

Calculus And Analytical Geometry

MTH 101



Virtual University of Pakistan

Knowledge beyond the boundaries

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Lecture # 20

Derivatives of Logarithmic and Exponential Functions and Inverse functions and their derivatives

- Derivative of the logarithmic function $f(x) = \log_b(x)$
- Derivative of the Natural log functions $f(x) = \ln(x)$
- Logarithmic Differentiation
- Derivatives of Irrational powers of x
- Derivatives of Exponential functions $f(x) = a^x$
- Inverse functions
- Derivatives of Inverse Functions

Derivative of the logarithmic function

$$y = f(x) = \log_b(x)$$

Recall the Logarithm of a real number. Let x, b be real, and an unknown real number y . What should y be so that if you raise y to b , you get x .

That is $b^y = x$

This is denoted as $y = f(x) = \log_b(x)$

b is called the *base*.

The equation above is read as “log base b of x is y .” We want to find the derivative of the log function. Before we do that, here are some formulas we will need that follow from the properties of log functions. You can refer to them in section 7.1 of your text.

Theorem 7.1.1

$$(a) \log_b 1 = 0$$

$$(d) \log_b \frac{a}{c} = \log_b a - \log_b c$$

$$(b) \log_b b = 1$$

$$(e) \log_b a^r = r \log_b a$$

$$(c) \log_b ac = \log_b a + \log_b c$$

$$(f) \log_b \frac{1}{c} = -\log_b c$$

Let's apply the derivative on both sides and use its definition to get

$$e = \lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}} \quad \therefore e \approx 2.71$$

$$\frac{dy}{dx} = \frac{d}{dx} [\log_b(x)] = \lim_{h \rightarrow 0} \frac{\log_b(x+h) - \log_b(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \log_b \left(\frac{x+h}{x} \right)$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \log_b \left(1 + \frac{h}{x} \right)$$

$$= \lim_{v \rightarrow 0} \frac{1}{vx} \log_b(1+v)$$

Last step comes from letting

$$v = \frac{h}{x} \text{ and then } v \rightarrow 0 \text{ as } h \rightarrow 0$$

$$= \frac{1}{x} \lim_{v \rightarrow 0} \frac{1}{v} \log_b(1+v)$$

$$= \frac{1}{x} \lim_{v \rightarrow 0} \log_b(1+v)^{\frac{1}{v}}$$

Using Theorem 7.1.1 part (e)

$$= \frac{1}{x} \log_b \left[\lim_{v \rightarrow 0} (1+v)^{\frac{1}{v}} \right]$$

Using the fact that log is continuous

$$= \frac{1}{x} \log_b e$$

Thus

$$\frac{d}{dx} [\log_b(x)] = \frac{1}{x} \log_b e, \quad x > 0$$

There exists a “change of base” formula which lets you convert your given log to a certain base to another. This is helpful because modern calculators have two log buttons, one with base 10, written as only $\log(x)$ and the other for the Natural log to the base e written as $\ln(x)$. To keep calculations easy, we can rewrite the above formula for the derivative as

$$\frac{d}{dx} [\log_b(x)] = \frac{1}{x \ln(b)}, \quad x > 0$$

If the base happens to be the number e , then we get from this formula

$$\frac{d}{dx} [\ln(x)] = \frac{1}{x}, \quad x > 0$$

So you see that the simplest formula for the derivative comes from base $b = e$. That's why $\ln(x)$ is a very important function in terms of Calculus

Derivative of the Natural log functions

$$f(x) = \ln(x)$$

We will generalize the derivative formula we just found for the log or $\ln$ function to include composition of functions where one is the $\ln$ function, and the other is some function $u(x) > 0$ and differentiable at x .

$$\frac{d}{dx} [\log_b(u)] = \frac{1}{u \ln(b)} \cdot \frac{du}{dx}$$

Using the chain rule here

$$\frac{d}{dx} [\ln(u)] = \frac{1}{u} \cdot \frac{du}{dx}$$

Example

Find $\frac{d}{dx}[\ln(x^2 + 1)]$

$$\begin{aligned}\frac{d}{dx}[\ln(x^2 + 1)] &= \frac{1}{(x^2 + 1)} \cdot \frac{d}{dx}[x^2 + 1] \\ &= \frac{1}{(x^2 + 1)} \cdot 2x = \frac{2x}{(x^2 + 1)}\end{aligned}$$

Example

$$\begin{aligned}\frac{d}{dx}\left[\ln\left(\frac{x^2 \sin(x)}{\sqrt{1+x}}\right)\right] \\ &= \frac{d}{dx}[\ln(x^2 \sin(x)) - \ln \sqrt{1+x}] = \frac{d}{dx}\left[2\ln(x) + \ln(\sin x) - \frac{1}{2}\ln(1+x)\right] \\ &= \frac{2}{x} + \frac{\cos(x)}{\sin(x)} - \frac{1}{2(1+x)}\end{aligned}$$

Logarithmic Differentiation Using log function properties, we can simplify differentiation of messy functions.

Example

$$y = \frac{x^2 \cdot \sqrt[3]{7x-14}}{(1+x^2)^4}$$

Apply $\ln$ to both sides and simplify the right side using $\ln$ properties

$$\ln y = \ln\left(\frac{x^2 \cdot \sqrt[3]{7x-14}}{(1+x^2)^4}\right) = 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

Differentiating both sides w. r. t. x gives

$$\begin{aligned}\frac{1}{y} \frac{dy}{dx} &= \frac{2}{x} + \frac{\frac{7}{3}}{7x-14} - \frac{8x}{1+x^2} \\ \Rightarrow \frac{dy}{dx} &= \left(\frac{2}{x} + \frac{\frac{7}{3}}{7x-14} - \frac{8x}{1+x^2}\right) \cdot y \\ \frac{dy}{dx} &= \left(\frac{2}{x} + \frac{\frac{7}{3}}{7x-14} - \frac{8x}{1+x^2}\right) \cdot \frac{x^2 \cdot \sqrt[3]{7x-14}}{(1+x^2)^4}\end{aligned}$$

Derivatives of Irrational powers of x

Remember that we proved the power rule for positive integers, then for all integers, then for rational numbers?

Now we prove it for all REAL numbers by proving it for the remaining type of real numbers, namely the irrationals.

Power Rule

$$\frac{d}{dx}[x^r] = r \cdot x^{r-1}$$

We will use ln to prove that the power rule holds for any real number r

Let $y = x^r$, r any real number

Proceed by differentiating this function using ln

$$\ln y = \ln x^r = r \ln x$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[r \ln x]$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{r}{x}$$

$$\frac{dy}{dx} = \frac{r}{x} \cdot y = \frac{r}{x} \cdot x^r = r \cdot x^{r-1}$$

Derivatives of Exponential functions $f(x) = b^x$

Now we want to find the derivative of the function $y = f(x) = b^x$

To do so, we will use logarithmic differentiation on $y = b^x$

$$\ln y = \ln b^x = x \ln b$$

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}[x \ln b]$$

$$\frac{1}{y} \frac{dy}{dx} = \ln b$$

$$\frac{dy}{dx} = y \cdot \ln b = b^x \cdot \ln b$$

$$\text{So } \frac{d}{dx} b^x = b^x \cdot \ln b$$

In general

$$\frac{d}{dx}[b^u] = b^u \cdot \ln b \cdot \frac{du}{dx}$$

If $b = e$

$$\frac{d}{dx} e^x = e^x \cdot \ln e = e^x$$

In general

$$\frac{d}{dx}[e^u] = e^u \cdot \ln e \cdot \frac{du}{dx} = e^u \cdot \frac{du}{dx}$$

Inverse Functions

We have talked about functions.

One way of thinking about functions was to think of a function as a process that does something to an input and then throws out an output.

So this way, when we put in a real number x , it comes out as some new number y , after some ACTION has taken place.

Now that question is, that is there another ACTION, that undoes the first action?

If f is a function that performs a certain action on x , is there a function g that undoes what f does?

Sometimes there is and sometime not. When there is a function g undoes what f does, then we say that g is an inverse function of f .

Here is how to determine if two given function are inverses of each other or not.

Definition 7.4.1

If the functions f and g satisfy the two conditions

$$f(g(x)) = x \text{ for every } x \text{ in the domain of } g$$

$$g(f(x)) = x \text{ for every } x \text{ in the domain of } f$$

then we say, that f is an inverse of g and g is an inverse of f or, alternatively that f and g are inverse functions.

Example

$$f(x) = 2x \text{ and } g(x) = \frac{1}{2}x$$

are inverse functions. This is obvious, but we can use def 7.4.1

$$f(g(x)) = 2\left(\frac{1}{2}x\right) = x$$

$$g(f(x)) = \frac{1}{2}(2x) = x$$

Inverses don't always exists for a given function.

There are conditions for it.

Most important one is that a function must be ONE to ONE to have an inverse.

Do you recall what One to One is??

Its when a function does not have two values for the same x value in the domain. Review this.

The inverse function to a function f is usually denoted as f^{-1} read as " f inverse" and

$$y = f^{-1}(x) \Rightarrow x = f(y)$$

Derivatives of Inverse Functions

Theorem 7.4.7

Suppose that the function f has an inverse and that the value of $f^{-1}(x)$ varies over an interval on which f has a nonzero derivative as x varies over an interval I . Then f^{-1} is differentiable on I and derivative of f^{-1} is given by the formula

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

The formula in this theorem can be simplified if you write

$$y = f^{-1}(x) \quad \Rightarrow \quad x = f(y)$$

$$\text{Then } \frac{dy}{dx} = (f^{-1})'(x) \quad \Rightarrow \quad \frac{dx}{dy} = f'(y) = f'(f^{-1}(x))$$

Plugging these values into the theorem we get a simpler formula

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Lecture # 21

Applications of Differentiation

- Related Rates
- Increasing Functions
- Decreasing Functions
- Concavity of functions

Related Rates

Related Rates are real life problems.

These involve finding the rate at which one quantity changes w.r.t another quantity.

For example, we may be interested in finding out how fast the polar ice caps are melting w.r.t the changes in temperature.

We may want to know how fast a satellite is changing altitude w.r.t changes in time, or w.r.t changes in gravity.

To solve problems involving related rates, we use the idea of derivatives, which measure the rate of change.

Example

Assume that oil spilled from a ruptured tanker spread in a circular pattern whose radius increases at a constant rate of 2 ft/sec. How fast is the area of the spill increasing when the radius of the spill is 60ft?

Let t = number of seconds elapsed from the time of the spill

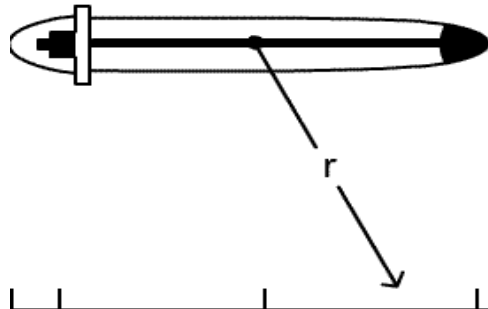
Let r = radius of the spill in feet after t seconds

Let A = area of the spill in square feet after t seconds

We want to find $\left. \frac{dA}{dt} \right|_{r=60}$ given that $\frac{dr}{dt} = 2 \text{ ft/sec}$

The spill is circular in shape so $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \Rightarrow \left. \frac{dA}{dt} \right|_{r=60} = 2\pi(60)(2) = 240\pi \text{ ft}^2 / \text{sec}$$



The following steps are helpful in solving related rate problems

1. Draw a figure and label the quantities that change
2. Identify the rates of change that are known and those that are to be found
3. Find an equation that relates the quantity whose rate of change is to be found to those quantities whose rates of change are known.
4. Differentiate the equation w.r.t the variable that quantities are changing in respect to. Usually Time.
5. Evaluate the derivative at appropriate points.

Example

A five foot ladder is leaning against a wall. It slips in such a way that its base is moving away from the wall at a rate of 2 ft/sec at the instant when the base is 4ft from the wall. How fast is the top of the ladder moving down the wall at that instant?

Let t = number of seconds after the ladder starts to slip

Let x = distance in feet from the base of the ladder

Let y = distance in feet from the top of the ladder to the floor.

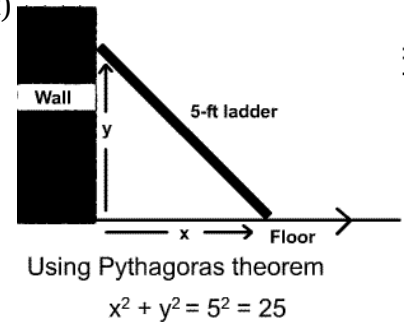
$\frac{dx}{dt}$ = rate of change of the base of the ladder (horizontal movement)

$\frac{dy}{dt}$ = rate of change of the top of the ladder (vertical movement)

We want $\left. \frac{dy}{dt} \right|_{x=4}$ Given That $\left. \frac{dx}{dt} \right|_{x=4} = 2 \text{ ft/sec}$

How do we relate the x and the y ?

Look at the picture again I see Pythagoras's theorem in here!



$$x^2 + y^2 = 25$$

Differentiating wrt t and using chain rule gives

$$2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}$$

$$\left. \frac{dy}{dt} \right|_{x=4} = -\frac{4}{3}(2) = -\frac{8}{3} \text{ ft/sec}$$

When $x = 4$, use the above equation to find corresponding y .

Increasing and decreasing functions

We saw earlier in the lectures that we can get an idea of the graph of a function by plotting a few values. But remember that we also said that this graph was an approximation as a few points may not give all the info.

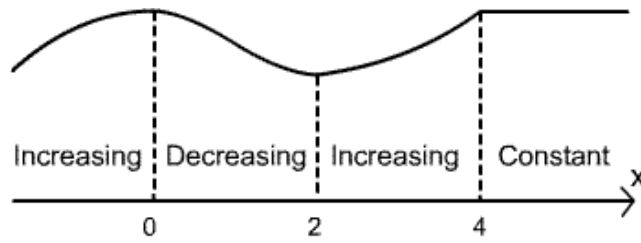
Now we will see that we can use derivatives to get accurate info about the behavior of the graph in an interval when we move from left to right.

Increasing function on an interval means that as we move from left to right in the x -direction, the y -values increase in magnitude.

Decreasing function on an interval means that as we move from left to right in the x -direction, the y -values decrease in magnitude.

An interval means that as we move from left to right in the x -direction, the y -values decrease in magnitude.

The graph of the function in this figure shows that the function is increasing on the intervals and $(-\infty, 0)$ and $[2, 4]$ decreasing on the interval $[0, 2]$.



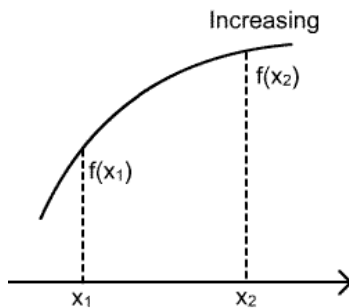
Let's make this idea concrete.

Definition 4.2.1

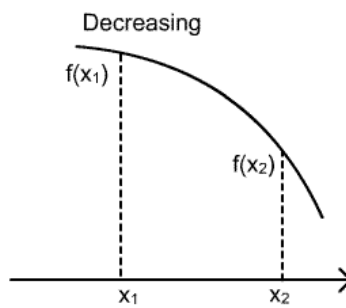
Let f be defined on an interval, and let x_1 and x_2 denote points in that interval.

- (a) f is increasing on the interval if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$
- (b) f is decreasing on the interval if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$
- (c) f is constant on the interval if $f(x_1) = f(x_2)$ for all x_1 and x_2

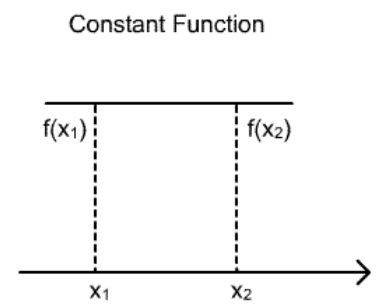
As shown in the figures below



$$f(x_1) < f(x_2) \text{ if } x_1 < x_2$$

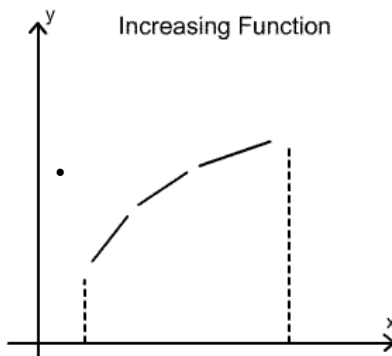


$$f(x_1) > f(x_2) \text{ if } x_1 < x_2$$

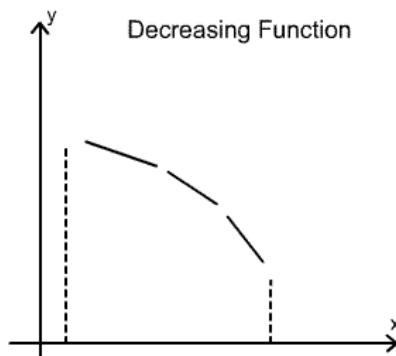


$$f(x_1) = f(x_2) \text{ for all } x_1, x_2$$

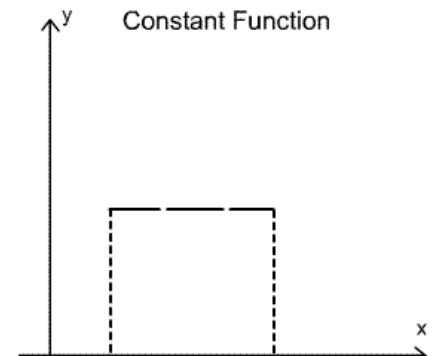
Let's take a few points on the 3 graphs in above figures and make tangent lines on these points. This gives



Tangent lines have positive slope



Tangent lines have negative slope



Tangent lines have zero slope

Note that incase where the graph was increasing, we get tangent line with positive slopes, decreasing we get negative slope, and constant gives 0 slope.

Let's formalize this idea

Theorem 4.2.1

Let f be a function that is continuous on a closed interval $[a,b]$ and differentiable on the open interval (a,b)

- (a) If $f'(x) > 0$ for every value of x in (a,b) , then f is increasing on $[a,b]$.
- (b) If $f'(x) < 0$ for every value of x in (a,b) , then f is decreasing on $[a,b]$.
- (c) If $f'(x) = 0$ for every value of x in (a,b) , then f is constant on $[a,b]$.

Example

Find the intervals on which the function is increasing and those on which **its** decreasing.

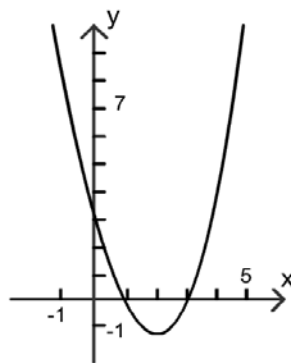
$$f(x) = x^2 - 4x + 3$$

Differentiating f gives

$$f'(x) = 2x - 4 = 2(x - 2)$$

$$f'(x) < 0 \Rightarrow 2(x - 2) < 0 \Rightarrow -\infty < x < 2$$

$$f'(x) > 0 \Rightarrow 2(x - 2) > 0 \Rightarrow 2 < x < +\infty$$



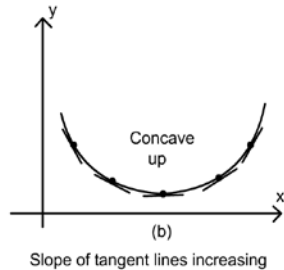
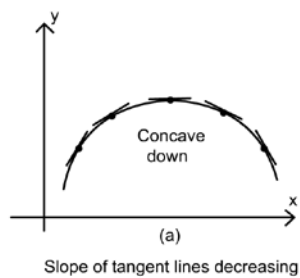
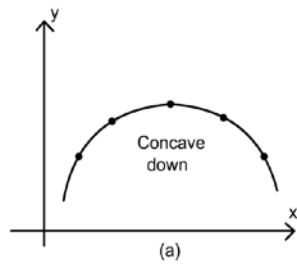
$$f(x) = x^2 - 4x + 3$$

Since f is continuous on $(2, +\infty)$, the function is actually increasing on the interval $[2, +\infty)$

Similarly it is decreasing on the interval $(-\infty, 2]$

The derivative is 0 at the point $x=2$

Since this is the only point not the interval we think of the point $x=2$ as the point where the transition occurs from decreasing to increasing in f .



Definition 4.2.3

Let f be differentiable on an interval

- (a) f is called concave up on the interval if f is increasing on the interval.
- (b) f is called concave down on the interval if f is decreasing on the interval.

Example

Find the open intervals on which the given function is concave up and on those on which it is concave down.

$$f(x) = x^2 - 4x + 3$$

$$f'(x) = 2x - 4$$

$$f''(x) = 2$$

THEOREM 4.2.2

- (a) If $f''(x) > 0$ on a open interval (a,b) then f is concave up on (a,b)
- (b) If $f''(x) < 0$ on a open interval (a,b) then f is concave down on (a,b)

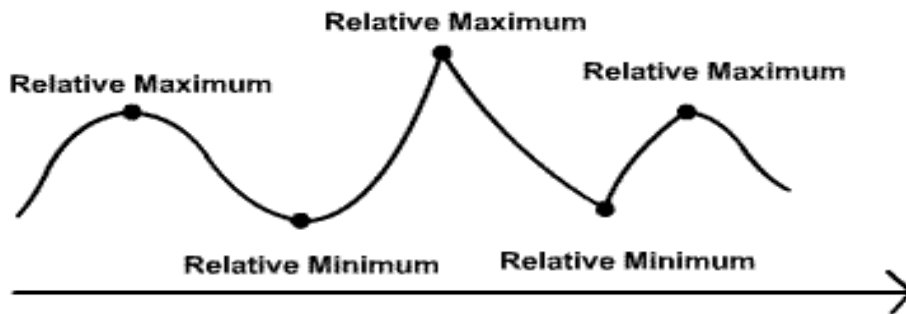
$$f''(x) = 2$$

Since f'' is greater than 0 for all x , the graph of this function is concave up on the interval $(-\infty, +\infty)$

Lecture # 22

Relative Extrema

- Relative Maxima
- Relative Minima
- Critical Points
- First Derivative test
- Second Derivative test
- Graphs of Polynomials
- Graphs of Rational functions

Relative Maxima

Most of the graphs we have seen have ups and downs, much like Hills and valleys on earth.

The Ups or the Hills are called relative Maxima.

The downs or the Valleys are called relative Minima

The reason we use the word relative is that just like a given Hill in a mountain range need not necessarily be the Highest point in the range. Similarly a given maxima in a graph need not be the maximum possible value in the graph.

Same goes for the relative minima.

In general, we may say that a given Hill is the highest one in some area. Look at relative maxima in a given interval.

Again, same is true for valleys and relative minima.

So when we talk about relative maxima and relative minima, we talk about them in the context of some interval.

Definition 4.3.1

A function f is said to have a relative maximum at x_0 if $f(x_0) \geq f(x)$ for all x in some open interval containing x_0

Definition 4.3.2

A function is said to have a relative minimum at x_0 if $f(x_0) \leq f(x)$ for all x in some open interval containing x_0 .

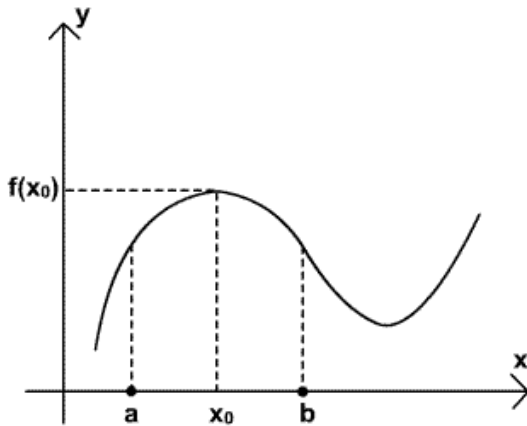
Definition 4.3.3

A function f is said to have a relative extremum at x_0 if it has either a relative maximum or relative minimum at x_0

Example

Here is a graph of a function f . This has a relative maximum in the interval (a, b) because from the graph it's obvious that

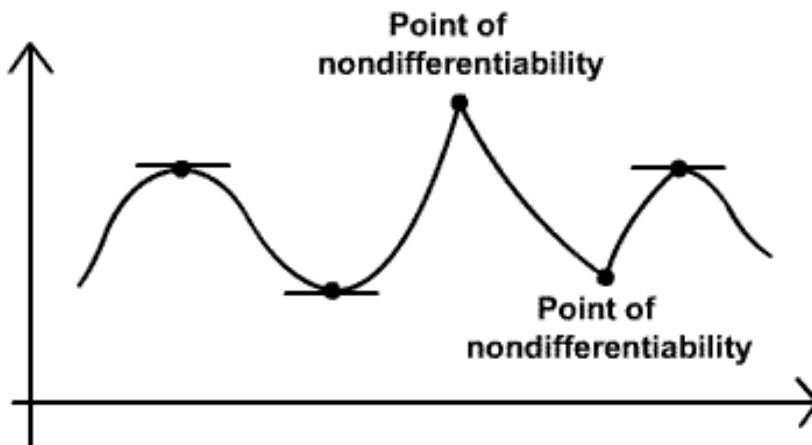
$$f(x_0) \geq f(x)$$



Function has a relative maximum in the interval (a, b)

Critical Points

It so happens that relative extrema can be viewed as transition points that separate the regions where a graph of a function is increasing from those where a graph is decreasing.



Here is a figure. This shows that relative extrema of a function occur at points where f has a horizontal tangent, or where the function is not differentiable.

Horizontal tangent means derivative = 0.

Non-differentiable means corners.

Theorem 4.3.4

If f has a relative extremum at x_0 , then
either $f'(x_0) = 0$ or f is not differentiable at x_0

Definition 4.3.5

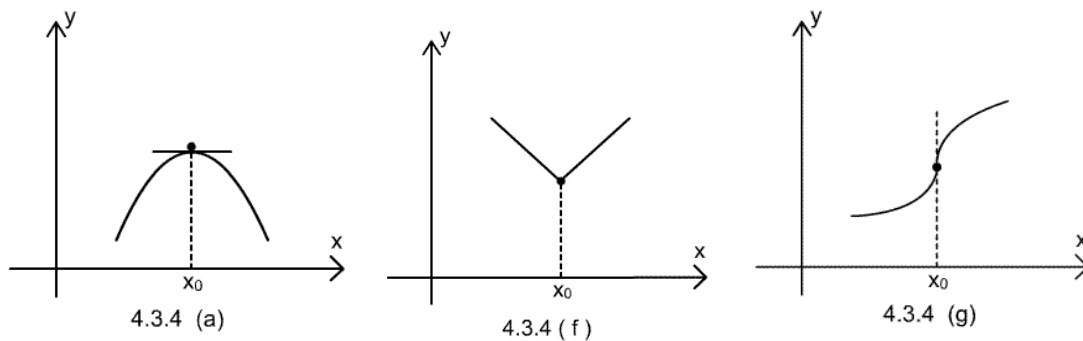
A critical point for a function f is any value of x in the domain of f at which $f'(x) = 0$ or at which f is not differentiable; the critical points where $f'(x) = 0$ are called stationary points of f .

So theorem 4.3.4 can be read as now with this new terminology As “The relative extrema of a function, if any, occur at critical points.”

Example

- a) x_0 here is a critical and stationary point as tangent line has slope 0
- f) x_0 here is a critical point and it has minimum value at that point but the tangent line is not defined at that point.
- g) x_0 is a critical point but not stationary as derivative does not exist

Here are the figures of the situations.



First Derivative test and Second Derivative test

Note that in (g) of the last figure, x_0 was a critical point, but there was not relative extrema there! This can happen.

So how do we know at which critical point a relative extrema occurs or not?

Here is a theorem for that.

THEOREM 4.3.6

(First Derivative Test)

- a) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0 , then f has a relative maximum at x_0
- b) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0 , then f has a relative minimum at x_0

- (c) If $f'(x)$ has the same sign [either $f'(x) > 0$ or $f'(x) < 0$] on an open interval extending left from x_0 and on an open interval extending right from x_0 , then f does not have a relative extremum at x_0

In short: “The relative extrema, if any, on an open interval where a function f is continuous and not constant occurs at those critical points where f' changes sign”

Example

Locate the relative extrema of

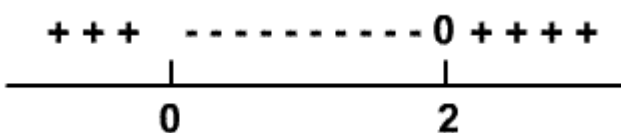
$$f(x) = 3x^{\frac{5}{3}} - 15x^{\frac{2}{3}}$$

$$f'(x) = 5x^{\frac{2}{3}} - 10x^{-\frac{1}{3}} = 5x^{\frac{1}{3}}(x-2) = \frac{5}{x^{\frac{1}{3}}}(x-2)$$

Note that there are two critical points, namely $x = 0$ and $x = 2$. Because at $x = 2$, the derivative $f' = 0$, and at $x = 0$, the derivative does not exist.

Now we need to know where there is a relative extrema by checking for the changing sign of f' at the 2 critical points.

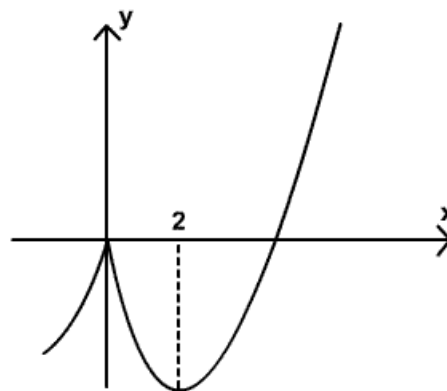
We use theorem 4.3.6 and draw a number line test



We see that there is relative maximum at 0

We see that there is relative minimum at 2

There is another test for finding extrema easier than the first derivative test.

**Theorem 4.3.7 (Second Derivative Test)**

Suppose that f is twice differentiable at a stationary point x_0

(a) If $f''(x_0) > 0$, then f has relative minimum at x_0 .

(b) If $f''(x_0) < 0$, then f has relative maximum at x_0 .

EXAMPLE

Locate the relative extrema of $f(x) = x^4 - 2x^2$

$$f'(x) = 4x^3 - 4x = 4x(x-1)(x+1)$$

$$f''(x) = 12x^2 - 4$$

Setting $f'(x) = 0$ gives stationary points $x = 0$ and $x = \pm 1$

Also,

$$f''(0) = -4 < 0$$

$$f''(1) = 8 > 0$$

$$f''(-1) = 8 > 0$$

There is a relative maximum at $x = 0$, and relative minima at $x = 1$ and $x = -1$

Graphs of Polynomials

In applied sciences and engineering, it is required many times to understand the behavior of a function.

Graphs are a good way to understand function behavior. But many times it is hard to graph the function.

So it is often necessary to understand the behavior in terms of maxima and minima and concavity etc.

We will look at how the stuff from the last lecture helps us in graphing polynomial and rational functions

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Graphs are a good way to understand function behavior. But many times it is hard to graph the function.

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We will look at how the stuff from the last lecture helps us in graphing polynomial and rational functions

Graphs of Polynomials

Let $P(x)$ be a polynomial function

- Calculate $P'(x)$ and $P''(x)$
- Using P' , determine the stationary points and the intervals of increase and decrease
- Use P'' to determine the inflection points and interval where P is concave up and concave down
- Plot all of the above and the x and y intercepts

Example

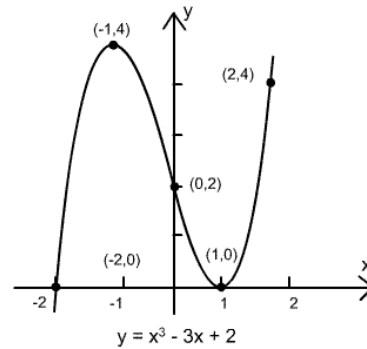
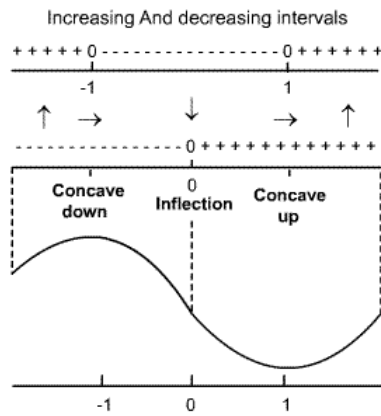
Sketch the graph of $P(x) = y = x^3 - 3x + 2$

$$\frac{dy}{dx} = 3x^2 - 3 = 3(x-1)(x+1)$$

$$\frac{d^2y}{dx^2} = 6x$$

You find the stationary points, inflection points.

x	$y = x^3 - 3x + 2$
-2	0
-1	4
0	2
1	0
2	4



Above figure Shows the intervals of increase/decrease and of concavity Y intercept at (0,2) Inflection point is at $x = 0$

Graph of Rational Functions

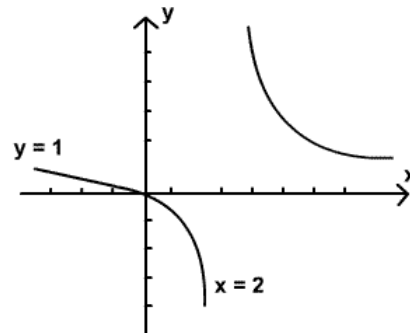
Rational function, remember, is a function defined by the ratio of two polynomials

$$R(x) = \frac{P(x)}{Q(x)}$$

Its obvious that if $Q(x) = 0$, then $R(x)$ has discontinuity at those values of x where $Q(x) = 0$

Consider the following graph of

$$f(x) = \frac{x}{x-2}$$



A line $x = x_0$ is called a vertical asymptote for the graph of a function f if $f(x) \rightarrow +\infty$ or $f(x) \rightarrow -\infty$ as x approaches x_0 from the right or from the left. A line $y = y_0$ is called a horizontal asymptote for the graph of f if

$$\lim_{x \rightarrow +\infty} f(x) = y_0 \quad \text{or} \quad \lim_{x \rightarrow -\infty} f(x) = y_0$$

Vertical asymptotes occur where the denominator is $\neq 0$

Example

Find horizontal and vertical asymptotes of

$$f(x) = \frac{x^2 + 2x}{x^2 - 1}$$

The vertical asymptotes occur at the points where $x^2 - 1 = 0$; these are the points $x = -1$ and $x = 1$,
Since

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{x^2 + 2x}{x^2 - 1} = \lim_{x \rightarrow +\infty} \frac{x^2}{x^2} = 1$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{x^2 + 2x}{x^2 - 1} = \lim_{x \rightarrow -\infty} \frac{x^2}{x^2} = 1$$

It follows that $y = 1$ is a horizontal asymptote.

