MTH644
Measure Theory

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Topic # 47

Theorem: Interval (a,∞) is measurable $\forall\ a\in\mathbb{R}$.

Proof:

Taking test set say $A\subseteq\mathbb{R}$ and we only need to prove:

$\because\ m^*(A) = m^*(A\cap U) + m^*(A\cap U')$

$\because\ m^*(a,b) = \ell(a,b) = b-a$

$$m^*(A) \geq m^*[A\cap(a,\infty)] + m^*[A\cap(-\infty,a]] \longrightarrow ①$$

let $A_\alpha = A\cap(a,\infty)$ and $A_\beta = A\cap(-\infty,a]$

$\therefore\ ① \Rightarrow\ m^*(A) \geq m^*(A_\alpha) + m^*(A_\beta) \longrightarrow ②$

Case I: $m^*(A) = +\infty$, then ② is obviously.

case II: if $m^*(A) < \infty$ and for $A\subseteq\mathbb{R}$ $\exists$ a seq. $\{I_n'\}_{n=1}^{\infty}$ such that $A\subseteq\bigcup_{n=1}^{\infty} I_n' \longrightarrow Ⓐ$

$\forall\ \varepsilon>0$, Then there exists another such sequence $\{I_n\}$ such that

$Ⓐ \Rightarrow\ m^*(A) \leq m^*(\bigcup_{n=1}^{\infty} I_n^*)$

$A\subseteq\bigcup_{n=1}^{\infty} I_n$ and $m^*(A)+\varepsilon > m^*(\bigcup_{n=1}^{\infty} I_n') \longrightarrow ⓐ$

$\because$ on $\mathbb{R}$; $m^*(I_n) = \ell(I_n)$

$$\therefore m^*\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} l(I_n) \longrightarrow (b)$$ $\rightarrow$ not disjoint

$$(a) \Rightarrow \sum_{n=1}^{\infty} l(I_n) < m^*(A) + \varepsilon \longrightarrow (3)$$

Let $M_n = I_n \cap (a,\infty)$ and $N_n = I_n \cap (-\infty, a]$

$$M_n \cap N_n = \{I_n \cap (a,\infty)\} \cap \{I_n \cap (-\infty, a]\}$$
$$= I_n \cap \{(a,\infty) \cap (-\infty, a]\}$$
$$M_n \cap N_n = I_n \cap \phi = \phi$$

Now $M_n \cup N_n = \{I_n \cap (a,\infty)\} \cup \{I_n \cap (-\infty, a]\}$
$$= I_n \cap \{(a,\infty) \cup (-\infty, a]\}$$
$$= I_n \cap \mathbb{R}$$
$$M_n \cup N_n = I_n$$
$$\Rightarrow l(I_n) = l(M_n) + l(N_n) \longrightarrow (4)$$

$\because$ $M_n \cap N_n = \phi$ $\forall n \in \mathbb{N}$

$\because A_\alpha = A \cap (a,\infty)$ and $A \subseteq \bigcup_{n=1}^{\infty} I_n$

$$\Rightarrow A_\alpha \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right) \cap (a,\infty)$$
$$\Rightarrow A_\alpha \subseteq (I_1 \cup I_2 \cup \cdots I_n) \cap (a,\infty)$$
$$\Rightarrow A_\alpha \subseteq \{I_1 \cap (a,\infty)\} \cup \{I_2 \cap (a,\infty)\} \cup \cdots$$
$$\Rightarrow A_\alpha \subseteq \bigcup_{n=1}^{\infty} \{I_n \cap (a,\infty)\}$$
$$\Rightarrow A_\alpha \subseteq \bigcup_{n=1}^{\infty} M_n$$
$$\Rightarrow m(A_\alpha) \le m^*\left(\bigcup_{n=1}^{\infty} M_n\right) \longrightarrow (c)$$

Similarly $A_\beta \subseteq \bigcup_{n=1}^{\infty} N_n$

$$\Rightarrow m(A_\beta) \le m^*\left(\bigcup_{n=1}^{\infty} N_n\right) \longrightarrow (d)$$

Adding (c) and (d)

$$m(A_\alpha) + m(A_\beta) \le m^*\left(\bigcup_{n=1}^{\infty} M_n\right) + m^*\left(\bigcup_{n=1}^{\infty} N_n\right)$$
$$\le \sum_{n=1}^{\infty} m^*(M_n) + \sum_{n=1}^{\infty} m^*(N_n)$$

$$m^*(A_\alpha) + m^*(A_\beta) \leq \sum_{n=1}^{\infty} l(M_n) + \sum_{n=1}^{\infty} l(N_n)$$
$$\leq \sum_{n=1}^{\infty} (l(M_n) + l(N_n))$$

$$m^*(A_\alpha) + m^*(A_\beta) \leq \sum_{n=1}^{\infty} l(I_n) \quad \text{From } (4)$$

$$\Rightarrow m^*(A_\alpha) + m^*(A_\beta) < m^*(A) + \varepsilon \quad \text{From } (3)$$

$\because$ $\varepsilon \to 0$ arbitrary ;

$\therefore$ $m^*(A_\alpha) + m^*(A_\beta) \leq m^*(A)$

or $m^*(A) \geq m^*(A_\alpha) + m^*(A_\beta)$

Hence proved ; (a, ∞) is measurable.

Topic # 48

Theorem: Open interval $(a,b) \subseteq \mathbb{R}$ is measurable.

proof: $\because$ (a,∞) is measurable

$\Rightarrow$ $(a,\infty)^c = (-\infty, a]$ is also measurable.

And $(-\infty, b-\frac{1}{n}]$ is also measurable $\forall$ $n \in \mathbb{N}$

and $(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b-\frac{1}{n}]$ is also measurable as $n \to \infty$

beacause being countable union of measurable set

$\Rightarrow (a,\infty) \cap (-\infty, b) = (a,b)$ i.e also measurable.

being intersection of two measurable set

Open set in $\mathbb{R}$:

Any open set in $\mathbb{R}$ say u can be written as countable union of disjoint open intervals

i.e $u = \bigcup_{n=1}^{\infty} \{I_n\}$ and each open intervals is measurable and countable union of measurable set is measurable.

Topic # 49

Theorem: A subset $\phi \neq E \subseteq \mathbb{R}$ is measurable $\Leftrightarrow$ for given $\varepsilon > 0$, $\exists$ an open set $U \supseteq E$ with $m^*(U-E) < \varepsilon$.

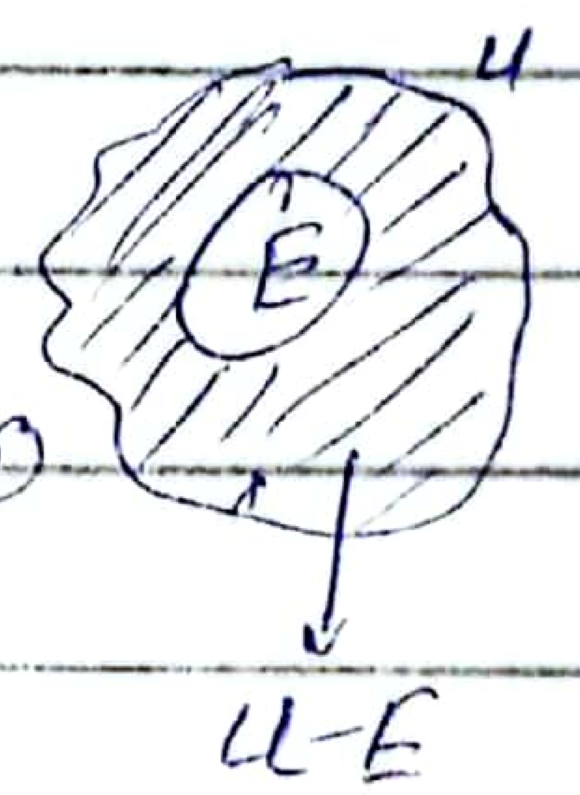
proof: Suppose E is measurable

Case I: If $m^*(E) < \infty$, $\forall \phi \neq E \subseteq \mathbb{R}$, there exists an open set U such that $E \subseteq U$ and for any $\varepsilon > 0$, $m^*(U) < m^*(E) + \varepsilon \longrightarrow (a)$

$E \subseteq U \Rightarrow U = E \cup (U-E)$

$\Rightarrow m^*(U) = m^*(E) + m^*(U-E)$

$\Rightarrow m^*(U-E) = m^*(U) - m^*(E) < \varepsilon$. from (a)



Case II: If $m^*(E) = \infty$

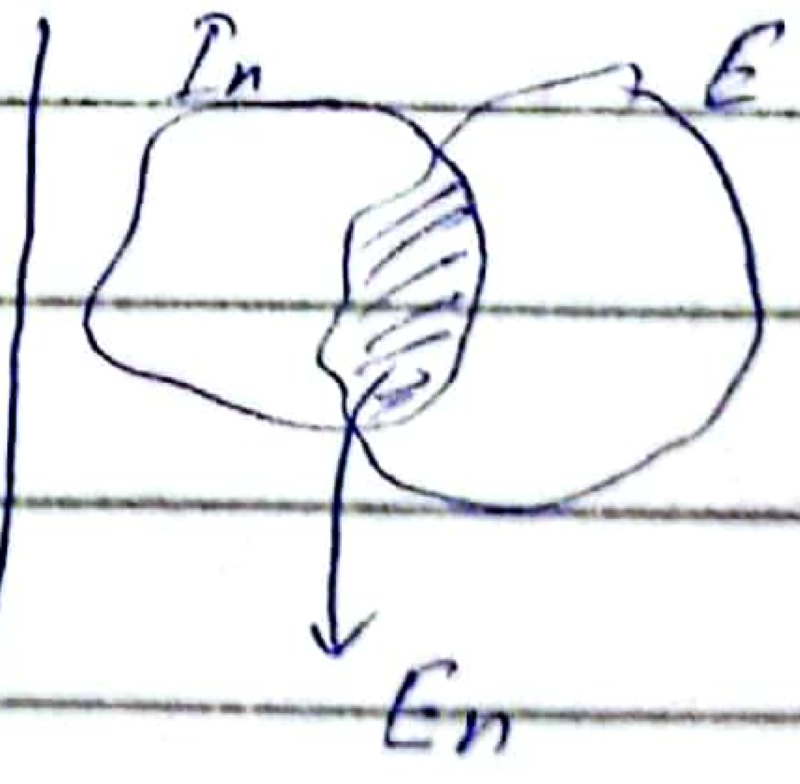
$\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$, where $l(I_n) < \infty \ \forall n \in \mathbb{N}$

and $I_j \cap I_k = \phi$, $j \neq k$

put $E_n = E \cap I_n \Rightarrow E_n \subseteq I_n \ \forall n \in \mathbb{N}$

$\Rightarrow m^*(E_n) \leq m^*(I_n) = l(I_n) < \infty$

By case I $\forall E_n$, $\exists$ an open set say "V_n" such that $E_n \subseteq V_n$, for $\varepsilon > 0$ and $m^*(V_n - E_n) < \frac{\varepsilon}{2^n} \longrightarrow (b)$

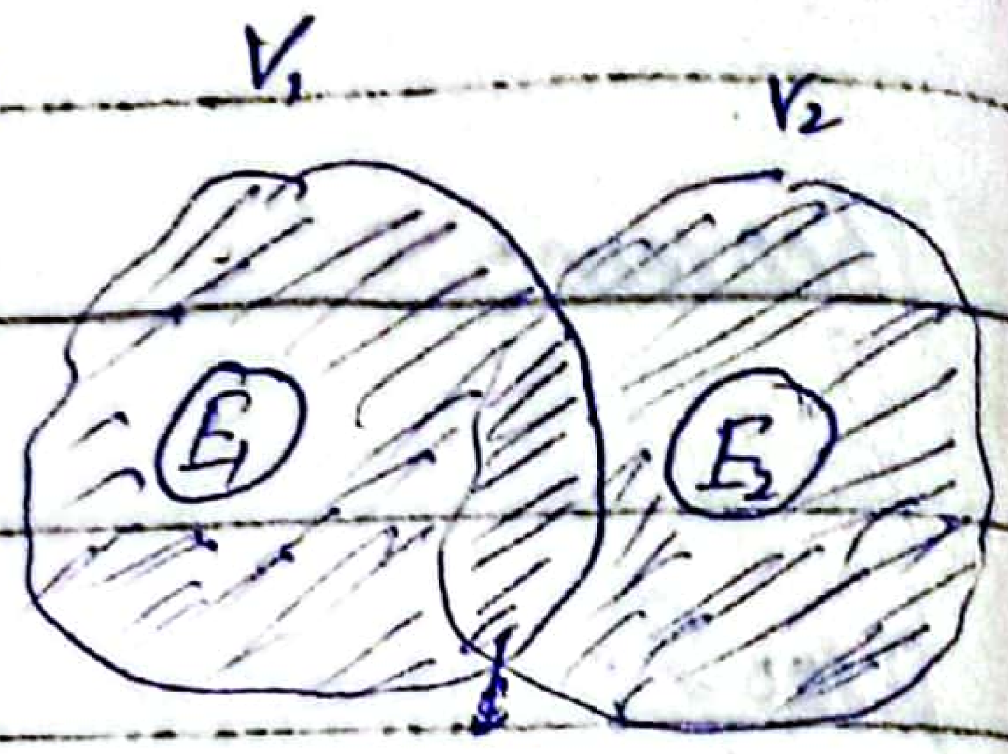


Also $V = \bigcup_{n=1}^{\infty} V_n$, an open set.

