Practice Questions lecture 23-25

TOPIC - 90 to 101

QNO:- 1*

What is a factor or quotient group. Give examples.

Solution:-

Let $(H, *)$ be a normal subgroup of the group $(G, *)$. Then $(G/H, \otimes)$ is called a quotient or factor group, where the operation $\otimes$ is defined on G/H by:

$$Hg_1 \otimes Hg_2 = H(g_1 * g_2).$$

If G is a finite group then G/H is also finite and

$$|G/H| = |G|/|H|.$$

For example:- The cyclic group of addition modulo n can be obtained from a group of integers under addition by identifying the elements that differ by a multiple of n.

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Q NO:- 2.

State and explain first isomorphism theorem.

Solution:-

First isomorphism theorem:-

Let K be the kernel of a group morphism $f: G \to H$. Then G/K is isomorphic to the image of f and the isomorphism $\psi: G/K \to \operatorname{Im} f$, is defined by $\psi(Kg) = f(g)$. This result is known as first isomorphism theorem. Now, let's prove this theorem.

Proof:-

The function ψ is well-defined on a coset by using one particular element in the coset so, we have to check ψ is well-defined; i.e. it doesn't matter which element we use.

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$\psi: G/K \to \text{Im} f$, $\psi(Kg) = f(g)$.

We have taken the right coset of Kg in place of G/K and it maps on the function $f(g)$.

If $Kg' = kg$ then $g' \equiv g \bmod K$. So, $g'g^{-1} = k \in K = \text{Ker} f$.

Hence, $g' = kg$ and $f'(g) = f(kg) = f(k)f(g) = e_H f(g) = f(g)$

So, $f'(g) = f(g)$.

So, we can say that if g' and g are under same modulo classes then they have the same identity element. Thus, the map ψ is well-defined.

*) The function ψ is morphic:-

The function ψ is morphism because

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$$\varphi(Kg_1 Kg_2) = \varphi(Kg_1g_2) = f(g_1g_2) = f(g_1)f(g_2)$$
$$= \varphi(Kg_1)\varphi(Kg_2).$$

*) If $\varphi(Kg) = e_H$ then

$f(g) = e_H$ and $g \in K$.

Hence only one element in the kernel of φ is the identity coset K, and φ is injective.

Finally $\mathrm{Im}\,\varphi = \mathrm{Im}\,f$, that is $\varphi^{-1}(f(g)) = Kg$, by the definition of φ. Therefore, φ is the required isomorphism between G/K and $\mathrm{Im}\,f$.

QNO:- 3.

Let $(G, \cdot)$ be a group with subgroup H.

For $a, b \in G$ what do you mean by a congruent to b modulo H?

Solution:-

Define of congruence modulo H:

Two integers a and b are congruent modulo H iff they have the same remainder when divided by H.

denoted by $a \equiv b \bmod H$.

Then according to this definition, we will say that a is congruent to b mod H if $a \bmod H = b \bmod H$. i.e, if divided by the subgroup H, the both elements a, b have the same remainder.

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Q NO:- 04.

Write down the two normal subgroups of

$\{1, -1, i, -i, j, -j, k, -k\}$.

Solution:-

1) $H = [\{1, -1\}, \times]$

2) $K = [\{i, -i\}, \times]$.

Q NO:- 5.

Define the Kernel of homomorphism and give example.

Solution:-

Let $h: G \longrightarrow G'$ be a homomorphism of groups.

The subgroup $h^{-1}[\{e'\}] = \{x \in G \mid h(x) = e'\}$ is the kernel of h.

And is denoted by $Ker(h)$.

Q NO:- 6*

Let y be the natural map of Z into Z_n given by $y(m) = r$, where r is the remainder given by the division algorithm when m is divided by n. Show that y is a homomorphism. Find Ker(y).

Solution:-

We need to show that $y(s+t) = y(s) + y(t)$ for $s,t \in Z$. Using the division algorithm, we suppose

$s = q_1 n + r_1$ —(1) and $t = q_2 n + r_2$ —(2)

where $0 \leq r_i < n$ for $i = 1, 2$.