Lecture # 23**Maximum and Minimum Values of Functions**

- Absolute Extrema
- Finding Absolute Extrema for a continuous function
- Summary of extreme behaviors of functions over
- Applied maximum and minimum problems
- Problems involving finite closed intervals
- Problems involving intervals that are non-finite and closed

Absolute Extrema

Previously we talked about relative maxima, relative minima of functions.

These were like the highest mountain and the deepest valley in a given vicinity or neighborhood.

Now we will talk about absolute maximum and minimum values of functions.

These are like the highest peak in a mountain range, and the deepest valley.

DEFINITION 4.6.1

If $f(x_0) \geq f(x)$ for all x in the domain of f , then $f(x_0)$ is called the absolute maximum value or simply the maximum value of f .

DEFINITION 4.6.2

If $f(x_0) \leq f(x)$ for all x in the domain of f , then $f(x_0)$ is called the absolute minimum value or simply the minimum value of f .

Absolute maximum means that the value is the maximum one over the entire domain of the function.

Absolute minimum means that the value is the minimum one over the entire domain of the function.

If we think of the Earth's surface as defining some function, then its absolute maximum will be Mt.

Everest, and absolute minimum will be the Marianna Trench in the Pacific Ocean near Hawaii.

Example

Consider the following picture of the graph of $f(x) = 2x+1$ on the interval $[0,3)$.

The minimum value is 1 at

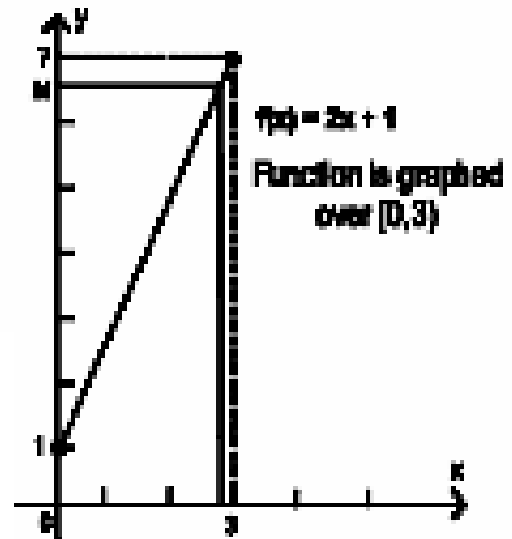
$x = 0$. But is there a maximum?

No. Because the function is defined on the interval $[0,3)$ which excludes the point $x = 3$. So note that you can get very close to 7 as the maximum value as you get very close to $x = 3$, but this is in a limiting process, and you can always get more closer to 7, and yet never EQUAL 7! So 7 looks like a max, but its NOT!

The question of interest given a function $f(x)$ is:
Does $f(x)$ has a maximum (minimum) value?
If $f(x)$ has a maximum (minimum) , what is it?
If $f(x)$ has a maximum value, where does it occur?

THEOREM 4.6.4
(Extreme-Value Theorem)

If a function f is continuous on a closed interval $[a,b]$, then f has both a maximum value and a minimum value on $[a,b]$



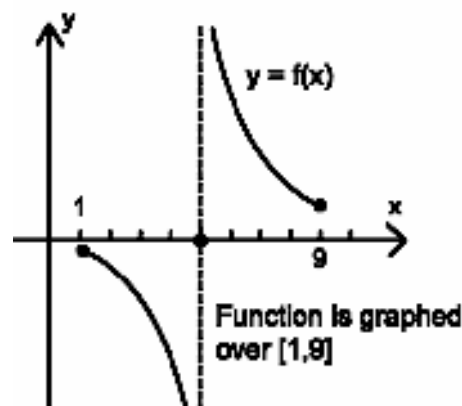
Won't prove this as its difficult. Just use it. This theorem doesn't tell us what the max and the min are, just the conditions on a function which will make it have a max or min

Example

Note that in the previous example we saw a function $f(x) = 2x+1$ which was defined over the interval $[0, 3)$. This one is a continuous function on that interval, but had no maximum because the interval was not closed!

Example

Here is another function graphed over the interval $[1,9]$. Although the interval is closed, the function not continuous on this interval as we can see from the graph, and so has no maximum or minimum values on that interval .



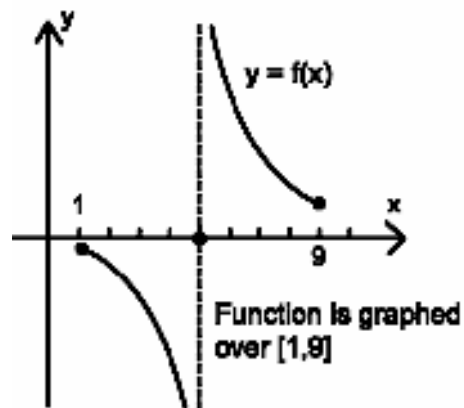
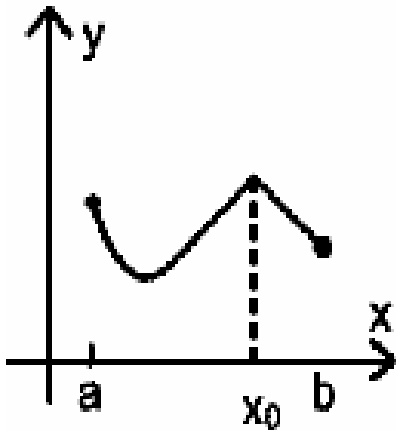
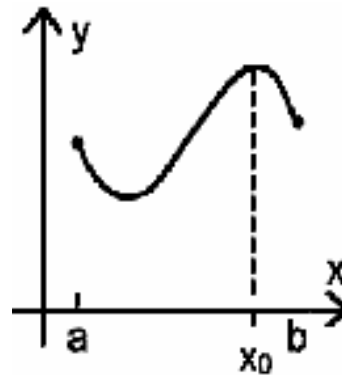
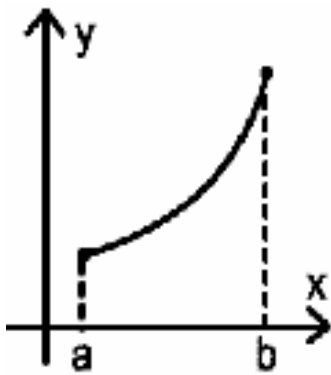
THEOREM 4.6.5

If a function f has an extreme value (either a maximum or a minimum) on an open interval (a,b) , then the extreme value occurs at a critical point of f .

Step 1: Find the critical points of f in (a,b)

Step 2: Evaluate f at all the critical points and the endpoint a and b

Step 3: The largest of the values in Step 2 is the maximum value of f on $[a,b]$ and the smallest is the minimum.



Example

Find the maximum and the minimum values of $f(x) = 2x^3 - 15x^2 + 36x$ on the interval $[1, 5]$

Since f is a polynomial, it's continuous and differentiable on the interval $(1, 5)$

$$f'(x) = 6x^2 - 30x + 36 = 0$$

$$f'(x) = (x-3)(x-2) = 0$$

So f' is 0 at $x = 2$ and $x = 3$. Max or min will occur at these two points or at the end points.

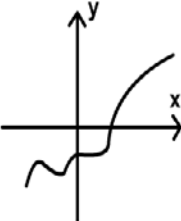
Evaluate the function at the critical points and the endpoints and we see that max is 55 at $x = 5$ and min is 23 at $x = 1$

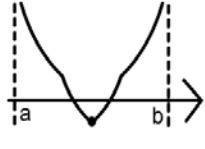
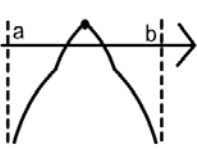
We want to know the max and min over $(-\infty, +\infty)$

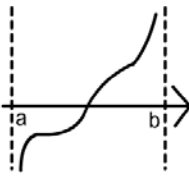
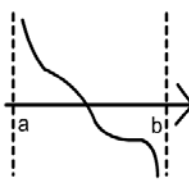
Here is how to find them for continuous functions

Limits	$\lim_{x \rightarrow -\infty} f(x) = +\infty$ $\lim_{x \rightarrow +\infty} f(x) = +\infty$	$\lim_{x \rightarrow -\infty} f(x) = -\infty$ $\lim_{x \rightarrow +\infty} f(x) = -\infty$
Conclusion if f is continuous	f has a minimum but no maximum on $(-\infty, +\infty)$	f has a maximum but no minimum on $(-\infty, +\infty)$
Graph		

Summary of extreme behaviors of functions over (a,b)

Limits	$\lim_{x \rightarrow -\infty} f(x) = -\infty$ $\lim_{x \rightarrow +\infty} f(x) = +\infty$	$\lim_{x \rightarrow -\infty} f(x) = +\infty$ $\lim_{x \rightarrow +\infty} f(x) = -\infty$
Conclusion if f is continuous	f has neither a maximum nor a minimum on $(-\infty, +\infty)$	f has neither a maximum nor a minimum on $(-\infty, +\infty)$
Graph		

Limits	$\lim_{x \rightarrow a^+} f(x) = +\infty$ $\lim_{x \rightarrow b^-} f(x) = +\infty$	$\lim_{x \rightarrow a^+} f(x) = -\infty$ $\lim_{x \rightarrow b^-} f(x) = -\infty$
Conclusion if f is continuous on (a,b)	f has a minimum but no maximum on (a,b)	f has a maximum but no minimum on (a,b)
Graph		

Limits	$\lim_{x \rightarrow a^+} f(x) = -\infty$ $\lim_{x \rightarrow b^-} f(x) = +\infty$	$\lim_{x \rightarrow a^+} f(x) = +\infty$ $\lim_{x \rightarrow b^-} f(x) = -\infty$
Conclusion if f is continuous on (a,b)	f has neither a maximum nor a minimum on (a,b)	f has neither a maximum nor a minimum on (a,b)
Graph		

Find the max and min values if any, of the function $f(x) = x^4 + 2x^3 - 1$ on the interval $(-\infty, +\infty)$

This is a continuous function on the given interval and

$$\lim_{x \rightarrow +\infty} (x^4 + 2x^3 - 1) = +\infty \quad \text{and} \quad \lim_{x \rightarrow -\infty} (x^4 + 2x^3 - 1) = +\infty$$

So f has a minimum but no maximum on $(-\infty, +\infty)$. By Theorem 4.6.5, the min must occur at a critical point. So

$$f'(x) = 4x^3 + 6x^2 = 2x^2(2x + 3) = 0$$

This gives $x = 0$, and $x = -3/2$ as the critical points. Evaluating gives $\min = -43/16$ at $x = -3/2$

Applied maximum and minimum problems

We will use what we have learnt so far to do some applied problems in OPTIMIZATION.

Optimization is the way efficiency is got in business, machines and even in nature in terms of animals competing for resources.

Problems involving continuous functions and Finite closed intervals

These are problems where the function is defined over a closed interval. These problems always have a solution because it is guaranteed by the extreme values theorem.

Example

Find the dimensions of a rectangle with perimeter 100 ft whose area is as large as possible.

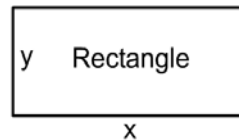
Let

$x = \text{length of the rectangle}$
in feet

$y = \text{width of the rectangle}$
in feet

$A = \text{area of the rectangle}$

$$\text{Perimeter} = 100\text{ft} = 2x + 2y$$



$$y = 50 - x$$

$$A = xy$$

$$A = x(50 - x)$$

$$A = 50x - x^2$$

Then $A = xy$

We want to maximize the area

$$A = x(50 - x) = 50x - x^2$$

$$\text{Perimeter} = 100\text{ft} = 2x + 2y \quad \text{or } y = 50 - x$$

Use this values of y in the equation $A = xy$ to get A a function of x

Because x represents a length, it cannot be negative and it cannot be a value that exceeds the perimeter of 100 ft. So we have the following constraints on x

$$0 \leq x \leq 50$$

So the question is of finding the max of $A = 50x - x^2$ on the interval $[0, 50]$. By what we have seen so far, that max must occur at the end points of this interval or at a critical point

Critical points:

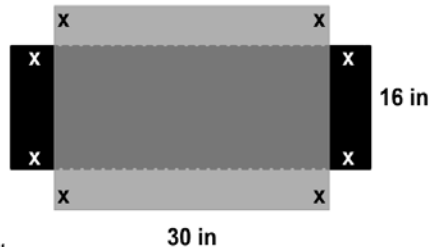
$$\frac{dA}{dx} = 50 - 2x = 0 \Rightarrow x = 25$$

So now we substitute $x = 0$, $x = 25$, and $x = 50$ into the function A to get the max 625 at the point $x = 25$.

Note that $y = 25$ also for $x = 25$. So the rectangle with perimeter 100 with the greatest area is a square with sides 25ft.

Example

An open box is to be made from a 16 inch by 30 inch piece of cardboard by cutting out squares of equal size from the 4 corners and bending up the sides. What size should the squares be to obtain a box with largest possible volume?



Let

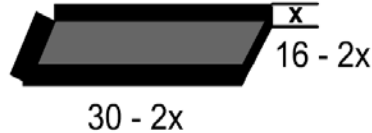
x = length of the sides of the squares to be cut out

V = volume of the resulting box.

We want to maximize the Volume V .

If we cut out the squares from the 4 corners of the cardboard, the resulting BOX will have dimensions

$(16 - 2x)$ by $(30 - 2x)$



$$\begin{aligned} V &= (\text{length})(\text{width})(\text{Height}) \\ &= (16 - 2x)(30 - 2x)x \\ &= 480x - 92x^2 + 4x^3 \\ 0 &\leq x \leq 8 \end{aligned}$$

s

So want to find max of the function on $[0, 8]$.

$$\frac{dV}{dx} = 480 - 184x + 12x^2 = 0 \Rightarrow x = \frac{10}{3} \text{ and } x = 12$$

Because $x = 12$ is out of $[0, 8]$, ignore it. Check V at the end points and at $x = 10/3$. We see then that $V = 19600/27$ is max when $x = 10/3$.

Lecture # 24

Newton's Method, Rolle's Theorem, and the Mean Value Theorem

- Newton's method for approximating solutions to $f(x)=0$
- Some difficulties with Newton's method
- Rolle's theorem
- Mean Value Theorem

Newton's method for approximating solutions to $f(x)=0$

We have seen in algebra that the solution for the equation $ax + b = 0$ is $x = -\frac{b}{a}$.

Similarly we have algebraic formulas for polynomial equation up to degree 5.

But there is no algebraic solution to the equation of the kind

$$x - \cos(x) = 0$$

For this equation and many like it, we settle for approximate solutions. How do we approximate the solutions? There are many methods, and one of them is Newton's method. Here is how Newton's method works to find approximate solutions to equations.

What does it mean for an equation to have a solution? or that $f(x) = 0$?

It means that we are looking for those x values, for which the corresponding y value or $f(x)$ is 0.

This means that the solutions are those points where the graph of the function crosses the x axis.

Suppose that $x = r$ is the solution we are looking for.

Let's approximate it by an initial guess called.

We draw a line tangent to the graph of the given function at the point.

If the tangent line is not parallel to the x -axis, then it will x_2 eventually intersect the x -axis at some

point x_1 which will generally be closer to r than x_2 . We will repeat the process, with a tangent line at x_2 that meets x axis at x_3 , and so on...

This is Newton's method.

We need a formula for this method.

Note that the tangent line at x_1

has the equation

$$y - f(x_1) = f'(x_1)(x_2 - x_1)$$

If $f'(x_1) \neq 0$, then the

line meets the x axis at $(x_2, 0)$

Plug this coordinate into above equation, we get

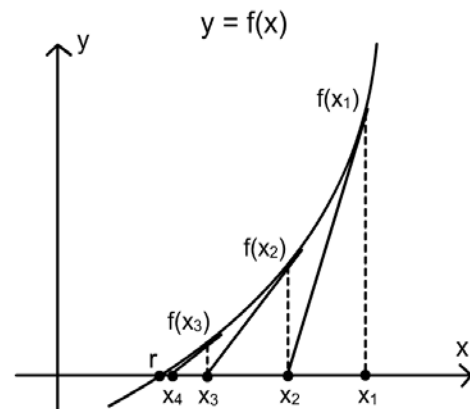
$$-f(x_1) = f'(x_1)(x_2 - x_1)$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Repeating this process for a third point $(x_3, 0)$ gives

$$-f(x_2) = f'(x_2)(x_3 - x_2)$$

$$\Rightarrow x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$



In general, then we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

There is limiting process involved here for finding the solutions. We get as close as we like to the solution.

Example

The equation $x = \cos(x)$ has a solution between 0 and 1. Approximate it using Newton's method.

Rewrite as $x - \cos(x)$ so our function is $f(x) = x - \cos(x)$. The derivative is

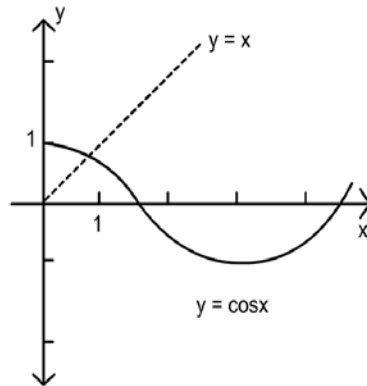
$$f'(x) = 1 + \sin(x)$$

So we have

$$x_{n+1} = x_n - \frac{x_n - \cos(x_n)}{1 + \sin(x_n)}$$

As our approximation formula

Here is a graph of the situation. From the graph it looks like the solution is closer to 1 than 0. So we will use $x_1 = 1$. So we get the following approximations.



$$x_2 = x_1 - \frac{x_1 - \cos(x_1)}{1 + \sin(x_1)} = 1 - \frac{1 - \cos(1)}{1 + \sin(1)} = 0.7503$$

$$x_3 = x_2 - \frac{x_2 - \cos(x_2)}{1 + \sin(x_2)} = 0.7503 - \frac{0.7503 - \cos(0.7503)}{1 + \sin(0.7503)} = 0.7391$$

You may continue if you will, but we will say that the solution is approximately $x \approx 0.7391$

Some difficulties with Newton's method

- Newton's method does not always work.
- If for some values of n $f'(x_n)$ then the formula for Newton's method involves division by 0 and we are out of business.

- Such a case will occur if the tangent line for some approximation has slope 0 or is parallel to the x axis.
- Sometimes the approximations don't converge to a solution.

Consider the equation $x^{\frac{1}{3}} = 0$

The only solution is $x = 0$. Let's approximate it by Newton's Method with initial approx $x_1 = 1$. We get the following Formula

$$x_{n+1} = x_n - \frac{(x_n)^{\frac{1}{3}}}{\frac{1}{3}(x_n)^{-\frac{2}{3}}} = -2x_n$$

Plug in $x_1 = 1$ and then the following approx to see that the values do not converge

Rolle's theorem

Rolle's Theorem says essentially that for a certain kind of function, if it crosses the x-axis at two points, then there is one point between those two points where the derivative of f is 0.

THEOREM 4.9.1 (Rolle's Theorem)

Let f be differentiable on (a,b) and continuous on $[a,b]$. If $f(a) = f(b) = 0$, then there is at least one point c in (a,b) where $f'(c) = 0$.

Example

The function $f(x) = \sin(x)$ is continuous and differentiable everywhere, hence continuous on $[0, 2\pi]$ and differentiable on $(0, 2\pi)$. Also, $f(0) = \sin(0) = 0$ and $f(2\pi) = \sin(2\pi) = 0$

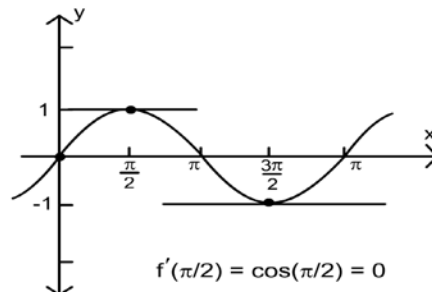
So the function satisfies the hypotheses of Rolle's Theorem. So there exists a point c in the interval $(0, 2\pi)$ such that

$$f'(c) = \cos(c) = 0$$

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$\Rightarrow$

$$y = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$



Here is a more tangible way to think of Rolle's theorem

I leave from Lahore to Islamabad.

When i start driving from Lahore, my velocity is 0, and when i reach Islamabad, my velocity is 0 as well.

Velocity is a continuous function on the interval $[0, 376]$.

Also, velocity is differentiable on $(0, 376)$ as its derivative acceleration is defined at each point on the velocity curve.

Hence during my drive from Lahore to ISB, there is some point on the motorway where the acceleration of the car was 0.

One could argue: What if you keep accelerating on the motorway!

Mean Value Theorem

This says basically that under the right conditions, a function will have the same slope for the tangent line at a point as that of a certain secant line.

THEOREM 4.9.2

(Mean-Value Theorem)

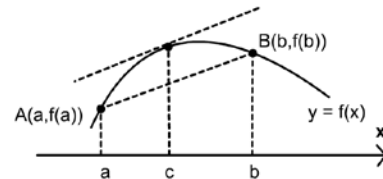
Let f be differentiable on (a,b) and continuous on $[a,b]$. Then there is at least one point c in (a,b) where

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof of MVT

From this figure we have the following:

Slope of Secant line joining A and B:



$v(x)$ = vertical distance between curve $f(x)$ and secant line through A and B

$$v(x) = f(x) - \left[\frac{f(b) - f(a)}{b - a}(x - a) + f(a) \right]$$

Since $f(x)$ is continuous $[a, b]$ and differentiable on (a, b) , so is $v(x)$ by its formula involving $f(x)$.

Also note that

$$v(a) = 0 \text{ and } v(b) = 0$$

So $v(x)$ satisfies the assumptions of Rolle's theorem on the interval $[a, b]$. So there is a point c in (a, b) such that $v'(c) = 0$. But note that

$$\begin{aligned}v'(x) &= f'(x) - \left[\frac{f(b) - f(a)}{b - a} \right] \\ \Rightarrow v'(c) &= f'(c) - \left[\frac{f(b) - f(a)}{b - a} \right]\end{aligned}$$

So by this last formula, at the point where $v'(c) = 0$, we have

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Lecture # 25

Integrations

In this lecture we will look at the beginnings of the other major Calculus problem.

- The Area Problem
- Anti-derivatives (Integration)
- Integration formulas
- Indefinite Integral
- Properties of Indefinite Integral

The Area Problem

Given a continuous and non negative function on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x -axis.

Instead of trying to solve a particular case like the one in the picture we just saw, we will generalize to solve this problem where the right end point will be any number x greater than or equal to a instead of just b . We will denote the area we are trying to find as $A(x)$ because this will be a function of x now as it depends on how far away x is from a .

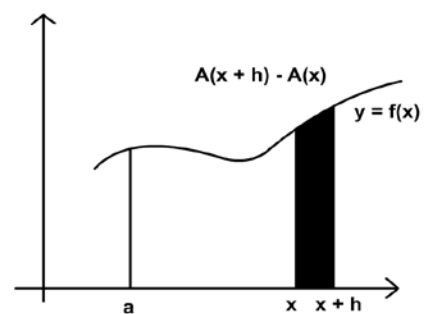
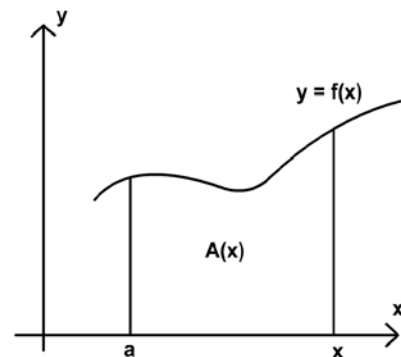
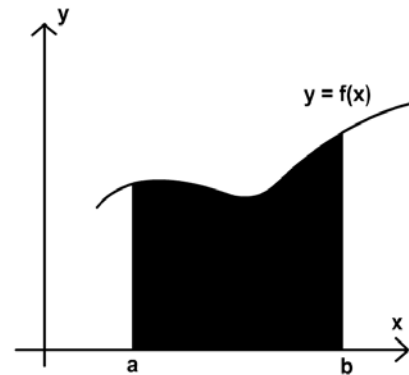
It was the idea of Newton and Leibniz that to find the unknown area $A(x)$, first find its derivative $A'(x)$ and use this derivative to determine what $A(x)$ is! Interesting approach. So we want to find out first.

$$A'(x) = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h}$$

Let's assume for now that $h > 0$

The top of the derivative quotient is the difference of the two areas $A(x)$ and $A(x+h)$

Let c be the midpoint of between x and $x+h$. Then the difference of areas can be approximated by the area of the rectangle with base length h and height $f(c)$.



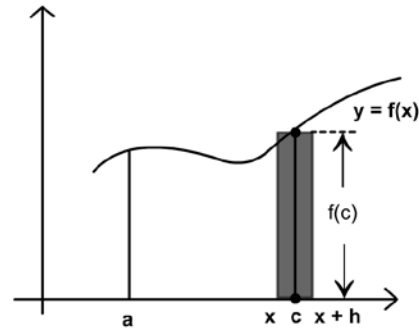
So we have

$$\frac{A(x+h) - A(x)}{h} \approx \frac{f(c) \cdot h}{h} = f(c)$$

Note that the error in the approximation from this rectangle in **A** will approach 0 as h goes to 0.

Then we have

$$A'(x) = \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} = \lim_{h \rightarrow 0} f(c)$$



As h goes to 0, c approaches x . Also, f is assumed to be a continuous function, so we have that $f(c)$ goes to $f(x)$ as c goes to x . Thus

$$\lim_{h \rightarrow 0} f(c) = f(x) \Rightarrow A'(x) = f(x)$$

So: The derivative of the area function $A(x)$ is the function whose graph forms the upper boundary of the region under which the area is to be found

Example

Find the area of the region under

the graph of $y = f(x) = x^2$

over the interval $[0, 1]$

Look at the situation over the interval $[0, x]$. Then we have from

the discussion that $A'(x) = x^2$

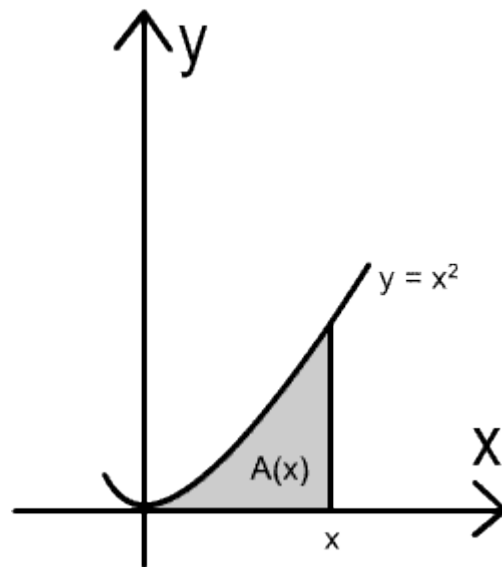
To find $A(x)$ we look for a

function whose derivative is x^2

This is called an antidifferentiation problem as we are trying to find $A(x)$ by undoing a differentiation.

A guess is the function

$$A(x) = \frac{1}{3}x^3$$



This is a formula for the areas function. So on the interval $[0, 1]$, we have $x = 1$ and our result is

$$A(1) = 1/3 \text{ units}$$

Anti-derivatives

Definition 5.2.1

A function F is called antiderivative of a function f on a given interval if $F'(x) = f(x)$ for all x in the interval.

Example

The functions $\frac{1}{3}x^3$, $\frac{1}{3}x^3 - \pi$, $\frac{1}{3}x^3 + C$ are all anti-derivatives of

$$f(x) = x^2 \text{ on the interval } (-\infty, +\infty)$$

As the derivative of each is $f(x) = x^2$

If $F(x)$ is any anti-derivative of $f(x)$, then so is $F(x) + C$ where C is a constant.

Here is a theorem

THEOREM 5.2.2

If $F(x)$ is any antiderivative of $f(x)$ on a given interval, then for any value of C the function $F(x) + C$ is also an antiderivative of $f(x)$ on that interval; moreover every antiderivative of $f(x)$ on the interval is expressible in the form $F(x) + C$, where C is constant.

Indefinite Integral

The process of finding anti-derivatives is called anti-differentiation or Integrations.

If there is some function F such that $\frac{d}{dx}[F(x)] = f(x)$

Then function of the form $F(x) + C$ are anti-derivatives of $f(x)$.

We denote this by $\int f(x)dx = F(x) + C$

The symbol $\int$ is called the integral sign and $f(x)$ is called the integrand.

It is read as the “Indefinite integral of $f(x)$ equals $F(x)$ ” $\int f(x)dx = F(x) + C$

The right side of the above equation is not a specific function but a whole set of possible functions.

That’s why we call it the Indefinite integral.

C is called the constant of integrations.

Example

As we saw earlier, the anti-derivatives of $f(x) = x^2$ are functions of the form So we can write

$$F(x) = \frac{1}{3}x^3 + C$$

The dx serves to identify the independent variable in the function involved in the integration.

Examples:

DERIVATIVE FORMULA	EQUIVALENT INTEGRATION FORMULA
$\frac{d}{dx}[x^3] = 3x^2$	$\int 3x^2 dx = x^3 + C$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dt}[\tan t] = \sec^2 t$	$\int \sec^2 t dt = \tan t + C$
$\frac{d}{dx}[u^{3/2}] = 3/2u^{1/2}$	$\int 3/2u^{1/2} du = u^{3/2} + C$

Example

From the table we just saw, we obtain the following results.

$$\int x^2 dx = \frac{x^3}{3} + C$$

$$\int x^3 dx = \frac{x^4}{4} + C$$

$$\int \frac{1}{x^5} dx = \int x^{-5} dx = -\frac{1}{4x^4} + C$$

Properties of Indefinite Integral

$$\frac{d}{dx} \left[\int f(x) dx \right] = f(x)$$

THEOREM 5.2.3

- a) A constant can be moved through an integral sign; that is,

$$\int cf(x) dx = c \int f(x) dx$$

- b) An antiderivative of a sum is the sum of the antiderivative; that is

$$\int [f(x) + g(x)] dx = \int f(x) dx + \int g(x) dx$$

c) An antiderivative of a difference is the difference of the antiderivative; that is

$$\int [f(x) - g(x)] dx = \int f(x) dx - \int g(x) dx$$

Example

Evaluate $\int 4 \cos(x) dx$

$$\int 4 \cos(x) dx = 4 \int \cos(x) dx = 4[\sin(x) + C] = 4 \sin(x) + K$$

Where $4C = K$

Example

$$\int (x^2 + x) dx$$

$$\int (x^2 + x) dx = \int x^2 dx + \int x \cdot dx = \frac{1}{3} x^3 + \frac{1}{2} x^2 + C$$

Generalized version of the Theorem 5.2.3 b and c

$$\begin{aligned} & \int [c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x)] dx \\ &= c_1 \int f_1(x) dx + c_2 \int f_2(x) dx + \dots + c_n \int f_n(x) dx \end{aligned}$$

Example

$$\begin{aligned} & \int (3x^6 - 2x^2 + 7x + 1) dx \\ &= 3 \int x^6 dx - 2 \int x^2 dx + 7 \int x dx + \int dx \\ &= \frac{3x^7}{7} - \frac{2x^3}{3} + \frac{7x^2}{2} + x + C \end{aligned}$$

Example

$$\begin{aligned} & \int \frac{\cos(x)}{\sin^2(x)} dx \\ &= \int \frac{1}{\sin(x)} \frac{\cos(x)}{\sin(x)} dx \quad . \\ &= \int \csc(x) \cot(x) dx = -\csc(x) + c \end{aligned}$$

Lecture # 26

Integration by substitution

This is like the Chain Rule we saw for differentiation. The idea is to integrate functions that are composition of different functions.

$$\frac{d}{du}[G(u)] = f(u)$$

Also this means that

$$\int f(u)du = \int \frac{d}{du}[G(u)]du = G(u) + C$$

Let U be a function of x . Then we have

$$\frac{d}{dx}[G(u)] = \frac{d}{du}[G(u)] \cdot \frac{du}{dx} = f(u) \frac{du}{dx}$$

Now if we apply the integral on both sides w . r . t x we get

$$\int \left(f(u) \frac{du}{dx} \right) dx = \int \frac{d}{dx}[G(u)] dx = G(u) + C$$

$\Rightarrow$

$$\int \left(f(u) \frac{du}{dx} \right) dx = \int f(u)du$$

Example

$$\int (x^2 + 1)^{50} \cdot 2x dx$$

Let $u = x^2 + 1$. Then $\frac{du}{dx} = 2x$.

