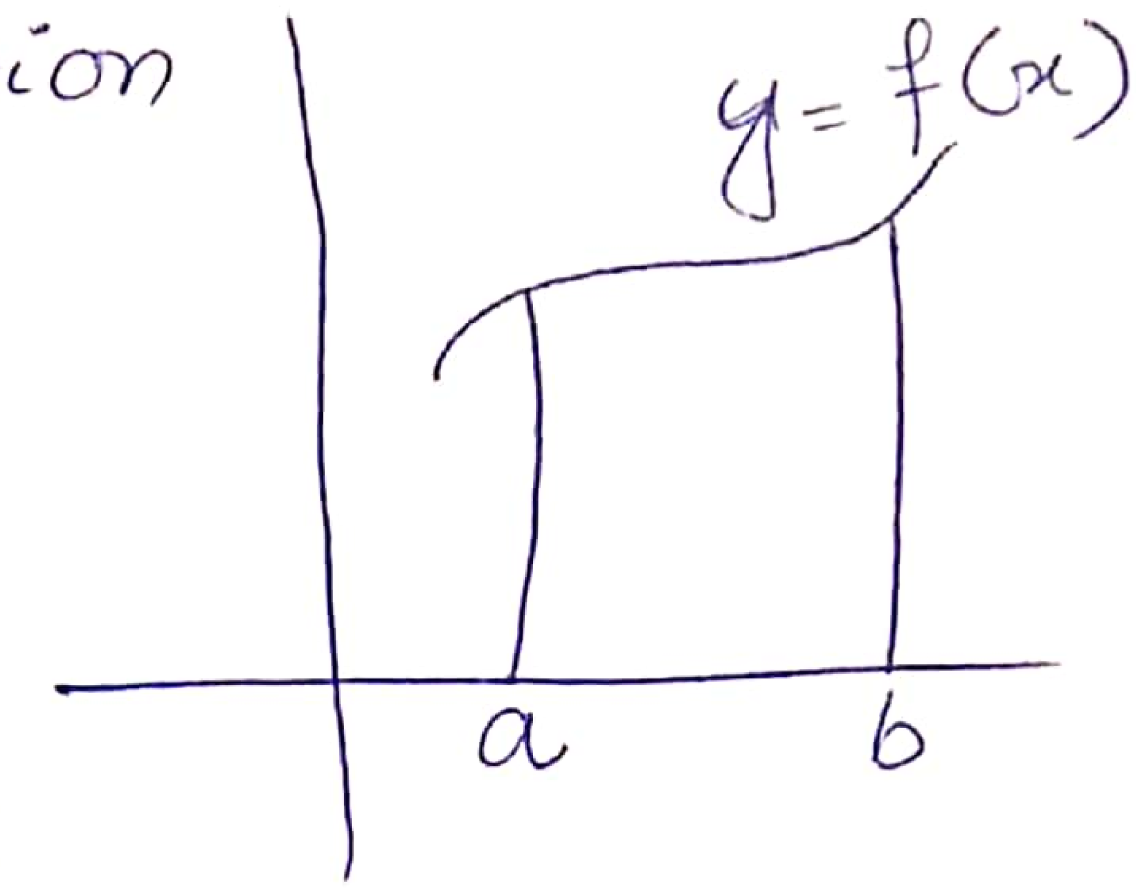
$$= \frac{56730}{6} + \frac{930}{2}$$

$$= 9455 + 465$$

$$= 9920$$

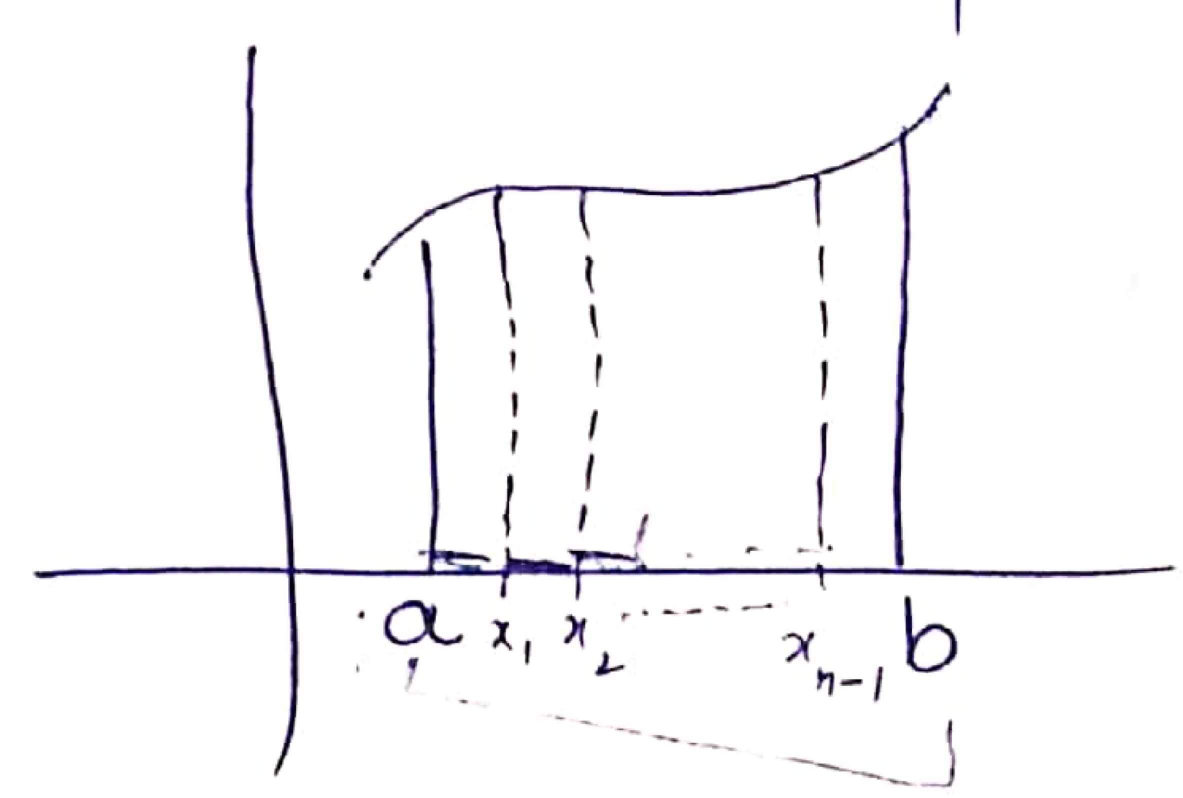
- In above Three Theorems (~~like~~, left part is called <u>open form</u> (like in 1st, $1+2+3+\cdots+n$) & right part is called <u>closed form</u> (like $\frac{n(n+1)}{2}$).

Lecture 28 Area as Limits.



In This figure, There is a region bounded below by x-axis, on The sides by The lines $x=a$ & $x=b$, Above by a curve of continuous function $f(x)=y$ which is also non-negative on interval $[a,b]$. Such an area can be calculated using anti-derivatives.

We can divide This graph/area/region into n equal parts of width $\frac{b-a}{n}$ by inserting $n-1$ equally spaced points between a & b $(x_1, x_2, x_3, \ldots, x_n)$.



$$\Delta x = \frac{b-a}{n}$$

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

$$= f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x$$

Left End Point:-

$$x_k^* = x_{k-1} = a + (k-1)\Delta x$$

Right End Point:-

$$x_k^* = x_k = a + k\Delta x$$

Mid Point:-

$$x_k^* = \frac{1}{2}(x_{k-1} + x_k) = a + \left(k - \frac{1}{2}\right)\Delta x$$

Example:- Use defination of area with x_k^* as right end point of each subinterval to find area under line $y = x$ over $[1, 2]$