If $r_1 + r_2 = q_3 n + r_3$ —(3) for $0 \leq r_3 < n$ then adding eq-(1) and -(2) we see that

$$s+t = (q_1+q_2+q_3)n + r_3,$$

such that $y(s+t) = r_3$. From eq - 1 and - 2 we see that $y(s) = r_1$ and $y(t) = r_2$. Eq - 3 shows that the sum $r_1 + r_2$ in Z_n is equal to r_3 also.

Q NO:- 01.

Let D be the additive group of all differential functions mapping $\mathbb{R}$ onto $\mathbb{R}$, and let F be the additive group of all functions mapping $\mathbb{R}$ into $\mathbb{R}$, then the map $h: D \to F$, where $h(f) = f'$. What is $Ker(h)$?

Solution:-

We can easily see that 'h' is a homo-morphism for $h(f+g) = (f+g)' = f'+g' = h(f)+h(g)$; the derivative of a sum is the sum of derivatives. Now,

$Ker(h)$ consists of all those functions f such that $f'=0$.

Thus $Ker(h)$ consists of all constant functions, which form a subgroup C of F. Let's find all the functions

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whose derivative is x^2 by 'h'. i.e., all those functions whose derivative is x^2. Now we know that $x^3/3$ is one of such functions.

By the theorem of Kernel of homomorphism which is stated as:

"Let h be a homomorphism from a group G into a group G'. Let K be the kernel of h. Then,

$aK = \{x \in G \mid h(x) = h(a)\} = h^{-1}[\{h(a)\}]$ and

$Ka = \{x \in G \mid h(x) = h(a)\} = h^{-1}[\{h(a)\}]$.

So, by this theorem, all such functions form the coset

$\frac{x^3}{3} + C$.

Q NO:- 02

Let $G = G_1 \times \dots \times G_i \times \dots \times G_n$ be a direct product of groups. The projection map $\pi_i : G \to G_i$ where $\pi_i(g_1, g_2, \dots, g_i, \dots g_n) = g_i$ is a homomorphism for each $i = 1, 2, \dots, n$.

Solution:-

Let's take any two arbitrary elements say:

$$\pi_1 = g_1, \dots, g_n \ , \ \pi_2 = g_1', \dots, g_n'.$$

Then their product will be given as:

$$\pi_1 \pi_2 = \{g_1 g_1', \dots, g_n g_n'\}.$$

This follows immediately the fact that the binary operation of G coincides in the ith component with the binary operation in G_i.

QNO:- 03,

We may sometimes write $Z/nZ = Z_n$.

Then how do you read Z_n.

Solution:-

We read Z_n as the residue classes of integers under modulo n.

Q NO:- 04.

Let $\phi : F \to R$ be defined by $\phi_c(f) = f(c)$ for all $f \in F$. Show that ϕ is homomorphism.

Solution:-

By definition the sum of two functions f and g is the function $f+g$ whose value at x is $f(x) + g(x)$. Thus, we have

$$\phi_c(f+g) = (f+g)c = f(c) + g(c) = \phi_c(f) + \phi_c(g),$$

So ϕ is a homo-morphism.

Q NO:-5.

Let H be a subgroup of G. Then show that H is normal if and only if
$(aH)(bH) = (ab)H, \ \forall \ a, b \in G.$

Solution:-

The above statement shows that if left and right cosets of H coincides, then the above equation $(aH)(bH) = (ab)H, \ \forall \ a, b \in G$ gives a well-defined binary operation on cosets.

Q NO:- 06

Show that the homomorphism $h: G \to G'$ is injective if and only if $Ker(h) = \{e\}$.

Solution:-

Suppose h is injective and let $x \in Ker(h)$.

Then $h(x) = e' = h(e)$. Hence $x = e$.

Conversely, Suppose that $Ker(h) = \{e\}$.

Then, $h(x) = h(y)$

$\Rightarrow h(xy^{-1}) = h(x)h(y^{-1})$

$\Rightarrow h(x)\,h(y)^{-1} = e'$

$\Rightarrow xy^{-1} \in Ker(h)$.

$\Rightarrow xy^{-1} = e$.

$\Rightarrow x = y$

Hence, h is injective.