Now we can rewrite the given problem as

$$\int (x^2 + 1)^{50} \cdot 2x dx = \int \left[u^{50} \frac{du}{dx} \right] dx = \int u^{50} du$$

$$\begin{array}{ccc} \uparrow & & \uparrow \\ \int \left[f(u) \frac{du}{dx} \right] dx & = & \int f(u) du \end{array}$$

$$\int u^{50} du = \frac{u^{51}}{51} + C = \frac{(x^2 + 1)^{51}}{51} + C, \text{ where } u = x^2 + 1$$

Caution: Don't feel tempted to just add 1 to the power 50 in the original problem!! That will be incorrect.

The reason is that the "Power Rule" for the integrals is applicable to function which are not a composition of others.

In this example, $(x^2 + 1)^{50}$ is a composition of two functions.

Here is a summary of the **general procedure** we need to follow to do integration by **u-substitution**.

Step 1: Make the choice for u , say $u = g(x)$

Step 2: Compute $du/dx = g'(x)$

Step 3: Make the substitution $u = g(x)$, $du = g'(x) dx$ in the original integral.

By this point, the whole original integral should be in terms of u and there should be no x 's in it.

Step 4: Evaluate the resulting integral

Step 5: Replace u by $g(x)$ so the final answer is in x .

- No hard and fast rule for choosing u .
- The choice of u should be such that the resulting calculus
- Results in a simplified integral.
- There may be more than one choice of u in some cases.

Only practice and experience makes things easier.

Think of this choice of u like playing chess. You need to choose it so that the future looks bright for problem solution!

Integrand is the derivative of a known function with a constant added or subtracted from the independent variable.

If the integrand is the derivative of a known function with a constant added or subtracted from the independent variable, then the substitution is the easiest.

Here is table 5.2.1 that we saw earlier. This will help recall BASIC integration formulas.

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
$\frac{d}{dx}[x] = 1$	$\int dx = x + C$
$\frac{d}{dx}\left[\frac{x^{r+1}}{r+1}\right] = x^r$	$\int x^r dx = \frac{x^{r+1}}{r+1} + C (r \neq -1)$
$\frac{d}{dx}[\sin x] = \cos x$	$\int \cos x dx = \sin x + C$
$\frac{d}{dx}[-\cos x] = \sin x$	$\int \sin x dx = -\cos x + C$

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
$\frac{d}{dx}[\tan x] = \sec^2 x$	$\int \sec^2 x dx = \tan x + C$
$\frac{d}{dx}[-\cot x] = \csc^2 x$	$\int \csc^2 x dx = -\cot x + C$
$\frac{d}{dx}[\sec x] = \sec x \tan x$	$\int \sec x \tan x dx = \sec x + C$
$\frac{d}{dx}[-\csc x] = \csc x \cot x$	$\int \csc x \cot x dx = -\csc x + C$

We want to integrate a function in which “The integrand is the derivative of a known function with a constant added or subtracted from the independent variable”

Example

$$\int \sin(x+9)dx = \int \sin(u)du = -\cos(u) + C = -\cos(x+9) + C$$

$$\begin{array}{c} \uparrow \\ u = x + 9 \end{array}$$

$$\frac{du}{dx} = 1 \Rightarrow du = 1 \cdot dx$$

The integrand sine is the derivative of cosine function

Example

$$\int (x-8)^{23} dx = \int u^{23} du = \frac{u^{24}}{24} + C = \frac{(x-8)^{24}}{24} + C$$

$$\begin{array}{c} \uparrow \\ u = x - 8 \end{array}$$

$$du = 1 \cdot dx = dx$$

Integrand is the derivative of a known function and a constant multiplies the independent variable

Example

$$\int \cos(5x)dx = \int \cos(u) \frac{du}{5} = \frac{1}{5} \int \cos(u)du = \frac{1}{5} \sin(u) + C = \frac{1}{5} \sin(5x) + C$$

$$\begin{array}{c} \uparrow \\ u = 5x \end{array}$$

$$du = 5dx \Rightarrow \frac{du}{5} = dx$$

Example

$$\int \sin^2(x) \cos(x) dx$$

Let $u = \sin(x)$, then

$$du = \cos(x) dx$$

So

$$\int \sin^2(x) \cos(x) dx = \int u^2 du = \frac{u^3}{3} + C = \frac{\sin^3(x)}{3} + C$$

Example

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$$

Let $u = \sqrt{x}$, then

$$du = \frac{1}{2\sqrt{x}} dx$$

So

$$\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx = \int 2 \cos(u) du = 2 \int \cos(u) du = 2 \sin(u) + C = 2 \sin(\sqrt{x}) + C$$

Complicated example

$$\int t^4 \sqrt{3-5t^5} dt \int (x^2+1)^{50} \cdot 2x dx = \int \left[u^{50} \frac{du}{dx} \right] dx = \int u^{50} du$$

What should the substitution be?

Well the idea is to get an expression to equal u so that when we differentiate it, we get a formula that involves a du and everything that was left over in x after the substitution.

$$\int t^4 \sqrt{3-5t^5} dt$$

$$u = 3-5t^5$$

$$du = -25t^4 dt \Rightarrow -\frac{1}{25} du = t^4 dt$$

So

$$\begin{aligned} \int t^4 \sqrt{3-5t^5} dt &= -\frac{1}{25} \int \sqrt[3]{u} du = -\frac{1}{25} \int u^{\frac{1}{3}} du = -\frac{1}{25} \frac{u^{\frac{4}{3}}}{\frac{4}{3}} + C \\ &= -\frac{3}{100} (3-5t^5)^{\frac{4}{3}} + C \end{aligned}$$

Lecture #. 27

Sigma Notation

- Sigma notation is used to write lengthy sums in compact form.
- Sigma or Σ is an Upper case letter in Greek.
- This symbol is called Sigma or summation as it is used to represent lengthy sums.

Here is an example of how this notation works.

Consider the sum

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

Note that every term in this sum is the square of an integer from 1 to 5.

Let's assign these integers a variable k and keep it in mind that this k can take on values from 1 to 5.

Then we can say that k^2 represent each of the elements in the sum. So we can write

this as $\sum_{k=1}^5 k^2$

“Summation of k^2 where k goes from 1 to 5

Example

$$\sum_{k=4}^8 k^3 = 4^3 + 5^3 + 6^3 + 7^3 + 8^3$$

$$\sum_{k=1}^5 2k = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) = 2 + 4 + 6 + 8 + 10$$

$$\sum_{k=0}^5 (-1)^k (2k+1) = 1 - 3 + 5 - 7 + 9 - 11$$

The number on the top are called the upper limits of the summation and the numbers at the bottom are called lower limits of the summation.

The letter k is called the index of the summation

It is not necessary that the letter k represents the index of summation. We could use i or j or m etc.

Example

$$\sum_{i=1}^4 \frac{1}{i}, \sum_{n=1}^4 \frac{1}{n}, \sum_{j=1}^4 \frac{1}{j},$$

All denote the sum

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

If the upper and lower limits are the same, then the summation reduces to just one

$$\text{term} \sum_{k=2}^2 k^3 = 2^3$$

If the expression to the right of the summation does not involve the index of the summation, then do the following

$$\sum_{i=1}^5 2 = 2 + 2 + 2 + 2 + 2$$

$$\sum_{k=3}^6 x^3 = x^3 + x^3 + x^3 + x^3$$

A sum can be written in more than one way with the Sigma notation if we change the limits of the summation

Example

The following summations all represent the sum of the first five positive integers

$$\sum_{k=1}^5 2k = 2 + 4 + 6 + 8 + 10$$

$$\sum_{k=0}^4 (2k + 2) = 2 + 4 + 6 + 8 + 10$$

$$\sum_{k=2}^6 (2k - 2) = 2 + 4 + 6 + 8 + 10$$

Changing the index of the summation

It is often necessary and useful to change a given Sigma notation for a sum to another sigma notation with different limits of summation

Example

Express $\sum_{k=3}^7 5^{k-2}$ in sigma notation so that the lower limit is 0 rather than 3.

Define a new summation index j by the following formula

$$j = k - 3 \Rightarrow k = j + 3$$

Then as k runs from 3 to 7, j runs from 0 to 4. So

$$\sum_{k=3}^7 5^{k-2} = \sum_{j=0}^4 j^{(j+3)-2} = \sum_{j=0}^4 j^{j+1}$$

You should check that the two actually represent the same sum by putting values into the index in both notations.

To represent a general sum, we will use letters with subscripts.

Example

$a_1 + a_2 + a_3$ represent the general sum with three terms

This can also be written as in sigma notation

$$\sum_{k=1}^3 a_k = a_1 + a_2 + a_3$$

General sum with n terms can be written as

$$\sum_{k=1}^n b_k = b_1 + b_2 + b_3 + \dots + b_n$$

Properties of Sigma notation

Here are a few properties of sigma notation that will be helpful later on

THEOREM 5.4.1

$$\begin{aligned} a) \quad & \sum_{k=1}^n c a_k = c \sum_{k=1}^n a_k \\ b) \quad & \sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k \\ c) \quad & \sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k \end{aligned}$$

Here are some sum formulas written in sigma notation that will be helpful later.

THEOREM 5.4.2

$$a) \sum_{k=1}^n k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$b) \sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$c) \sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$$

Example

$$\begin{aligned} \sum_{k=1}^{30} k(k+1) &= \sum_{k=1}^{30} (k^2 + k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k \\ &= \frac{(30)(31)(61)}{6} + \frac{30(31)}{2} = 9920 \end{aligned}$$



Theorem 5.4.2 a) and b)

In a formula involving summation like this one

$$\begin{aligned} \sum_{k=1}^n k^2 &= \frac{n(n+1)(2n+1)}{6} \\ \Rightarrow 1^2 + 2^2 + \dots + n^2 &= \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

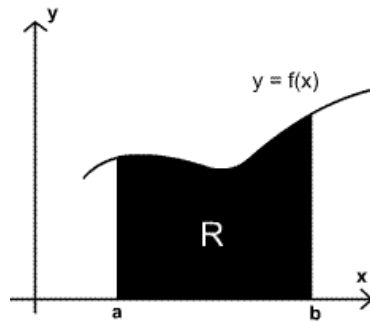
The left part is called the open form of the sum.

The right part is called the closed form of the sum.

Lecture # 28

Area as Limits

- Definition of Area
- Some technical considerations
- Numerical approx of area
- Look at the figure below



In this figure, there is a region bounded below by the x-axis, on the sides by the lines $x = a$ and $x = b$, and above by a curve or the graph of a continuous function $y = f(x)$ which is also non-negative on the interval $[a, b]$.

Earlier we saw that such an area can be computed using anti-derivatives.

Let's make the concept precise.

Recall that the slope of the tangent line was defined in terms of the limit of the slopes of secant lines.

Similarly, we will define area of a region R as limits of the areas of simpler regions whose areas are known.

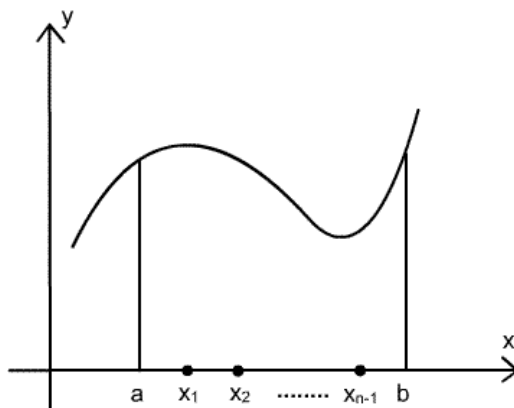
We will break up R into rectangles, and then find the area of each rectangle in R and then add all these areas up.

The result will be an approximation to the region R . Let's call it

If we let n go to infinity, the resulting will give a better and better approximation to R as the rectangle in R will get thinner and thinner, and the gaps will be filled in.

Here is the formal idea

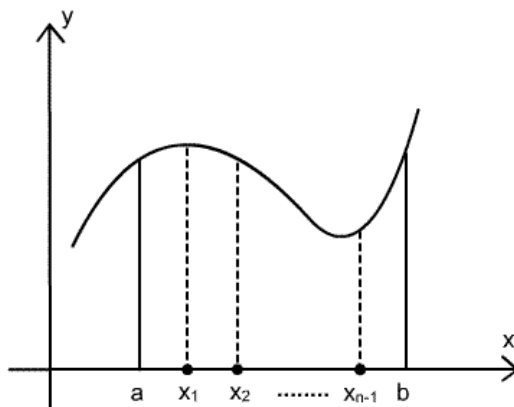
Choose an arbitrary positive integer n , and divide the interval $[a, b]$ into n subintervals of width $\frac{b-a}{n}$ by inserting $n-1$ equally spaced points between a and b say $x_1, x_2, \dots, x_{n-1}$



These points of subdivision form a regular partition of $[a, b]$ $a, x_1, x_2, \dots, x_{n-1}, b$

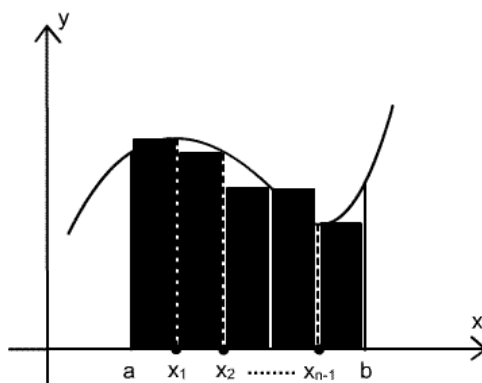
Next draw a vertical line through the points $x_1^*, x_2^*, \dots, x_n^*$

This will divide the region R into n strips of uniform width



Now we want to approximate the area of each strip by the area of a rectangle. For this, choose an arbitrary point in each subinterval ,

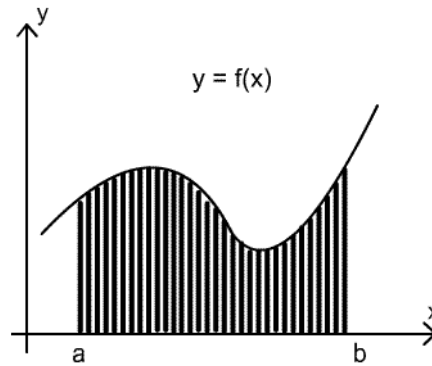
Over each subinterval construct a rectangle whose height is the values of the function f at the selected arbitrary point.



The union of these rectangles form the region R_n that we can regard as a reasonable

approx to the region R . The area of R_n can be got by adding the areas of all the rectangles forming it

If now we let n get big, the number of rectangles gets big, and the gaps btw the curve and the rectangles get filled in.



As n goes to infinity, the approx gets as good as the real thing.

So we define $A = \text{area}(R) = \lim_{n \rightarrow +\infty} [\text{area}(R_n)]$

For all the following work and computation, we will treat n as a POSITIVE integer

For computational purposes, we can write A in a different form as follows

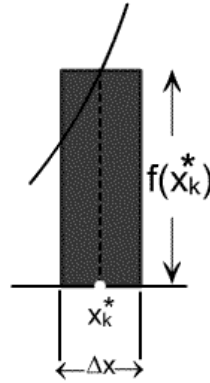
In the interval $[a, b]$, each of the approximation rectangles has width $\frac{b-a}{n}$

We call this delta x or $\Delta x = \frac{b-a}{n}$

The heights of the approximating rectangles are at the points $x_1^*, x_2^*, \dots, x_n^*$

So the approximating rectangles making up the region have areas

$$f(x_1^*)\Delta x, f(x_2^*)\Delta x, \dots, f(x_n^*)\Delta x$$



$$\text{Area of } k^{\text{th}} \text{ rectangle} = f(x_k^*) \cdot \Delta x$$

So the total area of R_n is given by

$$\text{area}(R_n) = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \dots + f(x_n^*)\Delta x$$

OR

$$\text{area}(R_n) = \sum_{k=1}^n f(x_k^*)\Delta x$$

So now A can be written as

$$A = \lim_{n \rightarrow +\infty} \sum_{k=1}^n f(x_k^*)\Delta x$$

This we will take to be the PRECISE def of the area of the region R

Some Technical Considerations

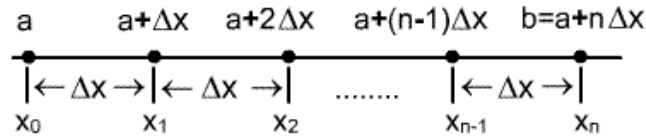
The points $x_1^*, x_2^*, \dots, x_n^*$ were chosen arbitrarily.

What if some other points were chosen? Would the resulting values for $f(x)$ at these points be different? And if so then the definition of area will not be well-defined?

It is proved in advanced courses that since f is continuous it doesn't matter what points are taken in a subinterval.

Usually the point x_k^* in a subinterval is chosen so that it is the left end point of the interval, the right end point of the interval or the midpoint of the interval.

Note that we divided the interval $[a, b]$ by the points $x_1, x_2, \dots, x_{n-1}$ with $x_0 = a$ and $x_n = b$ into equal parts of width Δx



This figure shows that

$$x_k = a + k\Delta x \text{ for } k=0,1,2 \dots n$$

SO

$$x_k^* = x_{k-1} = a + (k-1)\Delta x \quad \text{Left end point}$$

$$x_k^* = x_k = a + k\Delta x \quad \text{Right end point}$$

$$x_k^* = \frac{1}{2}(x_{k-1} + x_k) = a + (k - \frac{1}{2})\Delta x \quad \text{Midpoint}$$

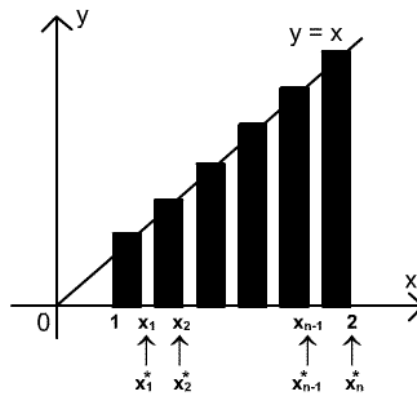
Example

Use the definition of the Area with x_k^* as the right end point of each subinterval to find the area under the line $y = x$ over the interval $[1,2]$.

Subdivide $[1,2]$ into n equal parts, then each part will have length

$$\Delta x = \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n} \quad \text{We are taking } x_k^* \text{ as the right end point so we}$$

$$\text{Have } x_k^* = a + k\Delta x = 1 + \frac{k}{n}$$



Thus, the k th rectangle has area

$$f(x_k^*)\Delta x = x_k^* \Delta x = \left(1 + \frac{k}{n}\right)\Delta x = \left(1 + \frac{k}{n}\right)\frac{1}{n}$$

And the sum of the areas of the n rectangles is

$$A = \lim_{n \rightarrow +\infty} \sum_{k=1}^n f(x_k^*)\Delta x = \lim_{n \rightarrow +\infty} \left(\frac{3}{2} + \frac{1}{2n}\right) = \frac{3}{2}$$

Note that the region we have computed the area of is a trapezoid with height $h = 1$ and bases $b_1 = 1$ and $b_2 = 2$. From basic geometry we have that

$$\text{Area of trapezoid} = A = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(1)(1 + 2) = \frac{3}{2}$$

Example

Same problem as before but with left end points. $[1, 2]$.

Use the definition of the Area with x_k^* as the left end point of each subinterval to find the area under the line $y = x$ over the interval $[1, 2]$.

Subdivide $[1, 2]$ into n equal parts, then each part will have length

$$\Delta x = \frac{b - a}{n} = \frac{2 - 1}{n} = \frac{1}{n}$$

We are taking x_k^* as the left end point so we

$$\text{Have } x_k^* = a + k\Delta x = 1 + \frac{k}{n}$$

Thus, the k th rectangle has area

$$f(x_k^*)\Delta x = x_k^* \Delta x = \left(1 + \frac{k}{n}\right)\Delta x = \left(1 + \frac{k}{n}\right)\frac{1}{n}$$

And the sum of the areas of the n rectangles is

$$\begin{aligned} \sum_{k=1}^n f(x_k^*)\Delta x &= \sum_{k=1}^n \left[\left(1 + \frac{k}{n}\right)\frac{1}{n} \right] = \sum_{k=1}^n \left[\left(\frac{1}{n} + \frac{k}{n^2}\right) \right] = \frac{1}{n} \sum_{k=1}^n 1 + \frac{1}{n^2} \sum_{k=1}^n k \\ &= \frac{1}{n} \cdot n + \frac{1}{n^2} \left[\frac{1}{2}n(n+1) \right] \end{aligned}$$

Note that the region we have computed the area of is a trapezoid with height $h = 1$ and bases $b_1 = 1$ and $b_2 = 2$. From basic geometry we have that

$$\text{Area of trapezoid} = A = \frac{1}{2}h(b_1 + b_2) = \frac{1}{2}(1)(1 + 2) = \frac{3}{2}$$

Example

Use the area definition with right endpoints of each subinterval to find the area under the parabola

$$y = 9 - x^2 \text{ over the interval } [0,3].$$

Do calculations from book on page 272 (end) and page 273 for example.

Numerical Approximations of Area

As we have all seen so far, the computations involved in computing the limits are tedious and long.

In some cases, it is even impossible to carry out the computations by the definition of the area.

In such cases, it is easier to get a GOOD approx for the area using large values of n and using a computer or a calculator.

Example

Use a computer or a calculator to find the area under the curve $y = 9 - x^2$ over the interval $[0,3]$ and $n = 10, 20$ and 50 .

Left end point approximation			Right end point approximation		Mid point approximation	
K	x_k^*	$9 - (x_k)^2$	x_k^*	$9 - (x_k)^2$	x_k^*	$9 - (x_k)^2$
1	0.0	9.000000	0.3	8.910000	0.15	8.977500
2	0.3	8.910000	0.6	8.640000	0.45	8.797500
3	0.6	8.640000	0.9	8.190000	0.75	8.437500
4	0.9	8.190000	1.2	7.560000	1.05	7.897500
5	1.2	7.560000	1.5	6.750000	1.35	7.177500
6	1.5	6.750000	1.8	5.760000	1.65	6.277500
7	1.8	5.760000	2.1	4.590000	1.95	5.197500
8	2.1	4.590000	2.4	3.240000	2.25	3.937500
9	2.4	3.240000	2.7	1.710000	2.55	2.497500
10	2.7	1.710000	3.0	0.000000	2.85	0.877500
$\Delta x \sum_{k=1}^n f(x_k^*) = 19.30500$			=16.605000		=18.022500	

And for different values of n we have the following approximations.

n	Left end point approximation	Right end point approximation	Mid point approximation
10	19.305000	16.605000	18.02250
20	18.663750	17.313750	18.005625
50	18.268200	17.728200	18.000900

Lecture # 29

The Definite Integral

- Definition of Definite Integral
- Definite Integral of continuous functions with nonnegative values
- Definite Integral of continuous functions with negative and positive values
- Definite Integral of functions with discontinuities
- Properties of the Definite Integral
- Inequalities involving Definite Integral

We have so far developed a definition of Area as limit of a bunch of rectangles with equal widths.

The question arises that why choose widths? The answer is that we don't have to.

We want now to have a definition of Area that includes the general case where the widths of the rectangles are not necessarily equal.

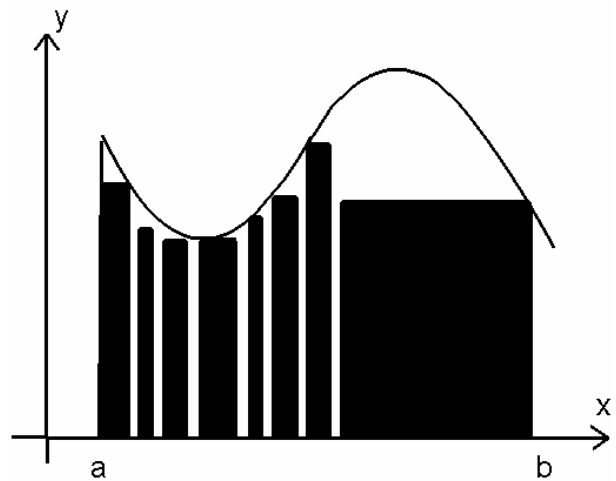
When the rectangles widths were equal note that the formula for the width was defined as

$$\Delta x = \frac{b-a}{n}$$

In this formula you can see that as n goes to infinity, the width Δx goes to zero

If the widths are not the same for all rectangles, then this is not necessarily the case.

Suppose that we have a rectangle construction in which we divide $[a,b]$ so that one half is continually subdivided into smaller Rectangles, while the right is left as one big rectangle.

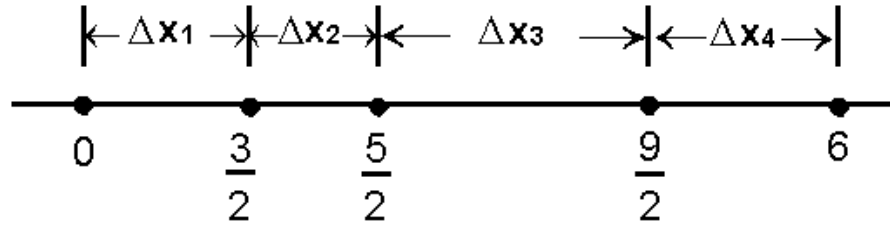


In this case, when n goes to ∞ , left part goes to 0, not the Right half.

Let's fix this problem.

Subdivide $[a,b]$ into n subinterval whose widths are $\Delta x_1, \Delta x_2, \Delta x_3, \dots, \Delta x_n$

The subintervals are said to partition the interval, and the largest subinterval width is called the mesh size of the partition. This is denoted by $\max \Delta x_k$



$$\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = 2$$

Read as “maximum of the Δx_k .”

In this figure

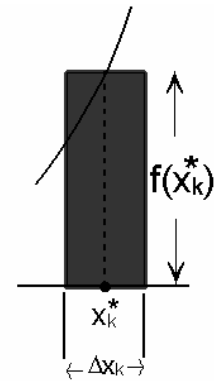
$$\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = 2$$

We see that

If $[a, b]$ is partitioned into n subintervals, and if x_k is an

arbitrary point in k th subinterval, then $f(x_k^*)\Delta x_k$

is the area of the rectangle with height $f(x_k)$ and width



Area of k^{th} rectangle = $f(x_k^*) \cdot \Delta x_k$

delta x_k , and $\sum_{k=1}^n f(x_k^*)\Delta x_k$ is the sum of the shaded rectangular areas

Now if we let $\max \Delta x_k \rightarrow 0$ the width of EVERY rectangle tends to 0 because none of them exceeds the max. So we have the area under the curve now defined more generally as

$$A = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*)\Delta x_k$$

DEFINITION 5.6.1

(Area under a curve)

If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the area under

the curve $y = f(x)$ over the interval $[a, b]$ is defined by $A = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*)\Delta x_k$

Definite Integral of continuous functions with nonnegative values

The limit in the definition we just saw is VERY important. It has special notation. We write it as

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

The expression on the left is called the definite integral of f from a to b . a and b are called the upper and lower limits of integration respectively.

There is a relationship btw this definite integral and the indefinite integral we talked about earlier.

We will see it later.

So with this notation we can say that

$$\int_a^b f(x)dx = \text{Area under the curve } y = f(x) \text{ over } [a, b].$$

Our goal will be to evaluate the definite integral efficiently instead of using the definition all the time.

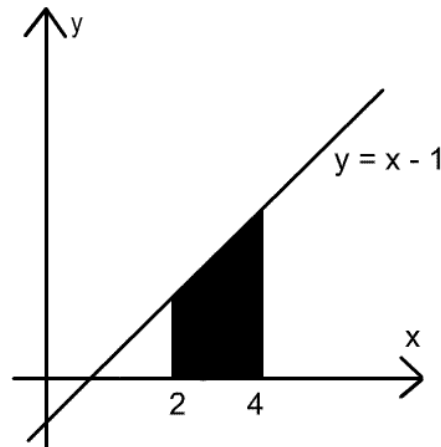
Example

$$\int_2^4 (x-1)dx$$

This represents the area under the curve $y = (x-1)$ over $[2,4]$.

Here is a picture of it.

Note that the region described



here is just a trapezoid with $h = 2$, and $b_1 = 1$ and $b_2 = 3$. So we can evaluate this integral from

$$\text{basic geometry as } \int_2^4 (x-1)dx = \frac{1}{2}(2)(1+3) = 4$$

The sum $\sum_{k=1}^n f(x_k^*) \Delta x_k$ is called the Riemann Sum in honor of the German

mathematician Bernhard Riemann.

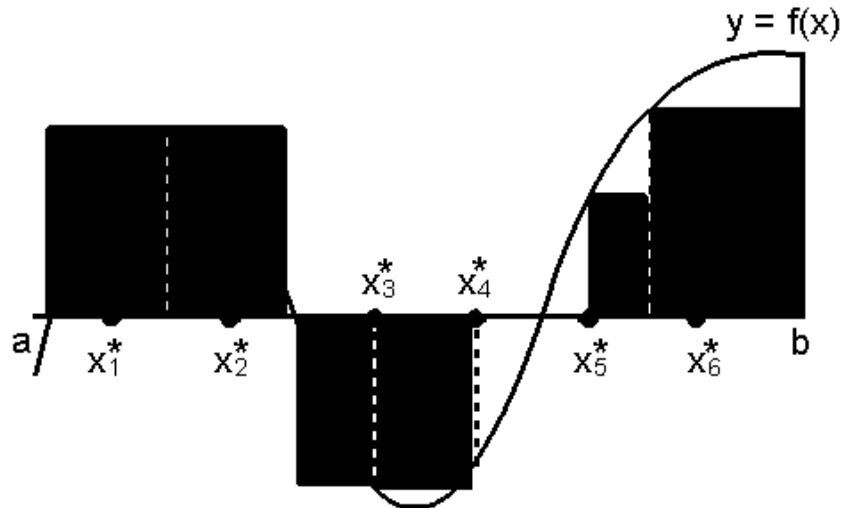
The formula for the area we just saw was for continuous and nonnegative functions. It involves a limit. The question is: does the limit always exist, so that we can always find the area ?

It is proved in advanced levels that for continuous and nonnegative functions, this limit always exists.

Definite Integral of continuous functions with negative and positive values

Now we want to extend our area definition to include continuous functions on $[a,b]$ that have both positive and negative values on $[a,b]$.

Look at the following figure

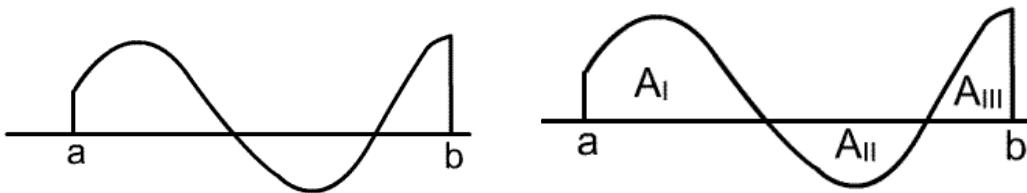


The rectangles below $[a,b]$ but above the curve $f(x)$ have some area just the way their counterparts above $[a,b]$ but below curve $f(x)$ do. The difference is that we can view the number representing the area of the rectangles below as negative values of $f(x)$ for some x_k^* . So the Riemann sum gives

$$\begin{aligned}\sum_{k=1}^6 f(x_k^*)\Delta x_k &= f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \dots + f(x_6^*)\Delta x_6 \\ &= A_1 + A_2 - A_3 - A_4 + A_5 + A_6 \\ &= (A_1 + A_2 + A_5 + A_6) - (A_3 + A_4)\end{aligned}$$

This says that the Riemann sum is the difference of two areas: the total area of rectangle above the x-axis, minus the total areas of rectangles below the x-axis.

Now as $\max \Delta x$ goes to 0, the situation still works out nicely with the large # of rectangles filling in any left over space between the earlier rectangles and the curve.



So we have the following definition

Definition 5.6.2

If the function f is continuous on $[a,b]$, and can assume both positive and negative values, then the **net signed area** A between $y = f(x)$ and the interval $[a,b]$ is defined by

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

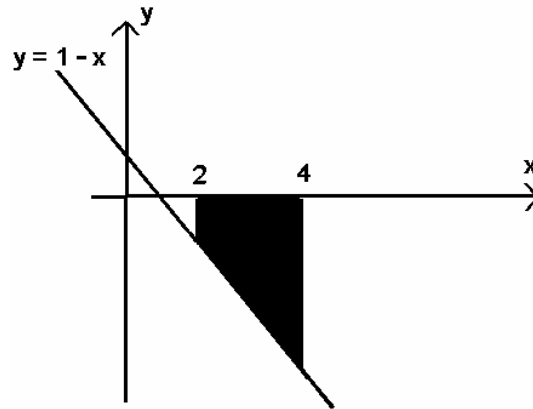
Net signed area means that the total difference of the two areas above and below may be negative. If that happens then we just treat the negative as representing the fact that the area below is larger than the area above.

EXAMPLE

$$\int_2^4 (1-x)dx$$

Geometrically, this region lies below the interval $[2,4]$. The region is a trapezoid and we can find the area as 4 using its dimensions. Since its below the x-axis we write

$$\int_2^4 (1-x)dx = -4$$



Definite Integral of functions with discontinuities

For the two definitions we have seen so far, f was a continuous function. This guaranteed that the area defined as a limit could always be found since the limits existed for continuous f .