$\Rightarrow \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap I_n) = E \cap \left(\bigcup_{n=1}^{\infty} I_n\right) = E \cap \mathbb{R} = E$

$$E = \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} V_n = V \qquad \because E_n \subseteq V_n \ \forall n \in \mathbb{N}$$

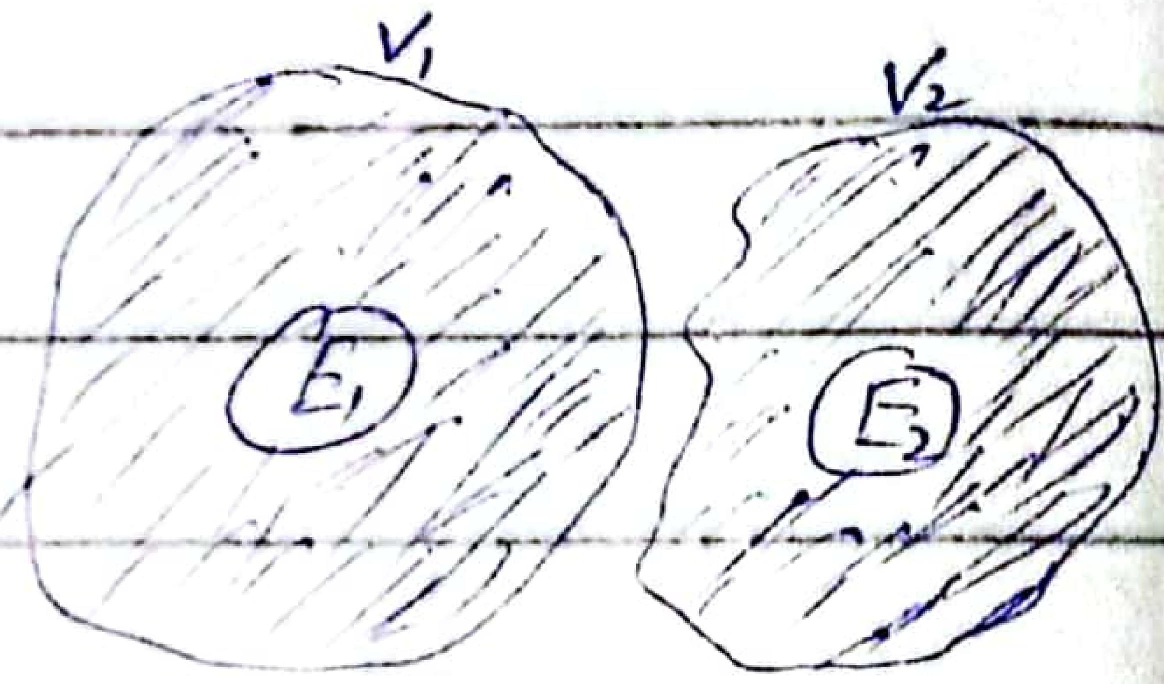
$$\Rightarrow V - E = \bigcup_{n=1}^{\infty} V_n - \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} (V_n - E_n)$$



$$\Rightarrow m^*(V-E) \leq \sum_{n=1}^{\infty} m^*(V_n - E_n)$$

From ⓑ $\Rightarrow m^*(V-E) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$

$(V_1 \cup V_2) = (E_1 \cup E_2)$



$$m^*(V-E) \leq \varepsilon\left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) = \varepsilon(1)$$

$$m^*(V-E) \leq \varepsilon$$

$(V_1 - E_1) \cup (V_2 - E_2)$

Conversely, let $\varepsilon = \frac{1}{n}$ (say) and

if $\exists$ an open set $O_n \supseteq E$ such that $E \subseteq O_n \ \forall n \in \mathbb{N}$

and $m^*(O_n - E) < \varepsilon = \frac{1}{n}$ $\longrightarrow$ Ⓐ

We will prove "E" is measurable.

put $G = \bigcap_{n=1}^{\infty} O_n \Rightarrow$ "G" is measurable being countable intersection of measurable open set.

$E \subseteq O_n \quad \forall n$ (given)

and $E \subseteq \bigcap_{n=1}^{\infty} O_n = G$

$$G - E \subseteq O_n - E$$

$\Rightarrow m^*(G-E) \leq m^*(O_n - E) < \frac{1}{n} = \varepsilon$

$\because \varepsilon \to 0$ arbitrary $m^*(G-E) = 0$.

$\Rightarrow G-E$ is measurable $\Rightarrow (G-E)'$ is also measurable.

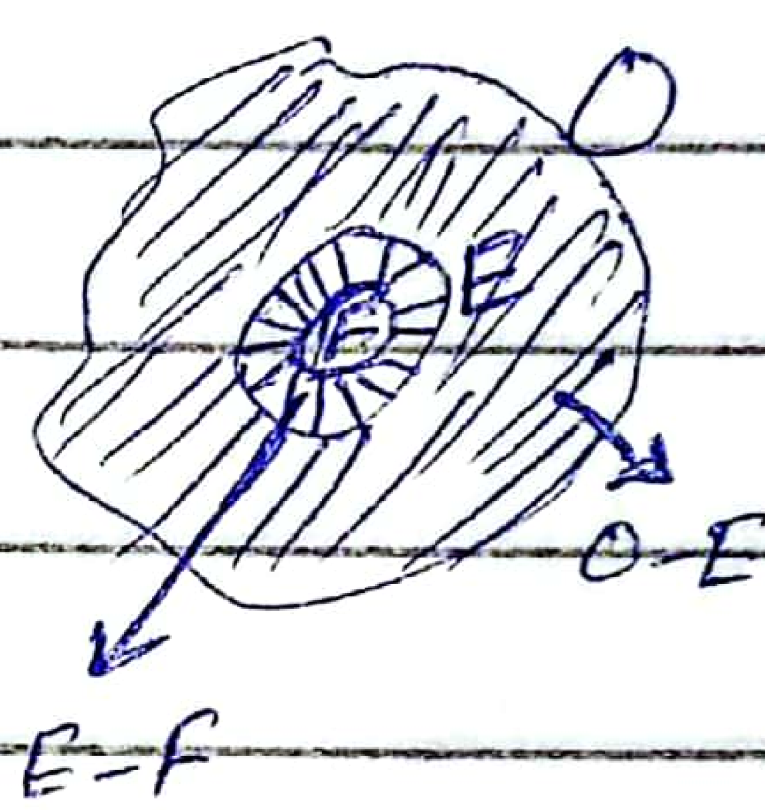
$\because E = G-(G-E) = G \cap (G-E)'$ is measurable being intersection of two measurable set.

Topic # 50

Theorem: For a subset "E" of $\mathbb{R}$ and given $\varepsilon > 0$ $\exists$ an open set $O \supseteq E$ and $m^*(O-E) < \varepsilon \Leftrightarrow \exists$ a closed set $F \subseteq E$ and $m^*(E-F) < \varepsilon$.

proof: For $\phi \neq E \subseteq \mathbb{R} \Rightarrow E' \subset \mathbb{R}$ $\exists$ for given $\varepsilon > 0$ $\exists$ an open set $O \supseteq E$ and

$m^*(O-E) < \varepsilon \longrightarrow (1)$



$O \supseteq E' \Rightarrow E'' \supseteq O' = F$ (say)

i.e. closed set

$\Rightarrow E \supseteq F \Rightarrow m^*(E-F) = m^*(E \cap F')$

$m^*(E-F) = m^*(F' \cap E)$

$m^*(E-F) = m^*(O-E') < \varepsilon$ from (1)

$\Rightarrow m^*(E-F) < \varepsilon$

Conversely for the set $E' \subseteq \mathbb{R}$, $\exists$ a closed set "F" and $\varepsilon > 0$ such that $F \subseteq E'$ and $m^*(E'-F) < \varepsilon \rightarrow (2)$ (given)

$F \subseteq E' \Rightarrow E = E'' \subseteq F' = O$ (open set) say.

$\Rightarrow m^*(O-E) = m^*(O \cap E') = m^*(E' \cap O)$

$\Rightarrow m^*(O-E) = m^*(E'-O') = m^*(E'-F) < \varepsilon$ from (2)

$m^*(O-E) < \varepsilon$ (Hence proved)

Topic #51

Theorem: Let $\phi \neq E \subseteq \mathbb{R}$, then the following statements are equivalent

(i) There is a G_δ set "G" with $E \subseteq G$, $m^*(G-E)=0$ → give

(ii) There is a F_σ set "F" with $F \subseteq E$, $m^*(E-F)=0$ → prove

proof: (i) ⇒ (ii) $\because \phi \neq E \subseteq \mathbb{R} \Rightarrow \phi \neq E' \subseteq \mathbb{R}$

$\therefore$ For $\phi \neq E'$ given $E' \subseteq G$, $m^*(G-E')=0$

$\therefore \quad 0 = m^*(G-E') = m^*(G \cap (E')') = m^*(G \cap E)$

$= m^*(E \cap G) = m^*(E \cap (G')') = m^*(E - G')$

$= m^*(E-F)$, where $G_\delta = G = \bigcup_{n=1}^{\infty} G_n$

$\Rightarrow G'_\delta = G' = \left(\bigcup_{n=1}^{\infty} G_n\right)'$

$\Rightarrow G'_\delta = \bigcap_{n=1}^{\infty} G'_n$

$\Rightarrow G'_\delta = \bigcap_{n=1}^{\infty} F'_n = F = F_\sigma$

(ii) ⇒ (i)

Conversely, again for $E' \subseteq \mathbb{R}$, with $F \subseteq E'$

$m^*(E'-F)=0$ (given).

$\therefore \quad 0 = m^*(E'-F) = m^*(E' \cap F') = m^*(F' \cap E')$

$= m^*(F'-E) = m^*(G-E)$

where $F'_\sigma = F' = \left(\bigcap_{k=1}^{\infty} F_k\right)' = \bigcup_{k=1}^{\infty} F'_k = \bigcup_{k=1}^{\infty} G_n = G = G_\delta$

(proved)

Topic #52

Lebesgue Measure:

Let "$\mathcal{M}$" be the class of measurable sets, which is σ-algebra, Then the Lebesgue measure of any set $E \subseteq \mathcal{M}$ is defined to be its Lebesgue outer measure.

$m^* \equiv m;\ \{A,\ m^*(A) = m^*(A \cap E) + m^*(A \cap E')$

Topic # 53

Theorem: let $\{E_i\}_{i=1}^{n}$ be a sequence a in measurable space "$\mathcal{M}$", then m is

(i) countably subadditive

(ii) finitely additive provided that $E_i \cap E_j = \phi,\ \forall\ i \neq j$.

(iii) countably additive if E_i's are pairwise disjoint.

(iv) "m" is monotonic.

proof: (i) All E_i's are measurable.

$\Rightarrow \bigcup_{i=1}^{n} E_i$ is also measurable, $\because$ "$\mathcal{M}$" is space of lebesgue measurable set;

$$\Rightarrow m\left(\bigcup_{i=1}^{n} E_i\right) = m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) = \sum_{i=1}^{\infty} m(E_i)$$

(ii) $\forall\ \phi \neq A \subseteq \mathbb{R}$ and for the sequence $\{E_i\}_{n=1}^{n}$ of pairwise disjoint measurable sets;

$$m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

put $A = \mathbb{R} \Rightarrow A\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} E_i$ $\quad A \cap E_i = \mathbb{R} \cap E_i$

$$\Rightarrow m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$$

(iii) $m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i)$

$\Rightarrow m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i)$ $\quad$ finitly additive.

iv. Monotonicity ; let $\phi \neq A, B \in \mathcal{M}$, such that $B \subseteq A$.

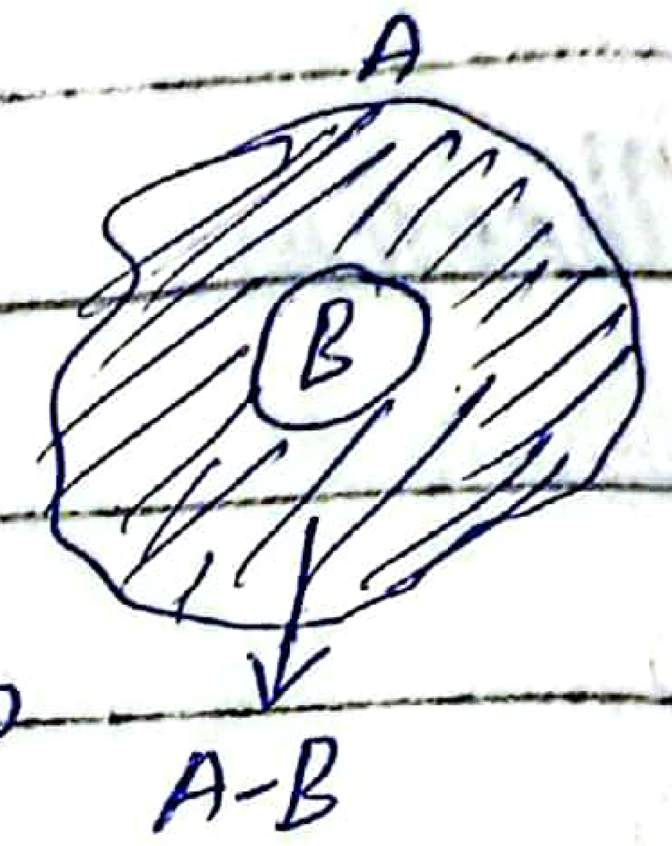
$\Rightarrow A = B \cup (A - B)$.

$\because m^*$ is finitily additive.

$\Rightarrow m^*(A) = m^*(B) + m^*(A-B)$

$\Rightarrow m^*(A) \geq m^*(B)$ $\because m^*(A-B) \geq 0$

$\Rightarrow m^*(B) \leq m^*(A)$



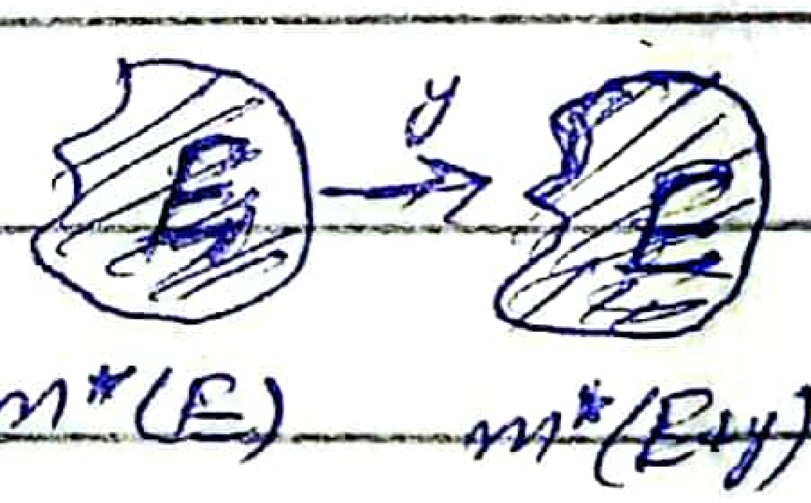
$\because B \cap (A-B) \neq \phi$

Topic # 54

Theorem: Lebesgue measure "m" is translation invariant.

proof: $\because$ Given that say $\phi \neq E \in \mathcal{M}$ is measurable.

