

### Solution File Lecture No 1-3

#### Question No 1:

Let p be “I am intelligent” and let q be “I have a book”, Give a simple verbal sentence which describes each of the following statement:

- (a)  $\neg p$
- (b)  $p \vee q$
- (c)  $q \wedge \neg p$

#### Solution:

- (a) I am not intelligent
- (b) I am intelligent or I have a book
- (c) I have a book but I am not intelligent

#### Question No 2:

Construct a truth table for the compound proposition  $[p \wedge (\neg p \rightarrow q)] \rightarrow q$

#### Solution:

p	q	$\neg p$	$\neg p \rightarrow q$	$p \wedge (\neg p \rightarrow q)$	$[p \wedge (\neg p \rightarrow q)] \rightarrow q$
T	T	F	T	T	T
T	F	F	T	T	F
F	T	T	T	F	T
F	F	T	F	F	T

#### Question No 3:

Convert into logical form and then write converse, inverse and contra positive of the following statement.

“If I study, then I shall pass the test”.

#### Solution:

Inverse: If I do not study, then I shall not pass the test.

Converse: If I shall pass the test then I study.

Contrapositive: if I shall not pass the test then I do not study.

**Question No 4:**

Let  $p$  = “Ahmad eats apple”,  $q$  = “Ahmad eats banana” and  $r$  = “Ahmad eats grapes”. Write each of the following in symbolic form:

- a) Ahmad eats apple or banana but not grapes.
- b) It is not true that Ahmad eats apple but not grapes.
- c) It is not true that Ahmad eats apples or bananas but not grapes.

**Solution:**

i)  $(p \vee q) \wedge \sim r$

ii)  $\sim (p \wedge \sim r)$

iii)  $\sim ((p \vee q) \wedge \sim r)$

**Question No 5:**

Assume that for the truth values  $p = F$  and  $q = T$ . Show that the proposition  $(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$  is true.

**Solution:**

$p$	$q$	$\sim p$	$\sim q$	$(p \wedge q)$	$(p \wedge \sim q)$	$\sim p \vee (p \wedge \sim q)$	$(p \wedge q) \vee (\sim p \vee (p \wedge \sim q))$
F	T	T	F	F	F	T	T

Hence the given proposition is true.

**Question No 6:**

Use truth table to verify the statement (how to prove it using laws)

$$q \vee (p \wedge q) \equiv q \wedge (p \rightarrow q)$$

**Solution:**

<b>p</b>	<b>q</b>	$(p \wedge q)$	$q \vee (p \wedge q)$	$(p \rightarrow q)$	$q \wedge (p \rightarrow q)$
T	T	T	T	T	T
T	F	F	F	F	F
F	T	F	T	T	T
F	F	F	F	T	F

Therefore from column 4&6,  $q \vee (p \wedge q) \equiv q \wedge (p \rightarrow q)$ .

**Question No 7:**

Do it by yourself

**Question No 8:**

Use De Morgan's laws to find the negation of

“Ahmad loves football and his brother loves cricket”

**Solution:**

Let  $p$ =Ahmed loves football

$q$ =His brother loves cricket

$p \wedge q$  = Ahmed loves football and His brother loves cricket

Using De Morgan's law

$\sim (p \wedge q) = \sim p \vee \sim q$  =Ahmed does not love football or his brother does not love cricket

**Question No 9:**

Do it by yourself.

## Lecture 4-6

**Question # 1:** Formulate the symbolic expression  $\sim (p \vee q) \rightarrow r$  in words using

p =: Today is Holiday  
q =: It is raining  
r =: We should study.

**Solution:**

If it is not true that today is holiday or it is raining, then we should study.

**Question # 2:** Check the validity of the following argument form using truth table:

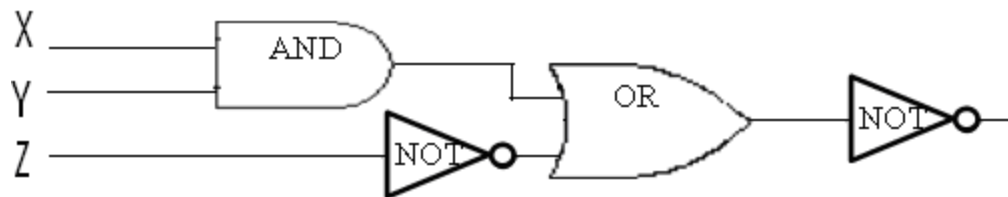
$$\begin{array}{l} p \rightarrow q \\ \frac{p \vee q}{\therefore p} \end{array}$$

**Solution:**

$p$	$q$	$p \rightarrow q$	$p \vee q$	$p$
$F$	$F$	$T$	$F$	$F$
$F$	$T$	$T$	$T$	$F$
$T$	$F$	$F$	$T$	$T$
$T$	$T$	$T$	$T$	$T$

Row 2 and 4 are critical rows, in second row conclusion is false so it is an invalid argument.