If f is not continuous, then the area may or may not be found as the limit of the Sum may or may not exist.

Here is a definition for such a function for the area.

DEFINITION 5.6.3 (Area under a curve)

If the function f is defined on the close interval $[a, b]$ then, f is called Riemann integrable on $[a, b]$ or more simply integrable on $[a, b]$ if the limit exists

$\lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$ If f is integrable on $[a, b]$, then we define The definite integral of f from a to b by

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*) \Delta x_k$$

So far we assumed that the upper limit of integration was greater than the lower limit.

Here is a definition that allows for the limits to be the same, and the case where the upper may be less than the lower.

DEFINITION 5.6.4

(a) If a is in the domain of f , we define

$$\int_a^a f(x)dx = 0$$

If f is integrable on $[a, b]$, then we define

$$\int_b^a f(x)dx = -\int_a^b f(x)dx$$

Properties of the definite Integral

Here are some properties of the definite integral

Theorem 5.6.5

If f and g are integrable on $[a, b]$ and if c is a constant, then cf , $f + g$, and $f - g$ are integrable on $[a, b]$ and

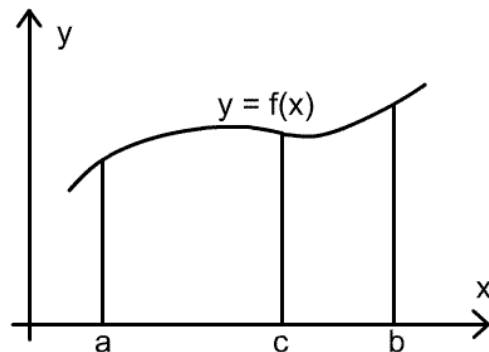
$$(a) \int_a^b cf(x)dx = c \int_a^b f(x)dx$$

$$(b) \int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

$$(c) \int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$$

Now look at this figure

It is clear from the picture that the area under the curve over $[a, b]$ can be split into a sum of two areas, one area from a to c , and the other area from c to b . Formally



Theorem 5.6.5

If f is integrable on a closed interval containing the three points a , b and c then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

no matter how the points are ordered.

Example

$$\int_1^5 f(x)dx = -1, \quad \int_3^5 f(x)dx = 3, \quad \int_3^5 g(x)dx = 4$$

Find

$$\int_1^3 f(x)dx$$

From thm5.6.6 with $a = 1$, $b = 5$, $c = 3$

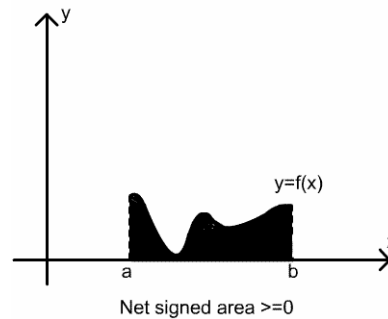
$$\int_1^5 f(x)dx = \int_1^3 f(x)dx + \int_3^5 f(x)dx$$

So

$$\int_1^3 f(x)dx = \int_1^5 f(x)dx - \int_3^5 f(x)dx = -1 - 3 = -4$$

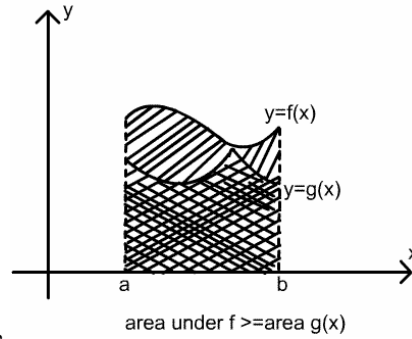
Inequalities involving definite integral

Look at this figure



This says that if a function is nonnegative (the graph does not go below x-axis) on $[a,b]$, then the net area between the curve and x-axis must be greater than or equal to 0.

Look at this figure



The graph of f does not go below that of g , or in other words $f(x) \geq g(x)$ over the interval and f and g are nonnegative, then area under f must be $\geq$ area under g over the interval

Theorem 5.6.7

(a) If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x) \geq 0$$

(b) If f and g are integrable functions on $[a, b]$ and $f(x) \geq g(x)$ for all x in

$$[a, b] \text{ then } \int_a^b f(x) \geq \int_a^b g(x)$$

Works with $<$, $>$ $\leq$ also

Example

Show that $\int_0^1 \frac{\cos(x)}{2x^3 - 5} dx$ is negative.

On the interval $[0, 1]$, $\cos(x) > 0$, but $2x^3 - 5 < 0$. So $f(x)$ as a whole is < 0 . So from theorem 5.6.7a), with $<$ instead of $\geq$ the integral is negative

Definition 5.6.8

A function f is said to be bounded on interval $[a, b]$ if there is a positive number M such that

$$-M \leq f(x) \leq M$$

for all x in $[a, b]$. Geometrically, this means that the graph of f on the interval $[a, b]$ lies between the lines $y = -M$ and $y = M$.

$y = x^2$ on the interval $[-2, 2]$ is bounded since its graph lies between the lines $y = 0$ and $y = 5$

Theorem 5.6.9

Let f be a function that is defined at all the points in the interval $[a, b]$.

- (a) If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.
- (b) If f is bounded on $[a, b]$ and has only finite many points of discontinuity on $[a, b]$ then f is integrable on $[a, b]$.
- (c) If f is not bounded on $[a, b]$, then f is not integrable on $[a, b]$.

Lecture # 30

First Fundamental Theorem of Calculus

- 1st fundamental theorem of calculus.
- Relationship between definite and indefinite integrals.
- Mean Values theorem for Integrals.
- Average Values of a function.

This is the 1st fundamental theorem of calculus.

Theorem 5.7.1(The first fundamental theorem of calculus)

If f is continuous on $[a, b]$ and if F is an antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

This theorem tells us how to evaluate EASILY the definite integral.

It says that find the anti-derivative of $f(x)$ first, call it $F(x)$, and then evaluate this function on the limits of the integration. Let's prove this.

Proof

We will use the Mean Value Theorem involving derivatives to prove the first fundamental theorem of calculus.

$$\int_a^b f(x)dx = F(b) - F(a)$$

Let us subdivide the given interval of integration namely $[a, b]$ into n subintervals using the points $x_1, x_2, \dots, x_{n-1}$ in the interval $[a, b]$ such that

$$a < x_1 < x_2 < \dots < x_{n-1} < b.$$

So now we have n subinterval of $[a, b]$. We can denote the widths of these intervals as

$$\Delta x_1, \Delta x_2, \dots, \Delta x_n. \text{ Where for example } \Delta x_2 = x_2 - x_1.$$

Since $F'(x) = f(x)$ for all x in $[a, b]$, it is obvious that $F(x)$ satisfies the requirements of the Mean Value Theorem involving derivatives on each of the subintervals. So by MVT, we can find points

$$x_1^*, x_2^*, \dots, x_n^*$$

In each of the respective subintervals $[a, x_1], [x_1, x_2], \dots, [x_{n-1}, b]$.

So we now make the following equations

$$\begin{aligned}
F(x_1) - F(a) &= F'(x_1^*)(x_1 - a) = f(x_1^*)\Delta x_1 \\
F(x_2) - F(x_1) &= F'(x_2^*)(x_2 - x_1) = f(x_2^*)\Delta x_2 \\
F(x_3) - F(x_2) &= F'(x_3^*)(x_3 - x_2) = f(x_3^*)\Delta x_3 \\
&\vdots \\
&\vdots \\
&\vdots \\
F(b) - F(x_{n-1}) &= F'(x_n^*)(b - x_{n-1}) = f(x_n^*)\Delta x_n
\end{aligned}$$

Adding up these equations we get

$$F(b) - F(a) = \sum_{k=1}^n f(x_k^*)\Delta x_k$$

Now increase n in such a way that

$$\max \Delta x_k \rightarrow 0$$

Since f is assumed to be continuous, we have the following result

$$F(b) - F(a) = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n f(x_k^*)\Delta x_k = \int_a^b f(x)dx$$

↑
THEOREM 5.6.9(a)

I did not apply the limits on the left because even if I did, the expression does not involve n and therefore nothing happens.

$F(b) - F(a)$ can also be written as $F(x)]_a^b$ so we have

$$\int_a^b f(x)dx = F(x)]_a^b$$

Example

Evaluate $\int_1^2 xdx$

$\int_1^2 xdx$. The function $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$. So we have

$$\int_1^2 xdx = \left[\frac{1}{2}x^2 \right]_1^2 = \frac{1}{2}(4) - \frac{1}{2}(1) = 2 - \frac{1}{2} = \frac{3}{2}$$

Here are a few properties of the bracket notation we just saw.

Prove these yourself

Properties

$$\begin{aligned}
[cF(x)]_a^b &= c[F(x)]_a^b \\
[F(x) + G(x)]_a^b &= [F(x)]_a^b + [G(x)]_a^b \\
[F(x) - G(x)]_a^b &= [F(x)]_a^b - [G(x)]_a^b
\end{aligned}$$

These are easy if you remember that this notation and the definite integral are the same thing!!

Relationship between definite and indefinite integrals

In applying the 1st theorem of Calc, it does not matter WHICH anti-derivative of f is used.

If F is any anti-derivative of f then all the others have form

$F(x) + C$ by theorem 5.2.2

Hence we have the following:

$$\begin{aligned} [F(x) + C]_a^b &= [F(b) + C] - [F(a) + C] \\ &= F(b) - F(a) = F(x) \Big|_a^b = \int_a^b f(x) dx \quad \dots(A) \end{aligned}$$

This shows that all anti-derivatives of f on $[a, b]$ give the same values for the definite integral

Now since

$$\int f(x) dx = F(x) + C$$

It follows from (A) that

$$\int_a^b f(x) dx = \left[\int f(x) dx \right]_a^b$$

So we can first evaluate the indefinite integral and use the limits as well to evaluate the definite integral

This relates the definite and the indefinite integrals.

Example

Using the 1st theorem of calculus, find the area under the curve $y = \cos(x)$ over the interval $[0, 2\pi]$

Since $\cos(x) \geq 0$ for $0 \leq x \leq \pi$, the area is

$$\begin{aligned} A &= \int_0^{\pi/2} \cos(x) dx = \left[\int \cos(x) dx \right]_0^{\pi/2} \\ &= \sin(x) \Big|_0^{\pi/2} = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1 \end{aligned}$$

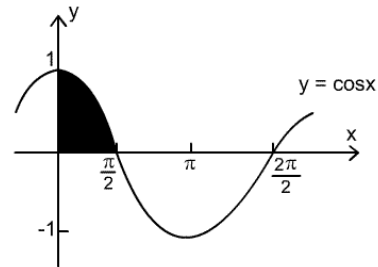


Figure shows that $\cos x \geq 0$ for $0 \leq x \leq \pi/2$

I deliberately chose $C = 0$ since we just saw that the value of $C=0$ doesn't change the answer.

Example

Evaluate $\int_0^3 (x^3 - 4x + 1) dx$

$$\begin{aligned}
 \int_0^3 (x^3 - 4x + 1)dx &= \left[\int (x^3 - 4x + 1)dx \right]_0^3 \\
 &= \left[\int x^3 dx - \int 4x dx + \int 1 dx \right]_0^3 \\
 &= \left[\frac{x^4}{4} - 4 \cdot \frac{x^2}{2} + x \right]_0^3 = \left(\frac{81}{4} - 18 + 3 \right) - (0) = \frac{21}{4}
 \end{aligned}$$

Example

Evaluate $\int_0^6 f(x)dx$ if $f(x) = \begin{cases} x^2 & x < 2 \\ 3x - 2 & x \geq 2 \end{cases}$

From theorem 5.6.6

$$\begin{aligned}
 \int_0^6 f(x)dx &\text{ if } f(x) = \int_0^2 f(x)dx + \int_2^6 f(x)dx \\
 &= \int_0^2 x^2 dx + \int_2^6 (3x - 2)dx \\
 &= \left[\frac{x^3}{3} \right]_0^2 + \left[\frac{3x^2}{2} - 2x \right]_2^6 = \left(\frac{8}{3} - 0 \right) + (42 - 2) = \frac{128}{3}
 \end{aligned}$$

Mean Values theorem for Integrals

Consider the picture.

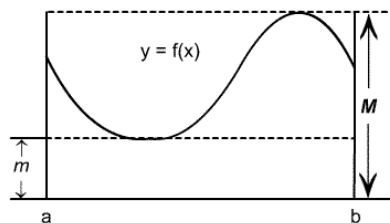
This figure shows a continuous function f on $[a, b]$.

Let M be the maximum values of f on $[a, b]$, and m the minimum.

Then if we draw a rectangle of height M and one

of height m , then its clear from the picture that the area under the curve of f is at least as large as the area of rectangle with height m AND no larger than the area of the rectangle with height M .

So we would like to say that there is some rectangle of a certain height for which the area under f = area of rectangle.

**THEOREM 5.7.2(The Mean-Value Theorem for Integrals)**

If f is continuous on a closed interval $[a, b]$, then there is at least one number x^* in $[a, b]$ such that

$$\int_a^b f(x)dx = f(x^*)(b - a)$$

Proof

By the extreme value theorem (theorem 4.6.4) f assumes a max M and a min m on $[a, b]$. So for all x in $[a, b]$

$$m \leq f(x) \leq M$$

$$\Rightarrow \int_a^b m dx \leq \int_a^b f(x) dx \leq \int_a^b M dx \quad \text{Theorem 5.6.7(b)}$$

$$\Rightarrow m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

$$\Rightarrow m \leq \frac{1}{b-a} \int_a^b f(x) dx \leq M \quad \text{Evaluating the end integral}$$

The last inequality states that $\frac{1}{b-a} \int_a^b f(x) dx$ is a number btw m and M on $[a, b]$

Since f is continuous and takes on all values in $[a, b]$, we can say using the Intermediate Values

theorem (theorem 2.7.9) that f must assume $\frac{1}{b-a} \int_a^b f(x) dx$ on $[a, b]$ for some point x^*

So we have

$$\frac{1}{b-a} \int_a^b f(x) dx = f(x^*)$$

$$\Rightarrow \int_a^b f(x) dx = f(x^*)(b-a)$$

Example

$f(x) = x^2$ is continuous on $[1, 4]$, the MVT for Integrals guarantees that there exists a number x^* in $[1, 4]$ such that

$$\int_1^4 x^2 dx = f(x^*)(4-1) = 3(x^*)^2$$

But

$$\int_1^4 x^2 dx = \left[\frac{x^3}{3} \right]_1^4 = 21$$

So

$$3(x^*)^2 = 21 \Rightarrow x^* = \sqrt{7}$$

Average Values of of a function**Definition 5.7.3**

If f is integrable on $[a, b]$ then the average value (or mean value) of f on $[a, b]$ is defined to be

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$$

If $y = f(x)$, then f_{avg} is also called the average value of y with respect to x over $[a, b]$.

Lecture # 31

Evaluating Definite Integral by Substitution.

Evaluating Definite Integrals.

- By substitution
- Approximation by Riemann Sums

Evaluating definite integral by substitution

We want to evaluate the following definite integral by substitution

$$\int_a^b f(x)dx$$

Method 1

Evaluate the indefinite integral $\int f(x)dx$ by substitution, and then use the relationship.

$$\int_a^b f(x)dx = \left[\int f(x)dx \right]_a^b$$

Method 2

First represent the definite integral in the form $\int_a^b f(x)dx = \int_a^b h(g(x))g'(x)dx$

And then make the substitution $u = g(x)$, $du = g'(x)dx$ directly into the definite integral. But now we have to change integration limits from x-limits to u-limits as follows

$$u = g(a), \text{ if } x = a$$

$$u = g(b), \text{ if } x = b$$

This leaves us with a new integral in terms of u

$$\int_a^b f(x)dx = \int_{g(a)}^{g(b)} h(u)du$$

This will be simpler if the choice of u is good.

Example

Evaluate $\int_0^2 x(x^2 + 1)^3 dx$

Solution:

Method 1

$$u = x^2 + 1 \text{ so } du = 2x dx$$

$$\int x(x^2 + 1)^3 dx = \frac{1}{2} \int u^3 du = \frac{u^4}{8} + C = \frac{(x^2 + 1)^4}{8} + C$$

So

$$\int_0^2 x(x^2 + 1)^3 dx = \left[\int x(x^2 + 1)^3 dx \right]_0^2 = \left[\frac{(x^2 + 1)^4}{8} \right]_0^2 = 78$$

Method 2

$$u = 1 \text{ if } x = 0$$

$$u = 5 \text{ if } x = 2$$

Thus

$$\int_0^2 x(x^2 + 1)^3 dx = \frac{1}{2} \int_1^5 u^3 du = \frac{u^4}{8} \Big|_1^5 = 78$$

Example

$$\int_0^{\frac{\pi}{4}} \cos(\pi - x) dx$$

Solution:

$$\text{Let } u = \pi - x \text{ so that } du = -dx$$

$$\text{Also } u = \pi \text{ if } x = 0, u = \frac{3\pi}{4} \text{ if } x = \frac{\pi}{4}$$

So,

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \cos(\pi - x) dx &= - \int_{\pi}^{\frac{3\pi}{4}} \cos(u) (du) = -\sin(u) \Big|_{\pi}^{\frac{3\pi}{4}} = -\left[\sin\left(\frac{3\pi}{4}\right) - \sin(\pi)\right] \\ &= -\left[\frac{1}{\sqrt{2}} - 0\right] = -\frac{1}{\sqrt{2}} \end{aligned}$$

Approximation by Riemann Sums

Recall that a Riemann Sum is the expression $\sum_{k=1}^n f(x_k^*) \Delta x_k$

which occurs when we try to approximate the area under the curve.

If we take the limits of this sum, we get the definite integral.

If we don't take the limits, then we can get a good approximation to the definite integral if n is relatively LARGE.

So we can say that for large n $\int_a^b f(x) dx \approx \sum_{k=1}^n f(x_k^*) \Delta x_k$

We need to use this approx when it's impossible to evaluate exactly.

It is a good idea in this case to use a regular partition of the interval $[a, b]$ which gives same width for each subinterval.

So we then write

$$\int_a^b f(x) dx \approx \sum_{k=1}^n f(x_k^*) \Delta x_k = \Delta x_k [f(x_1^*) + f(x_2^*) + \dots + f(x_n^*)]$$

This formula produces a left endpoint approximation, a right end point approximation and midpoint approximation, depending on the choice of the point x_k^*

Example

Approximate $\int_0^1 \sqrt{1-x^2} dx$ using the left endpoint, right end point, and the

midpoint approx each with $n=10$, $n=20$, $n=50$ and $n=100$ subintervals.

Solution:

For $n=10$ we can do this easily, but for others we should use a computer. We will do for $n=10$ using the ideas from previous lectures. Here is a table of results

n	Left endpoint approximation	Right endpoint approximation	Midpoint approximation
10	0.826129582	0.726129582	0.788102858
20	0.807116220	0.757116220	0.786357647
50	0.794567128	0.774567128	0.785641388
100	0.790104258	0.780104258	0.785484214

The exact value of the integral is $\frac{\pi}{4}$. This is consistent with the preceding computations , since

$$\frac{\pi}{4} \approx 0.785398163.$$

Lecture # 32
Second Fundamental Theorem of Calculus

Second Fundamental Theorem of Calculus

- Dummy Variable
- Definite Integrals with variable upper limit
- Second fundamental theorem of Calculus
- Existence of Ant derivatives for continuous functions
- Functions defined by Integrals

Dummy Variable

Here is some notational stuff.

If we change the letter for the variable of integration but don't change the limits, then the values of the definite integral are unchanged.

That is to say

$$\int_a^b f(x)dx, \int_a^b f(t)dt, \int_a^b f(y)dy$$

All have the same value

For this reason, we call the letter used for the variable of integration DUMMY variable

Example

$$\begin{aligned}\int_1^2 x^2 dx &= \left. \frac{x^3}{3} \right|_1^2 = \frac{26}{3} \\ \int_1^2 f(t) dt &= \left. \frac{t^3}{3} \right|_1^2 = \frac{26}{3} \\ \int_1^2 f(y) dy &= \left. \frac{y^3}{3} \right|_1^2 = \frac{26}{3}\end{aligned}$$

Definite Integrals with variable upper limit of integration

We will now consider definite integrals of the form $\int_a^x$ — where the upper limit is a variable rather

than a number.

In such integrals, we will use a different letter for the integration variable

This will distinguish between the limit of integration and the variable of integration.

Let's do an example

Example

Evaluate

$$\int_2^x t^2 dt$$

Solution:

$$\int_2^x t^2 dt = \left[\frac{t^3}{3} \right]_2^x = \frac{x^3}{3} - \frac{8}{3}$$

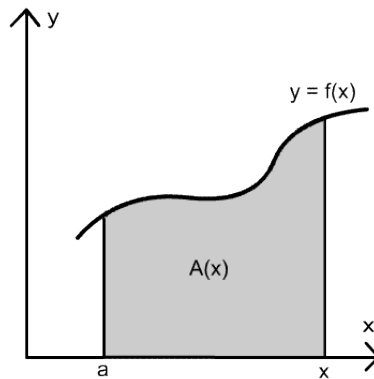
Note that the result in this case is a FUNCTION of x .

In all such integrals with an x as a limit, the final result is a function of x .

This is like when we said that instead of evaluating the derivative at a number, do it at a point x to get the derivative function.

Second fundamental theorem of Calculus

In lecture 25, we say that if f is a nonnegative continuous function over $[a, b]$, and $A(x)$ represents the area under the curve $y = f(x)$ over the interval $[a, x]$ as a function of x , then $A'(x) = f(x)$



Now Let's write $A(x)$ as a definite integral since that's how it is represented

$$A(x) = \int_a^x f(t) dt$$

If we now take the derivative w.r.t to x , we get

$$\frac{d}{dx}[A(x)] = A'(x) = \frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x)$$

This result holds for all continuous functions, and we can state it as a theorem

Theorem 5.9.1 (2nd fundamental theorem of Calculus)

Read as

- 1) If the integrand is continuous, then the derivative of a definite integral w.r.t its upper limit is equal to the integrand evaluated at the upper limit.
- 2) The derivative of a function representing the area under the curve of another continuous function f is equal to the function f on a given interval.

We usually write the theorem as

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x)$$

Example

Since $f(x) = x^3$ is continuous everywhere, evaluate the integral $\int_1^x t^3 dt$ to check the validity of

2nd fundamental theorem of calculus.

Solution:

By 2nd fundamental theorem of calculus:

$$\frac{d}{dx} \left[\int_1^x t^3 dt \right] = x^3$$

We check it by evaluating the integral

$$\int_1^x t^3 dt = \left[\frac{t^4}{4} \right]_1^x = \frac{x^4}{4} - \frac{1}{4}$$

Differentiating this gives x^3 .

Existence of Anti-derivatives of continuous functions:

If 'f' is a continuous function on an interval 'I' and 'a' is any point in 'I' then this formula

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x) \text{ tells us that } F(x) = \int_a^x f(t) dt \text{ is an}$$

anti-derivative of f on I. So now with the 2nd theorem of calculus, we can say that EVERY continuous function on an interval has an anti-derivative on that interval.

Functions defined by Integrals

So far we have been able to determine the anti-derivative of a given function by integration techniques.

But its not always possible to find the anti-derivative of a given function f which is continuous on an interval and thus is guaranteed to have an anti-derivative on that interval.

If this is the case, we get functions (anti-derivatives) that are defined in terms of integrals and nothing simpler that we have seen so far.

Example

$$\int_1^3 \frac{1}{x} dx$$

Solution:

This function is continuous on the interval $[1, 3]$ so it is integrable on $[1, 3]$. By the 2nd theorem of calculus, an antiderivative is

$$F(x) = \int_1^x \frac{1}{t} dt$$

Using the 1st theorem of calculus we can attempt to evaluate this integral

$$\int_1^x \frac{1}{t} dt = F(3) - F(1) = \int_1^3 \frac{1}{t} dt - \int_1^1 \frac{1}{t} dt = \int_1^3 \frac{1}{t} dt$$

But this is not a function like any we have seen so far.

In fact, $1/t$ cannot be integrated using polynomials, rational functions, or any other function we have seen so far.

But we can approximate this function by using numerical methods like the Riemann sums. Now this

formula $\int_1^x \frac{1}{t} dt$ defines a FUNCTION in terms of integrals.

Lecture # 33

Application to the Definite Integral

Application of the Definite Integral

- Area problem: Area Between two Curves
- Area between $y = f(x)$ and $y = g(x)$
- Area between $x = v(y)$ and $x = w(y)$

First Area problem

Area between two curves:

Area between $y=f(x)$ and $y=g(x)$

Suppose that f and g are continuous functions on an interval $[a, b]$ and

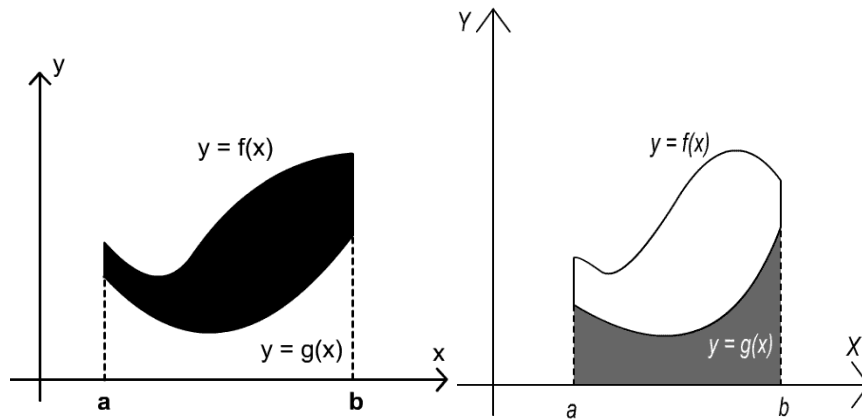
$f(x) \geq g(x)$ for $a \leq x \leq b$

(This means that the curve $y = f(x)$ is above $y = g(x)$ and that the two can touch but not cross).

Find the area A of the region bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$

If f and g are nonnegative on $[a, b]$ then we have

$$A = [\text{area under } f] - [\text{area under } g]$$



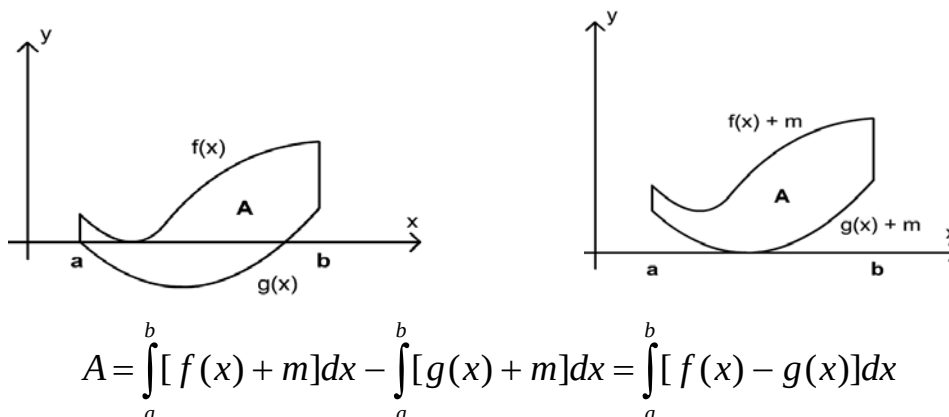
In terms of integrals we can say this as
$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

What if f and g take on negative values too?

This means that their graphs are below the x -axis for some values in $[a, b]$.

This can be remedied if we translate the two graphs by a constant so big that it shifts both f and g above the x -axis.

This shift does not affect the area between the two curves



The constants cancel each other in the calculations.

Here is the formal definition

DEFINITION 6.1. (Area Formula)

If f and g are continuous functions on the interval $[a, b]$, and $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x=a$, and on the right of the line $x=b$ is

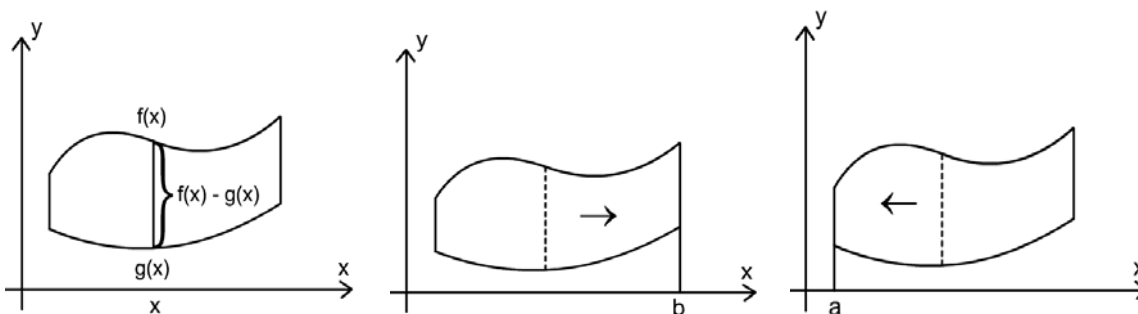
$$A = \int_a^b [f(x) - g(x)]dx$$

A few things to keep in mind.

If the region confined by two curves is complicated, then it may require some careful thought to determine the integrand.

It may be hard to find the limits of integration.

Here is a procedure we can follow



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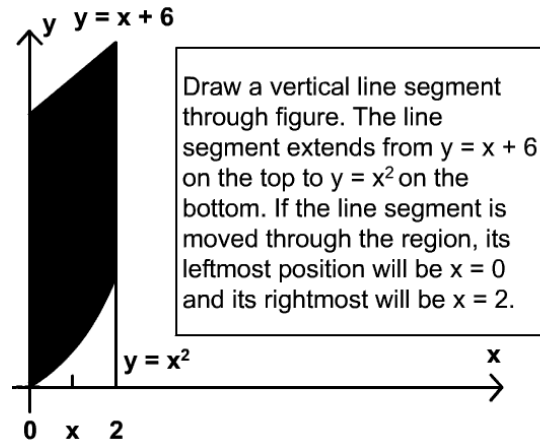
Example

Find the area of the region bounded above by $y = x + 6$, and below by

$y = x^2$ and on the sides by lines $x = 0$ and $x = 2$.

Solution:

The region and a vertical line through it are shown in this figure.



The line extends from $f(x) = x + 6$ on the top to $y = x^2$ at the bottom.

As the line moves left to right, its leftmost position will be $x = 0$ and right most will be $x = 2$

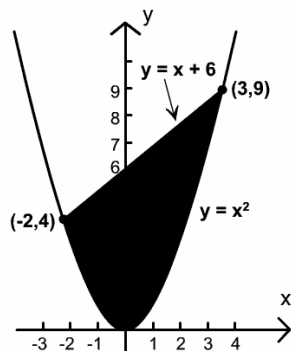
So we have

$$A = \int_0^2 [(x + 6) - x^2] dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_0^2 = \frac{34}{3} - 0 = \frac{34}{3}$$

Example

Find the area of the region that is enclosed between $y = x^2$ and $y = x + 6$.

solution:



There are no lines at the right or the left. So how do we find the limits of integration? Well here is a sketch of the region.

Note that the integration limits are defined by the points where the two curves intersect.

We can find these points by equating the two equations and solving for x

We get

$$x^2 = x + 6$$

$$\Rightarrow x^2 - x - 6 = 0$$

$$\Rightarrow (x + 2)(x - 3) = 0$$

$$\Rightarrow x = -2 \text{ and } x = 3$$

So now we have

$$A = \int_{-2}^3 [(x + 6) - x^2] dx = \left[\frac{x^2}{2} + 6x - \frac{x^3}{3} \right]_{-2}^3 = \frac{125}{6}$$

Example

Find the area of the region enclosed by $x = y^2$ and $y = x - 2$

Solution:

First we need to know where the two curves intersect.

Let's write down $y = x - 2$ as $x = y + 2$ and then equate

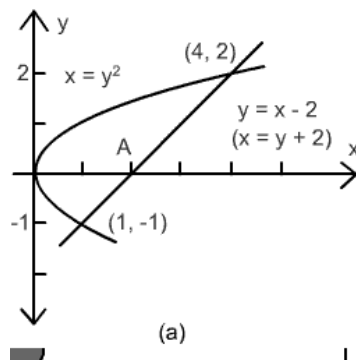
$$x = y^2 \text{ and } x = y + 2$$

This gives

$$y^2 = y + 2 \Rightarrow y^2 - y - 2 = 0 \Rightarrow y = -1 \text{ and } y = 2$$

Substituting these into the original equations will give us the desired x -values, which are $x = 1$ and $x = 4$ respectively

Here is the graph of the situation



From this figure, we see that the upper boundary or curve is defined by the equation $y = +\sqrt{x}$.

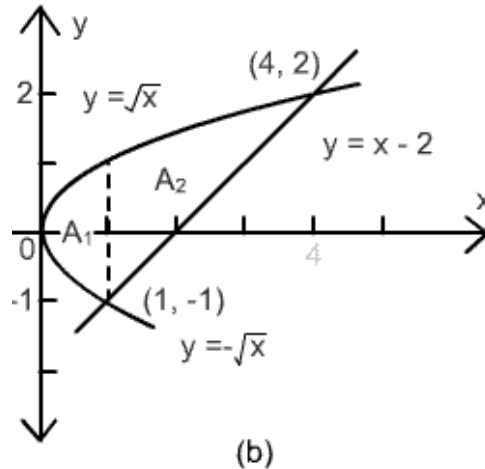
The lower boundary has two pieces

- 1) $y = -\sqrt{x}$ for x between -1 and 0
- 2) $y = x - 2$ for x between 1 and 4

This causes us problems when we try to determine $f(x) - g(x)$ for the integration.