$\because$ "E" is measurable $\Leftrightarrow$ for given $\varepsilon > 0$, $\exists$ an open set $U \supseteq E$, such that $m^*(U-E) < \varepsilon$

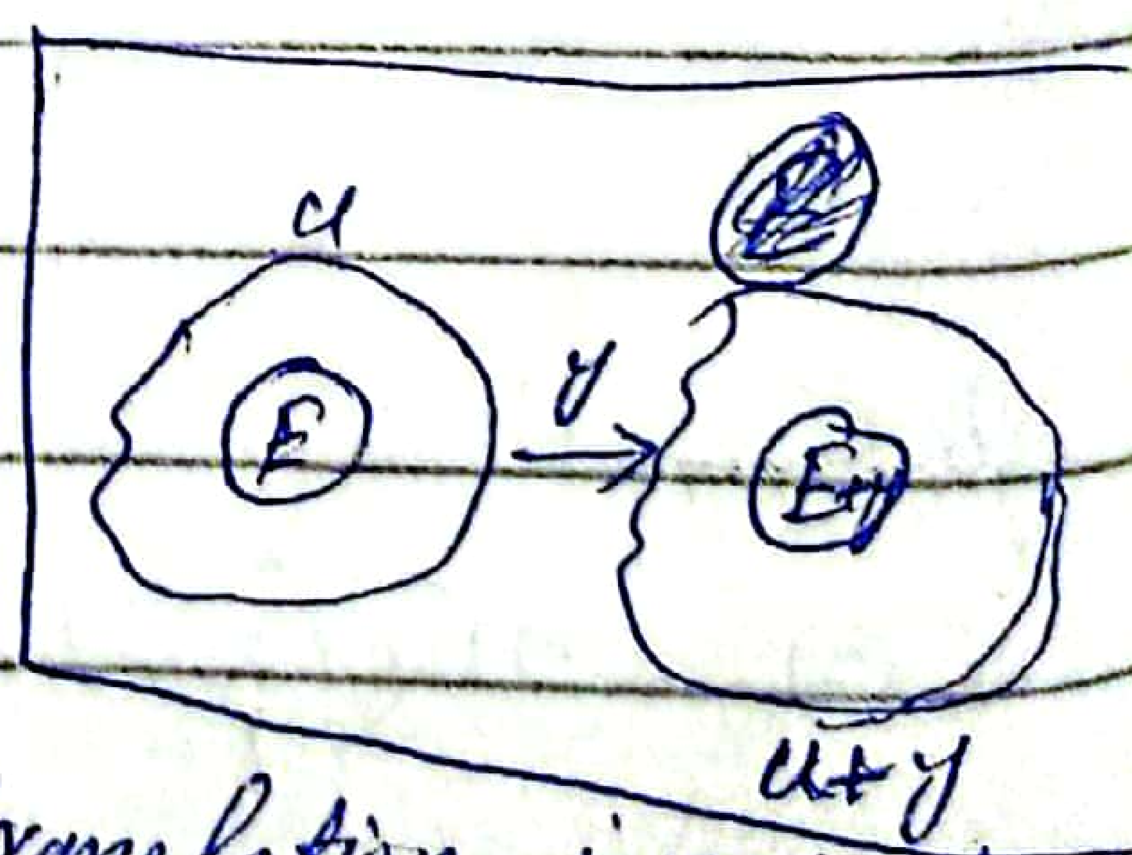


"U" is open set in $\mathbb{R} \Rightarrow$ "U+y" is also open in $\mathbb{R}$, $\forall y \in \mathbb{R}$

if $E \subseteq U \Rightarrow E + y \subseteq U + y$

$\Rightarrow (U+y) - (E+y) = (U-E) + y$

$\Rightarrow m^*[(U+y) - (E+y)] = m^*[(U-E) + y]$



$\because$ Lebesgue outer measure is translation invariant.

$= m^*(U-E) < \varepsilon$

But $m^* = m \Rightarrow m[(U+y) - (E+y)] = m(U-E) < \varepsilon$

$\Rightarrow$ "E+y" is also measurable $\Rightarrow$ "m" is translation invariant.

Topic # 55

Theorem: Suppose E_1 and E_2 are two measurable sets, then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

proof: If $m(E_1) = m(E_2) = \infty$, then ther result is obvious.

Case I: If $m(E_1) < \infty$, $m(E_2) < \infty$

$\because$ if E_1 is measurable, then for any $\phi \neq A \subseteq R$

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1') \longrightarrow ①$$

for particularly if $A = E_2$

$$① \Rightarrow m^*(E_2) = m^*(E_2 \cap E_1) + m^*(E_2 \cap E_1') \longrightarrow ②$$

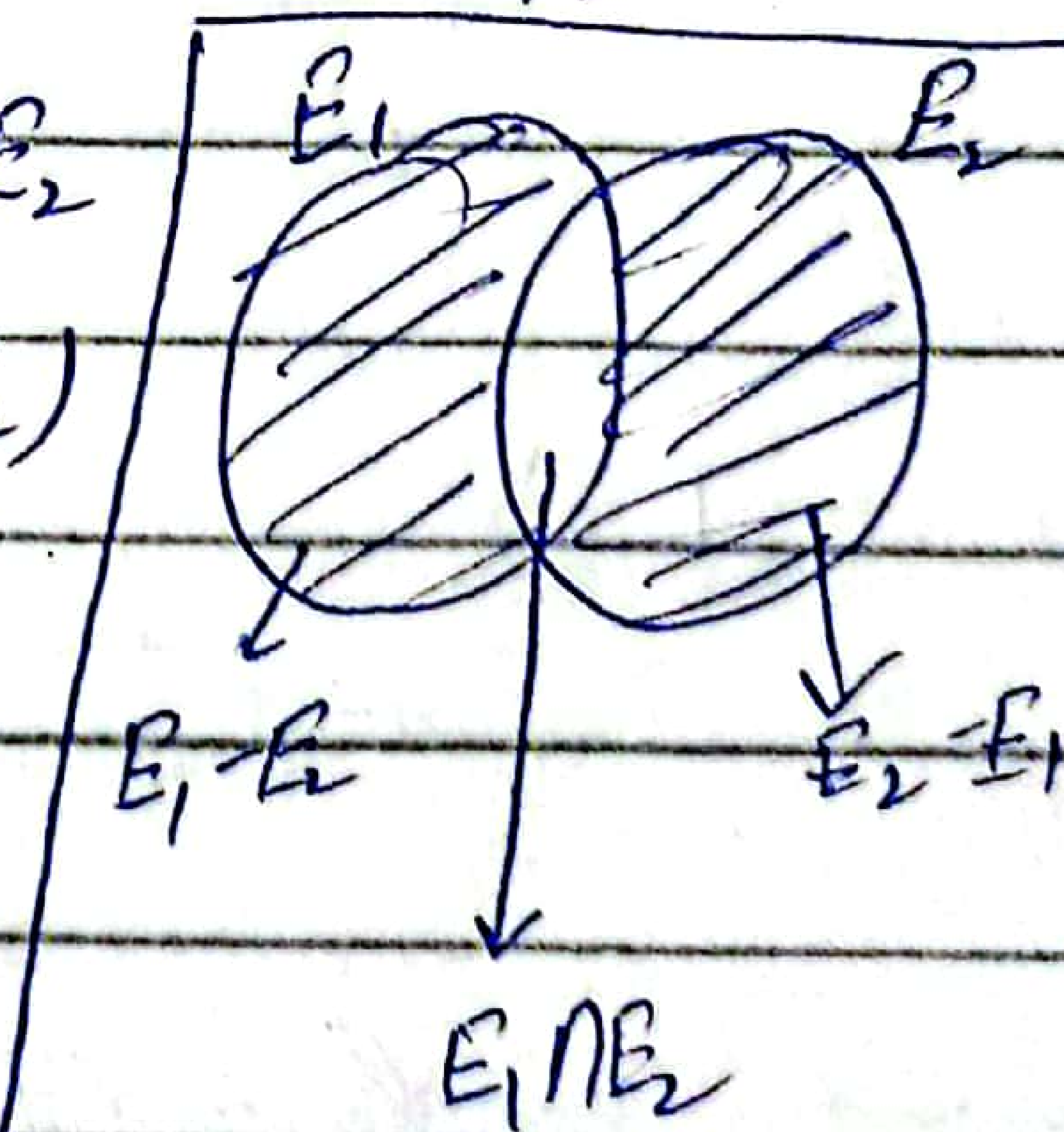
Similarly $E_2 \longleftrightarrow E_1$

$$m^*(E_1) = m^*(E_1 \cap E_2) + m^*(E_1 \cap E_2') \longrightarrow ③$$

Adding ② and ③ $\qquad (m^* = m)$

$$m(E_1) + m(E_2) = m(E_1 \cap E_2) + m(E_1 \cap E_2') + m^*(E_2 \cap E_1) + m(E_2 \cap E_1') \longrightarrow Ⓐ$$

$$E_1 \cup E_2 = (E_1 - E_2) \cup (E_2 - E_1) \cup E_1 \cap E_2$$
$$= (E_1 \cap E_2') \cup (E_2 \cap E_1') \cup (E_1 \cap E_2)$$



$$\Rightarrow m(E_1 \cup E_2) = m[(E_1 \cap E_2') \cup (E_2 \cap E_1') \cup (E_1 \cap E_2)]$$

"m" is finitely addative:

$$m(E_1 \cup E_2) = m(E_1 \cap E_2') + m(E_2 \cap E_1') + m(E_1 \cap E_2)$$

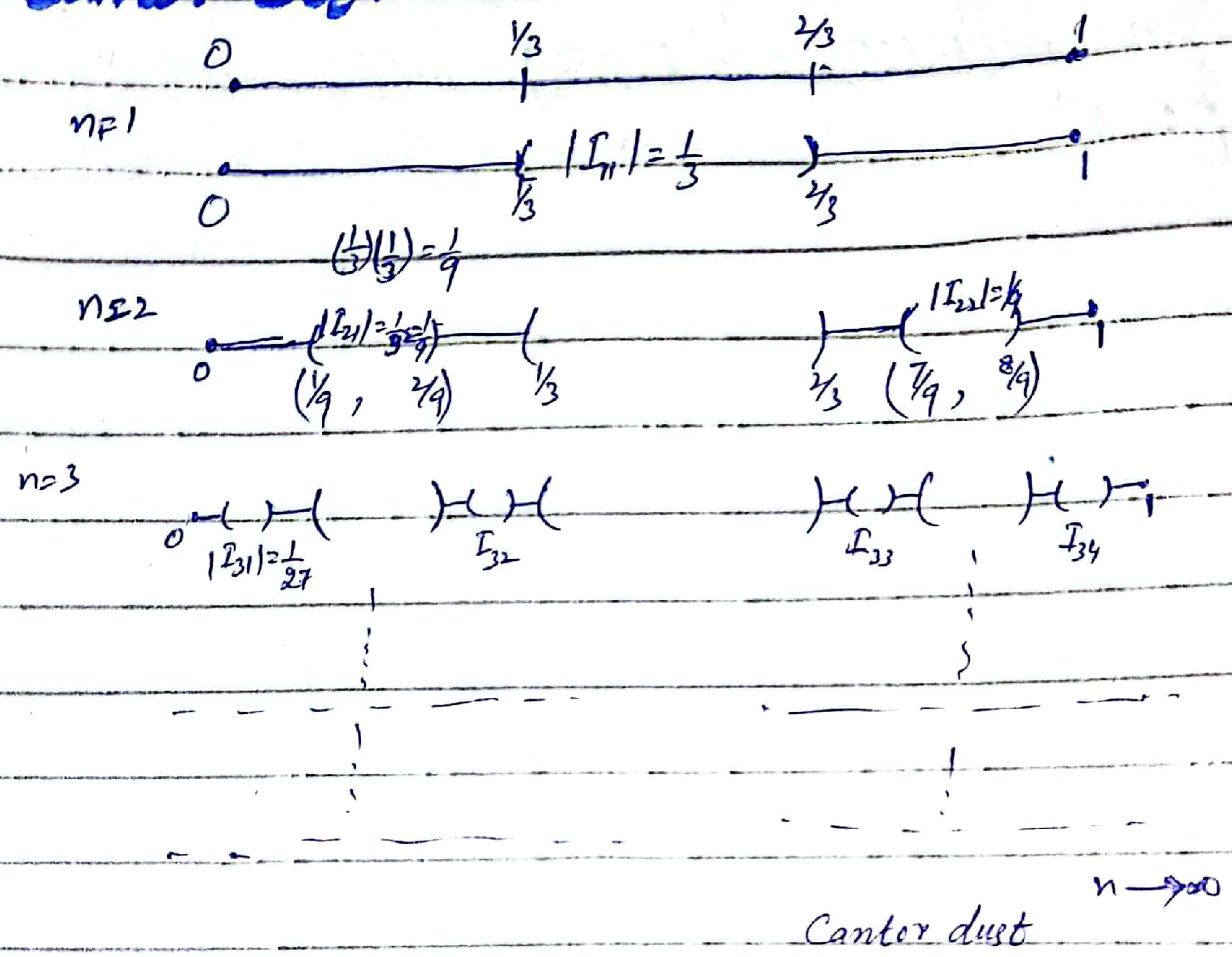
$$Ⓐ \Rightarrow m(E_1) + m(E_2) = m(E_1 \cup E_2) + m(E_1 \cap E_2)$$

proved

Topic # 56

Cantor Set:

$\mathbb{R} = \mathbb{R}^2 = \mathbb{R}^3$



# Step	# Dropped Interval	length	Interval
$n=1$	$1 = m = 2^{1-1}$	$l(I_{11}) = \frac{1}{3}$	$\left(\frac{1}{3}, \frac{2}{3}\right)$
$n=2$	$2 = m = 2^{2-1}$	$l(I_{21}) = l(I_{22}) = \frac{1}{3^2}$	$\left(\frac{1}{9}, \frac{2}{9}\right), \left(\frac{7}{9}, \frac{8}{9}\right)$
$n=3$	$4 = m = 2^{3-1}$	$l(I_{31}) = \text{---} = \frac{1}{3^3}$	
$\vdots$	$\vdots$	$\vdots$	
n	$m = 2^{n-1}$	$l(I_{n.}) = \text{---} = \frac{1}{3^n}$	

Dropped Interval $= G = \underbrace{I_{11}}_{n=1}, \underbrace{I_{21}, I_{22}}_{n=2}, \underbrace{I_{31}, I_{32}, I_{33}, I_{34}}_{n=3}, \text{---}$

$I_{n1}, I_{n2}, \text{------} I_{nm}, \text{---}$

$n \to \infty, \ m = 2^{n-1}$

$\therefore G = \{I_{11}\}, \{I_{21}, I_{22}\}, \{I_{31}, I_{32}, I_{33}, I_{34}\}, \text{------}$

$n \to \infty, \ m = 2^{n-1}$

$\Rightarrow \quad G = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}$

universal set.