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QNO:- 01.

Every subgroup H of an abelian group G is normal.

Solution:-

We need only to note that $gh = hg$ for all $g \in G$ and $h \in H$, So of course $ghg^{-1} = h$ where h belongs to the subgroup H and $g \in G$.

Q No:- 2.

The set of all inner automorphisms of G is a subgroup of Aut (G).

Solution:-

1) let $i_a, i_b \in Inn(G)$. Then,

$$i_a(i_b(x)) = a(i_b(x))a^{-1} = abxb^{-1}a^{-1} = abx(ab)^{-1} = i_{ab} \in Inn(G)$$

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2) The

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Solution:-

Let G

a, and

We claim

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group G is normal.

$gh = hg$ for

$ghg^{-1} = h$ where h

$g \in G$.

isms of G is

$x(ab)^{-1} = i_{ab} \in Inn(G)$

Hence the conjugation by b composed by conjugation by a is conjugation by ab.

2) The inverse of i_a is conjugation by $a' = a^{-1}$. $i_a((i_{a'})(x)) = i_a(a'x(a'^{-1})) = aa'xa'^{-1}a^{-1}$

$= aa'x(aa')^{-1} = x$.

Thus $Inn(G)$ is a subgroup of $Aut(G)$.

QNO:- 03.

A factor group of a group is cyclic.

Solution:-

Let G be a cyclic group with generator a, and let N be a normal subgroup of G.

We claim that the coset aN generates G/N.

We must compute all powers of aN. But this amounts to computing in G, all powers of representative a and all these powers give all the elements in G. Hence the powers of aN certainly give all cosets of N and G/N is cyclic.

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Q No:- 04.

Prove that $Aut(G_n) \cong U_n$.

Solution:-

An automorphism $\varphi: Z \to Z_n$ is determined by $\varphi(1)$ as for any integers k, $\varphi(k) = \varphi(1 + \cdots + 1) = \varphi(1) + \cdots + \varphi(1)$ $= k\varphi(1)$ must be a generator of Z_n.

Since isomorphism preserves order, $\varphi(1)$ must be a generator of Z_n. We have proved that the generators of Z_n are those integers $k \in Z_n$ for which $\gcd(k, n) = 1$. But these k are precisely the elements of $U_n = \{1, \omega, \ldots, \omega^{n-1} \mid \omega = e^{2\pi i/n}\}$.

In this way, each element a of U_n gives a distinct automorphism φ_a which is multiplication by a, and there are the automorphisms of Z_n.

Furthermore, $\mu : Aut(Z) \to U_n$ given by $\mu(\varphi_a) = a$ is a group isomorphism.

i) $\mu(\varphi_{ab}) = ab = \mu(\varphi_a)\mu(\varphi_b)$.

ii) $\mu(\varphi_a) = \mu(\varphi_b) \Rightarrow a = b$.

iii) $\mu(\varphi_a) = a$.

Q No:- 5.

Let H, a subgroup of a group G be a normal subgroup of G, then $ghg^{-1} \in H$ $\forall g \in G$ and $h \in H$.

Solution:-

Suppose that $gH = Hg$ $\forall g \in G$. Then $gh = h_1 g$. so $ghg^{-1} \in H$ $\forall g \in G$ and $h \in H$. Then:

$gHg^{-1} = \{ghg^{-1} \mid h \in H\} \subseteq H$ $\forall g \in G$. We claim that actually $gHg^{-1} = H$. We must show that $H \subseteq gHg^{-1}$ $\forall g \in G$.

Let $h \in H$. Replacing g by g^{-1} in the relation $ghg^{-1} \in H$, we obtain: $g^{-1}h(g^{-1})^{-1} = g^{-1}hg = h_1$ where $h_1 \in H$.

Consequently, $gHg^{-1} = H$ for all $g \in G$

Conversely if $gHg^{-1} = H$ for all $g \in G$, then $ghg^{-1} = h_1$, so $gh = h_1 g \in Hg$ and $gH \subseteq Hg$.

But also, $g^{-1}Hg = H$ giving $g^{-1}Hg = h_2$, so that $hg = gh_2$ and $Hg \subseteq gH$.