Instead we will divide the region into two parts and evaluate the area of each region and then add together.

Here is how we will divide it



$$f(x) = +\sqrt{x}, g(x) = -\sqrt{x}, a = 0, b = 1$$

$$\begin{aligned} A_1 &= \int_0^1 [\sqrt{x} - (-\sqrt{x})] dx = 2 \int_0^1 \sqrt{x} dx \\ &= 2 \left[\frac{2}{3} x^{\frac{3}{2}} \right]_0^1 = \frac{4}{3} \end{aligned}$$

$$f(x) = +\sqrt{x}, g(x) = x - 2, a = 1, b = 4$$

$$\begin{aligned} A_2 &= \int_1^4 [\sqrt{x} - (x - 2)] dx = \int_1^4 (\sqrt{x} - x + 2) dx \\ &= \left[\frac{2}{3} x^{\frac{3}{2}} - \frac{1}{2} x^2 + 2x \right]_1^4 = \frac{19}{6} \end{aligned}$$

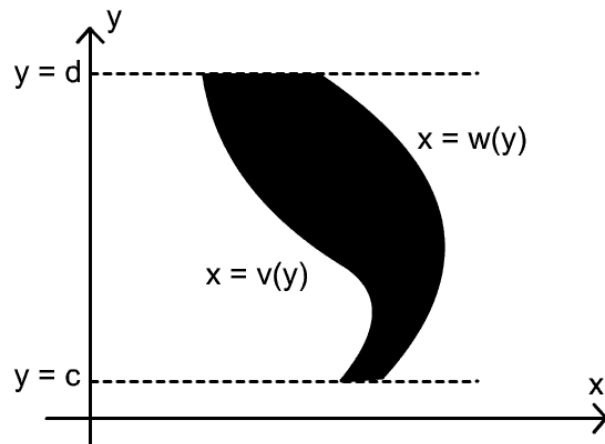
Total area or region is $19/6 + 4/3 = 9/2$

Area between $x = v(y)$ and $x = w(y)$

We can avoid splitting the region if we integrate w.r.t y instead of x .

Second Area problem

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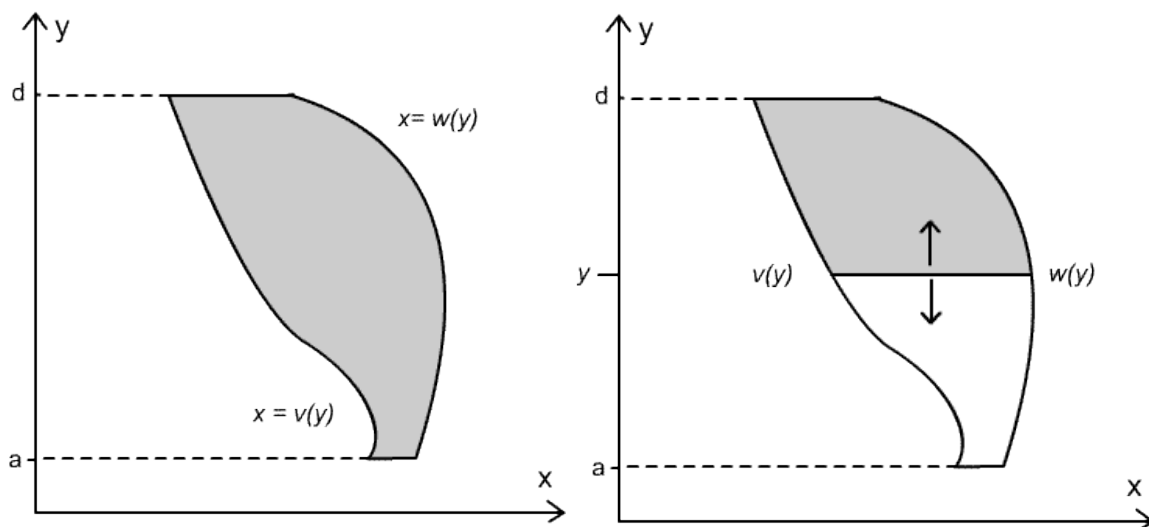
Just as we derived an answer earlier in the lecture for the first area problem for regions confined by two curves, we get the following definition

DEFINITION 6.1.2 (Area Formula)

If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)] dy$$

The procedure for finding the integrand and limits of integration is the same as earlier. Here are two figures to help you see this



Example

Find the area of the region confined by $x = y^2$ and $y = x - 2$ or $x = y + 2$

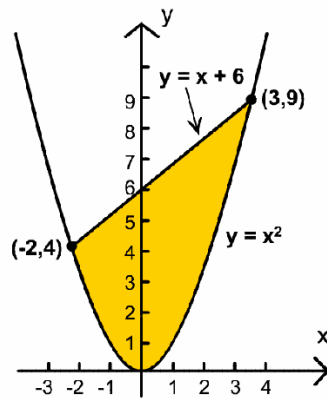
Solution:

Left boundary is $x = y^2$

Right boundary is $x = y + 2$

Limits are from -1 to 2

$$A = \int_{-1}^2 [(y + 2) - y^2] dy = \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 = \frac{9}{2}$$



The area is shown in the figure ab

Lecture # 34

Volume by Slicing; Disks and Washers

- Definite Integrals to find Volumes of three dimensional solids
- Cylinders
- The method of Slicing
- Volumes by cross sections perpendicular to the x axis
- Volumes by cross sections perpendicular to the y axis
- Volumes of solids of revolution by:
 - i. Volumes by Disks perpendicular to x axis
 - ii. Volumes by Disks perpendicular to y axis
 - iii. Volumes by washers perpendicular to x axis
 - iv. Volumes by washers perpendicular to y axis

Cylinders

How to go from 0-d to 3-d and beyond in terms of a cube.

We can get other 3d solids by moving 2d planes in a certain fashion in 3d space.

If I move a 2d circle along a line that is perpendicular to the circle, I will get a cylinder.

If I move a circle with a hole in it, or a washer, along a line perpendicular to the washer, then I will get a cylinder with a hole in it.

In general, if I move a 2d plane in a direction along a line perpendicular to the region, then I get a RIGHT CYLINDER.

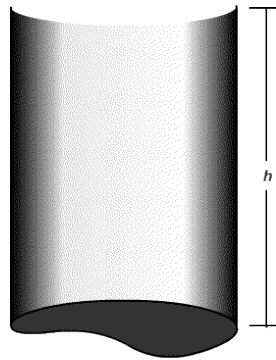


Every solid object has Volume, Volume is the 3d area!

We can find the volume of a right cylinder generated by moving a 2d plane by the following formula

$$V = A.h$$

Where A is the area of the 2d object, and h is the height of the cylinder



$$\text{Volume} = A \cdot h$$

So the Volume of a cylinder is the area of a cross section of the cylinder multiplied by the height of the cylinder.

But there are 3d solids that are not right cylinders and are neither made up of finitely many right cylinders.

So we cannot use the formulas for the area of the right cylinders to find the area of such an object. In this case we use the technique of slicing.

The method of Slicing

Here is such a 3d solid



This is not made up of finitely many right cylinders.

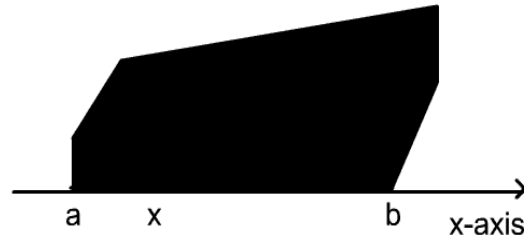
We find the volume of this object by slicing.

Impose an x axis on the solid.

Then we can imagine that the solid is bounded to the left by a plane perpendicular to the x -axis at $x = a$ and to the right by a plane at $x = b$



The problem with finding the volume of this non-cylindrical solid is clear now: at any point along the x -axis, the cross sections perpendicular to the x -axis will have different areas!

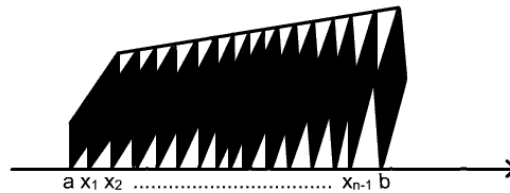


Let's call the area of a cross section at some arbitrary point x $A(x)$

Now Let's divide the interval $[a,b]$ into n subintervals of width $\Delta x_1, \Delta x_2, \dots, \Delta x_n$ By inserting the points $x_1, x_2, \dots, x_n$ between a and b .

We can now pass a plane perpendicular to the x -axis through each of these points.

These planes will subdivide the solid into SLICES $S_1, S_2, \dots, S_n$.



Consider a typical slice S_k as shown in the figure

If this slice is very thin, then the cross section

of this slice S_k will not vary too much in the

interval Δx_k which defines the thickness of the

slice S_k . That is, the cross section will have the same shape in the interval if Δx_k is thin.

So let's choose a point x_k^* in the interval Δx_k and take a cross section of the slice at this point. Then the area of this cross section will be $A(\Delta x_k)$

The volume of the slice S_k then will be approximately

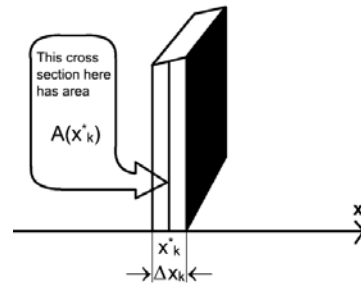
$$V_k \approx A(x_k^*)\Delta x_k$$

We can use this formula for all the THIN slices we make to get their respective volumes.

The total volume of the solid can be approximated by

$$V = V_1 + \dots + V_n \approx \sum_{i=1}^n A(x_k^*)\Delta x_k$$

Now increase the number of slices. Then the slices will become VERY THIN and the approx will get better and better. So we can take the limit to get



$$\max \Delta x_k \rightarrow 0$$

$$V = \lim_{\max \Delta x_k \rightarrow 0} \sum_{i=1}^n A(x_k^*) \Delta x_k = \int_a^b A(x) dx$$

Volumes by cross sections perpendicular to the x axis

VOLUME FORMULA

Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If , for each x in $[a,b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x) dx$$

Volumes by cross sections perpendicular to the y axis

VOLUME FORMULA

Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If , for each y in $[c,d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

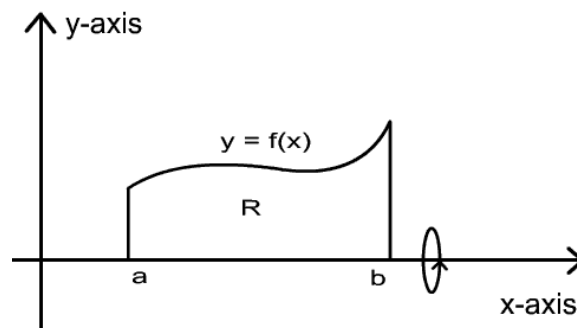
$$V = \int_c^d A(y) dy$$

Volumes of solids of revolution by:

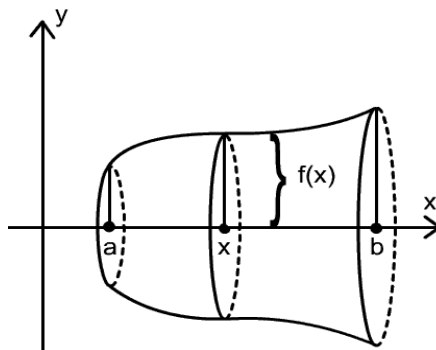
- I. Disks perpendicular to x-axis

Let f be a nonnegative and continuous function on $[a,b]$

Let R be the region bounded above by the graph of f and on the sides by lines $x = a$ and $x = b$



If this solid is revolved around the x-axis, it generates a solid with a circular cross sections.



The cross section at x has radius $f(x)$ so the area of this cross section is

$$A(x) = \pi [f(x)]^2$$

So from the formulas we just saw we can write for the volume of this solid:

$$V = \int_a^b \pi [f(x)]^2 dx$$

Example 2

Find the volume of solid that is obtained when the region under the curve $y = \sqrt{x}$ over the $[1, 4]$ is revolved about the x-axis .

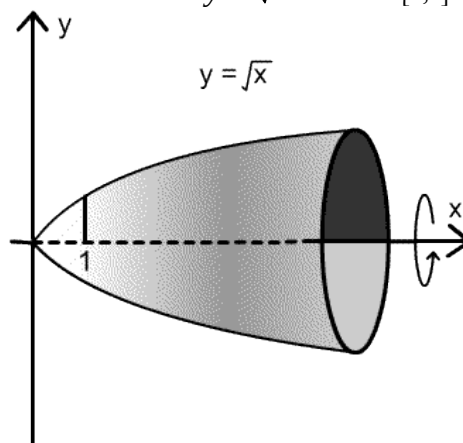
Solution:

From $V = \int_a^b \pi [f(x)]^2 dx$, the volume is

$$V = \int_1^4 \pi [\sqrt{x}]^2 dx = \int_1^4 \pi x dx$$

on evaluating

$$V = \frac{15\pi}{2}$$



Example 3

Derive the formula for the volume of a sphere of radius r .

Solution:

As indicated in the figure , a sphere of radius r can be generated by revolving the upper half of the circle

$$x^2 + y^2 = r^2$$

About the x-axis. Since upper half of this circle is graph of

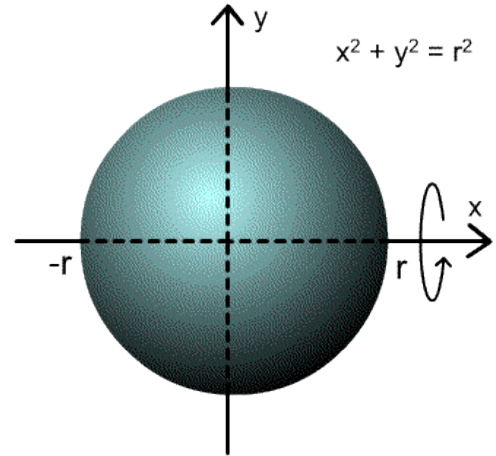
$$y = f(x) = \sqrt{r^2 - x^2}$$

it follows from above formula that the volume of the sphere is

$$V = \int_a^b [f(x)]^2 dx = \int_{-r}^r \pi(r^2 - x^2) dx$$

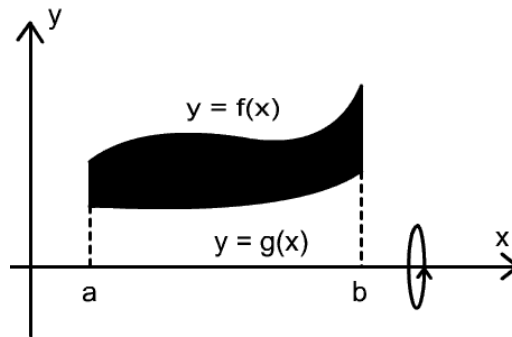
On evaluating

$$V = \frac{4}{3} \pi r^3$$



Volumes by washers perpendicular to x axis

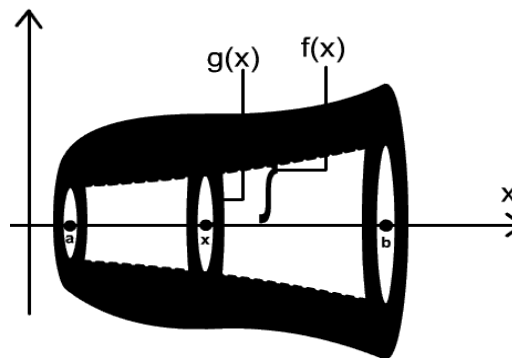
Now consider this picture



Here R is a region confined by the graph of two non-negative continuous functions f and g with f greater than or equal to g on the interval $[a, b]$

When this region is revolved around the x-axis, we get a solid with a hole in the center

This solid has a cross section that looks like a washer.



This washer cross section has two radii defined by the function f and g as is in the picture.

So the area of this cross section will be

$$A(x) = \pi [f(x)]^2 - \pi [g(x)]^2 = \pi ([f(x)]^2 - [g(x)]^2)$$

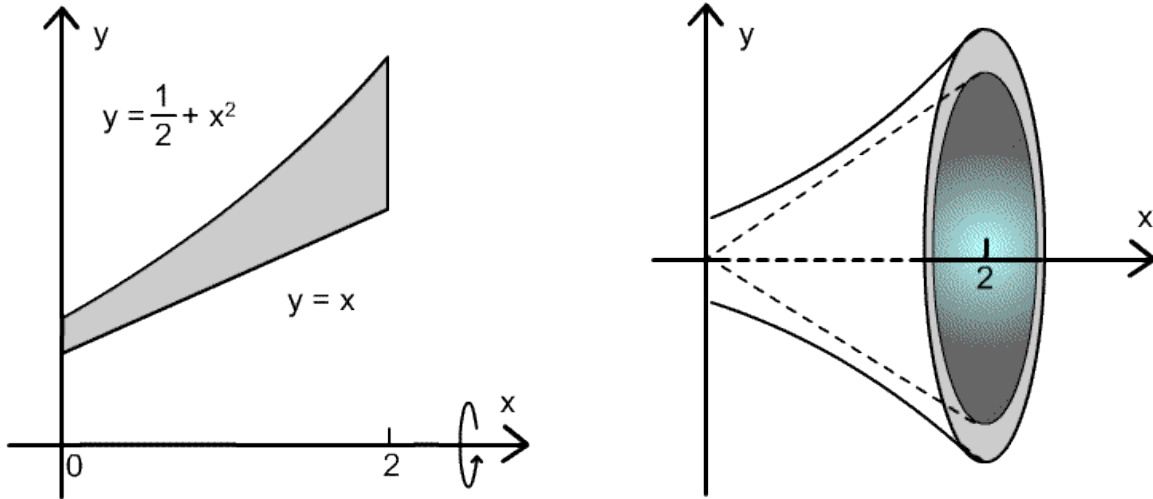
And the volume of the solid now can be expressed

$$V = \int_a^b \pi \left([f(x)]^2 - [g(x)]^2 \right) dx$$

Example 4

Find the volume of the solid generated when the region between the graph of

$f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis.

**Solution;**

From

$$V = \int_a^b \pi \{ [f(x)]^2 - [g(x)]^2 \} dx$$

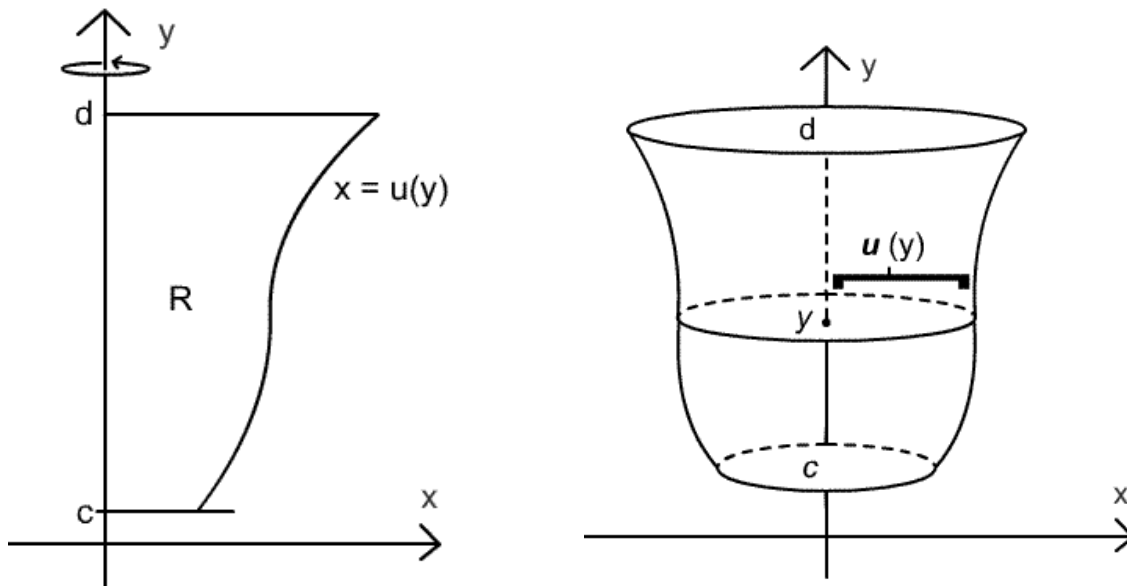
the volume is

$$\begin{aligned} V &= \int_0^2 \pi \{ [f(x)]^2 - [g(x)]^2 \} dx = \int_0^2 \pi \left(\left[\frac{1}{2} + x^2 \right]^2 - x^2 \right) dx \\ &= \int_0^2 \pi \left(\frac{1}{4} + x^4 \right) dx = \frac{69\pi}{10} \quad (\text{on evaluating}) \end{aligned}$$

Volume by Disks perpendicular to y axis

Just as we found the volumes of solids generated by revolving a region around the x-axis, we can find the volume of solids generated by revolving a region around the y-axis in which the cross section is a disk.

Here is how in a picture

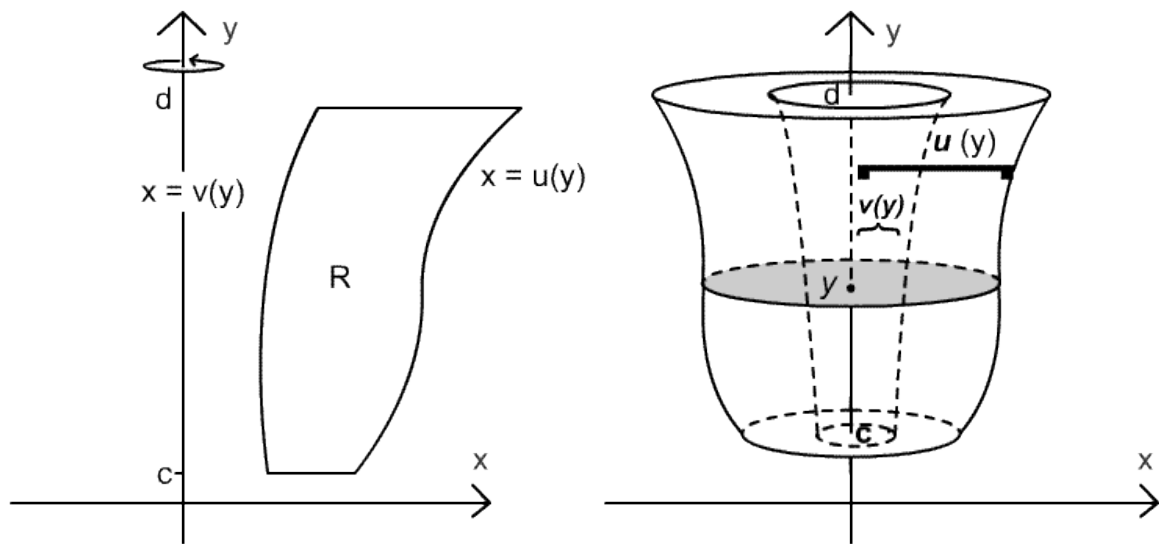


The formula in this case for volume is the following

$$V = \int_c^d \pi [u(y)]^2 dy$$

Volume by Washers perpendicular to y axis

Just as we found the volumes of solids generated by revolving a region around the x-axis, we can find the volume of solids generated by revolving a region around the y-axis in which the cross section is a washer



The formula expressing it is $V = \int_c^d \pi ([u(y)]^2 - [v(y)]^2) dy$

Lecture # 35

Volume by Cylindrical Shells

- Cylindrical Shells
- Volume of a Cylindrical Shell
- Cylindrical Shells centered on the y-axis
- Volume through the surface area of a surface created from a cylindrical shell

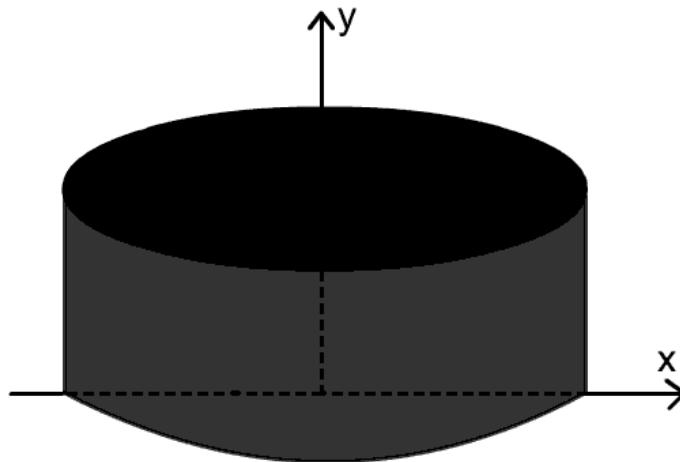
Cylindrical Shells

If we take a washer – a disk with a hole in it – and extend it UP, we generate a solid called a cylindrical shell.

This is a solid confined by two concentric right circular cylinders

The Volume of a cylindrical shell can be expressed as

$$\begin{aligned}
 V &= (\text{area of cross section}) \cdot (\text{height}) = (\pi r_2^2 - \pi r_1^2)h \\
 &= \pi (r_2 + r_1)(r_2 - r_1)h \\
 &= 2\pi \left[\frac{1}{2}(r_2 + r_1) \right] \cdot h \cdot (r_2 - r_1)
 \end{aligned}$$



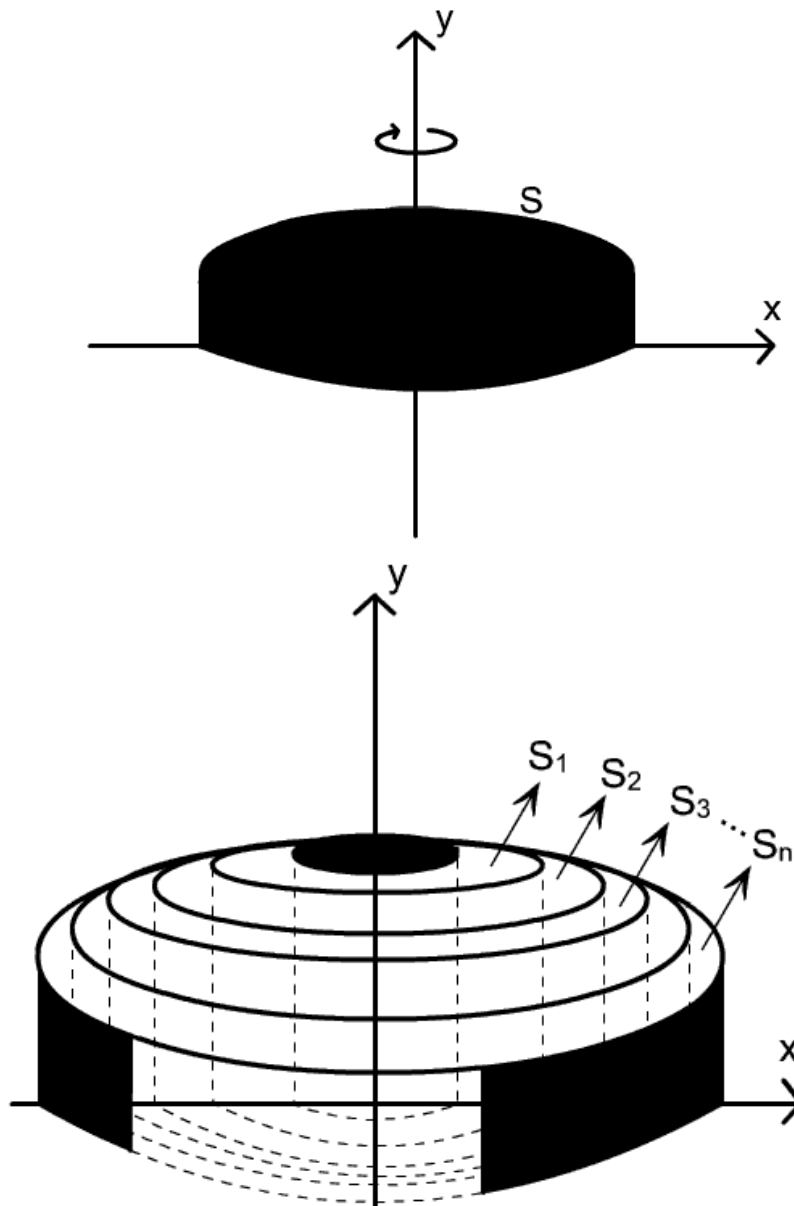
Let's rewrite this rearrangement of the Volume of a cylindrical shell as

$$V = 2\pi (\text{average radius}) \cdot (\text{height}) \cdot (\text{thickness})$$

Now we can use this formula to compute the Volume of a Solid generated by revolution of a surface around an axis.

Consider the following

R is a region bounded by the graph of $f(x)$ on top, below by the x-axis, and to the left by $x = a$, and right by $x = b$



When we revolve R around the x -axis, we get the solid S as in the picture.

To find the volume of S , here is what we do.

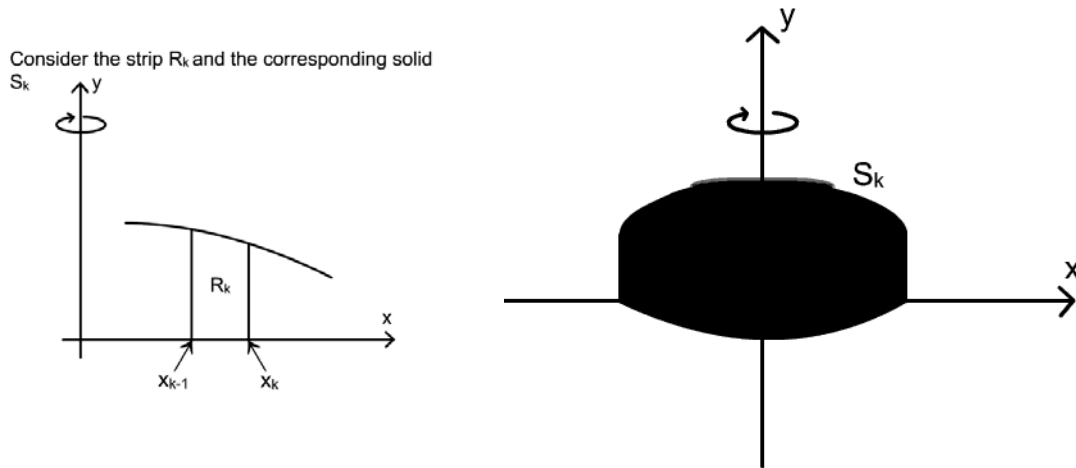
Subdivide $[a, b]$ into n subintervals with widths $\Delta x_1, \dots, \Delta x_n$ by inserting the points $x_1, \dots, x_n$ between a and b .

If we draw vertical lines from these points to the graph of f , we get subdivisions $R_1, \dots, R_n$ of the region R .

Revolving these strips around the y -axis gives solids $S_1, \dots, S_n$

So we can do the following to get the volume of S

$$V(S) = V(S_1) + \dots + V(S_n)$$



Consider the strip R_k and the corresponding solid S_k .

The solid S_k is not necessarily a cylindrical shell as it may have a curve upper surface depending on the graph of f .

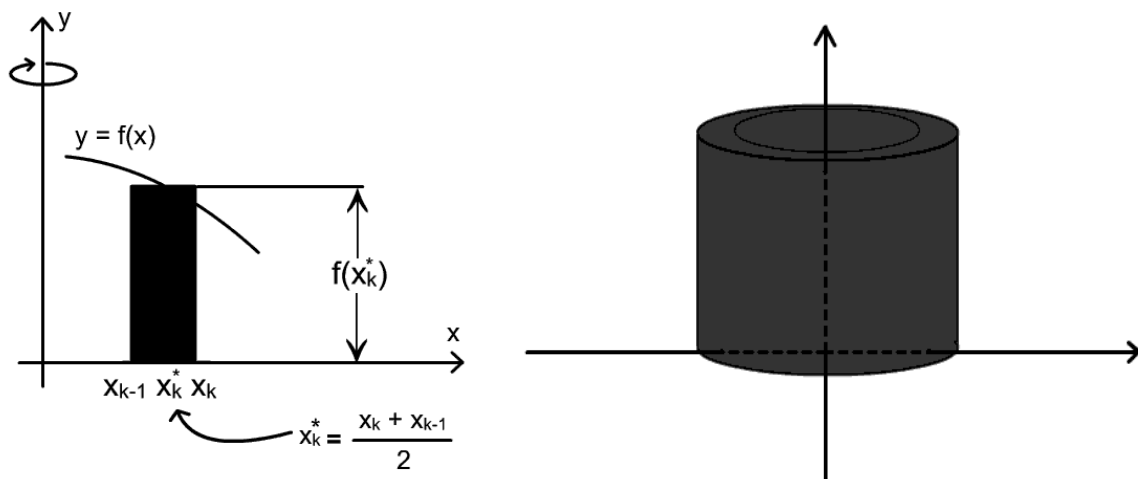
However, if the width of Δx_k is very small, then we can get a good approx to the region R_k by the rectangle of width Δx_k and height

$f(x_k^*)$ where

$$x_k^* = \frac{x_k + x_{k-1}}{2}$$

Is the midpoint of the interval $[x_{k-1}, x_k]$

This in turn will give us a cylindrical shell when we revolve it around y-axis. which will be a good approx to the solid.