$\therefore$ Cantor Set $= C = [0,1] - G = G'$

$l = 1$

Topic#57

Measure of Cantor Set:

$\because$ "G" is open, being countable union of disjoint open intervals $\Rightarrow$ "G" is measurable.

$\because C = G'$ = closed and is also measurable.

$$m(G) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right) \quad \because \text{"}m\text{" is countably additive}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} m(I_{nm}) = \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} l(I_{nm})$$

expanding firstly by "n"

$$= \sum_{m=1}^{2^{1-1}} l(I_{1m}) + \sum_{m=1}^{2^{2-1}} l(I_{2m}) + \sum_{m=1}^{2^{3-1}} l(I_{3m}) + \cdots$$

$$= \underbrace{l(I_{11})} + \underbrace{l(I_{21}) + l(I_{22})} + \underbrace{l(I_{31}) + l(I_{32}) + l(I_{33}) + l(I_{34})} + \cdots$$

$$= \frac{1}{3} + \left(\frac{1}{3^2} + \frac{1}{3^2}\right) + \left(\frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3}\right) + \cdots$$

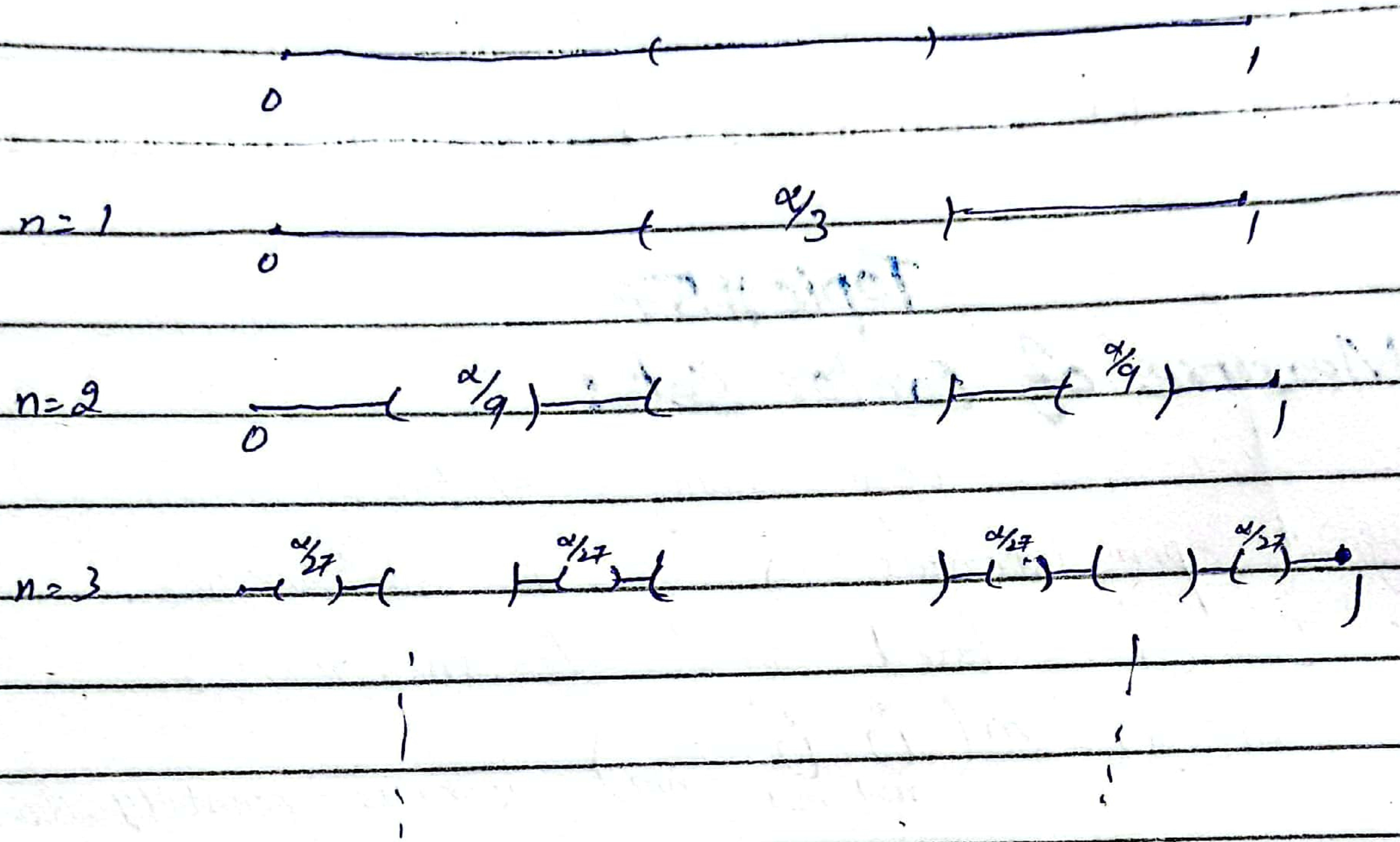
$$m(G) = \frac{1}{3} + \frac{2}{3^2} + \frac{4}{3^3} + \cdots \qquad \left\{ a = \frac{1}{3}, \ r = \frac{2}{3} \right.$$

$$S_\infty = \frac{a}{1-r} = \frac{1/3}{1-2/3} = 1$$

$\boxed{m(G) = 1}$ $\because [0,1] = G \cup C$ (disjoint) $\Rightarrow m[0,1] = m(G) + m(C) \Rightarrow \boxed{m(C) = 0}$

Topic # 58

Generalized Cantor Set and its Measure:



$$m(G) = \frac{\alpha}{3} + \left(\frac{\alpha}{9} + \frac{\alpha}{9}\right) + \left(\frac{\alpha}{27} + \frac{\alpha}{27} + \frac{\alpha}{27} + \frac{\alpha}{27}\right) + \cdots$$

$$m(G) = \frac{\alpha}{3} + \frac{2\alpha}{3^2} + \frac{4\alpha}{3^3} + \cdots$$

$$\begin{cases} a = \alpha/3 \,, \quad r = 2/3 \\ S_\infty = \dfrac{a}{1-r} = \dfrac{\alpha/3}{1-2/3} = \dfrac{\alpha/3}{1/3} = \alpha \end{cases}$$

$\boxed{m(G) = \alpha}$, $\because [0,1] = G \cup C$ disjoint

$\Rightarrow m[0,1] = m(G) + m(C)$

$1 = \alpha + m(C)$

$\Rightarrow \boxed{m(C) = 1 - \alpha}$

Topic # 59

Theorem: Let $\{E_n, \uparrow\}_{n=1}^{\infty}$ be an increasing sequence of measurable sets. Then

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$$

proof: $\because$ Given that $(E_n, \uparrow)$, such that

$\Rightarrow E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ of measurable sets

Defining; $F_1 = E_1$, $F_2 = E_2 - E_1$, $F_3 = E_3 - E_2$, ...

$F_n = E_n - E_{n-1}$, where $\{F_n\}_{n=1}^{\infty}$ is a seq. of disjoint measurable sets

$$\Rightarrow \bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n} E_k \Rightarrow \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k$$

disjoint + measurable.

$$\therefore m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} F_k\right)$$

$$= \sum_{k=1}^{\infty} m(F_k) = \lim_{n\to\infty} \sum_{k=1}^{n} m(F_k)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{n} F_k\right) = \lim_{n\to\infty} m\left(\bigcup_{k=1}^{n} E_k\right)$$

$$= \lim_{n\to\infty} m^*(E_n) \quad \text{(proved)}$$

Topic # 60

Theorem: Let $\{E_n, \downarrow\}_{n=1}^{\infty}$ be a decreasing sequence of measurable sets such that $m(E_k) < \infty$ for some $k \in \mathbb{N}$, Then

$$m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$$

proof: Let $m(E_1) < \infty$, Say (without lose of generality)

$\because \langle E_n, \downarrow \rangle \Rightarrow E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$

$\Rightarrow \phi = E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_2 \subseteq \cdots E_1 - E_n \subseteq \cdots$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty}(E_1 - E_n)\right) = \lim_{n\to\infty} m^*(E_1 - E_n)$$

$$\bigcup_{n=1}^{\infty}(E_1 - E_n) = \bigcup_{n=1}^{\infty}(E_1 \cap E_n')$$

$$= (E_1 \cap E_1') \cup (E_1 \cap E_2') \cup \cdots$$

$$= E_1 \cap [E_1' \cup E_2' \cup \cdots]$$

$$= E_1 \cap [E_1 \cap E_2 \cap \cdots]'$$

$$= E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n\right)'$$

$$= E_1 - \left(\bigcap_{n=1}^{\infty} E_n\right)$$

sets Relation

$$m^*\left(E_1 - \bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_1 - E_n)$$

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_1) - \lim_{n\to\infty} m^*(E_n)$$

$$\cancel{m^*(E_1)} - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \cancel{m^*(E_1)} - \lim_{n\to\infty} m^*(E_n)$$

$$\Rightarrow m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n) \quad \text{(proved)}$$

~~Lect~~ Topic # 61

Borel Set:

e.g. $G_\delta = \bigcap_{n=1}^{\infty} U_n$; $F_\sigma = \bigcup_{n=1}^{\infty} F_n$

Definition: A set "E" is said to be a Borel set if it can be expressed as countable union or intersection of open or closed sets.

e.g. $\{x\} = \bigcap_{n=1} \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ G_δ Borel sets.

Topic # 62

Borel Measure Space:

Restriction of Lebesgue measure "m" to the class of Borel set "B" is said to be Borel measure space. $(\mathbb{R}, B, m)$

Theorem: Every Borel set is measurable.

proof: Every open or closed set is measurable $\Rightarrow$ countable union or intersection of closed or open set is also measurable.

$\therefore$ Borel set is also measurable being union or intersection of closed or open sets.

Topic # 63

Theorem: Countable set are Borel sets with zero measure.

proof: Case I: Singleton:

$\forall\ a \in \mathbb{R},\ \{a\} \subseteq \bigcap_{n=1}^{\infty}\left(a-\frac{1}{n}, a+\frac{1}{n}\right)$
$; n \in \mathbb{N}$

$$\text{Countable} \begin{cases} \downarrow \\ \text{finite} \longleftrightarrow \text{infinite} \\ \mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{Q} \cdots \end{cases}$$

$\because \left\{\left\{a-\frac{1}{n}, a+\frac{1}{n}\right\}, \downarrow\right\}$ is decreasing sequence

with $m(E_n) = \frac{2}{n} < \infty$

where $E_n = \left(a-\frac{1}{n}, a+\frac{1}{n}\right)$

$$\Rightarrow m\{a\} = m\left\{\bigcap_{n=1}^{\infty} E_n\right\} = \lim_{n\to\infty} m(E_n)$$

$$= \lim_{n\to\infty} l(E_n)$$

$$= \lim_{n\to\infty}\left(\frac{2}{n}\right) = \frac{2}{\infty}$$

$$= 0$$

Finite Set

$$\{x_1, x_2, x_3, \cdots x_n\} = \bigcup_{i=1}^{n}\{x_i\}$$

$$\Rightarrow m\left(\bigcup_{i=1}^{n}\{x_i\}\right) = \sum_{i=1}^{n} m\{x_i\} = 0$$

Infinite Set:

$$\{x_1, x_2, x_3, \cdots\cdots\} = \bigcup_{i=1}^{\infty}\{x_i\}$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty}\{x_i\}\right) = \sum_{i=1}^{\infty} m\{x_i\} = 0$$

Topic # 64

Sum Modulo 1:

Let $x, y \in A = [0,1)$ and we define sum modulo "1" of x and y as

$$x \hat{+} y = \begin{cases} x+y & \text{if } x+y<1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

Translation Modulo 1:

Let $E \subseteq A = [0,1)$ and $y \in A$, we define a translation modulo "1" of "E" as:

$$E \hat{+} y = \{ \underbrace{x \hat{+} y}_{\text{set of all "Sum Modulo "1""}} , x \in E \}.$$

Topic # 65

Theorem: Let $E \subset A = [0,1)$ be a measurable set, Then for ~~any~~ each $y \in A$. The set $E \hat{+} y$ is measurable and $m(E) = m(E \hat{+} y)$.

proof: Suppose $E_1 = E \cap [0, 1-y)$ and

$$E_2 = E \cap [1-y, 1)$$

$\because$ E, $[0, 1-y)$, $[1-y, 0)$ are measurable

$\Rightarrow$ E_1 and E_2 are also measurable being the intersection of two measurable sets.

Also $E_1 \cap E_2 = \{E \cap [1-y, 1)\} \cap \{E \cap [0, 1-y)\}$

$$= E \cap \{[0, 1-y) \cap [1-y, 1)\}$$

$$= E \cap \phi = E$$

and $E_1 \cup E_2 = E \cup \phi = E$

$\Rightarrow m(E) = m(E_1) + m(E_2) \longrightarrow (1)$

"m" is finitely additive.

Let $y_1 \in A$, if $y_1 \in E_1$

$\Rightarrow y_1 < 1 - y \Rightarrow y_1 + y < 1 \Rightarrow y_1 \hat{+} y = y_1 + y$

$E_1 + y$

$\Rightarrow E_1 \hat{+} y = E_1 + y \longrightarrow (a)$

if $y_2 \in E_2 \Rightarrow y_2 \geq 1 - y \Rightarrow y_1 + y \geq 1$

$\Rightarrow y_2 \hat{+} y = y_2 + y - 1$

$\Rightarrow E_2 \hat{+} y = E_2 + (y-1) \longrightarrow (b)$

$\because$ "m" is translation invariant.

$\Rightarrow E_1 + y$ and $E_2 + (y-1)$ are measurable.

$\Rightarrow E_1 \hat{+} y$ and $E_2 \hat{+} (y-1)$ are measurable.