$[a, b]$

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

For right end point

$$x_k^* = a + k\Delta x$$

here $a = 1$ & $\Delta x = \frac{1}{n}$

$$x_k^* = 1 + k\left(\frac{1}{n}\right)$$

$$= 1 + \frac{k}{n}$$

$$f(x_k^*)\Delta x = x_k^* \Delta x$$

$$= \left(1 + \frac{k}{n}\right)\left(\frac{1}{n}\right)$$

$$= \left(\frac{1}{n} + \frac{k}{n^2}\right)$$

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n} + \frac{k}{n^2} \right)$$

$$= \lim_{n\to\infty} \left[\sum_{k=1}^{n} \frac{1}{n} + \sum_{k=1}^{n} \frac{k}{n^2} \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \sum_{k=1}^{n} 1 + \frac{1}{n^2} \sum_{k=1}^{n} k \right]$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \cdot n + \frac{1}{n^2} \cdot \frac{n(n-1)}{2} \right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{(n-1)}{2n} \right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{n}{2n} - \frac{1}{2n} \right]$$

$$= \lim_{n\to\infty} \left[1 + \frac{1}{2} - \frac{1}{2n} \right]$$

$$= 1 + \frac{1}{2} - \frac{1}{2(\infty)}$$

$$= \frac{2+1}{2} - \frac{1}{\infty}$$

$$= \frac{3}{2} - 0 \Rightarrow \boxed{\frac{3}{2}} \quad \text{Ans.}$$

Example:- Same problem as before but with left end point.

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$$

Left end point

$$x_k^* = x_{k-1} = a + (k-1)\Delta x$$

here $a = 1$ & $\Delta x = \frac{1}{n}$

$$x_k^* = 1 + (k-1)\frac{1}{n}$$

$$x_k^* = 1 + \frac{k-1}{n}$$

now

$$f(x_k^*)\Delta x = x_k^* \Delta x$$

$$= \left(1 + \frac{k-1}{n}\right)\frac{1}{n}$$

$$= \frac{1}{n} + \frac{k-1}{n^2}$$

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n} + \frac{k-1}{n^2}\right)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{n} + \frac{k}{n^2} - \frac{1}{n^2}\right)$$

$$= \lim_{n\to\infty}\left[\sum_{k=1}^{n}\frac{1}{n}+\sum_{k=1}^{n}\frac{k}{n^2}-\sum_{k=1}^{n}\frac{1}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{k=1}^{n}1+\frac{1}{n^2}\sum_{k=1}^{n}k-\frac{1}{n^2}\sum_{k=1}^{n}1\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\cdot n+\frac{1}{n^2}\cdot\frac{n(n-1)}{2}-\frac{1}{n^2}\cdot n\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{n-1}{2n}-\frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{n}{2n}-\frac{1}{2n}-\frac{1}{n}\right]$$

$$= \lim_{n\to\infty}\left[1+\frac{1}{2}-\frac{1}{2n}-\frac{1}{n}\right]$$

$$= 1+\frac{1}{2}-\frac{1}{2(\infty)}-\frac{1}{\infty}$$

$$= \frac{2+1}{2}-\frac{1}{\infty}-0$$

$$= \frac{3}{2}-0-0$$

$$= \boxed{\frac{3}{2}}$$ Ans.

Example:- Use area defination with right end points of each subinterval to find area under parabola $y = 9 - x^2$ over $[0, 3]$.

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

right end point:-

$$x_k^* = a + k\Delta x$$

here $a = 0$ & $\Delta x = \frac{3}{n}$

$$x_k^* = 0 + k\left(\frac{3}{n}\right) = \frac{3k}{n}$$

Now $f(x_k^*)\Delta x = (9 - x_k^{*2})\Delta x$

$$= \left(9 - \left(\frac{3k}{n}\right)^2\right)\frac{3}{n} \Rightarrow \left[9 - \frac{9k^2}{n^2}\right]\frac{3}{n}$$

$$= \frac{27}{n} - \frac{27k^2}{n^3}$$

$$\text{Area} = \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{27}{n} - \frac{27k^2}{n^3}\right)$$