The volume of S_k can now be approximated using the width height $f(x_k^*)$, average radius x_k^* of the cylindrical shell as Δx_k

$$V(S_k) \approx \text{volume of approximating cylindrical shell} = 2\pi x_k^* f(x_k^*) \Delta x_k$$

The volume of the whole solid S will now be just the sum

$$V(S) = \sum_{k=1}^n 2\pi x_k^* f(x_k^*) \Delta x_k$$

We can take the limit and get the exact Volume of S as

$$V(S) = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n 2\pi x_k^* f(x_k^*) \Delta x_k = \int_a^b 2\pi x f(x) dx$$

In this process, we see that we have a formula to compute volume of a solid which is got from revolving a region around the Y-AXIS

Cylindrical Shells centered on the y-axis

VOLUME FORMULA

Let R be a plane region bounded above by a continuous curve $y = f(x)$ below by x -axis, and on the left and right, respectively, by the lines $x = a$ and $x = b$. Then the volume of the solid generated by revolving R about the y -axis is given by

$$V = \int_a^b 2\pi x f(x) dx$$

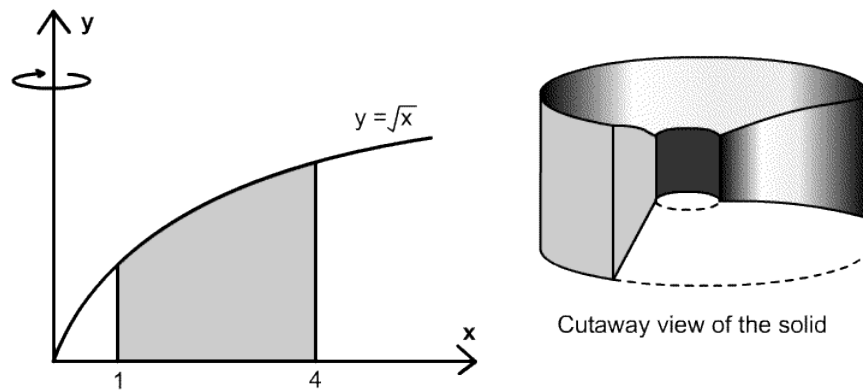
Example 1

Use cylindrical shells to find the volume of the solid generated by the region enclosed between $y = \sqrt{x}$, $x = 1$, $x = 4$ and the x -axis is revolved about the y -axis.

Solution:

Since $f(x) = \sqrt{x}$, $a = 1$, and $b = 4$ and the above volume formula yields

$$\begin{aligned} V &= \int_1^4 2\pi x \sqrt{x} dx = 2\pi \int_1^4 x^{\frac{3}{2}} dx \\ V &= 2\pi \left[\frac{2}{5} x^{\frac{5}{2}} \right]_1^4 = \frac{124\pi}{5} \end{aligned}$$



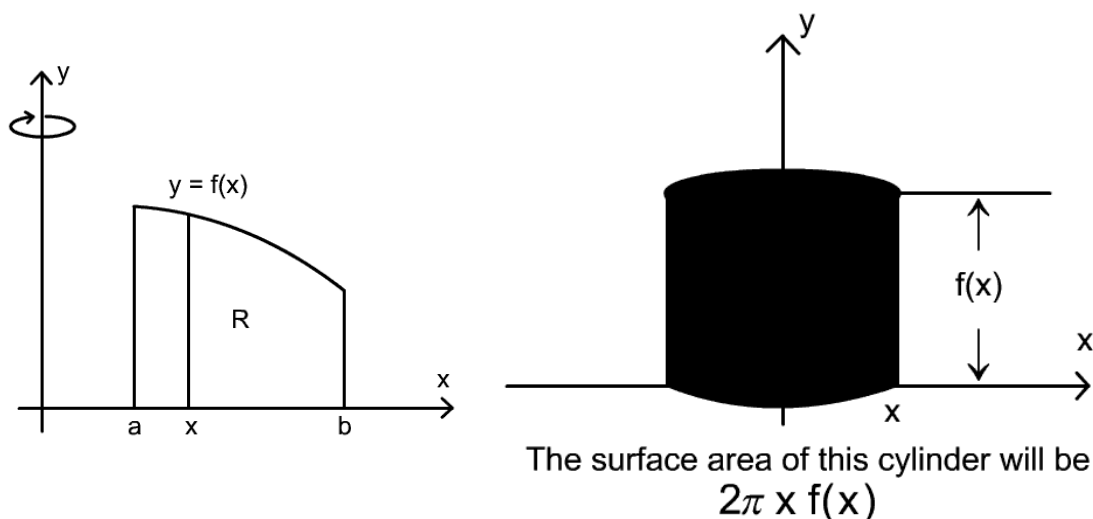
Volume through the surface area of a surface created from a cylindrical shell

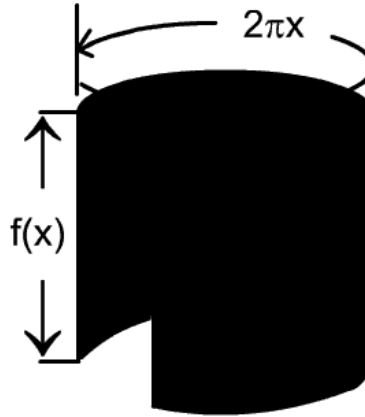
A bit more general and perhaps easier to conceptualize method can be worked out for cylindrical shells

Instead of thinking of the solid S_k as having a width Δx_k suppose that it has no width

This would mean that S_k , will be a straight forward right circular cylinder having radius x , height $f(x)$.

The surface area of this cylinder will be $2\pi x f(x)$.





The surface area of this cylinder will be

$$2\pi x f(x)$$

This is exactly the integrand in the Volume formula we saw earlier. So this really means that :

VOLUME **V** BY CYLINDRICAL SHELLS IS THE INTEGRAL OF THE SURFACE AREA GENERATED BY AN ARBITRARY SECTION OF THE REGION **R** TAKEN PARALLEL TO THE AXIS ABOUT WHICH **R** IS REVOLVED.

This view of volume by cylindrical shells helps us do calculation in more general setting where for example the lower boundary may not be interval

Example 2

Use cylindrical shells to find the volume of the solid when the region **R** in the first quadrant enclosed between $y = x$, and $y = x^2$

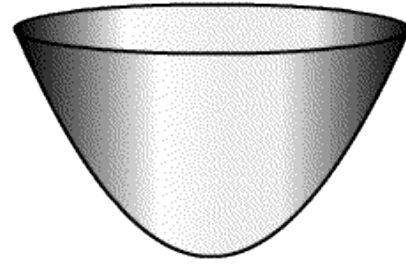
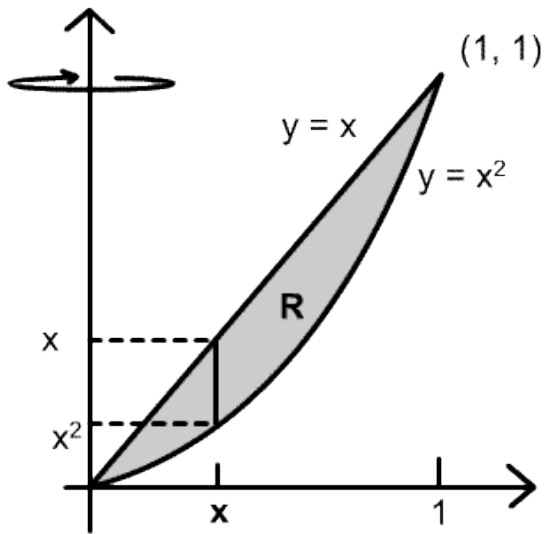
Is revolved about the y -axis

Solution:

At each x in $[0, 1]$ the cross section of **R** parallel to the y -axis generates a cylindrical surface of height $x - x^2$ and radius x . Since the area of the surface is $2\pi x(x - x^2)$. Thus the volume of the solid is given by the following formula and the figure is also below

$$V = \int_0^1 2\pi x(x - x^2) dx = 2\pi \int_0^1 (x^2 - x^3) dx$$

$$V = 2\pi \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{\pi}{6}$$



Lecture # 36

Length of a Plane Curve

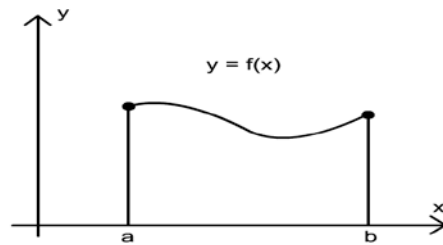
Arc Length:

We all know how to find the length of a line. In this section, we will develop the ways of finding the length of curves, or lines that twist and turn. The curves we look at will be graphs of functions. The functions we look at will be such that will be continuous on a given interval. Such functions are called **smooth** functions, and their graphs **smooth curves**.

So in this lecture, we are concerned with the ARC LENGTH PROBLEM

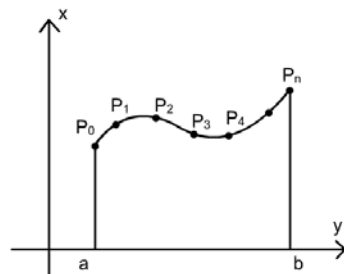
Arc Length Problem:

Suppose that f is a smooth function on the interval $[a, b]$. Find the arc length L of the curve $y = f(x)$ over the interval $[a, b]$.

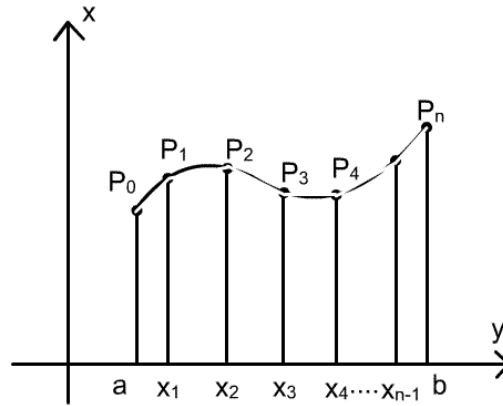


Here question arises that **How can we measure the arc length?**

Arc length is just the length of a line that is not straight, but can be rounded in various places. Sometimes we want to measure something with a measuring tape, something that is straight. But often the tape hangs loosely away from the straight object. In such cases our result is not accurate, since if the tape is loose then it will show some extra length. Similarly, if we have a curve surface, we can not measure its length with a straight meter rod accurately. Consider this curve in the figure



Now say I want to measure the length from P_0 to P_1 . Let's say I use a straight stick. The stick will be rigid and straight, and so I will miss some length that is curved between the points P_0 and P_1 . If P_0 and P_1 are so close that distance between them become straight then straight stick can be used. Now I can do the same for all the lengths between all the points in the figure.



The line segments joining the points P_i $i = (1 \text{ to } n)$ form a polygonal path and it's clear that this path is a good approximation of the length of the curve.

Well, it should also be clear that just as we have done so far, if we let the number of these segments joining all points between P_0 to P_n increase, we will get a better approximation to the curve.

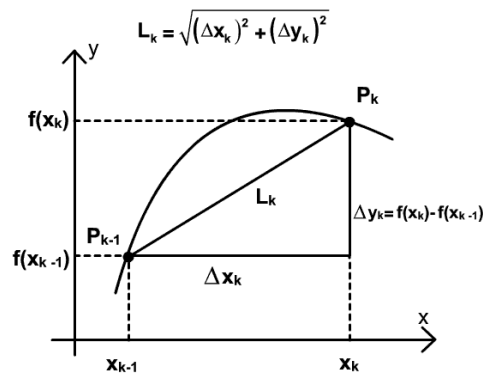
We can do it in the following way

Note in the figure that to each point P_i , there corresponds a point x_i on the x-axis in the interval $[a, b]$

Let's assign the distance between the x_i 's as $\Delta x_1, \Delta x_2, \dots, \Delta x_n$.

Lets look at the k_{th} line segment. We will call it L_k . Here is what it looks like.

Figure 6.4.3



We want to find its length. Using the distance formula, we get

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

From the figure 6.4.3, it's clear that $\Delta y_k = f(x_k) - f(x_{k-1})$

So we get

$$L_k = \sqrt{(\Delta x_k)^2 + (f(x_k) - f(x_{k-1}))^2}$$

The Mean Value Theorem (Theorem 4.9.2) can be applied here.

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \text{ for some point } x_k^* \text{ between } x_k \text{ and } x_{k-1}$$

$$\Rightarrow f(x_k) - f(x_{k-1}) = f'(x_k^*)(x_k - x_{k-1}) = f'(x_k^*)\Delta x_k$$

So, we can write our first equation as

$$L_k = \sqrt{(\Delta x_k)^2 + (f'(x_k^*))^2 (\Delta x_k)^2}$$

$$L_k = \sqrt{1 + (f'(x_k^*))^2} \Delta x_k$$

So, this is the length of one segment.

The length of the WHOLE polygonal path will be

$$\sum_{k=1}^n L_k = \sum_{k=1}^n \sqrt{1 + (f'(x_k^*))^2} \Delta x_k$$

Just as we have done before, if we increase the number of divisions of the interval $[a, b]$ in such a way that the widest of the subinterval goes to 0, we get

$$L = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n \sqrt{1 + (f'(x_k^*))^2} \Delta x_k \quad \text{which we will call arc length of the curve given by the function } f(x).$$

Of course, the above equation defines an integral $L = \int_a^b \sqrt{1 + (f'(x))^2} dx$

Arc Length Formulas:

If f is a smooth function on $[a, b]$, then the arc length L of the curve $y = f(x)$ from $x = a$ to $x = b$ is defined by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2} dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Lecture # 37

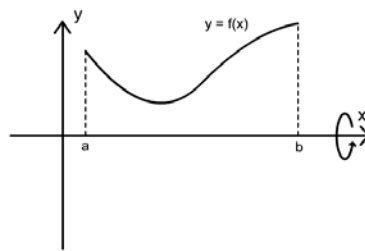
Area of a surface of Revolution

We will study in this lecture

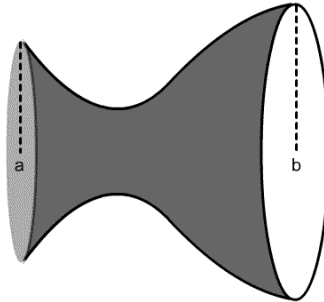
- Definition of Surface Area
- Surface Area Formulas

Surface Area Problem:

Let f be a smooth, non negative function on $[a, b]$. Find the area of the surface generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis.



By revolving this curve we get a 3-dimensional solid as



Surface area is roughly the area covered by the surface of a solid in 3-dimensional space.

But we need a more precise definition.

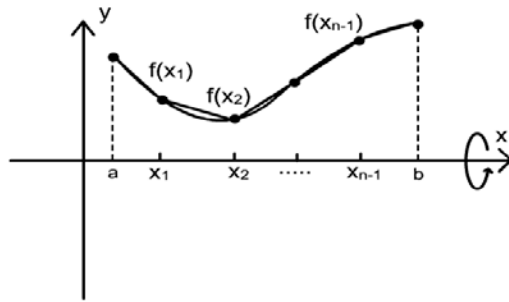
Say we have the graph of the function f as stated earlier in 6.5.1 area problem.

As usual, divide the interval $[a, b]$ into subinterval with widths $\Delta x_1, \Delta x_2, \dots, \Delta x_n$ having x-coordinates

$a, x_1, \dots, x_{n-1}, b$.

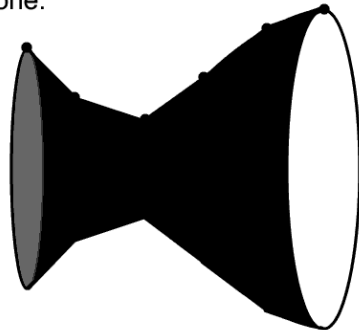
This in turn subdivides the curve into “sub-curves” which can be approximated by a polygonal path made up of line segments joining the end points of the “sub-curves”

Figure 6.5.2 a)



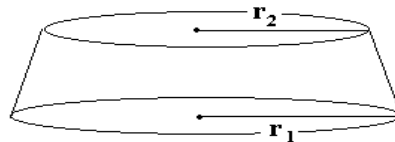
If we now rotate this polygonal path around the x-axis, we get a solid that is approximately the same in properties to the one we had by rotating the graph of f around the x-axis.

Surface generated by the polygonal path is made of parts, each of which is a frustum of a cone.



This approximated solid is made up of frustums of a cone. A frustum of a cone is the solid that you get if you chop off the pointed top of a cone.

FRUSTUM



It has two radii and a slanted length.

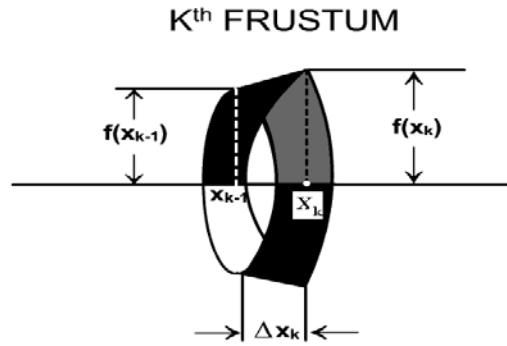
Its surface area is measured by $S = \pi (r_1 + r_2)l$

In our case of the solid, the radii correspond to the height of the points of subdivision on the curve in an interval, and l is the length of the line segment (arc length) in a given interval.

So we can use this formula to find the surface area of each frustum on the approx. solid, and then add them all up.

To get the surface area of the original solid, if we let the number of our subdivisions increase without bound, then we get the surface area of the solid. Here is how it is done.

Lets consider the k -th frustum of our approx. solid.



Its surface area can be found using the formula for the frustum

$$S_k = \pi [f(x_{k-1}) + f(x_k)] \sqrt{(\Delta x)^2 + [f(x_k) - f(x_{k-1})]^2}$$

$$\text{as } r_1 = f(x_{k-1})$$

$$r_2 = f(x_k)$$

$$l = \sqrt{(\Delta x)^2 + [f(x_k) - f(x_{k-1})]^2}$$

The formula for l is just that of arc length so we can write it as $\sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$

So

$$S_k = \pi [f(x_{k-1}) + f(x_k)] \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

By the Intermediate Value Theorem, there exists a point x_k^{**} in the interval $[x_{k-1}, x_k]$ such that

$$\frac{1}{2} [f(x_{k-1}) + f(x_k)] = f(x_k^{**})$$

So, we can re-write our equation for S_k as $S_k = 2\pi f(x_k^{**}) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$

The area of the approximated solid is then $\sum_{k=1}^n S_k = \sum_{k=1}^n 2\pi f(x_k^{**}) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$

As we have done before, let the largest width on the x-axis approach 0 to get

$$S = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n 2\pi f(x_k^{**}) \sqrt{1 + [f'(x_k^*)]^2} \Delta x_k$$

$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

Surface Area Formulas:

Let f be a smooth, nonnegative function on $[a, b]$. Then the surface area S generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is

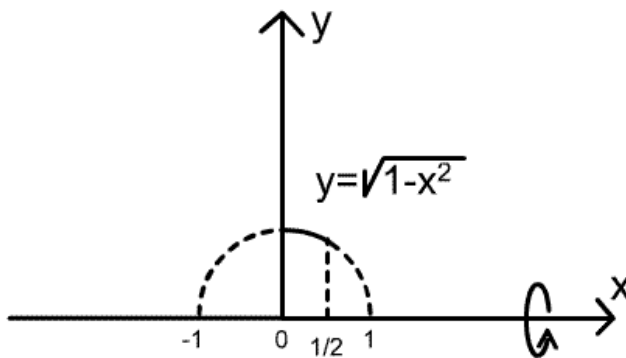
$$S = \int_a^b 2\pi f(x) \sqrt{1 + [f'(x)]^2} dx$$

Example:

Find the surface area of the portion of the sphere generated by revolving the curve

$$y = \sqrt{1-x^2} \quad 0 \leq x \leq \frac{1}{2} \quad \text{about the x-axis}$$

Solution: Since



$$f(x) = \sqrt{1-x^2}$$

$$f'(x) = -\frac{x}{\sqrt{1-x^2}}$$

Thus from surface area formula

$$S = \int_0^{1/2} 2\pi\sqrt{1-x^2} \sqrt{1+\frac{x^2}{1-x^2}} dx = \int_0^{1/2} 2\pi dx = 2\pi x \Big|_0^{1/2} = \pi$$

Lecture # 38**Work and Definite Integral**

In this lecture we will discuss

- Work done by a constant force
- Work done by a variable force
- Fluid Pressure
- Pascal's Principle

Work done by a constant force

If an object moves a distance d along a line while a CONSTANT force F is acting on it, then work W done on the object by the force F is defined as

$$W = F \cdot d \Rightarrow \text{Work} = \text{Force} \times \text{distance}$$

Distance unit is meter represented by m

Force units are Pounds (lbs), Dynes or Newton (N).

One Dyne = the force needed to give a mass of 1 gram an acceleration of 1 cm/s^2

1 Newton = the force needed to give a mass of 1 Kg an acceleration of 1 m/s^2

The most common units of work are

- foot-pounds(ft lb)
- dyne-centimeters(dyne cm)
- newton-meter(N m)

One newton-meter is also called a joule(j)

Example

An object moves 5 ft along a line while subjected to a force of 100 lbs in its direction of motion. The work done is

$$W = F \cdot d = 100(5) = 500 \text{ ft.lbs.}$$

Work done by a variable force

- So far the force was a constant force.
- What if it's changing constantly?
- The equation of Work we just saw will not work.
- We need Calculus

6.7.1 PROBLEM

Suppose that an object moves in the positive direction along a coordinate line while subject to a force $F(x)$, in the direction of motion, whose magnitude depends on the coordinate x . Find the work done by the force when the object moves over an interval $[a, b]$.



In this figure, we have a block subjected to the force of a compressed spring. As the block moves from a to b , the spring gets un-compressed and the force it applied diminishes. So here is a case where the force varies with the position x of the spring. So the force $F(x)$ is a function of the position of the block on the x -axis. We need to define work done by a variable force. This, in turn, will give us the answer to calculate it as well through **definite integral**.

As before we subdivide the interval $[a, b]$ into subintervals with coordinates

$a, x_1, \dots, x_{n-1}, b$ and widths Δx_i etc

Work and definite integral

Let's consider the k th interval and the force F over it.

If the interval $[x_{k-1}, x_k]$ is small, then the force F will be almost constant on it.

We can approx F on this interval by $F(x_k^*)$ where x_k^* is a point in the k th interval. The width of this interval will be Δx_k

So the work done on this interval is $W_k = F(x_k^*)\Delta x_k$

So the work done over the whole interval is $\sum_{k=1}^n W_k = \sum_{k=1}^n F(x_k^*)\Delta x_k$

Let the largest subinterval go to 0 to get $W = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n F(x_k^*)\Delta x_k = \int_a^b F(x)dx$

DEFINITION 6.7.2

If an object moves in the positive direction over the interval $[a,b]$ while subjected to a variable force $F(x)$ in the direction of motion, then the work done by the force is

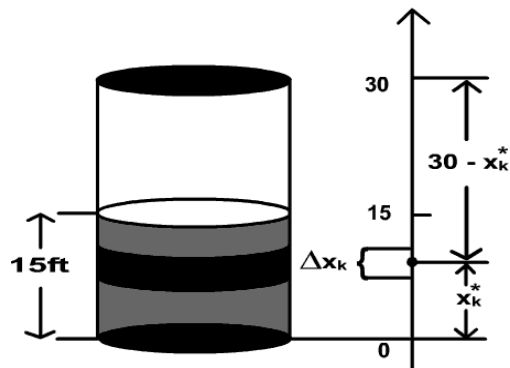
$$W = \int_a^b F(x)dx$$

Example

A cylindrical water tank of radius 10 ft and height 30 ft is half filled with water. How much force is needed to pump all the water over the upper rim of the tank?

Solution

Introduce a coordinate line as shown in figure, imagine the water to be divided into n thin layers with thicknesses $\Delta x_1, \Delta x_2, \dots, \Delta x_n$



$(30 - x_k^*)$ represent approximate distance covered by the k^{th} layer when it moves above the rim

How much force is required to move the K^{th} layer of water above the rim?

The force required to move the Kth layer equals the weight of the layer, which can be found by multiplying its volume by weight density of water

$$\begin{aligned} \left[\begin{array}{c} \text{force to} \\ \text{move the} \\ k^{\text{th}} \text{ layer} \end{array} \right] &= (\pi r^2 \Delta x_k) \left[\begin{array}{c} \text{weight density} \\ \text{of water} \end{array} \right] \\ &= (\pi (10)^2 \Delta x_k) (62.4) \\ &= 6240 \pi \Delta x_k \end{aligned}$$

$$W_k \approx (30 - x_k^*) 6240 \pi \Delta x_k$$

and the work W required to pump all layers will be approximately

$$W = \sum_{k=1}^n W_k \approx \sum_{k=1}^n (30 - x_k^*) (6240 \pi) \Delta x_k$$

To find the exact value of the work we take the limit as $\max \Delta x_k \rightarrow 0$. This yields

$$\begin{aligned} W &= \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n (30 - x_k^*) (6240 \pi) \Delta x_k \\ &= \int_0^{15} (30 - x) (6240 \pi) dx \\ &= 6240 \left(30x - \frac{x^2}{2} \right) \bigg|_0^{15} \\ &= 2,106,000 \pi \text{ ft} / \text{lb} \approx 6,616,194 \text{ ft} / \text{lb} \end{aligned}$$

Pressure is defined as Force per unit Area.

$$P = F / A = \rho h$$

Where p is weight density and h is depth below surface of fluid.

Fluid Pressure

If a flat surface of area A is submerged horizontally in a fluid at a depth h , then the fluid exerts a force F perpendicular to the surface

This is given by $F = phA$

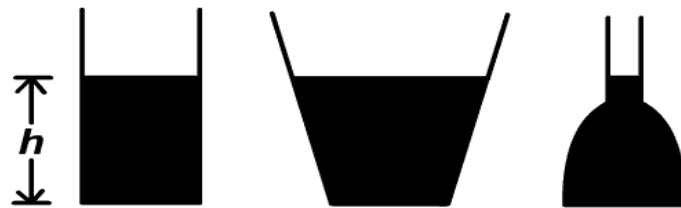
p is the weight density of the fluid.

p for water is 62.4 lbs/ft³

It is a fact from Physics that the shape of the container containing the fluid does not in any way effect F .

If the three containers here have the same area for their bases and the fluid has the same height, then the F on the bases will be equal.

Figure 6.8.1

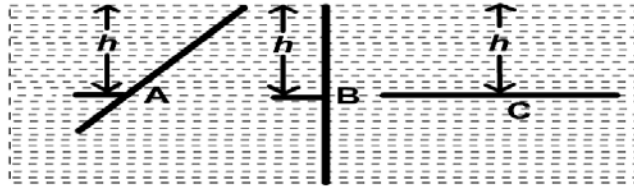


It is a physical fact that the force F does not depend on the shape of the container.

Pascal's Principle

Fluid pressure is the same in all directions at a given height.

Figure 6.8.2



By Pascal's Principle the fluid pressure at point A ,B and C is the same

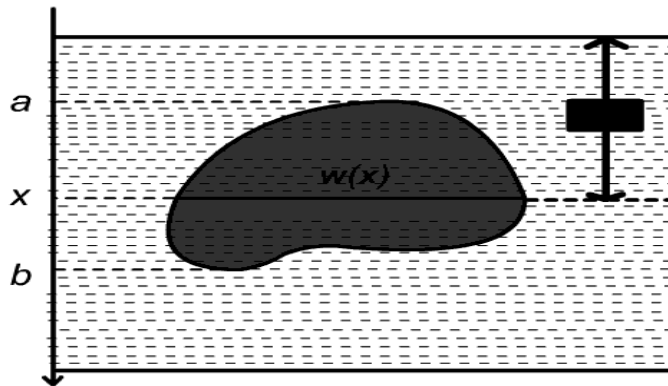
If a flat surface is submerged horizontally, then the total force on its face can be measured easily since the pressure is the same at all the points since the height of these points does not vary.

Suppose we submerge a flat surface in a fluid VERTICALLY.

Then at each point along the height of the surface, the pressure will be different and therefore Force will be different.

Here is where we need calculus

Look at the picture



We have a surface submerged in a tank.

There is VERTICAL axis on the tank

The surface is confined by $x = a$ and $x = b$

There is a height function of x that measures the depth of a point on the surface from the top of the tank.

There is a width function of x that measures the width $w(x)$ of the surface at the height $h(x)$.

At different height (depth) Force on a section of the surface will be different.

We want to know the total force on the flat surface.

Subdivide the interval $[a,b]$ into subintervals of various widths.

Using the k th interval, which form a rectangle that will be used to approximate the force on that interval on the surface?

The smaller the interval, the more accurate the approximation.

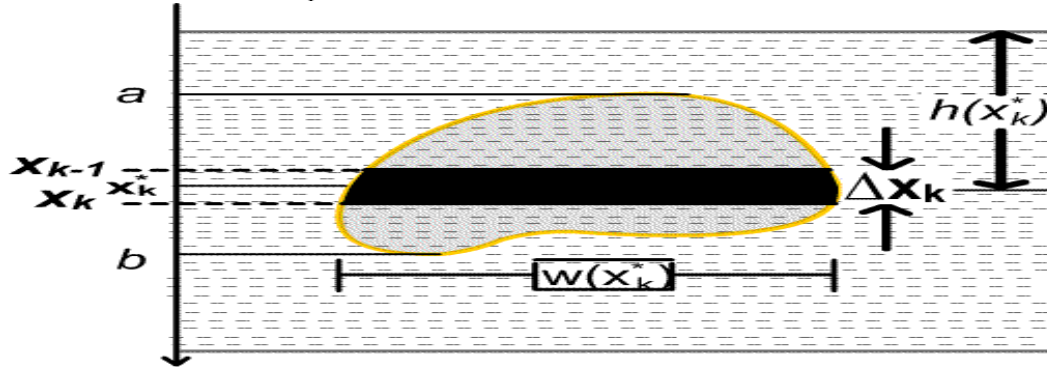
The approximation will be made by $F_k \approx ph(x_k^*)w(x_k^*)\Delta x_k$

p = fluid density

$h(x_k^*)$ = depth

$w(x_k^*)\Delta x_k$ = area of rectangle

• F_k is the force on the strip of width almost zero on the surface



In the figure, imagine the rectangle to shrink to the dashed line.

Hence, the total force in this fashion will be $F = \sum_{k=1}^n F_k \approx \sum_{k=1}^n ph(x_k^*)w(x_k^*)\Delta x_k$

Make the largest interval go to 0 to get $F = \lim_{\max \Delta x_k \rightarrow 0} \sum_{k=1}^n ph(x_k^*)w(x_k^*)\Delta x_k = \int_a^b ph(x)w(x)dx$

Formula For Fluid Force

Assume that a flat surface is immersed vertically in a liquid of weight density P and that the submerged portion extends from $x=a$ to $x=b$ on a vertical x -axis.

For $a \leq x \leq b$, let $w(x)$ be the width of the surface at x and let $h(x)$ be the depth of the point x . Then

the total **fluid pressure** on the surface is $F = \int_a^b ph(x)w(x)dx$

Lecture # 39

Improper Integral

In this lecture we will study

- Integrals over Infinite Interval
- Integrals whose Integrands become Infinite

Integrals over Infinite Interval

As we saw before, for a given continuous function f , the definite integral is $\int_a^b f(x)dx$

It is assumed that the interval $[a, b]$ is finite.

What if we look at $[a, +\infty)$ and the corresponding integral $\int_a^{+\infty} f(x)dx$

In this case, we define what is called an improper integral over an infinite interval.

What does it mean to integrate all the way to $+\infty$?

The answer will be clear if we define this integral as a limit in the following way.

$$\int_a^{+\infty} f(x)dx = \lim_{l \rightarrow +\infty} \int_a^l f(x)dx$$

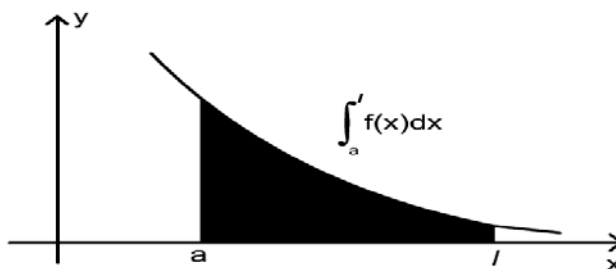
What this does is to first turn the integral into more familiar form of over a finite interval, and then we let the upper limits of the interval approach $+\infty$ and see what happens to the answer we had got earlier.

If this limit exists, then we say that the Improper Integral **Converges**, and the value of the limit is assigned to the integral.

If the limit does not exist, then we say that the Improper Integral **Diverges**, and no finite value is assigned.

Let's get some geometric ideas to understand things.