$\Rightarrow m(E_1 \hat{+} y) = m(E_1 + y) = m(E_1)$

Similarly

$m(E_2 \hat{+} y) = m(E_2 + y - 1) = m(E_2)$

$\because E_1 \cup E_2 = E$, $E_1 \cap E_2 = \phi$

$\Rightarrow E \hat{+} y = (E_1 + y) \cup (E_2 + y)$ and $(E_1 + y) \cap (E_2 + y) = \phi$

"$E \hat{+} y$" being union of two disjoint measurables sets its measurable.

$\Rightarrow m(E \hat{+} y) = m(E_1 \hat{+} y) + m(E_2 \hat{+} y)$

$= m(E_1 + y) + m(E_2 + y)$

$= m(E_1) + m(E_2) = m(E)$

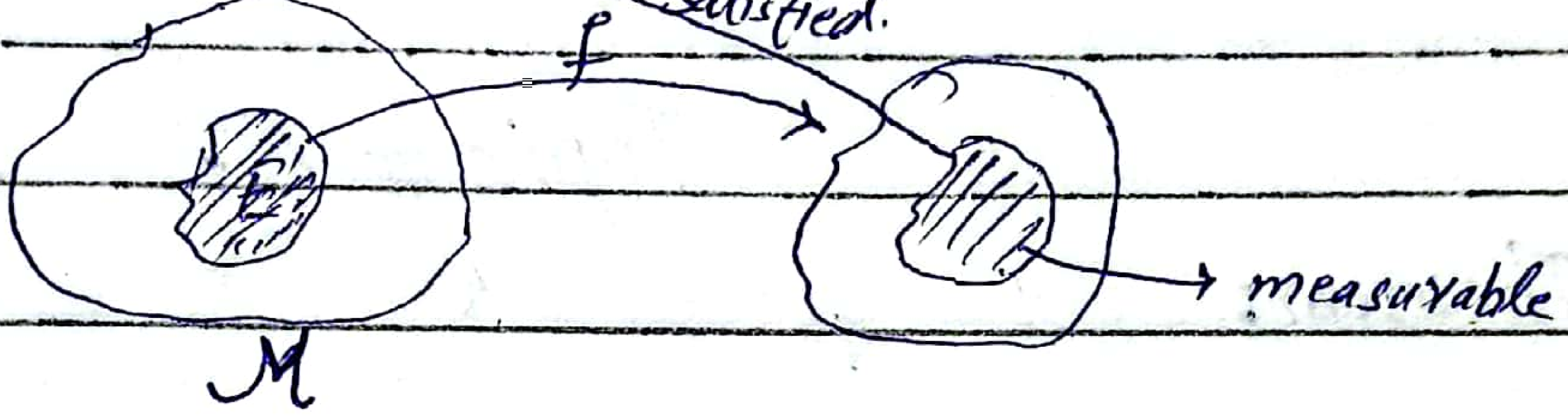
$\Rightarrow$ "m" is also invariant w.r.t translation modulo "1".

Topic # 66

Measurable Function:

Let "$\mathcal{M}$" be the measurable space (σ-algebra of measurable sets): a function $f: E \to \mathbb{R}$ is Lebesgue measurable function on $\phi \neq E \in \mathcal{M}$ if $\{x \in E : f(x) > a\} \in \mathcal{M}$ for every $a \in \mathbb{R}$.

satisfied.



For Example ①: Suppose "f" is real valued identically constant function $f(x) = \beta$, Then

$$\{x : f(x) > a\} = \begin{cases} \mathbb{R} & \text{if } \beta > a \\ \phi \in \mathcal{M} & \end{cases}$$

$$\because \mathbb{R}\ \phi \in \mathcal{M}$$

$\Rightarrow$ "f" is measurable.

Example ②:

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$\{x : f(x) > a\} = \begin{cases} \phi \\ A \neq X \\ X = A \end{cases}$$

$\phi, A, X \in \mathcal{M} \Rightarrow$ "f" is measurable.

Example (3): $(\mathbb{R}, \mathcal{B}, m) \to$ (Borel σ-algebra)

$f(x) = x$

$\Rightarrow \{x : f(x) > a\} = \begin{cases} (a, +\infty) \in \mathcal{B} \\ \text{i.e always measurable} \end{cases}$

$\Rightarrow$ "f" is measurable.

Topic # 67

Theorem: Let $f: E \to \mathbb{R} \cup \{\pm\infty\}$, $\phi \neq E \in \mathcal{M}$, Then
(i) "f" is measurable $\Leftrightarrow$ (ii) $\{x \in E : f(x) \geq a\}$ is measurable for all $a \in \mathbb{R}$ $\Leftrightarrow$ (iii) $\{x \in E : f(x) < a\}$ is measurable $\forall$ $a \in \mathbb{R}$ $\Leftrightarrow$ (iv) $\{x \in E : f(x) \leq a\}$ is measurable $\forall$ $a \in \mathbb{R}$.

Proof: (i) $\Rightarrow$ (ii) "f" is measurable $\Rightarrow \{x \in E : f(x) > a\} \in \mathcal{M}$

$$\{x \in E : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \left\{x \in E : f(x) > a - \frac{1}{n}\right\}$$

Let $y \in \{x \in E : f(x) \geq a\} \Rightarrow f(y) \geq a, \forall$ $a \in \mathbb{R}$

$\Rightarrow f(y) > a - \frac{1}{n}$, $\forall$ $n \in \mathbb{N}$

$\Rightarrow y \in \left\{x \in E : f(x) > a - \frac{1}{n}\right\}$, $\forall$ $n \in \mathbb{N}$

$$\Rightarrow y \in \bigcap_{n=1}^{\infty} \left\{x \in E : f(x) > a - \frac{1}{n}\right\}$$

$$\{x \in E : f(x) \geq a\} \subseteq \bigcap_{n=1}^{\infty} \left\{x \in E : f(x) > a - \frac{1}{n}\right\} \to (a)$$

Let conversely; $t \in \bigcap_{n=1}^{\infty} \left\{x \in E : f(x) > a - \frac{1}{n}\right\}$

$\Rightarrow f(t) > a - \frac{1}{n}, \forall n \in \mathbb{N}$

$\Rightarrow f(t) \geq a, \forall a \in \mathbb{R}$ ~~$\Rightarrow t \in$~~

$\Rightarrow t \in \{x \in E : f(x) \geq a\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\} \subseteq \{x : f(x) \geq a\}$ ⓑ

From ⓐ and ⓑ

$\Rightarrow \{x \in E : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$

$\because \{x \in E : f(x) > a - \frac{1}{n}\}$ is measurable.

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$ is also measurable, $\forall n \in \mathbb{N}$

$\Rightarrow \{x \in E : f(x) \geq a\}$ is also measurable.

Topic # 68

(ii) $\Rightarrow$ (iii) $\{x \in E : f(x) \geq a\} \in \mathcal{M} \Rightarrow \{x \in E : f(x) < a\} \in \mathcal{M}$.

$\{x \in E : f(x) < a\} = \{x \in E : f(x) \geq a\}^c \quad \forall a \in \mathbb{R}$

$\because \{x \in E : f(x) \geq a\} \in \mathcal{M} \Rightarrow \{x \in E : f(x) \geq a\}^c \in \mathcal{M}$

$\Rightarrow \{x \in E : f(x) < a\} \in \mathcal{M}$ i.e. is measurable.

Topic # 69

(iii) $\Rightarrow$ (iv) $\underbrace{\{x \in E : f(x) < a\}}_{\in \mathcal{M}} \Rightarrow \{x \in E : f(x) \leq a\} \in \mathcal{M}$.

We show that $\{x \in E : f(x) \leq a\}$

$= \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$

Let $y \in \{x \in E : f(x) \leq a\}$ $\longrightarrow$ (a)

$\Rightarrow f(y) \leq a \quad , \forall a \in \mathbb{R}$

$\Rightarrow f(y) < a + \frac{1}{n} \quad , \forall n \in \mathbb{N}$

$\Rightarrow y \in \{x \in E : f(x) < a + \frac{1}{n}\} , \forall n \in \mathbb{N}$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$ $\longrightarrow$ (B)

From (a) and (B) $\Rightarrow \{x \in E : f(x) \leq a\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$

let $z \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$

$\Rightarrow z \in \{x \in E : f(x) < a + \frac{1}{n}\} \Rightarrow f(z) < a + \frac{1}{n}, \forall n \in \mathbb{N}$

$\Rightarrow f(z) \leq a \ \forall a \in \mathbb{R} \Rightarrow z \in \{x \in E : f(x) \leq a\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} \subseteq \{x \in E : f(x) \leq a\}$

$\therefore \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} = \{x \in E : f(x) \leq a\}$

Given that $\{x \in E : f(x) < a + \frac{1}{n}\} \in \mathcal{M}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} \in \mathcal{M}$

$\Rightarrow \{x \in E : f(x) \leq a\} \in \mathcal{M}$. i.e is measurable.

Topic # 70

(iv) $\Rightarrow$ (i); $\{x \in E : f(x) \leq a\} \in \mathcal{M} \Rightarrow$ "f" is measurable.

$f : E \to \mathbb{R} \cup \{+\infty\}$, $\{x \in E : f(x) < a\} \in \mathcal{M}$,

$\{x \in E : f(x) \leq a\} \in \mathcal{M}$, $\quad \forall a \in \mathbb{R}$

$= \{x \in E : f(x) \leq a\}^c = \{x \in E : f(x) > a\} \in \mathcal{M}$

$\Rightarrow$ "f" is measurable.

Topic # 71

Theorem: Let $\phi \neq E \in \mathcal{M}$, $f : E \to \mathbb{R} \cup \{\pm\infty\}$, then $\{x \in E : f(x) = a\} \in \mathcal{M}$, $\forall\, a \in \mathbb{R}$.

Proof: Case I: $a \in \mathbb{R}$

$$\Rightarrow \{x \in E : f(x) = a\} = \{x \in E : f(x) \geq a\} \cap \{x \in E : f(x) < a\}$$

$\because$ $\{x \in E : f(x) \geq a\}$ and $\{x \in E : f(x) < a\} \in \mathcal{M}$.

$$\Rightarrow \{x \in E : f(x) \geq a\} \cap \{x \in E : f(x) < a\} \in \mathcal{M}$$

$$\Rightarrow \{x \in E : f(x) = a\} \in \mathcal{M} \text{ (measurable)}$$

Case II: If $a = \infty$, Then $\{x \in E : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\}, \forall n \in \mathbb{N}$

Let $y \in \{x \in E : f(x) = \infty\} \Rightarrow y \in \{x \in E : f(x) > n\}$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\}$, $\forall\, n \in \mathbb{N}$

$$\Rightarrow \{x \in E : f(x) = \infty\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\} \quad (a)$$

Now let $t \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\}, \forall n \in N$

$\Rightarrow t \in \{x \in E : f(x) > n\}, \forall n \in N$

$\Rightarrow f(t) > n, \forall n \in N$

$\Rightarrow t \in \{x \in E : f(x) = \infty\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\} \subseteq \{x \in E : f(x) = \infty\} \longrightarrow$ (b)

From (a) and (b)

So, $\{x \in E ; f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\}$

Case III: If $a = -\infty$, Then $\{x \in E : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}$

let $z \in \{x \in E : f(x) = -\infty\} \Rightarrow z \in \{x \in E : f(x) < -n\}$

$\Rightarrow z \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}, \forall n \in N$

$\Rightarrow \{x \in E : f(x) = -\infty\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\} \longrightarrow$ (a)

Now let $s \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}, \forall n \in N$

$\Rightarrow s \in \{x \in E : f(x) < -n\}, \forall n \in N$

$\Rightarrow f(s) < -n, \forall n \in N$

$\Rightarrow s \in \{x \in E : f(x) = -\infty\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\} \subseteq \{x \in E : f(x) = -\infty\} \longrightarrow$ (b)

From ⓐ and ⓑ

So, $\{x \in E : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}$

Topic # 72

Positive and Negative parts of Extented Real Valued Function:

$$|f(x)| = \begin{cases} +f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

$$f^{+} = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

$$= \max\{f(x), 0\} = f \vee 0$$

$$f^{-} = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

$$= \max\{-f(x), 0\} = -f \vee 0$$

where $f : A \to \mathbb{R} \cup \{\pm\infty\}$, $A \neq \phi$, $\forall x \in A$

Topic # 73

Theorem: Show that

(i) $f^{+} = \frac{f + |f|}{2}$, (ii) $f^{-} = \frac{|f| - f}{2}$, (iii) $f = f^{+} - f^{-}$

(iv) $|f| = f^{+} + f^{-}$, (v) $\max(f, g) = \frac{(f+g) + |f-g|}{2}$

(vi) $\min(f, g) = \frac{(f+g) - |f-g|}{2}$

proof:

(i) if $f(x) \geq 0 \Rightarrow f^{+}(x) = f(x) \Rightarrow f^{+} = f$

$$|f(x)| = |f|(x) = f(x)$$

$$\therefore \frac{f+|f|}{2}(x) = \frac{f(x)+|f|(x)}{2} = \frac{f(x)+f(x)}{2} = f(x) = f^+(x)$$

If $f(x) < 0 \Rightarrow f^+(x) = 0$, $\quad |f|(x) = |f(x)| = -f(x)$

$$\therefore \frac{f+|f|}{2}(x) = \frac{f(x)+|f|(x)}{2} = \frac{f(x)-f(x)}{2} = 0 = f^+(x)$$

$$\Rightarrow \frac{f+|f|}{2} = f^+ \quad \text{(proved)}$$

(ii) If $f(x) \geq 0 \Rightarrow f^-(x) = 0$, $\quad |f(x)| = |f|(x) = +f(x)$

$$\therefore \frac{|f|-f}{2}(x) = \frac{|f|(x)-f(x)}{2} = \frac{+f(x)-f(x)}{2} = 0 = f^-(x)$$