$$= \lim_{n\to\infty}\left[\frac{27}{n}\sum_{k=1}^{n} 1 - \frac{27}{n^3}\sum_{k=1}^{n} k^2\right]$$

$$= \lim_{n\to\infty}\left[\frac{27}{n}\cdot n - \frac{27}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[27 - \frac{27}{n^2}\frac{(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty}\left[27 - \frac{27(2n^2+3n+1)}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left[27 - \frac{54n^2+81n+27}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left[27 - \frac{54n^2}{6n^2} - \frac{81n}{6n^2} - \frac{27}{6n^2}\right]$$

$$= \lim_{n\to\infty}\left[27 - \frac{54}{6} - \frac{81}{6n} - \frac{27}{6n^2}\right]$$

$$= \left(27 - \frac{54}{6} - \frac{81}{6(\infty)} - \frac{27}{6(\infty)}\right)$$

$$= 27 - 9 - 0 - 0$$

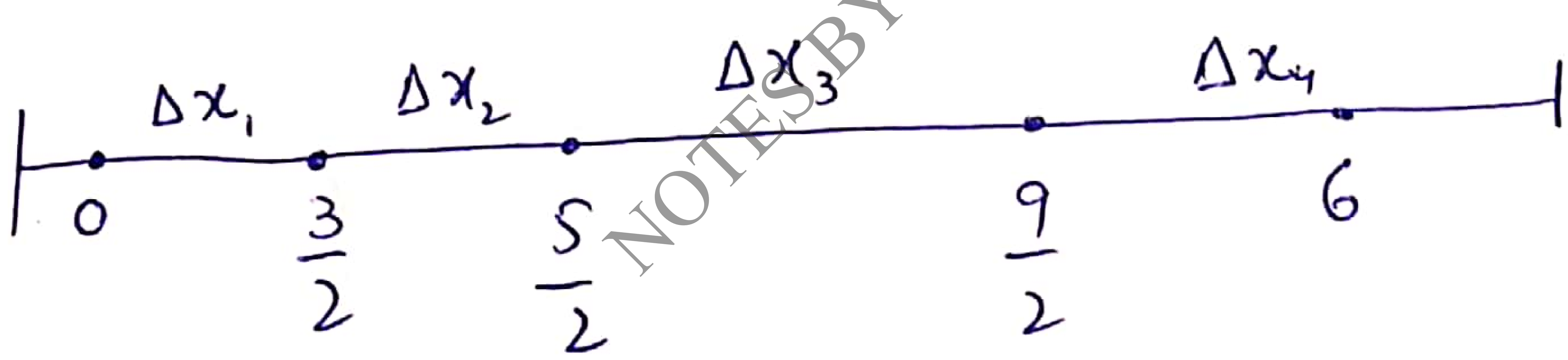
$$= \boxed{18}$$ Ans.

Lecture 29 The Definite Integral

As the formula for width is $\Delta x = \frac{b-a}{n}$

If the width between subintervals is not equal then this formula changes.

If we want to find width of the largest subinterval, then it is called <u>mesh size</u> & represented by <u>$\max \Delta x_k$</u> Read as <u>"maximum of the Δx_k"</u>. For example



$$\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = \frac{4}{2} = 2$$

Theorem:- Area under curve

If function f is continuous on $[a,b]$ & if $f(x) \geq 0$ for all x in $[a,b]$, then area under curve over $[a,b]$ is $A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$

can also be written as $\int_a^b f(x)\,dx$

$\sum_{k=1}^{n} f(x_k^*)\Delta x_k$ is called Riemann Sum.

$\int_a^b f(x)\,dx$ is called definite integral of f from a to b & b, a are called upper & lower limits respectively.