The integral $\int_a^l f(x)dx$ represents the area under the curve of $f(x)$ over $[a, l]$



Evaluate $\int_1^{+\infty} \frac{dx}{x^2} = \int_1^{+\infty} \frac{1}{x^2} dx$

We begin by replacing the infinite upper limit with a finite upper limit l

$$\begin{aligned} \int_1^l \frac{dx}{x^2} &= \left[-\frac{1}{x} \right]_1^l \\ &= -\frac{1}{l} - \left(-\frac{1}{1} \right) = 1 - \frac{1}{l} \end{aligned}$$

Thus

$$\begin{aligned} \int_1^{+\infty} \frac{dx}{x^2} &= \lim_{l \rightarrow +\infty} \int_1^l \frac{dx}{x^2} = \lim_{l \rightarrow +\infty} \left(1 - \frac{1}{l} \right) \\ &= 1 - \frac{1}{\infty} = 1 - 0 = 1 \end{aligned}$$

EXAMPLE

Evaluate $\int_1^{+\infty} \frac{dx}{x} = \int_1^{+\infty} \frac{1}{x} dx$

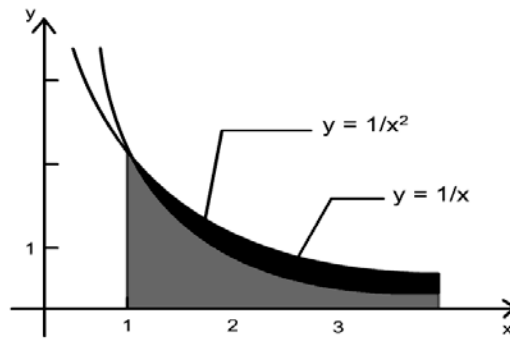
$$\begin{aligned} \int_1^{+\infty} \frac{dx}{x} &= \lim_{l \rightarrow +\infty} \int_1^l \frac{dx}{x} \\ &= \lim_{l \rightarrow +\infty} \left[\ln |x| \right]_1^l \\ &= \lim_{l \rightarrow +\infty} \ln |l| = +\infty \end{aligned}$$

What's happening here in these examples?

In the first case with $f(x) = 1/x^2$, we get finite answer over the same interval?

In the second case we get a divergent limit and so we were unable to calculate the area under graph of $1/x$ over $[1, +\infty)$

Look at the graphs of the two functions



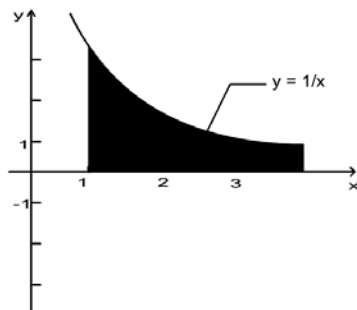
We can see geometrically that the graph of $1/x^2$ is approaching $y = 0$ much faster than that of $1/x$. Algebraically also, if you divide 1 by the square of a number, the result is much smaller than if you divide by the number itself.

For example $1/2 > 1/4$ and $1/8 > 1/64$

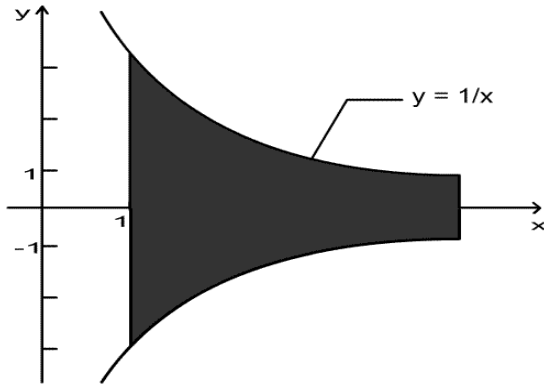
So the idea is that as x goes to $+\infty$, $1/x^2$ goes to 0 much faster than $1/x$, so much so that when we attempt to find the area under the graph over the infinite interval $[1, +\infty)$ the first is convergent, and the other is divergent.

Lets think about Volume of second case.

Lets rotate the graphs of $1/x$ over $[1, +\infty)$ around the x-axis.



We get solid of revolution that look like funnels with no lower point as shown in figure below.



We would like to find the volume of this solid.
 The cross section is a disk with radius $f(x)$.
 For $f(x) = 1/x$, we get for volume

$$\begin{aligned}
 & \lim_{l \rightarrow +\infty} \int_1^l \pi [f(x)]^2 dx \\
 &= \lim_{l \rightarrow +\infty} \int_1^l \pi \left[\frac{1}{x} \right]^2 dx = \lim_{l \rightarrow +\infty} \int_1^l \frac{\pi}{x^2} dx \\
 &= \lim_{l \rightarrow +\infty} \left[-\frac{\pi}{x} + \frac{\pi}{1} \right] = \pi
 \end{aligned}$$

So we can find out how much paint can be held in this solid, but we cannot paint the inside of the solid!!!

We can also have an **improper integral** of this type: $\int_{-\infty}^b f(x) dx = \lim_{l \rightarrow -\infty} \int_l^b f(x) dx$

Integrals over Infinite Interval

We saw earlier that if a function f is not bounded on an interval $[a, b]$, then f is not integrable on $[a, b]$

The integral

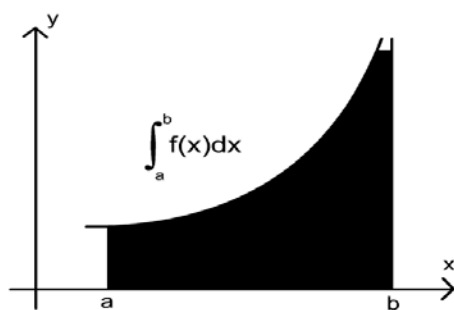
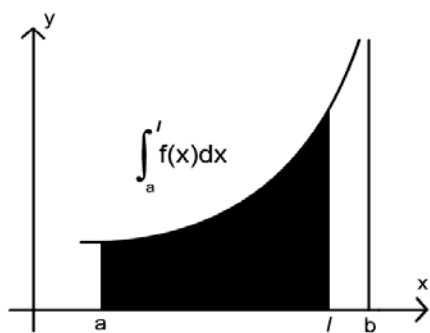
$$\int_0^3 \frac{1}{(x-2)^{\frac{2}{3}}} dx \text{ is unbounded at } x = 2 \text{ in } [0, 3].$$

We can get around this problem by doing the following

If f is continuous on $[a, b)$ but does not have a limit from the left then we define the improper integral as a limit in this way:

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

Geometrically it can be represented as



If f is continuous on $(a, b]$ but fails to have a limit as x approaches a from the right, then we define the improper integral as:

$$\int_a^b f(x) dx = \lim_{l \rightarrow a^+} \int_l^b f(x) dx$$

If f is continuous on $[a, b]$ except that at some point c such that $a < c < b$, $f(x)$ becomes infinite as x goes to c from left or right.

If both improper integrals $\int_a^c f(x) dx$ and $\int_c^b f(x) dx$ **Converge** then we say that the improper

integral $\int_a^b f(x) dx$ Converges as we define $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$

EVALUATE: $\int_1^4 \frac{dx}{(x-2)^{2/3}}$

Solution: The integrand approaches $+\infty$ as $x \rightarrow 2$ so we solve it as

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \int_1^2 \frac{dx}{(x-2)^{2/3}} + \int_2^4 \frac{dx}{(x-2)^{2/3}}$$

$$\int_1^4 \frac{dx}{(x-2)^{2/3}} = \lim_{l \rightarrow 2^-} \int_1^l \frac{dx}{(x-2)^{2/3}} = \lim_{l \rightarrow 2^-} [3(l-2)^{1/3} - 3(1-2)^{1/3}] = 3$$

$$\lim_{l \rightarrow 2^+} \int_l^4 \frac{dx}{(x-2)^{2/3}} = \lim_{l \rightarrow 2^+} [3(4-2)^{1/3} - 3(l-2)^{1/3}] = 3\sqrt[3]{2}$$

Lecture # 40

L'Hopital's Rule and Indeterminate forms

- L'Hopital's Rule and $0/0$
- Indeterminate form of type ∞/∞
- Indeterminate form of type $0 \cdot \infty$
- Indeterminate forms of type 0^0 , ∞^0 and 1^∞ and $\infty - \infty$

If we look at $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$ and $\lim_{x \rightarrow 0} \frac{\sin x}{x}$

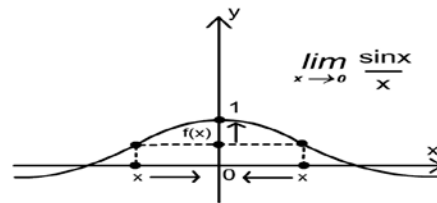
The top and the bottom both approach 0 and so these types of limits are called indeterminate forms of type $0/0$.

This kind of limit can converge in which case it will have finite real values, or it can diverge.

The value, if it converges, is not obvious right away.

The first expression can be factored and bottom cancelled to get a finite value and the second expression requires geometrical observation as shown below

$$\begin{aligned} & \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} \\ &= \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{(x - 2)} \\ &= \lim_{x \rightarrow 2} (x + 2) = 4 \end{aligned}$$



We need a more general concept that work almost always.

L'Hopital's Rule provide us solution that we are looking for

THEOREM 10.2.1 (L'Hopital's Rule for Form $0/0$)

Let $\lim$ stand for one of the *limits*

$$\lim_{x \rightarrow a} \quad \lim_{x \rightarrow a^+} \quad \lim_{x \rightarrow a^-} \quad \lim_{x \rightarrow +\infty} \quad \lim_{x \rightarrow -\infty}$$

and suppose that $\lim f(x) = 0$ and $\lim g(x) = 0$. If $\lim [f'(x) / g'(x)]$ has a finite value L , or if this *limit* is $+\infty$ or $-\infty$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

How we can use this theorem, here are some steps to be followed

STEP1:

Check that $\lim [f(x)/g(x)]$ is an intermediate form. If it is not, then L'Hopital's rule cannot be used.

STEP2:

Differentiate f and g separately.

STEP3:

Find $\lim [f'(x)/g'(x)]$. If this limit is finite, $+\infty$ or $-\infty$, then it is equal to $\lim [f(x)/g(x)]$

EXAMPLE

Use L'Hopital's rule to evaluate

$$\text{a) } \lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} \quad \text{b) } \lim_{x \rightarrow 0} \frac{\sin 2x}{x}$$

Since

$$\lim_{x \rightarrow 2} (x^2 - 4) = 0 \quad \text{and} \quad \lim_{x \rightarrow 2} (x - 2) = 0$$

the given limit is an intermediate form of type 0/0. Thus, L'Hopital's rule applies and we can write

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{(x^2 - 4)}{x - 2} &= \lim_{x \rightarrow 2} \frac{\frac{d}{dx} [x^2 - 4]}{\frac{d}{dx} [x - 2]} \\ &= \lim_{x \rightarrow 2} \frac{2x}{1} = 4 \end{aligned}$$

$$\text{b) } \lim_{x \rightarrow 0} \frac{\sin 2x}{x}$$

Since

$$\lim_{x \rightarrow 0} \sin 2x = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} x = 0$$

the given limit is an intermediate form of type 0/0. Thus, L'Hopital's rule applies and we can write

$$\begin{aligned}\lim_{x \rightarrow 0} \frac{\sin 2x}{x} &= \lim_{x \rightarrow 0} \frac{\frac{d}{dx}[\sin 2x]}{\frac{d}{dx}[x]} \\ &= \lim_{x \rightarrow 0} \frac{2 \cos 2x}{1} = 2\end{aligned}$$

EXAMPLE

Evaluate

$$\lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{\cos x}$$

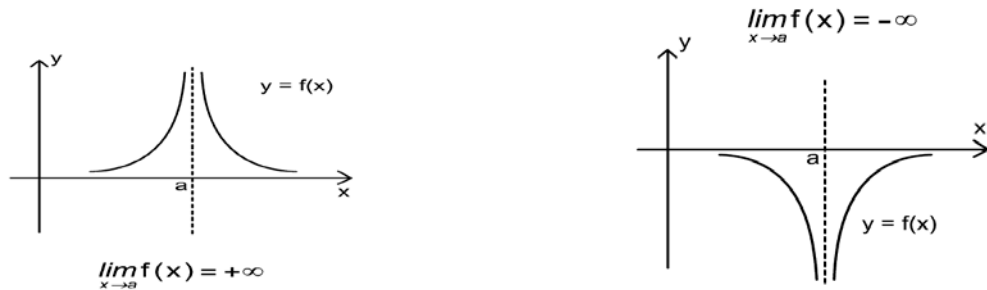
$$\lim_{x \rightarrow \pi/2} (1 - \sin x) = 0 \quad \lim_{x \rightarrow \pi/2} \cos x = 0$$

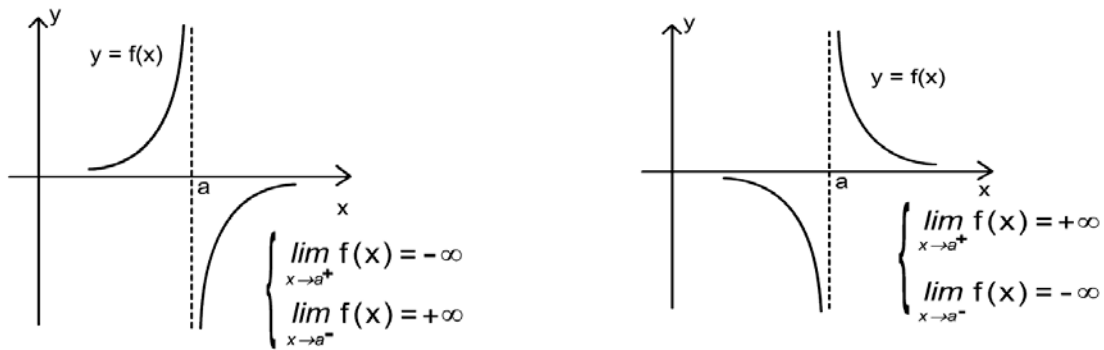
the given limit is an intermediate form of type 0/0. Thus by , L'Hopital's rule

$$\begin{aligned}\lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{\cos x} &= \lim_{x \rightarrow \pi/2} \frac{\frac{d}{dx}[1 - \sin x]}{\frac{d}{dx}[\cos x]} \\ &= \lim_{x \rightarrow \pi/2} \frac{-\cos x}{-\sin x} = \frac{0}{-1} = 0\end{aligned}$$

Some Notation

In the four parts of figure below,





The one-sided limits are either $+\infty$ or $-\infty$. When we want to indicate that one of these four situations occurs without specifying which one, we shall write

Indeterminate form of type ∞ / ∞

An indeterminate form of type ∞/∞ is a limit i.e. $\lim f(x)/g(x)$ in which $\lim f(x) = \infty$ and $\lim g(x) = \infty$. Some examples are

$$\lim_{x \rightarrow 0^+} \frac{\ln x}{\csc x} \quad \begin{array}{l} \text{Numerator} \rightarrow -\infty \\ \text{Denominator} \rightarrow +\infty \end{array}$$

The following version of L'Hopital Rule is used for problems like this

THEOREM 10.3.1 (L'Hopital's Rule for Form ∞/∞)

Let $\lim$ stand for one of the limits

$$\lim_{x \rightarrow a}, \quad \lim_{x \rightarrow a^+}, \quad \lim_{x \rightarrow a^-}, \quad \lim_{x \rightarrow +\infty}, \quad \lim_{x \rightarrow -\infty}$$

and suppose that $\lim f(x) = \infty$ and $\lim g(x) = \infty$. If $\lim [f'(x) / g'(x)]$ has a finite value L , or if this limit is $+\infty$ or $-\infty$, then

$$\lim \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$$

EXAMPLE

Evaluate $\lim_{x \rightarrow +\infty} \frac{x}{e^x}$

$$\lim_{x \rightarrow +\infty} x = +\infty \quad \text{and} \quad \lim_{x \rightarrow +\infty} e^x = +\infty$$

so that the given *limit* is an indeterminate form of type ∞/∞ . Thus, by L'Hopital's Rule

$$\lim_{x \rightarrow +\infty} \frac{x}{e^x} = \lim_{x \rightarrow +\infty} \frac{\frac{d}{dx}[x]}{\frac{d}{dx}[e^x]} = \lim_{x \rightarrow +\infty} \frac{1}{e^x} = 0$$

If $\lim f(x)=0$ and $g(x)=\infty$, then a product limit like $\lim f(x)g(x)$ is of the type $0 \cdot \infty$

Limit problems of this type can be converted to the form of $0/0$ by writing

$$f(x)g(x) = \frac{f(x)}{1/g(x)}$$

EXAMPLE

Evaluate $\lim_{x \rightarrow 0^+} x \ln x$

Since the given problem is an indeterminate form of type $0 \cdot \infty$. We shall convert the problem to the form ∞/∞ and apply L'Hopital rule as follows:

$$\begin{aligned} \lim_{x \rightarrow 0^+} x \ln x &= \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} \\ &= \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0 \end{aligned}$$

Indeterminate forms of type 0^0 , ∞^0 and 1^∞ and $\infty - \infty$

Limits of the form $\lim_{x \rightarrow a} f(x)^{g(x)}$ give rise to indeterminate form of type 0^0 , ∞^0 and 1^∞ .

All these types are treated by first introducing a dependent variable

$$y = f(x)^{g(x)}$$

and then calculating

$$\begin{aligned}\lim_{x \rightarrow a} \ln y &= \lim_{x \rightarrow a} \left[\ln \left(f(x)^{g(x)} \right) \right] \\ &= \lim_{x \rightarrow a} [g(x) \ln f(x)]\end{aligned}$$

once the value of $\lim_{x \rightarrow a} \ln y$ is known, it is simple to determine

$$\lim_{x \rightarrow a} y = \lim_{x \rightarrow a} f(x)^{g(x)}$$

EXAMPLE

Show that $\lim_{x \rightarrow 0} (1+x)^{1/x} = e$

Since

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = 1 \quad \text{and} \quad \lim_{x \rightarrow 0} 1/x = \infty$$

the given limit is an indeterminate form of type 1^∞ . As discussed above, we introduce a dependent variable

$$y = (1+x)^{1/x}$$

and take the natural logarithm of both sides

$$\ln y = \ln(1+x)^{1/x} = \frac{1}{x} \ln(1+x) = \frac{\ln(1+x)}{x}$$

The limit

$$\lim_{x \rightarrow 0} \ln y = \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$$

is an indeterminate form of type 0/0, so by L'Hopital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0} \ln y &= \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} \\ &= \lim_{x \rightarrow 0} \frac{1/(1+x)}{1} = 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \ln y &= \lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} \\ &= \lim_{x \rightarrow 0} \frac{1/(1+x)}{1} = 1 \end{aligned}$$

Since $\ln y \rightarrow 1$ as $x \rightarrow 0$, it follows from the continuity of the natural exponential function that $e^{\ln y} \rightarrow e^1$ or equivalent, $y \rightarrow e$ as $x \rightarrow 0$.
Therefore

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = e$$

Lecture # 41**Sequences and Monotone Sequences**

- Definition of a sequence
- Graphs of sequences
- Limit of a sequence
- Recursive sequence
- Testing for monotonicity
- Eventually monotonic sequences
- Convergence of monotonic sequence
- More on convergence intuitively

Definition of a sequence

A sequence in math is a succession of numbers e.g. $2, 4, 6, 8, \dots$ & $1, 2, 3, 4, \dots$

The numbers in a sequence are called terms of a sequence

Each term has a positional name, like 1st term, 2nd term etc and we can write them as a_1, a_2 etc.

It is convenient to write a sequence as a formula

$2, 4, 6, 8, \dots$ can be written as a formula as $\{2n\}_{n=1}^{+\infty}$

EXAMPLE

List the first five terms of the sequence

$$\{2^n\}_{n=1}^{+\infty}$$

Substituting $n = 1, 2, 3, 4, 5$ into the formula we get

$$2^1, 2^2, 2^3, 2^4, 2^5$$

or equivalently,

$$2, 4, 8, 16, 32$$

We will write a sequences like $a_1, a_2, a_3, \dots$ as $\{a_n\}_{n=1}^{+\infty}$

Here is a formal definition of a sequence

DEFINITION 11.1.1

A sequence or infinite sequence is a function whose domain is the set of positive integers; that is ,

$$\{a_n\}_{n=1}^{+\infty}$$

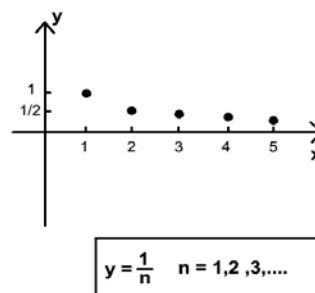
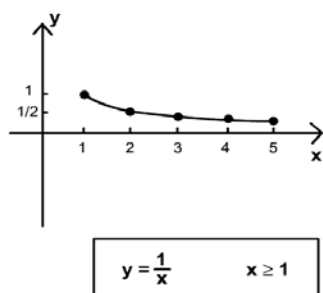
which is an alternative notation for the function

$$f(n) = a_n \quad \text{where } n = 1, 2, 3, \dots$$

Graphs of Sequences

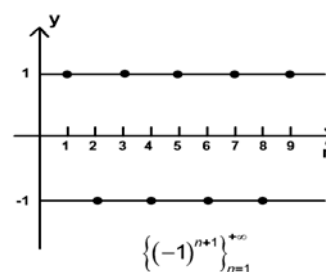
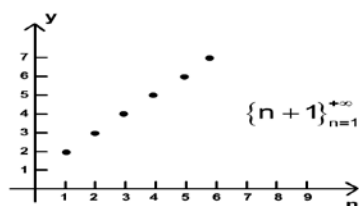
As we see that sequences are functions, we can talk about the graphs of them.

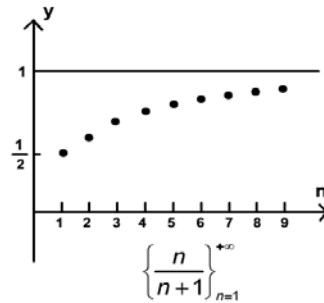
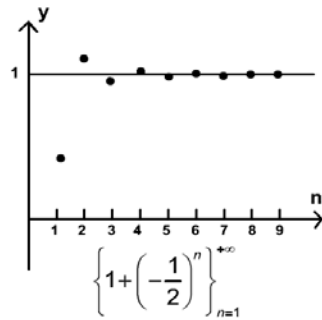
The graph of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{+\infty}$ is the graph of the function equation $y = 1/n$ for $n = 1, 2, 3, \dots$



Limit of a sequence

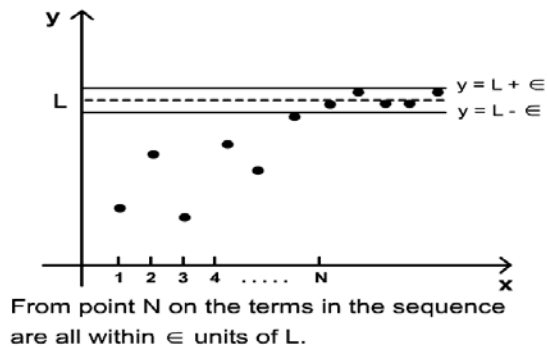
Here are graphs of four sequences, each of which behaves differently as n increases





Some of these graphs have the concept of a limit in them.

A sequence a_n converges to a limit L if for any positive number ϵ there is a point in the sequence after which all terms lie between $L + \epsilon$ and $L - \epsilon$.



DEFINITION 11.1.2

A sequence $\{a_n\}$ is said to *converge* to the limit L if given any $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$$\lim_{n \rightarrow +\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to *diverge*.

As we apply limit the following two sequences converge

$$\lim_{n \rightarrow +\infty} \frac{n}{(n+1)} = 1$$

and

$$\lim_{n \rightarrow +\infty} \left(1 + \left(-\frac{1}{2}\right)^n\right) = 1$$

The following theorem shows that the familiar properties of limits apply to sequence

THEOREM 11.1.3

Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2 respectively, and c is a constant. Then

$$a) \lim_{n \rightarrow +\infty} c = c$$

$$b) \lim_{n \rightarrow +\infty} ca_n = c \lim_{n \rightarrow +\infty} a_n = cL_1$$

$$c) \lim_{n \rightarrow +\infty} (a_n + b_n) = \lim_{n \rightarrow +\infty} a_n + \lim_{n \rightarrow +\infty} b_n = L_1 + L_2$$

$$d) \lim_{n \rightarrow +\infty} (a_n - b_n) = \lim_{n \rightarrow +\infty} a_n - \lim_{n \rightarrow +\infty} b_n = L_1 - L_2$$

$$e) \lim_{n \rightarrow +\infty} (a_n b_n) = \lim_{n \rightarrow +\infty} a_n \lim_{n \rightarrow +\infty} b_n = L_1 L_2$$

$$f) \lim_{n \rightarrow +\infty} \left(\frac{a_n}{b_n} \right) = \frac{\lim_{n \rightarrow +\infty} a_n}{\lim_{n \rightarrow +\infty} b_n} = \frac{L_1}{L_2} \quad (\text{If } L_2 \neq 0)$$

Determine whether the sequence converges or diverges

$$\left\{ \frac{n}{2n+1} \right\}_{n=1}^{+\infty}$$

Dividing numerator and denominator by n yields

$$\begin{aligned} \lim_{n \rightarrow +\infty} \frac{n}{2n+1} &= \lim_{n \rightarrow +\infty} \frac{1}{2+1/n} \\ &= \frac{\lim_{n \rightarrow +\infty} 1}{\lim_{n \rightarrow +\infty} (2+1/n)} \end{aligned}$$

$$\begin{aligned} &= \frac{\lim_{n \rightarrow +\infty} 1}{\lim_{n \rightarrow +\infty} 2 + \lim_{n \rightarrow +\infty} 1/n} \\ &= \frac{1}{2+0} = \frac{1}{2} \end{aligned}$$

Thus the sequence converges to $1/2$

Recursive Sequence

Some sequences are defined by specifying one or more initial terms and giving a formula that relates each subsequent term to the term that precedes it, such sequences are said to be defined recursively.

EXAMPLE

Let $\{a_n\}$ be the sequence defined by $a_1 = 1$ and the recursion formula

$$a_{n+1} = \frac{1}{2}(a_n + 3/a_n) \quad \text{for } n \geq 1$$

$$a_2 = \frac{1}{2}(a_1 + 3/a_1) = \frac{1}{2}(1+3) = 2 \quad \boxed{n=1}$$

$$a_3 = \frac{1}{2}(a_2 + 3/a_2) = \frac{1}{2}(2 + \frac{3}{2}) = \frac{7}{4} \quad \boxed{n=2}$$

$$a_4 = \frac{1}{2}(a_3 + 3/a_3) = \frac{1}{2}(\frac{7}{4} + \frac{12}{7}) = \frac{97}{56} \quad \boxed{n=3}$$

Monotonicity and testing for monotonicity**DEFINITION 11.2.1**

A sequence $\{a_n\}$ is called

Increasing if

$$a_1 < a_2 < a_3 < \dots < a_n < \dots$$

Nondecreasing if

$$a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq \dots$$

Decreasing if

$$a_1 > a_2 > a_3 > \dots > a_n > \dots$$

Nonincreasing if

$$a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$$

A sequence that is either nondecreasing or nonincreasing is called monotone, and a sequence that is increasing or decreasing is called strictly monotone. Observe that a strictly monotone is monotone, but not conversely.

EXAMPLE

$\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, \dots, \frac{n}{n+1}, \dots$ is increasing

$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, \dots$ is decreasing

$1, 1, 2, 2, 3, 3, \dots$ is nondecreasing

$1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \dots$ is nonincreasing

All four of these sequences are monotone, but the sequence

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots, (-1)^{n+1} \frac{1}{n}, \dots$$

is not monotone. The first and second sequences are strictly monotone.

Testing for monotonicity

Monotone sequences are classified as follows

Difference between successive terms	Classification
$a_{n+1} - a_n > 0$	Increasing
$a_{n+1} - a_n < 0$	Decreasing
$a_{n+1} - a_n \geq 0$	Nondecreasing
$a_{n+1} - a_n \leq 0$	Nonincreasing

Monotone Sequence with positive terms are classified as follows

Ratio of successive terms	Classification
$a_{n+1} / a_n > 1$	Increasing
$a_{n+1} / a_n < 1$	Decreasing
$a_{n+1} / a_n \geq 1$	Nondecreasing
$a_{n+1} / a_n \leq 1$	Nonincreasing

If $f(n) = a_n$ is the n th term of a sequence, and if f is differentiable for $x \geq 1$ then we have the following results

Derivative of f for $x \geq 1$	Classification of the sequence with $f(x) = a_x$
$f'(x) > 0$	Increasing
$f'(x) < 0$	Decreasing
$f'(x) \geq 0$	Nondecreasing
$f'(x) \leq 0$	Nonincreasing

EXAMPLE

Show that $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$

is an increasing sequence

It is intuitively clear that the sequence is increasing. To prove that this is so, let

$$a_n = \frac{n}{n+1}$$

We can obtain a_{n+1} by replacing n by $n+1$ in this formula. This yields

$$a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2}$$

Thus for $n \geq 1$

$$\begin{aligned} a_{n+1} - a_n &= \frac{n+1}{n+2} - \frac{n}{n+1} \\ &= \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+1)(n+2)} \\ &= \frac{1}{(n+1)(n+2)} > 0 \end{aligned}$$

which proves that the sequence is increasing, so sequence is strictly monotone.

EXAMPLE

Show that the sequence

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}, \dots$$

is increasing by examining the ratio of successive terms.

As shown earlier

$$a_n = \frac{n}{n+1} \quad \text{and} \quad a_{n+1} = \frac{n+1}{n+2}$$

Thus

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{(n+1)/(n+2)}{n/(n+1)} \\ &= \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2 + 2n + 1}{n^2 + 2n}\end{aligned}$$

Since the numerator exceeds the denominator, the ratio exceeds 1, that is $a_{n+1}/a_n > 1$ for $n \geq 1$. This proves that the sequence is increasing.

Eventually monotonic sequences

A sequence a_n is eventually monotone if there is some integer N such that the sequence is monotone for $n \geq N$.

EXAMPLE

Show that the sequence $\left\{ \frac{10^n}{n!} \right\}_{n=1}^{+\infty}$ is eventually decreasing.

We have

$$a_n = \frac{10^n}{n!} \quad \text{and} \quad a_{n+1} = \frac{10^{n+1}}{(n+1)!}$$

so

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{10^{n+1}/(n+1)!}{10^n/n!} \\ &= \frac{10^{n+1}n!}{10^n(n+1)!} = 10 \frac{n!}{(n+1)n!} \\ &= \frac{10}{n+1}\end{aligned}$$

This is clear that $a_{n+1}/a_n < 1$ for all $n \geq 10$, so the sequence is eventually decreasing.

Convergence of monotonic sequence:**THEOREM 11.2.2**

If $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq \dots$ is a nondecreasing sequence, then there are two possibilities:

a) There is a constant M , called a upper bound for the sequence, such that $a_n \leq M$ for all n , in which case the sequence converges to limit L satisfying $L \leq M$

b) No upper bound exists, in which case

$$\lim_{n \rightarrow +\infty} a_n = +\infty$$

THEOREM 11.2.3

If $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$ is a nonincreasing sequence, then there are two possibilities:

a) There is a constant M , called a lower bound for the sequence, such that $a_n \geq M$ for all n , in which case the sequence converges to limit L satisfying $L \geq M$

b) No lower bound exists, in which case

$$\lim_{n \rightarrow +\infty} a_n = -\infty$$

EXAMPLE

Show that the sequence $\left\{ \frac{n}{n+1} \right\}_{n=1}^{+\infty}$ converges.

We showed earlier that the given sequence is increasing (hence nondecreasing). It is evident that the number $M=1$ is an upper bound for the sequence since

$$a_n = \frac{n}{n+1} < 1 \quad n = 1, 2, \dots$$

Thus, by Theorem 11.2.2 the sequence converges to some limit L such that $L \leq M = 1$. This is indeed the case since

$$\lim_{n \rightarrow +\infty} \frac{n}{n+1} = \lim_{n \rightarrow +\infty} \frac{1}{1+1/n} = 1$$

An Intuitive View of Convergence:

Informally stated, the convergence or divergence of a sequence does not depend on the behavior of the “initial terms” of the sequence, but rather on the behavior of the “tail end”. Thus for sequence $\{a_n\}$ to converge to a limit L , it does not matter if the initial terms are far from L , just so the terms in the sequence are eventually arbitrarily close to L . This being the case, one can add, delete, or alter finitely many terms without affecting the convergence, divergence, or the limit (if it exists).

Lecture # 42

Infinite Series

Definition 11.3.1

An Infinite series is an expansion that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots + u_k + \dots$$

The numbers $u_1, u_2, \dots$ are called the terms of the series.

What does it mean to add infinitely many terms?

Can we add $1+2+3+4+\dots$ All the way to infinity?

This is physically and mentally impossible!