If $f(x) < 0 \Rightarrow f^-(x) = -f(x)$, $\quad |f(x)| = |f|(x) = -f(x)$

$$\therefore \frac{|f|-f}{2}(x) = \frac{|f|(x)-f(x)}{2} = \frac{-f(x)-f(x)}{2} = -f(x) = f^-(x)$$

$$\Rightarrow \frac{|f|-f}{2} = f^- \quad \text{(proved)}$$

(iii) $f = f^+ - f^-$

If $f(x) \geq 0 \Rightarrow f^+(x) = f(x)$, $f^-(x) = 0$

$$\therefore (f^+ - f^-)(x) = f^+(x) - f^-(x) = f(x) - 0 = f(x) = f$$

$$\Rightarrow f^+ - f^- = f \quad \text{(proved).}$$

(iv) $|f| = f^+ + f^-$

If $f(x) \geq 0 \Rightarrow f^+(x) = f(x)$, $\quad f^-(x) = 0$

$$\therefore (f^+ + f^-)(x) = f^+(x) + f^-(x) = f(x) + 0 = f(x) = |f(x)|$$

$$\Rightarrow f^+ + f^- = |f| \quad \text{(proved)}$$

(v) $\max(f,g) = \dfrac{(f+g)+|f-g|}{2}$

$$|f-g| = \begin{cases} +(f-g), & \text{if } \geq 0 \\ -(f-g), & \text{if } < 0 \end{cases}$$

$$|f-g|(x) = |f(x)-g(x)| = \begin{cases} f(x)-g(x); & f(x) \geq g(x) \\ g(x)-f(x); & f(x) < g(x) \end{cases}$$

Case I: If $f(x) \geq g(x) \Rightarrow f(x)-g(x) \geq 0$

$\Rightarrow |f(x)-g(x)| = f(x)-g(x)$

$\therefore \dfrac{(f+g)+|f-g|}{2}(x) = \dfrac{f(x)+g(x)+|f(x)-g(x)|}{2}$

$= \dfrac{f(x)+g(x)+f(x)-g(x)}{2} = f(x)$

Note: $f(x) \geq g(x)$ then $\max(f,g) = f(x)$

$\Rightarrow \dfrac{(f+g)+|f-g|}{2} = \max(f,g)$

Case II: If $f(x) < g(x) \Rightarrow f(x)-g(x) < 0$

$\Rightarrow |f(x)-g(x)| = -(f(x)-g(x)) = g(x)-f(x)$

and if $f(x) < g(x)$ then $\max(f,g) = g(x)$

$\therefore \dfrac{(f+g)+|f-g|}{2}(x) = \dfrac{f(x)+g(x)+|f(x)-g(x)|}{2}$

$= \dfrac{f(x)+g(x)+g(x)-f(x)}{2} = g(x)$

$\Rightarrow \dfrac{(f+g)+|f-g|}{2} = \max(f,g)$ (proved)

(vi) $\min(f,g) = \dfrac{(f+g) - |f-g|}{2}$

$$|f-g|(x) = |f(x) - g(x)| = \begin{cases} f(x) - g(x) ; & f(x) \geq g(x) \\ g(x) - f(x) ; & f(x) < g(x) \end{cases}$$

Case I: If $f(x) \geq g(x) \Rightarrow f(x) - g(x) \geq 0$

$\Rightarrow |f(x) - g(x)| = f(x) - g(x)$ and

If $f(x) \geq g(x) \Rightarrow \min(f,g) = g(x)$

$$\therefore \frac{(f+g) - |f-g|}{2}(x) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$

$$= \frac{f(x) + g(x) - (f(x) - g(x))}{2}$$

$$= \frac{f(x) + g(x) - f(x) + g(x)}{2} = g(x)$$

$$\Rightarrow \frac{(f+g) - |f-g|}{2} = \min(f,g)$$

Case II: If $f(x) < g(x) \Rightarrow f(x) - g(x) < 0$

$\Rightarrow |f(x) - g(x)| = g(x) - f(x)$ and

If $f(x) < g(x) \Rightarrow \min(f,g) = f(x)$

$$\therefore \frac{(f+g) - |f-g|}{2}(x) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$

$$= \frac{f(x) + g(x) - g(x) - f(x)}{2} = f(x)$$

$$\Rightarrow \frac{(f+g) - |f-g|}{2} = \min(f,g)$$

(proved).

Topic # 74

Characteristic Function:

Let $\phi \neq X$ and $A \subseteq X$, then characteristic function $\chi_A : X \to R$ is defined and given by:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

If $A = \phi$, $\Rightarrow \chi_A(x) = 0 = \underset{\text{zero function}}{0(x)}$, $\forall\ x \in X$

$\Rightarrow \chi_\phi = 0$

And if $A = X \Rightarrow \chi_A(x) = 1 = \underset{\text{Identity function}}{1(x)}\ \forall\ x \in X$

$\Rightarrow \chi_X = 1$

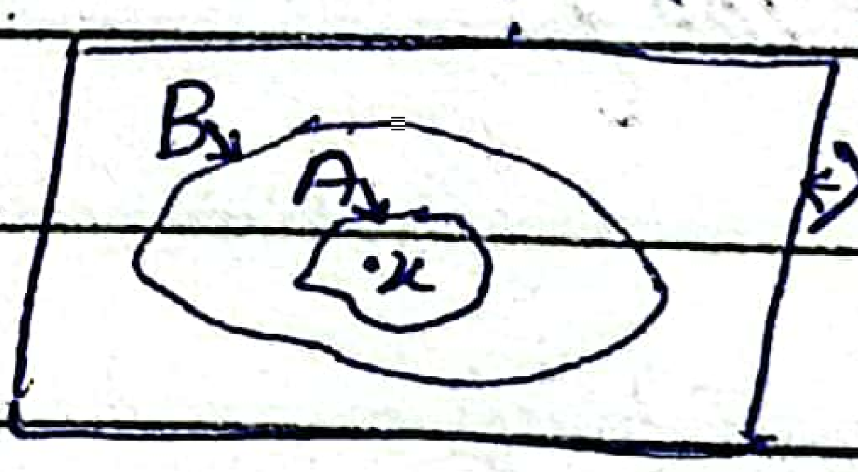
Topic # 75

Theorem: Prove that if $A \subset B$, then $\chi_A \leq \chi_B$

Proof: Case I; If $x \in A$ and

$A \subseteq B \Rightarrow x \in B$

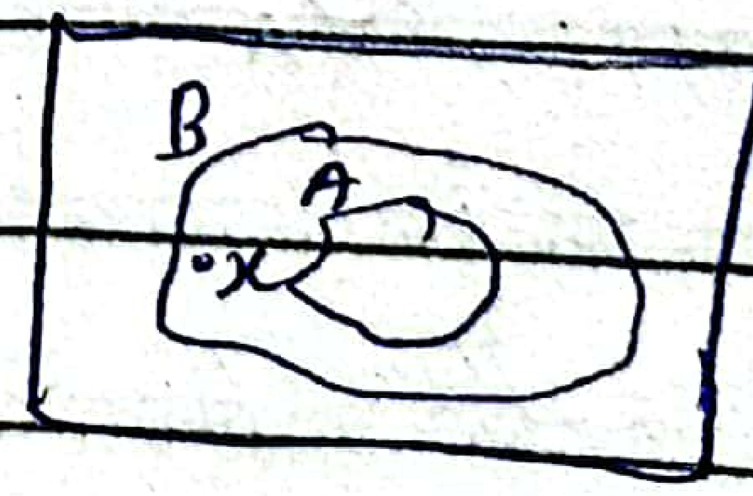
$\Rightarrow \chi_A = 1 = \chi_B \Rightarrow \chi_A \leq \chi_B$



Case II; If $x \in B$ but $x \notin A$

$\Rightarrow \chi_A = 0$ and $\chi_B = 1$

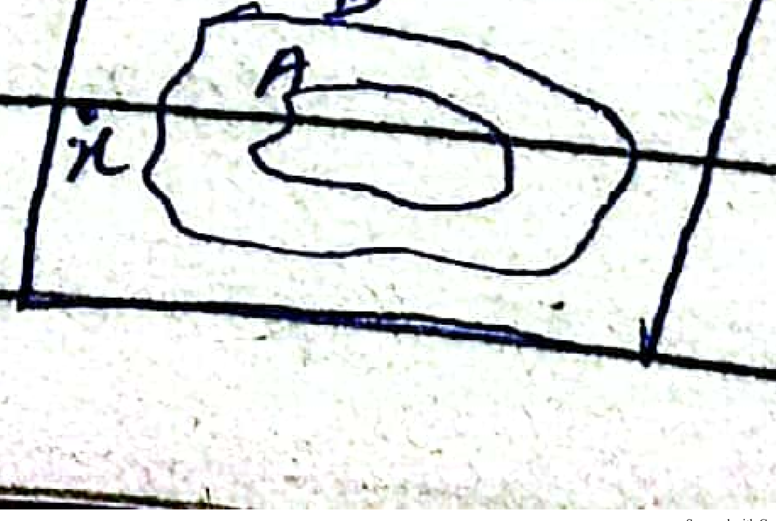
$\Rightarrow \chi_A \leq \chi_B$



Case III; If $x \notin A$ nor $x \notin B$

$\Rightarrow \chi_A = 0 = \chi_B \Rightarrow \chi_A \leq \chi_B$

(proved)



Topic # 76

Theorem: Show that $\chi_{A\cap B} = \chi_A \cdot \chi_B$

Proof: If $x \in A\cap B \Rightarrow x \in A$ and $x \in B$

$$\Rightarrow \chi_{A\cap B} = 1 = \chi_A = \chi_B = 1\cdot 1 = \chi_A \cdot \chi_B$$

$$\text{If } x \notin A\cap B \Rightarrow \begin{cases} x\in A \text{ but } x\notin B \Rightarrow \chi_A = 1, \chi_B = 0 \\ x \in B \text{ but } x \notin A \Rightarrow \chi_B = 1, \chi_A = 0 \\ x\notin A \text{ and } x \notin B \Rightarrow \chi_A = \chi_B = 0 \end{cases}$$

$x \notin A\cap B \Rightarrow \chi_{A\cap B} = 0$ $\quad \{ \because \chi_A \cdot \chi_B = 0 = O(x)$

For all, $x \notin A\cap B$; $\chi_{A\cap B} = 0 = O(x) = \chi_A \cdot \chi_B$

$\chi_{A\cap B} = \chi_A \cdot \chi_B$ (proved)

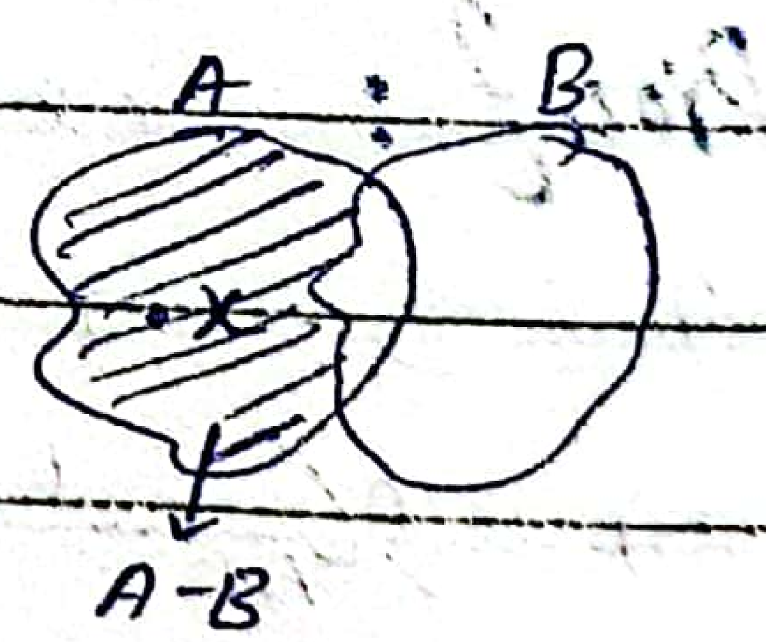
Topic # 77

Theorem: Show that $\chi_{A-B} = \chi_A - \chi_{A\cap B}$.

Proof: Case I: If $x \in A-B$

$\Rightarrow x\in A$ but $x \notin B, x\notin A\cap B$

$\Rightarrow \chi_{A-B} = 1$



$$\chi_A - \chi_{A\cap B} = 1 - 0 = 1 = \chi_{A-B}$$

$$\text{Case II If } x \notin A-B \Rightarrow \begin{cases} x\notin A, x\notin B, x\notin A\cap B \rightarrow (a) \\ x\notin A, x\in B, x\notin A\cap B \rightarrow (b) \\ x\notin A, x\in B, x\in A\cap B \rightarrow (c) \end{cases}$$

$\Rightarrow \chi_{A-B} = 0 = O(x)$

$$\chi_A - \chi_{A\cap B} = \begin{cases} (a) \Rightarrow 0-0=0 \\ (b) \Rightarrow 0-0=0 \\ (c) \Rightarrow 1-1=0 \end{cases}$$

$\therefore$ For all cases; $\chi_{A-B} = \chi_A - \chi_{A\cap B}$ (proved)

Topic # 78

Theorem: Show that $\chi_{A\times B} = \chi_A \cdot \chi_B$

Proof: $A\times B = \{(x,y) : x\in A, y\in B\}$

Case I:

$(x,y) \in A\times B \Rightarrow \begin{cases} x\in A \text{ and } y\in B \\ \Rightarrow \chi_A = 1 = \chi_B \end{cases}$

$\Rightarrow \chi_{A\times B}(x,y) = 1$

$\therefore \chi_{A\times B} = \chi_A \cdot \chi_B$

Case II:

If $(x,y) \notin A\times B \Rightarrow \begin{cases} x\in A \text{ but } y\notin B \rightarrow (a) \\ x\notin A \text{ but } y\in B \rightarrow (b) \\ x\notin A \text{ and } y\notin B \rightarrow (c) \end{cases}$

$\Rightarrow \chi_{A\times B}(x,y) = 0$

Also : $\chi_A \cdot \chi_B = \begin{cases} (a) \Rightarrow 1\cdot 0 = 0 \\ (b) \Rightarrow 0\cdot 1 = 0 \\ (c) \Rightarrow 0\cdot 0 = 0 \end{cases}$