Example:- Evaluate $\int_2^4 (x-1)\,dx$

This represents area under curve $y = x-1$ over $[2, 4]$.

$$\int_2^4 x\,dx - \int_2^4 1\,dx$$

$$= \left|\frac{x^2}{2}\right|_2^4 - \left|x\right|_2^4 + c$$

$$= \left(\frac{4^2}{2} - \frac{2^2}{2}\right) - (4-2)$$

$$= \left(\frac{16}{2} - \frac{4}{2}\right) - 2$$

$$= (8 - 2) - 2$$

$$= 6 - 2$$

$$= 4 \quad \text{Ans.}$$

Example:- $\int_2^4 (1-x)\,dx$

$$= \int_2^4 1\,dx - \int_2^4 x\,dx$$

$$= \left| x \right|_2^4 - \left| \frac{x^2}{2} \right|_2^4$$

$$= (4-2) - \left(\frac{4^2}{2} - \frac{2^2}{2} \right)$$

$$= 2 - \frac{16}{2} + \frac{4}{2}$$

$$= 2 - 8 + 2$$

$$= -4 \quad \text{Ans.}$$

-ve sign just shows that area below x-axis is larger than above.

Defination:-

- If a is in the domain of f.

$$\int_a^a f(x)\,dx = 0$$

- If f is integrable on $[a, b]$, then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

⇒ Properties of definite integral

- $\int_a^b c\, f(x)\, dx = c \int_a^b f(x)\, dx$
- $\int_a^b [f(x) + g(x)]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$
- $\int_a^b [f(x) - g(x)]\, dx = \int_a^b f(x)\, dx - \int_a^b g(x)\, dx$

Theorem:- If f is integrable on a closed interval containing three points a, b & c then

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

Example:- $\int_1^5 f(x)\, dx = -1$, $\int_3^5 f(x)\, dx = 3$

$\int_3^5 g(x)\, dx = 4$, $\int_1^3 f(x)\, dx = ?$

$$\int_1^5 f(x)\, dx = \int_1^3 f(x)\, dx + \int_3^5 f(x)\, dx$$

$$-1 = \int_1^3 f(x)\, dx + 3$$

$$-4 = \int_1^3 f(x)\, dx$$

Theorem:-

If f is integrable on $[a, b]$ & $f(x) \geq 0$ for all x in $[a, b]$ Then $\int_a^b f(x) \geq 0$

If f & g are integrable functions on $[a, b]$ & $f(x) \geq g(x)$ for all x in $[a, b]$ Then

$$\int_a^b f(x) \geq \int_a^b g(x)$$

works with $>, <, \leq$ also.

Example:- Show that $\int_0^1 \frac{\cos x}{2x^3 - 5} dx$ is negative.

On interval $[0, 1]$, $\cos x > 0$ but $2x^3 - 5 < 0$, So $f(x)$ as a whole is < 0

So integral is negative.

Defination:- Function is said to be bounded on $[a, b]$ if there is a +ve number M such that

$$-M \leq f(x) \leq M$$

so $y = -M$ & $y = M$

Theorem:-

Let f is defined at all points in $[a,b]$

- If f is continuous on $[a,b]$ Then it is also integrable on $[a,b]$.
- If f is bounded on $[a,b]$ & has only finite many points of discontinuity on $[a,b]$ Then f is integrable on $[a,b]$.
- If f is not bounded on $[a,b]$ Then f is not integrable on $[a,b]$.

Lecture 30

First Fundamental Theorem of Calculus.

Theorem:- If f is continuous on $[a,b]$ & if F is an antiderivative of f on $[a,b]$, Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

$$F(b) - F(a) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$$

$$= \int_a^b f(x)\,dx$$

or

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b$$

Example:- Evaluate $\int_1^2 x\,dx$

$$= \frac{x^2}{2}\Big]_1^2 \Rightarrow \frac{(2)^2}{2} - \frac{(1)^2}{2}$$

$$= \frac{4}{2} - \frac{1}{2} \Rightarrow \frac{4-1}{2}$$

$$= \frac{3}{2} \quad \text{Ans.}$$

Properties

$$\left[c\,F(x)\right]_a^b = c\left[F(x)\right]_a^b$$

$$\left[F(x)+G(x)\right]_a^b = \left[F(x)\right]_a^b + \left[G(x)\right]_a^b$$

$$\left[F(x)-G(x)\right]_a^b = \left[F(x)\right]_a^b - \left[G(x)\right]_a^b$$

As

$$\int f(x)\,dx = F(x)+c$$

$$\int_a^b f(x)\,dx = \left[\int f(x)\,dx\right]_a^b$$

Example:- Find area under curve $y = \cos x$ over $[0, \pi]$.