But we can compute an infinite sum using the idea of limits

Look at this decimal expansion

$$0.333333\dots$$

We can rewrite this as an infinite series

$$0.3+0.03+0.003+0.0003+\dots$$

Or as

$$\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \dots$$

Note that we already know that $0.333\dots$ is really $1/3$

So our sum of the above series should also be $1/3$

Let's work out a definition for the sum of an infinite series. Consider the following sequences of (finite) sums

$$s_1 = \frac{3}{10} = 0.3$$

$$s_2 = \frac{3}{10} + \frac{3}{10^2} = 0.33$$

$$s_3 = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} = 0.333$$

The sequence of numbers s_1, s_2 , etc. can be viewed as a succession of approximations to the “sum” of the infinite series. We know that the answer is $1/3$, so it is better be that the approximations are close to this at the start, and get closer and closer to this number as more and more terms are added together. This suggests that if we take the limit of the sequence of approximations, we should get $1/3$. Note that the n th term of the approximating series is

$$s_n = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \dots + \frac{3}{10^n}$$

Taking its limit gives (limit of the n th partial sum of the series)

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \left(\frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \dots + \frac{3}{10^n} \right)$$

Now we do some algebraic manipulation here

$$S_n = \frac{3}{10} + \frac{3}{10^2} + \dots + \frac{3}{10^n} \quad \dots(A)$$

$$\lim_{n \rightarrow +\infty} S_n = \lim_{n \rightarrow +\infty} \left(\frac{3}{10} + \frac{3}{10^2} + \dots + \frac{3}{10^n} \right)$$

$$\frac{1}{10} S_n = \frac{3}{10^2} + \frac{3}{10^3} + \dots + \frac{3}{10^n} + \frac{3}{10^{n+1}} \quad \dots(B)$$

Subtracting (B) from (A)

$$S_n - \frac{1}{10} S_n = \frac{3}{10} - \frac{3}{10^{n+1}}$$

$$\frac{9}{10} S_n = \frac{3}{10} \left(1 - \frac{1}{10^n} \right)$$

$$S_n = \frac{1}{3} \left(1 - \frac{1}{10^n} \right)$$

Since $1/10^n \rightarrow 0$ as $n \rightarrow +\infty$, it follows that

$$\lim_{n \rightarrow +\infty} S_n = \lim_{n \rightarrow +\infty} \frac{1}{3} \left(1 - \frac{1}{10^n} \right) = \frac{1}{3}$$

Now we can formalize the idea of a sum of an infinite series

DEFINITION 11.3.2

Let $[S_n]$ be the sequence of partial sums of the series $u_1 + u_2 + u_3 + \dots + u_k + \dots$. If the sequence $[S_n]$ converges to a limit S , then the series is said to converge, and S is called the **sum** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to diverge: A divergent series has no sum.

Example

Determine whether the series

$$1 - 1 + 1 - 1 + 1 - 1 + \dots$$

converges or diverges. If it converges, find the sum.

$$\begin{aligned} s_1 &= 1 \\ s_2 &= 1 - 1 = 0 \\ s_3 &= 1 - 1 + 1 = 1 \\ s_4 &= 1 - 1 + 1 - 1 = 0 \end{aligned}$$

and so forth. Thus the sequence of partial sum is

$$1, 0, 1, 0, 1, 0, \dots$$

Since this is a divergent sequence, the given series diverges and consequently has no sum.

Geometric Series

A geometric series is one of the form

$$a + ar + ar^2 + \dots + ar^{k-1} + \dots$$

Each term is obtained by multiplying the previous one by a constant number, r , and this number is called the ratio for the series.

Examples are

$$1+2+4+8+\dots$$

$$1+1+1+1+\dots$$

THEOREM 11.3.3

A geometric series

$$a + ar + ar^2 + \dots + ar^{k-1} + \dots \quad (a \neq 0)$$

converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series converges, then the sum is

$$\frac{a}{1-r} = a + ar + ar^2 + \dots + ar^{k-1} + \dots$$

Example

The series

$$5 + \frac{5}{4} + \frac{5}{4^2} + \dots + \frac{5}{4^{k-1}} + \dots$$

is a geometric series with $a = 5$ and $r = 1/4$.

Since $|r| = 1/4 < 1$, the series converges and the sum is

$$\frac{a}{1-r} = \frac{5}{1-1/4} = \frac{20}{3}$$

Harmonic Series

A Harmonic series is of the type

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

It may look like this series converges as each successive term is smaller than the first, but in fact this series diverges!

Convergence Tests

Here are some tests that are used to determine if a series converges or diverges. These tests are applied on the k th term of a series (general term), and not on the n th partial sum as we did earlier to derive the definition of the sum of an infinite series.

THEOREM 11.4.1 (The Divergence Test)

a) If $\lim_{k \rightarrow +\infty} u_k \neq 0$, then the series $\sum u_k$
diverges

b) If $\lim_{k \rightarrow +\infty} u_k = 0$, then the series $\sum u_k$
may either **converge** or **diverge**.

This theorem tells us when a series diverges

The alternative form of part (a) is sufficiently important that we state it separately

THEOREM 11.4.2

If the series $\sum u_k$ converges, then

$$\lim_{k \rightarrow +\infty} u_k = 0$$

Example

The series

$$\sum_{k=1}^{\infty} \frac{k}{k+1} = \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{k}{k+1} + \dots$$

Diverges since

$$\lim_{k \rightarrow \infty} \frac{k}{k+1} = \lim_{k \rightarrow \infty} \frac{1}{1 + \frac{1}{k}} = 1 \neq 0$$

Algebraic Properties of Infinite Series**THEOREM 11.4.3**

- a) If $\sum u_k$ and $\sum v_k$ are convergent series, then $\sum(u_k + v_k)$ and $\sum(u_k - v_k)$ are convergent series and the sums of these series are related by

$$\sum_{k=1}^{\infty} (u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty} (u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

- b) If c is a nonzero constant, then the series $\sum u_k$ and $\sum cu_k$ both *converge* or both *diverge*. In this case of *convergence*, the sums are related by

$$\sum_{k=1}^{\infty} cu_k = c \sum_{k=1}^{\infty} u_k$$

- c) Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer K , the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \dots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \dots$$

both converge or both diverge.

Example

Find the sum of the series

$$\sum_{k=1}^{\infty} \left(\frac{3}{4^k} - \frac{2}{5^{k-1}} \right)$$

Solution: The series

$$\sum_{k=1}^{\infty} \frac{3}{4^k} = \frac{3}{4} + \frac{3}{4^2} + \frac{3}{4^3} + \dots$$

is a convergent series ($a = 3/4$, $r = 1/4$) and the series

$$\sum_{k=1}^{\infty} \frac{2}{5^{k-1}} = 2 + \frac{2}{5} + \frac{2}{5^2} + \frac{2}{5^3} + \dots$$

is also a convergent series ($a = 2$, $r = 1/5$).

Thus, from Theorem 11.4.3(a) and 11.3.3 the given series converges and

$$\sum_{k=1}^{\infty} \frac{3}{4^k} - \sum_{k=1}^{\infty} \frac{2}{5^{k-1}} = \frac{3/4}{1-1/4} - \frac{2}{1-1/5} = -\frac{3}{2}$$

THEOREM 11.4.4

(The Integral Test)

Let $\sum u_k$ be a series with positive terms, and let $f(x)$ be the function that results

when k is replaced by x in the formula for u_k .

If f is decreasing and continuous on

the interval $[a, +\infty]$, then

$$\sum_{k=1}^{\infty} u_k \quad \text{and} \quad \int_a^{\infty} f(x) dx$$

both converge or both diverge.

This theorem allows us to study the convergence of a series by studying a related integral.

This integral is the improper integral over the interval $[1, +\infty)$ and the function is the k th term of the series.

It is interesting that the integral represents a continuous phenomenon, while the summation a discrete one!

EXAMPLE

Use the integral test to determine whether the following series converge or diverge

$$\sum_{k=1}^{\infty} \frac{1}{k}$$

If we replace k by x in the general term $1/k$, we obtain the function $f(x) = 1/x$.

Since

$$\begin{aligned} \int_1^{+\infty} \frac{1}{x} dx &= \lim_{l \rightarrow +\infty} \int_1^l \frac{1}{x} dx \\ &= \lim_{l \rightarrow +\infty} [\ln l - \ln 1] = +\infty \end{aligned}$$

the integral diverges and consequently so does the series

p-Series

p-series or **hyper-harmonic series** is an infinite series of the form.

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \dots \quad \text{where } p = 1$$

$$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots \quad \text{where } p = \frac{1}{2}$$

The following theorem tells us whether a p-series converges or diverges

THEOREM 11.4.5
(Convergence of p-series)

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{k^p} + \dots$$

converges if $p > 1$ and diverges if $0 < p \leq 1$

EXAMPLE

$$1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \dots + \frac{1}{\sqrt[3]{k}} + \dots$$

Since $p = \frac{1}{3} < 1$ so p-series diverges

Lecture # 43

ADDITIONAL CONVERGENCE TESTS

In this section we shall develop some additional convergence tests for series with positive terms.

THEOREM 11.5.1

(The Comparison Test)

Let $\sum a_k$ and $\sum b_k$ be series with nonnegative terms and suppose that

$$a_1 \leq b_1, a_2 \leq b_2, a_3 \leq b_3, \dots, a_k \leq b_k$$

(a) If the “bigger series” $\sum b_k$ converges then the “smaller series” $\sum a_k$ also converges.

(b) If the “smaller series” $\sum a_k$ diverges, then the “bigger series” $\sum b_k$ also diverges.

This test basically compares two series with each other, and if the series in which each term is bigger than that of the other series converges, then so does the smaller one.

If the series in which each term is smaller than that of the other series diverges, then so does the bigger one.

We will see more of this later. For now, we shall use the comparison test to develop some other tests that are easier to apply.

THEOREM 11.5.2

(The Ratio Test)

Let $\sum u_k$ be a series with positive terms and suppose that $\rho = \lim_{k \rightarrow \infty} \frac{u_{k+1}}{u_k}$

- (a) If $\rho < 1$, the series converges.
- (b) If $\rho > 1$ or $\rho = +\infty$, the series diverge.
- (c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

Use the ratio test to determine whether the following series converge or diverge.

a) $\sum_{k=1}^{\infty} \frac{1}{k!}$

b) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

c) $\sum_{k=1}^{\infty} \frac{k^k}{k!}$

d) $\sum_{k=1}^{\infty} \frac{(2k)!}{4^k}$

a)

$$\begin{aligned}\rho &= \lim_{k \rightarrow +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \rightarrow +\infty} \frac{1/(k+1)!}{1/k!} \\ &= \lim_{k \rightarrow +\infty} \frac{k!}{(k+1)!} = \lim_{k \rightarrow +\infty} \frac{k!}{(k+1)k!} \\ &= \lim_{k \rightarrow +\infty} \frac{1}{(k+1)} = 0 < 1\end{aligned}$$

The series converges.

b)

$$\begin{aligned}\rho &= \lim_{k \rightarrow +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \rightarrow +\infty} \frac{k+1}{2^{k+1}} \cdot \frac{2^k}{k} \\ &= \frac{1}{2} \lim_{k \rightarrow +\infty} \frac{k+1}{k} = \frac{1}{2} < 1\end{aligned}$$

The series converges

c)

$$\begin{aligned}\rho &= \lim_{k \rightarrow +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \rightarrow +\infty} \frac{(k+1)^{k+1}}{(k+1)!} \cdot \frac{k!}{k^k} \\ &= \lim_{k \rightarrow +\infty} \frac{(k+1)^k}{k^k} = \lim_{k \rightarrow +\infty} \left(1 + \frac{1}{k}\right)^k \\ &= e > 1\end{aligned}$$

The series diverges

d)

$$\begin{aligned}\rho &= \lim_{k \rightarrow +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \rightarrow +\infty} \frac{[2(k+1)]!}{4^{k+1}} \cdot \frac{4^k}{2k!} \\ &= \lim_{k \rightarrow +\infty} \left(\frac{(2k+2)!}{(2k)!} \cdot \frac{1}{4} \right) \\ &= \frac{1}{4} \lim_{k \rightarrow +\infty} (2k+2)(2k+1) = +\infty\end{aligned}$$

The series diverges

Determine whether the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \dots + \frac{1}{2k-1} + \dots$$

converges or diverges.

The ratio test is of no help since

$$\begin{aligned}\rho &= \lim_{k \rightarrow +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \rightarrow +\infty} \frac{1}{2(k+1)-1} \cdot \frac{2k-1}{1} \\ &= \lim_{k \rightarrow +\infty} \frac{2k-1}{2k+1} = 1\end{aligned}$$

However, the integral test proves that the series diverges since

$$\begin{aligned}\int_1^{+\infty} \frac{dx}{2x-1} &= \lim_{l \rightarrow +\infty} \int_1^l \frac{dx}{2x-1} \\ &= \lim_{l \rightarrow +\infty} \left[\frac{1}{2} \ln(2x-1) \right]_1^l = +\infty\end{aligned}$$

So, this series diverges.

Sometimes the following result is easier to apply than ratio test

Theorem 11.5.3

(The Root Test)

Let $\sum u_k$ be a series with positive terms and suppose that

- (a) If $\rho < 1$, the series converges.
- (d) If $\rho > 1$ or $\rho = +\infty$, the series diverge.
- (e) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

Example

$$\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1} \right)^k$$

Use the root test to determine whether the following series converge or diverge.

Solution: The series diverges, since $\rho = \lim_{k \rightarrow \infty} (u_k)^{1/k} = \lim_{k \rightarrow \infty} \frac{4k-5}{2k+1} = 2 > 1$

We have not really talked about the comparison test so far. To apply comparison test, we use an alternative version of this test that is easier to work with.

INFORMAL PRINCIPLE 11.6.1

Constant terms in the denominator of u_k can usually be deleted without affecting the convergence or divergence of the series.

Example

Use the above informal principle to help guess whether the following series converge or diverge

$$\sum_{k=1}^{\infty} \frac{1}{2^k + 1}$$

Solution: Deleting the constant 1 suggests that $\sum_{k=1}^{\infty} \frac{1}{2^k + 1}$ & $\sum_{k=1}^{\infty} \frac{1}{2^k}$ behaves alike.

The modified series is a convergent geometric series, so the given series is likely to converge.

INFORMAL PRINCIPLE 11.6.2

If a polynomial in k appears as a factor in the numerator or denominator of u_k , all but the highest power of k in the polynomial may usually be deleted without affecting the convergence or divergence of the series.

Example

Use the above principle to help guess whether the following series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3 + 2k}}$ converge or diverge.

Solution: Deleting the term $2k$ suggests that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3 + 2k}}$ & $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k^3}} = \sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$

behaves alike

Since the modified series is a convergent p -series ($p=3/2$) the given series is likely to converge.

The Limit Comparison Test

The following result can be used to establish convergence or divergence by examining the limit of the ratio of the general term of the series in question with general term of a series whose convergence properties are known.

THEOREM 11.6.3

(The Limit Comparison Test)

Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that $\rho = \lim_{k \rightarrow \infty} \frac{a_k}{b_k}$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge.

Example

Use the limit comparison test to determine whether the following series $\sum_{k=1}^{\infty} \frac{1}{2k^2 - k}$

converge or diverge.

Solution: The given series behaves like the series $\sum_{k=1}^{\infty} \frac{1}{2k^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{k^2}$

$a_k = \frac{1}{2k^2 - k}$ $b_k = \frac{1}{2k^2}$ which is constant times a convergent p-series.

Thus the given series is likely to converge.

By applying theorem 11.6.3 with $\rho = \lim_{k \rightarrow +\infty} \frac{a_k}{b_k} = \lim_{k \rightarrow +\infty} \frac{2k^2}{2k^2 - k} = \lim_{k \rightarrow +\infty} \frac{2}{2 - 1/k} = 1$

Since ρ is finite and positive, it follows from Theorem 11.6.3 that the given series converge.

The Comparison Test

We shall now discuss some techniques for applying the comparison test (Theorem 11.5.1).

There are two basic steps required to apply the comparison test to a series $\sum u_k$ of positive terms:

- Guess whether the series $\sum u_k$ converges or diverges.
- Find a series that proves the guess to be correct. Thus if the guess is divergence, we must find a divergent series whose terms are “smaller” than the corresponding terms of $\sum u_k$, and if the guess is convergence,

we must find a convergent series whose terms are “bigger” than the corresponding terms of $\sum u_k$.

Example

$$\sum_{k=1}^{\infty} \frac{1}{k - 1/4}$$

Use the comparison test to determine whether the following series converge or diverge.

Solution:

$$\text{As } \frac{1}{k - 1/4} > \frac{1}{k}$$

Note that the series behaves like the divergent harmonic series, and hence is likely to diverge. Thus our goal is to find a divergent series that is “smaller” than the given series. We can do this by dropping the constant $-1/4$ in the denominator, thereby decreasing the size of the general term:

$$\text{for } k=1, 2, \dots$$

Thus the given series diverges by the comparison test, since $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges.

Lecture # 44

Alternating Series; Conditional convergence

- So far we have seen infinite series that have positive terms only.
- We have also defined the Limit and the Sum of such infinite series
- Now we look at series that have terms which have alternating signs, known as **Alternating series**

Example

- 1) $1-1+1-1+\dots$
- 2) $1+2-3+4-\dots$

More generally, an alternating series has one of the two following forms

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \dots \quad (1)$$

$$\sum_{k=1}^{\infty} (-1)^k a_k = -a_1 + a_2 - a_3 + a_4 - \dots \quad (2)$$

Note that all the terms a_k are to be taken as being positive.

The following theorem is the key result on convergence of alternating series.

Theorem 11.7.1

(Alternating Series Test)

An alternating series of either form (1) or (2) converges if the following two conditions are satisfied:

- (a) $a_1 > a_2 > a_3 > \dots > a_k > \dots$
- (b) $\lim_{k \rightarrow +\infty} a_k = 0$

This theorem tells us when an alternating series converges.

Example

Use the alternating series test to show that the following series converge

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$$

Solution:

The two conditions in the alternating series test are satisfied since

$$a_k = \frac{1}{k} > \frac{1}{k+1} = a_{k+1}$$

$$\text{and } \lim_{k \rightarrow +\infty} a_k = \lim_{k \rightarrow +\infty} \frac{1}{k} = 0$$

Note that this series is the Harmonic series, but alternates. It's called the Alternating Harmonic Series.

The harmonic series diverges. The alternating Harmonic series converges.

Now we will look at the errors involved in approximating an alternating series with a partial sum.

Theorem 11.7.2

If an alternating series satisfies the conditions of the alternating series test, and if the sum S of the series is approximated by the n th partial sum S_n , thereby resulting in an error of $S - S_n$, then

$$|S - S_n| < a_{n+1}$$

Moreover, the sign of error is the same as that of the coefficient of a_{n+1} in the series.

Example

The alternating series

$$1 - 1/2 + 1/3 - 1/4 + \dots + (-1)^{k+1} 1/k + \dots$$

satisfies the condition of the alternating series test; hence the series has a sum S , which we know must lie between any two successive partial sums. In particular, it must lie between

$$S_7 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 = 319/420$$

And

$$S_8 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 - 1/8 = 533/840$$

So

$$533/840 < S < 319/420$$

If we take $S = \ln 2$ then

$$\begin{aligned} 533/840 < \ln 2 < 319/420 \\ 0.6345 < \ln 2 < 0.7596 \end{aligned}$$

The value of $\ln 2$, rounded to four decimal places, is .6931, which is consistent with these inequalities. It follows from Theorem 11.7.2 that

$$| \ln 2 - S_7 | = | \ln 2 - 319/420 | < a_8 = 1/8$$

And

$$| \ln 2 - S_8 | = | \ln 2 - 533/840 | < a_9 = 1/9$$

Absolute and Conditional Convergence

The series

$$1 - 1/2 - 1/2^2 + 1/2^3 + 1/2^4 - 1/2^5 - 1/2^6 + \dots$$

does not fit in any of the categories studied so far, it has mixed signs, but is not alternating. We shall now develop some convergence tests that can be applied to such series.

Definition 11.7.3

A series $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots + u_k + \dots$

is said to converge absolutely, if the series of absolute values

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \dots + |u_k| + \dots \text{ converges.}$$

Example

The series

$$1 - 1/2 - 1/2^2 + 1/2^3 + 1/2^4 - 1/2^5 - 1/2^6 + \dots$$

Converges absolutely since the series of absolute values

$$1 + 1/2 + 1/2^2 + 1/2^3 + 1/2^4 + 1/2^5 + 1/2^6 + \dots$$

is a convergent geometric series.

On the other hand, the alternating harmonic series

$$1 - 1/2 + 1/3 - 1/4 + 1/5 - \dots$$

does not converge absolutely since the series of absolute values

$$1 + 1/2 + 1/3 + 1/4 + 1/5 + \dots$$

diverges.

Absolute convergence is of importance because of the following theorem.

Theorem 11.7.4

If the series $\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \dots + |u_k| + \dots$

Converges, then so does the series $\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots + u_k + \dots$

In other words, if a series converges absolutely, then it converges.

Since the series

$$1 - 1/2 - 1/2^2 + 1/2^3 + 1/2^4 - 1/2^5 - 1/2^6 + \dots$$

Converges absolutely. It follows from Theorem 11.7.4 that the given series converges.

Example

Show that the series $\sum_{k=1}^{\infty} \frac{\cos k}{k^2}$ converges.

Solution: Since $|\cos k| \leq 1$ for all k ,

$$\left| \frac{\cos k}{k^2} \right| \leq \frac{1}{k^2}$$

thus

$$\left| \sum_{k=1}^{\infty} \frac{\cos k}{k^2} \right| \sum_{k=1}^{\infty} \frac{\cos k}{k^2}$$

converges by the comparison test, and consequently

$$\sum_{k=1}^{\infty} \frac{\cos k}{k^2} \quad \text{converges.}$$

If $\sum |u_k|$ diverges, no conclusion can be drawn about the convergence or divergence of $\sum u_k$

For example, consider the two series

$$1 - 1/2 + 1/3 - 1/4 + \dots + (-1)^{k+1} 1/k + \dots \quad (A)$$

$$-1 - 1/2 - 1/3 - 1/4 - \dots - 1/k - \dots \quad (B)$$

Series (A), the alternating harmonic series, converges, whereas series (B), being a constant times the harmonic series, diverges.

Yet in each case the series of absolute values is

$$1 + 1/2 + 1/3 + \dots + 1/k + \dots$$

which diverges. A series such as (A), which is convergent, but not absolutely convergent, is called **conditionally convergent**.

Theorem 11.7.5

(Ratio Test for Absolute Convergence)

Let $\sum u_k$ be a series with nonzero terms and suppose that $\rho = \lim_{k \rightarrow \infty} \frac{|u_{k+1}|}{|u_k|}$

(a) If $\rho < 1$, the series $\sum u_k$ converges absolutely and therefore converges.

(b) If $\rho > 1$ or $\rho = +\infty$, then the series $\sum u_k$ diverges.

(c) If $\rho = 1$, no conclusion about convergence or absolute convergence can be drawn from this test.

EXAMPLE

The series

$$\sum_{k=1}^{\infty} (-1)^k \frac{2^k}{k!}$$

converges absolutely since

$$\begin{aligned} \rho &= \lim_{k \rightarrow \infty} \frac{|u_{k+1}|}{|u_k|} = \lim_{k \rightarrow \infty} \frac{2^{k+1}}{(k+1)!} \cdot \frac{k!}{2^k} \\ &= \lim_{k \rightarrow \infty} \frac{2}{k+1} = 0 < 1 \end{aligned}$$

Power Series in x

If $c_0, c_1, c_2, \dots$ are constants and x is a variable, then a series of the form

$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \dots + c_k x^k + \dots$ is called a power series in x .

Some examples of power series in x are

$$\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

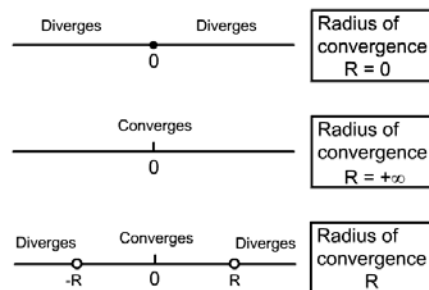
Theorem 11.8.1

For any power series in x , exactly one of the following is true:

- The series converges only for $x=0$
- The series converges absolutely (and hence converges) for all real values of x .
- The series converges absolutely (and hence converges) for all x in some finite open interval $(-R, R)$, and diverges if $x < -R$ or $x > R$. At either of the points $x = R$ or $x = -R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

Radius and Interval of Convergence

Theorem 11.8.1 states that the set of values for which a power series in x converges is always an interval centered at 0; we call this the interval of convergence, corresponding to this interval series has radius called radius of convergence.



Example:

Find the interval of convergence and radius of convergence of the following power series.

$$\sum_{k=1}^{\infty} x^k$$

Solution: We shall apply the ratio test for absolute convergence. We have

$$\rho = \lim_{k \rightarrow \infty} \left| \frac{u_{k+1}}{u_k} \right| = \lim_{k \rightarrow \infty} \left| \frac{x^{k+1}}{x^k} \right| = \lim_{k \rightarrow \infty} |x| = |x|$$

So the ratio test for absolute convergence implies that the series converges absolutely if $\rho = |x| < 1$ and diverges if $\rho = |x| > 1$. The test is inconclusive if $|x| = 1$ (i.e. $x = 1$ or $x = -1$), so convergence at these points must be investigated separately. At these points the series becomes

$$\sum_{k=0}^{\infty} 1^k = 1 + 1 + 1 + 1 + \dots \quad x=1$$

$$\sum_{k=0}^{\infty} (-1)^k = 1 - 1 + 1 - 1 + \dots \quad x=-1$$

Both of which diverge; thus, the interval of convergence for the given power series is $(-1, 1)$, and the radius of convergence is $R = 1$.

Power series in x-a

$$\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_k(x-a)^k + \dots$$

This series is called power series in x-a. Some examples are

$$\sum_{k=0}^{\infty} \frac{(x-1)^k}{k+1} = 1 + \frac{(x-1)}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + \dots \quad a=1$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k (x+3)}{k!} = 1 - (x+3) + \frac{(x+3)^2}{2!} - \frac{(x+3)^3}{3!} + \dots \quad a=-3$$

Theorem 11.8.1

For any power series in $\sum c_k (x-a)^k$, exactly one of the following is true:

- (a) The series converges only for $x=a$
- (b) The series converges absolutely (and hence converges) for all real values of x .
- (c) The series converges absolutely (and hence converges) for all x in some finite open interval $(a-R, a+R)$, and diverges if $x < a-R$ or $x > a+R$. At either of the points $x = a-R$ or $x = a+R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

It follows from this theorem that now interval of convergence is centered at $x=a$.

Lecture # 45

Taylor and Maclaurin Series

One of the early applications of calculus was the computation of approximate numerical values for functions such as $\sin x$, $\ln x$, and e^x . One common method for obtaining such values is to approximate the function by polynomial, then use that polynomial to compute the desired numerical values.

Problem

Given a function f and a point a on the x -axis, find a polynomial of specified degree that best approximates the function f in the “vicinity” of the point a .

Suppose that we are interested in approximating a function f in the vicinity of the point $a=0$ by a polynomial

$$P(x) = c_0 + c_1x + c_2x^2 + \dots + c_nx^n \quad (1)$$

Because $P(x)$ has $n+1$ coefficients, it seems reasonable that we should be able to impose $n+1$ conditions on this polynomial to achieve a good approximation to $f(x)$. Because the point $a=0$ is the center of interest, our strategy will be to choose the coefficients of $P(x)$ so that the value of P and its first n derivatives are the same as the value of f and its first n derivatives at $a=0$. By forcing this high degree of “match” at $a=0$, it is reasonable to hope that $f(x)$ and $P(x)$ will remain close over some interval (possibly quite small) centered at $a=0$. Thus, we shall assume that f can be differentiated n times at 0, and we shall try to find the coefficients in (1) such that

$$f(0) = p(0), f'(0) = p'(0), f''(0) = p''(0), \dots, f^n(0) = p^n(0) \quad (2)$$

We have

$$p(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots + c_nx^n$$

$$p'(x) = c_1 + 2c_2x + 3c_3x^2 + \dots + nc_nx^{n-1}$$

$$p''(x) = 2c_2 + 3 \cdot 2c_3x + \dots + n(n-1)c_nx^{n-2}$$

$$p'''(x) = 3 \cdot 2c_3 + \dots + n(n-1)(n-2)c_nx^{n-3}$$

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$$p^n(x) = n(n-1)(n-2)\dots(1)c_n$$

Thus, to satisfy (2) we must have

$$f(0) = p(0) = c_0$$

$$f'(0) = p'(0) = c_1$$

$$f''(0) = p''(0) = 2c_2 = 2!c_2$$

$$f'''(0) = p'''(0) = 3 \cdot 2c_3 = 3!c_3$$

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$$f^n(0) = p^n(0) = n(n-1)(n-2)\dots(1)c_n = n!c_n$$

Which yields the following values for the coefficients of $P(x)$?

$$c_0 = f(0), c_1 = f'(0), c_2 = \frac{f''(0)}{2!}, c_3 = \frac{f'''(0)}{3!}, \dots, c_n = \frac{f^n(0)}{n!}$$

MACLAURIN POLYNOMIALS

If f can be differentiated n times at 0, then we define the n th Maclaurin Polynomial for f to be

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

This polynomial has the property that its value and the values of its first n derivatives match the value of $f(x)$ and its first n derivatives when $x = 0$.

Example

Find the Maclaurin polynomials P_0 , P_1 , P_2 , P_3 , and P_n for e^x .

Solution: Let $f(x) = e^x$, Thus

$$f'(x) = f''(x) = f'''(x) = \dots = f^n(x) = e^x$$

and

$$f(0) = f'(0) = f''(0) = f'''(0) = \dots = f^n(0) = e^0 = 1$$

Therefore,

$$p_0(x) = f(0) = 1$$

$$p_1(x) = f(0) + f'(0)x = 1 + x$$

$$p_2(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 + x + \frac{x^2}{2!}$$

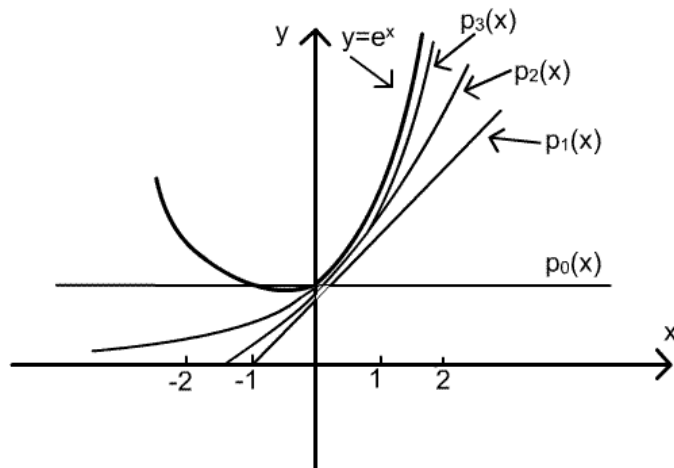
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$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^n(0)}{n!}x^n$$

$$= 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$



Graphs of e^x and first four Maclaurin polynomials are shown here. Note that the graphs of $P_1(x)$, $P_2(x)$, $P_3(x)$ are virtually indistinguishable from the graph of e^x near the origin, so these polynomials are good approximations of e^x near the origin.

But away from origin it does not give good approximation.

To obtain polynomial approximations of $f(x)$ that have their best accuracy near a general point $x=a$, it will be convenient to express polynomials in powers of $x-a$, so that they have the form

$$P(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots + c_n(x-a)^n$$

DEFINITION 11.9.2

If f can be differentiated n times at 0, then we define the n th Taylor polynomial for f about $x = a$ to be

$$p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor and Maclaurin Series

For a fixed value of x near a , one would expect that the approximation of $f(x)$ by its Taylor polynomial $P_n(x)$ about $x=a$ should improve as n increases, since increasing n has the effect of matching higher and higher derivatives of $f(x)$ with those of $P_n(x)$ at $x=a$. Indeed, it seems plausible that one might be able to achieve any desired degree of accuracy by choosing n sufficiently large; that is, the value of $P_n(x)$ might actually converge to $f(x)$ as $n \rightarrow \infty$.

DEFINITION 11.9.3

If f has derivatives of all orders at a , then we define the Taylor Series for f about $x = a$ to be

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \dots + \frac{f^{(k)}(a)}{k!} (x-a)^k + \dots$$

In the special case where $a=0$, the **Taylor series** for f is called the **Maclaurin series** for f .

Find the Maclaurin Series for

a) e^x b) $\sin x$

a) The n th Maclaurin polynomial for e^x is

$$\sum_{k=0}^n \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

Thus, the Maclaurin series for e^x is

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^k}{k!} + \dots$$

b) Let $f(x) = \sin x$,

$$f(x) = \sin x \quad f(0) = 0$$

$$f'(x) = \cos x \quad f'(0) = 1$$

$$f''(x) = -\sin x \quad f''(0) = 0$$

$$f'''(x) = -\cos x \quad f'''(0) = -1$$

Since $f^{(4)}(x) = \sin x = f(x)$, the pattern 0, 1, 0, -1 will repeat over and over as we evaluate successive derivatives at 0.

Therefore , the successive Maclaurin polynomials for $\sin x$ are

$$p_0(x) = 0$$

$$p_1(x) = 0 + x$$

$$p_2(x) = 0 + x + 0$$

$$p_3(x) = 0 + x + 0 - x^3/3!$$

$$p_4(x) = 0 + x + 0 - x^3/3! + 0$$

$$p_5(x) = 0 + x + 0 - x^3/3! + 0 + x^5/5!$$

$$p_6(x) = 0 + x + 0 - x^3/3! + 0 + x^5/5! + 0$$

$$p_7(x) = 0 + x + 0 - x^3/3! + 0 + x^5/5! + 0 - x^7/7!$$

Because of the zero terms, each even-numbered Maclaurin polynomial [after $p_0(x)$] is the same as the odd-number Maclaurin polynomial; that is

$$\begin{aligned} p_{2n+1}(x) &= p_{2n+2}(x) \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} \\ &\quad (n = 0, 1, 2, 3, \dots) \end{aligned}$$

Thus, the Maclaurin series for $\sin x$ is

$$\begin{aligned} &\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + \dots \end{aligned}$$