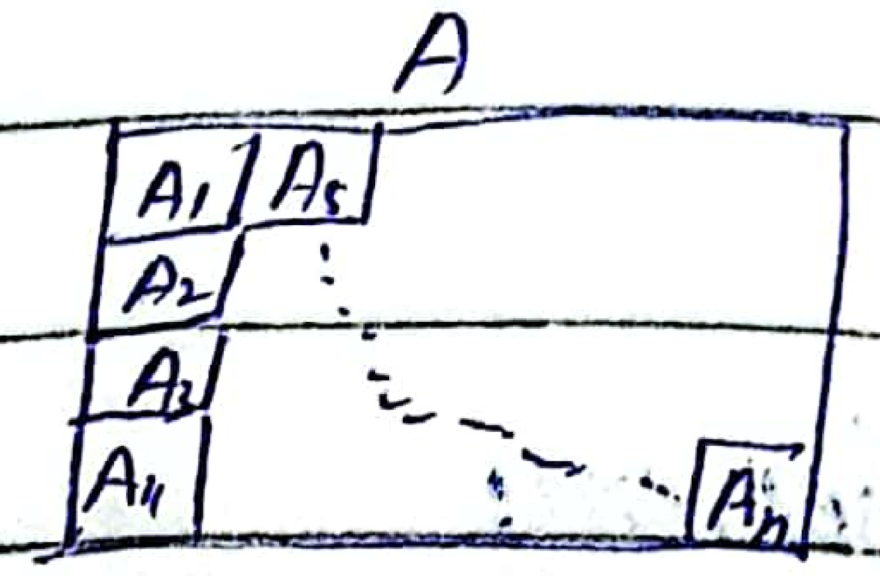
In all cases ; $\chi_{A\times B} = \chi_A \cdot \chi_B$ (proved)

Topic # 79

Theorem: If $A = \bigcup_{n=1}^{\infty} A_n$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of X. Then show that $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

Case I:

Proof: If $x \in A \Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$



$\Rightarrow x \in A_k$ exactly for $n = k$

$x \notin \bigcup_{n=1}^{\infty} A_n - \{A_k\}$

$\Rightarrow \chi_{A_k}(x) = 1,\ \chi_{A_n}(x) = 0,\ \forall\ n \neq k$

$x \in A \Rightarrow \chi_A(x) = 1$

$$\therefore\ \chi_A(x) = 1 = 0 + 0 + \cdots + \overset{n=k}{1} + 0 + 0 + \cdots$$

$$\chi_A(x) = 1 = \sum_{n=1}^{\infty} \chi_{A_n}(x) \Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$$

Case II: $x \notin A = \bigcup_{n=1}^{\infty} A_n \Rightarrow x \notin A_n,\ \forall\ n \in N$

$\Rightarrow \chi_A(x) = 0$ $\quad \Big\{ \Rightarrow \chi_{A_n}(x) = 0,\ \forall\ n \in N$

$$= 0 + 0 + 0 + \cdots + 0 + \cdots = \sum_{n=1}^{\infty} \chi_{A_n}(x)$$

$$\Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_{n}} \quad \text{(proved)}$$

Topic # 80

Theorem: let $\phi \neq A \subseteq X$ and for characteristic function $\chi_A : X \to \mathbb{R}$, χ_A is measurable $\Leftrightarrow$ "A" is measurable.

Proof: Suppose "χ_A" is measurable (function)

$\Rightarrow \forall\ a \in \mathbb{R}$ is set; $\{x : \chi_A(x) > a\}$ is measurable

In particular for $a = 0$; $\{x : \chi_A(x) > 0\} = A$

By definition of χ_A

$\therefore\ A \in \mathcal{M}$ (measurable)

Conversely, let $A \in \mathcal{M}$ (measurable) and we prove "χ_A" is measurable.
In other words we want to prove the $\{x : \chi_A(x) \geq a\} \in \mathcal{M}, \forall, a \in \mathbb{R}$

$$\{x : \chi_A(x) \geq a\} = \begin{cases} \phi & \text{if } a \geq 1 \\ A & \text{if } 0 \leq a < 1 \\ A \cup A^c & \text{if } a < 0 \end{cases}$$

(i) $a \geq 1$, $\chi_A(x) \geq 1 \Rightarrow$ there does not exist $x \in A$ such that $\chi_A(x) \geq 1$
(ii) $0 \leq a < 1$, $\chi_A(x) \geq 0.9$ (say)
(iii) $a < 0$ (say -2), $\chi_A(x) \geq -2$

In all cases; if $A \in \mathcal{M}$ (measurable)
$\Rightarrow A^c \in \mathcal{M}$ (measurable)
$\Rightarrow A \cup A^c \in \mathcal{M}$ (measurable)
$\phi \in \mathcal{M}$ (measurable)
$\Rightarrow \{x : \chi_A(x) \geq a\} \in \mathcal{M}$ (measurable)
So "χ_A" is measurable. (Proved)

Topic # 81

Algebra of Measurable Functions:

If "f" and "g" are two extended real valued and measurable functions defined on same domain $\phi \neq D$ then (i) $f+k$, (ii) kf, (iii) $f \pm g$, (iv) fg, (v) $\frac{f}{g}, g \neq 0$, (vi) $|f|$, (vii) $f \vee g$ (viii) $f \wedge g$ are measurable functions.

Topic # 82 & 83

Proof: (i) "$f+K$" is measurable, $K \in \mathbb{R}$

$\because$ "f" is measurable $\Rightarrow \{x : f(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

$\Rightarrow \{x : f(x) > a-K\} \in \mathcal{M}$ (measurable)

$\Rightarrow \{x : f(x)+K > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

$\Rightarrow$ "$f+K$" is measurable.

83

(ii) "Kf" is measurable, $\forall K \in \mathbb{R}$

$f \to Kf$
Scaler composition

Proof: $\because$ "f" is measurable.

$\Rightarrow \{x : f(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

(a) If $K > 0$; $\because f(x) > a, \forall a \in \mathbb{R}$

$\Rightarrow f(x) > \frac{a}{K} \Rightarrow Kf(x) > a, \forall a \in \mathbb{R}$

$\therefore \{x : Kf(x) > a\} \in \mathcal{M} \ \forall a \in \mathbb{R}$

$\Rightarrow$ "Kf" is measurable.

(b) If $K < 0$; $f(x) > a, \forall a \in \mathbb{R}$ (given)

$\Rightarrow f(x) > \frac{a}{K} \Rightarrow K f(x) < a \quad (\because K<0)$

$\Rightarrow \{x : Kf(x) < a\} \in \mathcal{M}$

$\Rightarrow$ "Kf" is measurable.

(c) If $K = 0 \Rightarrow \{x : Kf(x) > a\}, \forall a \in \mathbb{R}$

$= \{x : 0 f(x) > a\}, \forall a \in \mathbb{R}$

$= \{x : 0 > a\}, \forall a \in \mathbb{R}$

$$\{x : 0 > a\} = \begin{cases} \text{"}D\text{"} \xrightarrow{} \exists x : f(x) > a & ; \ a < 0 \\ \phi & ; \ a > 0 \end{cases}$$

"D" is measurable (given) and ϕ is measurable.

$\therefore \{x : Kf(x) > a\}$ is measurable.

$\Rightarrow$ "Kf" is measurable.

Topic # 84

(iii) "$f+g$" is measurable.

proof: We have to prof "$f+g$" is measurable.

$\Rightarrow \{x : (f+g)(x) > a\} \in \mathcal{M}$ (To prove)

$\{x : (f+g)(x) > a\} = \{x : f(x) + g(x) > a\}$

$= \{x : f(x) > a - g(x)\}, \ \forall a \in \mathbb{R}$

$\because \{f(x), a - g(x) \in \mathbb{R} \Rightarrow \exists p \in \mathbb{Q}$ such that

$f(x) > p > a - g(x)\}$

because "$\mathbb{Q}$" is dense in $\mathbb{R}$: $\bar{\mathbb{Q}} = \mathbb{R}$

So $\{x : (f+g)(x) > a\} = \bigcap_{p \in \mathbb{Q}} \{x : f(x) > p\} \cap \{x : a - g(x) < p\}$

i.e. our claim.

Let $y \in \{x : (f+g)(x) > a\} \Rightarrow y \in \{x : f(x) + g(x) > a\}$

$\Rightarrow f(y) + g(y) > a \Rightarrow f(y) > a - g(y)$

$\Rightarrow$ there exist($\exists$) $p \in \mathbb{Q}$ such that $f(y) > p > a - g(y)$

$\bar{\mathbb{Q}} = \mathbb{R}$

$\Rightarrow f(y) > p$ and $a - g(y) < p$

$\Rightarrow y \in \{x : f(x) > p\}$ and $y \in \{x : a - g(x) < p\}$

$\Rightarrow y \in \{x : f(x) > p\} \cap \{x : a - g(x) < p\}$

$\Rightarrow y \in \bigcup_{p \in \mathbb{Q}} [\{x : f(x) > p\} \cap \{x : a - g(x) < p\}]$

$\Rightarrow \{x : (f+g)(x) > a\} \subseteq \bigcup_{p \in \mathbb{Q}} \left[\{x : f(x) > p\} \cap \{x : a - g(x) < p\}\right]$ → (A)

Now let $t \in \bigcup_{p \in \mathbb{Q}} \left[\{x : f(x) > p\} \cap \{x : a - g(x) < p\}\right]$

$\Rightarrow t \in \{x : f(x) > p\} \cap \{x : a - g(x) < p\}$ for some $p \in \mathbb{Q}$

$\Rightarrow t \in \{x : f(x) > p\}$ and $t \in \{x : a - g(x) < p\}$

$\Rightarrow f(t) > p$ and $a - g(t) < p \Rightarrow p > a - g(t)$

$\Rightarrow f(t) > p > a - g(t) \Rightarrow f(t) > a - g(t)$

$\Rightarrow f(t) + g(t) > a \Rightarrow (f+g)(t) > a$

$\Rightarrow t \in \{x : (f+g)(x) > a\}$

$\Rightarrow \bigcup_{p \in \mathbb{Q}} \left[\{x : f(x) > p\} \cap \{x : a - g(x) < p\}\right] \subseteq \{x : (f+g)(x) > a\}$ → (B)

From (A) and (B) $\Rightarrow \{x : (f+g)(x) > a\}$

$= \bigcup_{p \in \mathbb{Q}} \left[\{x : f(x) > p\} \cap \{x : a - g(x) < p\}\right]$

$\because$ $\begin{cases} \text{"}f\text{" is measurable} \Rightarrow \{x : f(x) > p\} \in \mathcal{M} \\ \text{"}g\text{" is measurable} \Rightarrow \{x : a - g(x) < p\} \in \mathcal{M} \end{cases}$

$\Rightarrow \{x : f(x) > p\} \cap \{x : a - g(x) < p\} \in \mathcal{M}$, for $p \in \mathbb{Q}$

$\Rightarrow \bigcup_{p \in \mathbb{Q}} \left[\{x : f(x) > p\} \cap \{x : a - g(x) < p\}\right] \in \mathcal{M}$

$\Rightarrow \{x : (f+g)(x) > a\} \in \mathcal{M}$ (measurable)

$\Rightarrow$ "$f+g$" is measurable function.

Now we have to prove "$f-g$" is measurable.

$\Rightarrow \because g$ is measurable $\Rightarrow (-1)g = -g$ is also M.

$\Rightarrow f+(-1)g = f-g$ is also measurable.

(proved)

Topic#85

(iv) "fg" is measurable.

proof: $fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right]$

Firstly we prove "f^2" is measurable.

$\Rightarrow$ we need to prove $\{x : f^2(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$.

$y = x^2$ (Say function)

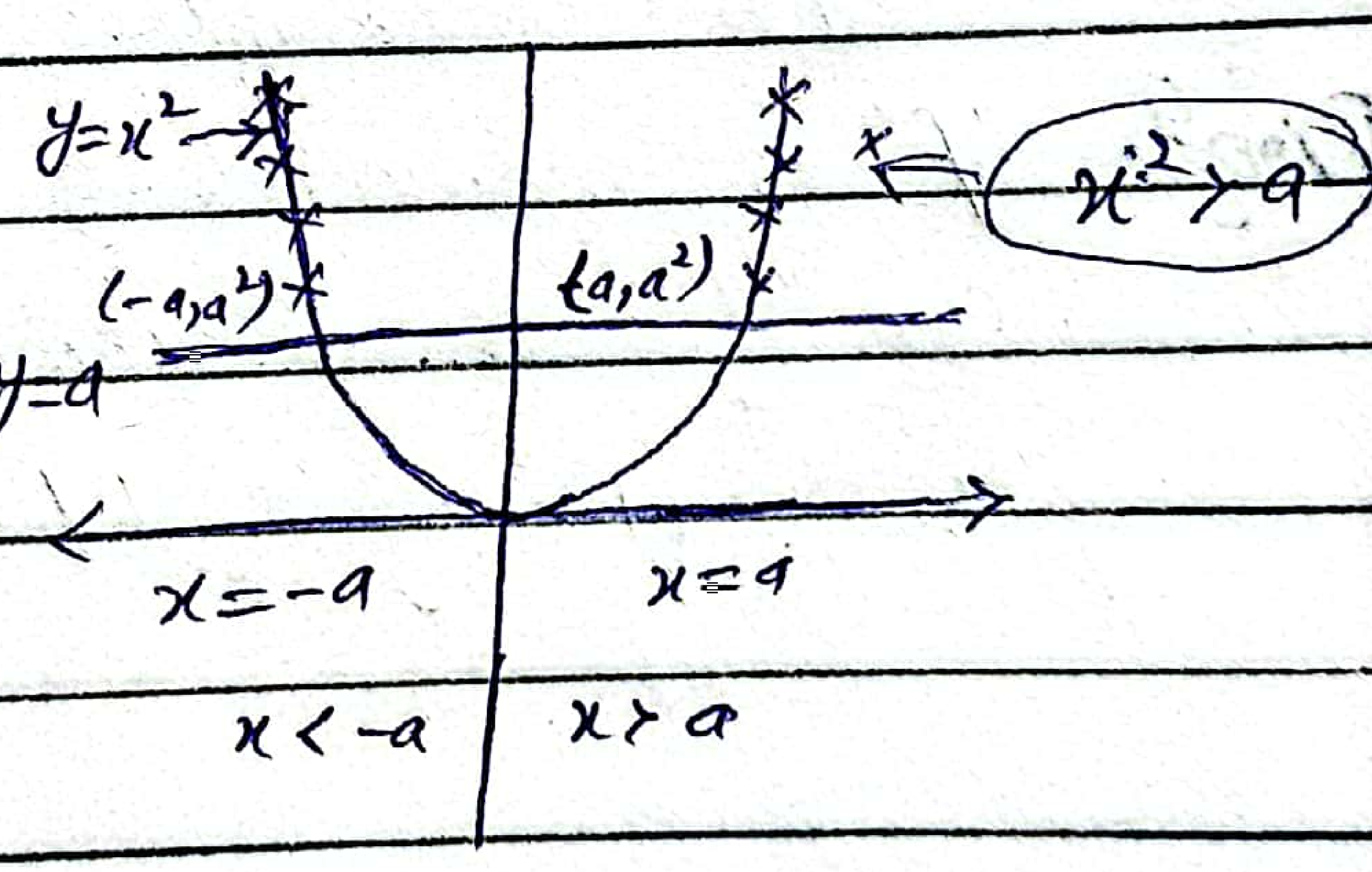
$y = a$ (line)

$\Rightarrow y^2 = a^2$

$y = x^2$

$a^2 = x^2$

$\Rightarrow x = \pm a$



$\left\{ \begin{array}{l} f^2(x) > a \\ \Rightarrow f(x) > a, \\ f(x) < -a \end{array} \right.$

$$\{x : f^2(x) > a\} = \begin{cases} \{x : f(x) > \sqrt{a}\} \cup \{x : f(x) < -\sqrt{a}\} & ; a \geq 0 \\ \text{All Domain's } (D) & ; a < 0 \end{cases}$$

$\because$ "f" is measurable $\Rightarrow \{x : f(x) > \sqrt{a}\} \in \mathcal{M}$

and $\{x : f(x) < -\sqrt{a}\} \in \mathcal{M}$

$\Rightarrow \{x : f(x) > \sqrt{a}\} \cup \{x : f(x) < -\sqrt{a}\} \in \mathcal{M}$

and "Domain" is given to be measurable. i.e $D \in \mathcal{M}$

$\therefore \{x : f^2(x) > a\} \in \mathcal{M} \Rightarrow$ "f^2" is measurable.