since $\cos x \geq 0$ for $0 \leq x \leq \frac{\pi}{2}$

$$A = \int_0^{\pi/2} \cos x\,dx$$

$$= \left|\sin x\right|_0^{\pi/2} \Rightarrow \sin\left(\frac{\pi}{2}\right) - \sin(0)$$

$$= 1 - 0 \Rightarrow 1 \quad \text{Ans.}$$

Example:- Evaluate $\int_0^3 (x^3 - 4x + 1)\,dx$

$$= \int_0^3 x^3\,dx - \int_0^3 4x\,dx + \int_0^3 1\,dx$$

$$= \left|\frac{x^4}{4}\right|_0^3 - \left|\frac{4x^2}{2}\right|_0^3 + |x|_0^3$$

or $$\left|\frac{x^4}{4} - \frac{4x^2}{2} + x\right|_0^3$$

$$= \left(\frac{(3)^4}{4} - 2(3)^2 + 3\right) - \left(\frac{0^4}{4} - \frac{4(0)}{2} + 0\right)$$

$$= \frac{81}{4} - 18 + 3 \Rightarrow \frac{21}{4}$$ Ans.

Example:- Evaluate $\int_0^6 f(x)\,dx$ if

$$f(x) = \begin{cases} x^2 & x < 2 \\ 3x-2 & x \geq 2 \end{cases}$$

$$\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx$$

$$= \int_0^2 x^2\,dx + \int_2^6 (3x-2)\,dx$$

$$= \left(\frac{x^3}{3}\right]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^6$$

$$= \left(\frac{2^3}{3} - \frac{0^3}{3}\right) + \left(\frac{3}{2}(6^2 - 2^2) - 2(6-2)\right)$$

$$= \frac{8}{3} + \frac{3}{2}(32) - 2(4)$$

$$= \frac{8}{3} + 48 - 6 \Rightarrow \frac{8}{3} + 42 \Rightarrow \frac{128}{3}$$

Mean Value Theorem:-

$$\int_a^b f(x)dx = f(x^*)(b-a)$$

if f is continuous on a closed interval $[a,b]$, then there is at least one number x^* in $[a,b]$. &

Proof:-

$$m \leq f(x) \leq M$$

if f assumes a max M & a min m on $[a,b]$

$$m \leq f(x) \leq M$$

$$\int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx$$

$$m\,|x|_a^b \leq \int_a^b f(x)\,dx \leq M(x)\Big|_a^b$$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

$$m \leq \int_a^b f(x)\,dx \cdot \frac{1}{b-a} \leq M$$

Central value is a number b/w m & M. So by intermediate Theorem assume $\frac{1}{b-a}\int_a^b f(x)$ on $[a,b]$ for some

point $f(x^*)$

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(x^*)$$

$$\int_a^b f(x)\,dx = f(x^*)(b-a)$$

Example:- $f(x)=x^2$ is continuous on $[1,4]$ The MVT for integral guarantees That There exist a number x^*

Such That

$$\int_1^4 x^2\,dx = f(x^*)(4-1)$$

$$\left.\frac{x^3}{3}\right|_1^4 = 3(x^*)^2$$

$$\frac{4^3}{3} - \frac{1^3}{3} = 3(x^*)^2$$

$$\frac{64-1}{3} = 3(x^*)^2$$

$$\frac{\cancel{63}^{\,\cancel{21}\,7}}{3\times3} = x^{*2}$$

$$\sqrt{7} = x^*$$

Average Value Theorem:-

If f is integrable on $[a, b]$ Then The average value (or mean value) of f on $[a,b]$ is defined to be

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\,dx$$

if $y = f(x)$ Then f_{avg} is also called average value of y w.r.t x over $[a,b]$.