$\because f, g$ is measurable $\Rightarrow (f \pm g)$ is also measurable.

$\Rightarrow (f \pm g)^2$ is measurable

$\Rightarrow \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right] = fg$ is also measurable.

($\because kf$ is also measurable) $\forall k \in \mathbb{R}$.

Topic # 86

Q prove that if "f" is measurable function, then "$\frac{1}{f}$" is also measurable.

Proof: "f" is measurable $\Rightarrow \{x : f(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$

To prove "$\frac{1}{f}$" is measurable, we show that

$\{x : \frac{1}{f}(x) > a\} \in \mathcal{M}, \forall a \in \mathbb{R}$.

$$\frac{1}{f}(x) = \frac{1}{f(x)} > a$$

$$\left\{x : \frac{1}{f}(x) > a\right\} = \begin{cases} \{x : f(x) < \frac{1}{a}\} \cap \{x : f(x) > 0\} ; & a > 0 \\ \{x \in D : f(x) > 0\} ; & a = 0 \\ \{x : f(x) > 0\} \cup \left[\{x : f(x) < \frac{1}{a}\} \cap \underbrace{\{x : f(x) < 0\}}_{\phi}\right] ; & a < 0 \end{cases}$$

All the sets on R.H.S are measurable.

$\Rightarrow \{x : \frac{1}{f}(x) > a\}$ is also measurable, $\forall a \in \mathbb{R}$.

$\Rightarrow$ "$\frac{1}{f}$" is measurable.

(v) $\frac{f}{g}$, $g \neq 0$, is measurable.

proof: $\frac{f}{g}$ is also measurable because $\frac{f}{g} = \frac{1}{g} \cdot f$ is measurable being product of two measurable functions.

Topic # 87

(vi) $|f|$ is measurable.

proof: $\because$ "f" is measurable $\Rightarrow \{x : f(x) > a\} \in \mathcal{M}$, $\forall a \in \mathbb{R}$

To prove $|f|$ is measurable, we need to show:

$\{x : |f|(x) > a\} \in \mathcal{M}$, $\forall a \in \mathbb{R}$

$$\{x : |f|(x) > a\} = \{x : |f(x)| > a\}$$

$$\{x : |f|(x) > a\} = \begin{cases} \underbrace{\{x : f(x) > a\}}_{\in \mathcal{M}} \cup \underbrace{\{x : f(x) < -a\}}_{\in \mathcal{M}}, & \forall a \geq 0 \\ D \in \mathcal{M} & , a < 0 \end{cases}$$

In all cases on R.H.S, we get a measurable set. $\Rightarrow \{x : |f|(x) > a\} \in \mathcal{M}$ (measurable), $\forall a \in \mathbb{R}$

$\Rightarrow$ $|f|$ is also measurable.

(vii) $f \vee g = \max(f, g)$ is measurable.

proof: $\because f \vee g = \max(f, g) = \frac{1}{2}[(f+g) + |f-g|]$

So $f \vee g$ is measurable because $(f+g)$ and $|f-g|$ is measurable and "kf" is measurable, $\forall\, k \in \mathbb{R}$.

(viii) $f \wedge g = \min(f, g)$ is measurable.

proof: $\because f \wedge g = \min(f, g) = \frac{1}{2}[(f+g) - |f-g|]$

So $f \wedge g$ is measurable because $(f+g)$ and $|f-g|$ is measurable and "kf" is measurable, $\forall\, k \in \mathbb{R}$.

Topic # 88

Restriction of a function:

Let $f : X \to Y$ be a function and $A \subset X$, Then $g : A \to Y$ defined as $g(x) = f(x)$, $x \in A$ is called the restriction of "f" to $A \subset X$ and given by $f_A = g$.

Theorem: Let "f" be an extended real valued measurable function on "D". Let $A \subset D$, Then restriction of "f" to "A" is also measurable and $A \in \mathcal{M}$.

Proof: $\because$ "f" is measurable $\Rightarrow \{x \in D : f(x) > a\} \in \mathcal{M}\ \forall\, a \in \mathbb{R}$

For any $a \in \mathbb{R}$, To prove "f_A" is measurable we have $\{x \in A : f_A(x) > a\} \in \mathcal{M}$

Note: $\{x \in A : f_A(x) > a\} = \{x \in D : f(x) > a\} \cap A$

On R.H.S both the sets are measurable

$\Rightarrow \{x \in A : f_A(x) > a\} \in \mathcal{M}$

$\because$ Intersection of two measurable sets is measurable.

$\Rightarrow$ "f_A" is measurable.

Topic # 89

Theorem: An extended real valued function defined on a set of measure zero is measurable.

Proof: Let $\phi \neq E \subseteq \mathbb{R}$, such that $m^*(E) = 0$ and $f : E \to \mathbb{R} \cup \{\pm\infty\}$, To prove "$f$" is measurable, we need to show that

$\{x \in E : f(x) > a\} \in \mathcal{M}$, $\forall\ a \in \mathbb{R}$.

$\because \{x \in E : f(x) > a\} \subset E$

$\Rightarrow m^*\{x \in E : f(x) > a\} \leq m^*(E)$

$\because$ By monotonicity of m^*.

$\Rightarrow m^*\{x \in E : f(x) > a\} \leq 0$ ..... (given)

But by non-negativity of "m^*"

$\Rightarrow m^*\{x \in E : f(x) > a\} \geq 0$

$\Rightarrow m^*\{x \in E : f(x) > a\} = 0$, $\forall\ a \in \mathbb{R}$.

$\Rightarrow \{x \in E : f(x) > a\} \in \mathcal{M} \;;\; \forall\, a \in \mathbb{R}$

So "f" is measurable.

Topic # 90

Theorem: Let $\{f_n\}$ be a sequence of extended real valued measurable function on "D", then:

(i) $\sup\limits_{n\in N} f_n$, $\inf\limits_{n\in N} f_n$ are measurable.

(ii) $\max\limits_{1\le i\le n}(f_i)$, $\min\limits_{1\le i\le n}(f_i)$ are measurable.

(iii) $\overline{\lim\limits_{n\in N}} f_n$, $\underline{\lim\limits_{n\in N}} f_n$ are measurable.

(iv) If $\lim\limits_{n\to\infty} f_n = f$, Then "$f$" is measurable.

Proof: Let $f = \inf\limits_{n\in N} f_n$ and To prove f is measurable.

Then we have to show ~~that~~ $\{x : f(x) > a\} \in \mathcal{M}$, $\forall\, a \in \mathbb{R}$

We claim: $\{x : f(x) > a\} = \bigcap\limits_{n=1}^{\infty} \{x : f_n(x) > a\}$

Let $y \in \{x : f(x) > a\} \Rightarrow f(y) > a$

$\because f = \inf\limits_{n\in N} f_n \Rightarrow f(y) \le f_n(y)$, $\forall\, n \in N$

$\because f(y) > a \Rightarrow a < f(y) \le f_n(y)$, $\forall\, n \in N$

$\Rightarrow f_n(y) > a \quad \forall\, n \in N \Rightarrow y \in \{x : f_n(x) > a\} \;\forall\, n \in N$

$\Rightarrow y \in \bigcap\limits_{n=1}^{\infty} \{x : f_n(x) > a\}$

$\Rightarrow \{x : f(x) > a\} \subseteq \bigcap\limits_{n=1}^{\infty} \{x : f_n(x) > a\} \longrightarrow (A)$

Now let $t \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\}$

$\Rightarrow t \in \{x : f_n(x) > a\}, \forall a \in \mathbb{R}, \forall n \in \mathbb{N}$

$\Rightarrow f_n(t) > a, \forall n \in \mathbb{N}$

$\Rightarrow a < f_n(t), \forall n \in \mathbb{N}$

$\Rightarrow$ "a" is lower bound $\{f_1(t), f_2(t), \ldots\}$

But $f = \inf_{n \in \mathbb{N}} f_n$ is also its lower bound.

$\Rightarrow a < \inf_{n \in \mathbb{N}} f_n, \forall n \in \mathbb{N}$

$\Rightarrow \cancel{f_n(t) > a} \inf_{n \in \mathbb{N}} f_n > a \Rightarrow f(t) > a$

$f(t) > a \Rightarrow t \in \{x : f(t) > a\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\} \subset \{x : f(x) > a\} \longrightarrow$ B

From A and B $\{x : f(x) > a\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\}$

So $f = \inf_{n \in \mathbb{N}} f_n$ is measurable.

Now let $h = \sup_{n \in \mathbb{N}} f_n$

The set $\{x : h(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\}$ is measurable. So $h = \sup_{n \in \mathbb{N}} f_n$ is measurable.

Topic# 91

$\max_{1\le i\le n} f_i$, $\min_{1\le i\le n} f_i$ are measurable.

$\{f_n\}_{n=1}^{\infty}$ is seq. of measurable function.

Proof: Let $f = \max_{1\le i\le n} f_i$, To prove "f" is measurable, we need to show that $\{x: f(x) > a\} \in \mathcal{M}$ and we claim

$$\{x: f(x) > a\} = \bigcup_{i=1}^{n} \{x: f_i(x) > a\} \ \forall a \in \mathbb{R}.$$

Let $y \in \{x: f(x) > a\} \Rightarrow f(y) > a$

$\because f(x) = \max_{1\le i\le n} f_i$ So there exist $j = \{1, 2, \ldots, n\}$

$\Rightarrow f(x) = f_j(x) \Rightarrow f(y) = f_j(y) > a$

$\Rightarrow y \in \{x: f_j(x) > a\}$, for some $j = \{1, 2, \ldots, n\}$

Hence $y \in \bigcup_{i=1}^{n} \{x: f_i(x) > a\}$.

Now let $y \in \bigcup_{i=1}^{n} \{x: f_i(x) > a\}$.

Then $y \in \{x: f_k(x) > a\}$ for some $k = \{1, 2, \ldots, n\}$

$\Rightarrow f_k(y) > a$, By definition $f_i(y) \le \max f_i(y) = f(y)$

$\forall i = 1, 2, \ldots n.$

and $\because f_k(y) \le f(y) \Rightarrow f(y) > a$

$\Rightarrow y \in \{x: f(x) > a\}.$

The countable union of measurable sets is measurable. So $f = \max_{1\le i\le n} f_i$ is measurable.

$\overline{\lim}_{n\in N} \rightarrow$ Limit superior

$\underline{\lim}_{n\in N} \rightarrow$ Limit inferior

Let $g = \min_{1\le i\le n} f_i$, To prove "$g$" is measurable we need to show that $\{x : g(x) > a\} \in \mathcal{M}$ $\forall a \in \mathbb{R}$

and we claim $\{x : g(x) > a\} = \bigcap_{i=1}^{n} \{x : f_i(x) > a\}$

The countable intersection of measurable sets is measurable. So $g = \min_{1\le i\le n} f_i$ is measurable.

Topic # 92

Show that $\overline{\lim}_{n\in N} f_n$, $\underline{\lim}_{n\in N} f_n$ are measurables.

Proof:

$$\overline{\lim}_{n\in N} f_n = \inf_{n\in N} \left(\sup_{k\ge n} f_k \right) \rightarrow (a)$$

$$\sup_{k\ge n} f_k = \sup\{f_k, f_{k+1}, f_{k+2}, \ldots\}, \; k = 1, 2, 3 \ldots$$

$$= \begin{cases} \sup\{f_1, f_2, f_3, \ldots\} = F_1 \text{ is measurable being "sup" of measurable functions.} \\ \sup\{f_2, f_3, f_4, \ldots\} = F_2, \ldots \end{cases}$$

$\Rightarrow \{F_n\}_{n=1}^{\infty}$ is a sequence of measurable functions

$\Rightarrow \inf_{n\in N} \{F_n\}_{n=1}^{\infty}$ is also measurable

$\Rightarrow \overline{\lim}_{n\in N} f_n = \inf_{n\in N} \{F_n\} = \inf_{n\in N} \left(\sup_{k\ge n} f_k \right)$ is also measurable

Similarly $\underline{\lim}_{n\in\mathbb{N}} f_n = \sup_{n\in\mathbb{N}} \left(\inf_{k\geq n}\right)$ is measurable.

Topic # 93

If $\lim_{n\to\infty} f_n = f$, then "$f$" is measurable.

Proof: $\because \lim_{n\to\infty} f_n = \overline{\lim}_{n\in\mathbb{N}} f_n = \underline{\lim}_{n\in\mathbb{N}} f_n = f$ $\left\{ \{f_n\}_{n=1}^{\infty} \downarrow \text{Limit (given)} \right.$

But $\overline{\lim}_{n\in\mathbb{N}} f_n$ and $\underline{\lim}_{n\in\mathbb{N}} f_n$ are measurable.

$\Rightarrow$ "f" is also measurable.

MTH644 – Measure Theory Handouts

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