**Question # 3:** Write the output of each gate in the following circuit



**Solution:**

$X$	$Y$	$Z$	$X \wedge Y$	$\sim Z$	$(X \wedge Y) \vee \sim Z$	$\sim((X \wedge Y) \vee \sim Z)$
0	0	0	0	1	1	0
0	0	1	0	0	0	1
0	1	0	0	1	1	0
0	1	1	0	0	0	1
1	0	0	0	1	1	0
1	0	1	0	0	0	1
1	1	0	1	1	1	0
1	1	1	1	0	1	0

**Question # 4:** Construct a truth table to determine whether the following statement is logically equivalent or not:

$$(p \wedge q) \vee r \text{ and } p \wedge (q \vee r)$$

**Solution:**

$p$	$q$	$r$	$p \wedge q$	$(p \wedge q) \vee r$	$q \vee r$	$p \wedge (q \vee r)$
$F$	$F$	$F$	$F$	$F$	$F$	$F$
$F$	$F$	$T$	$F$	$T$	$T$	$F$
$F$	$T$	$F$	$F$	$F$	$T$	$F$
$F$	$T$	$T$	$F$	$T$	$T$	$F$
$T$	$F$	$F$	$F$	$F$	$F$	$F$
$T$	$F$	$T$	$F$	$T$	$T$	$T$
$T$	$T$	$F$	$T$	$T$	$T$	$T$
$T$	$T$	$T$	$T$	$T$	$T$	$T$

Columns 5 and 7 tells that they are not equivalent

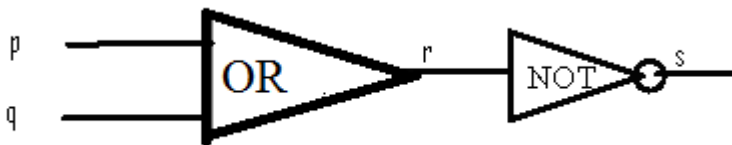
**Question # 5:** By using the laws of logic verify the following logical equivalence:

$$\sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) \equiv p$$

**Solution:**

$$\begin{aligned}
 \sim((\sim p \wedge q) \vee (\sim p \wedge \sim q)) \vee (p \wedge q) &\equiv \sim(\sim p \wedge (q \vee \sim q)) \vee (p \wedge q) && \text{Distributive Law} \\
 &\equiv \sim(\sim p \wedge 1) \vee (p \wedge q) && \text{Negation Law} \\
 &\equiv \sim(\sim p) \vee (p \wedge q) && \text{Identity Law} \\
 &\equiv p \vee (p \wedge q) && \text{Double Negation Law} \\
 &\equiv p && \text{Absorption Law}
 \end{aligned}$$

**Question # 6:** Construct the input/output table for  $\square (p \vee q)$

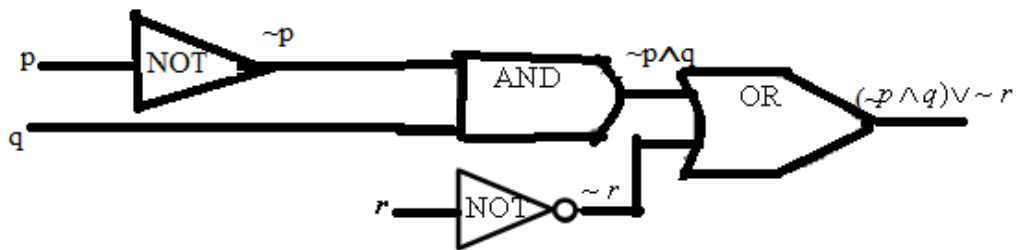


Solution:

Inputs			Output
p	q	$(p \vee q) = r$	$\sim(p \vee q) = s$
1	1	1	0
1	0	1	0
0	1	1	0
0	0	0	1

**Question # 7:** Construct circuit for the Boolean expression  $(\sim p \wedge q) \vee \sim r$ .

Solution:



## Solution File of Practice Questions Lecture No 7-9

### Solution No 1:

$$U = \{a, b, c, d, e, f, g\}$$

$$A = \{a, c, e, g\}$$

$$B = \{d, e, f, g\}$$

$$A \cup B = \{a, c, e, g\} \cup \{d, e, f, g\} = \{a, c, d, e, f, g\}$$

$$A \cap B = \{a, c, e, g\} \cap \{d, e, f, g\} = \{e, g\}$$

$$B - A = \{d, e, f, g\} - \{a, c, e, g\} = \{d, f\}$$

$$A^c = U - A = \{a, b, c, d, e, f, g\} - \{a, c, e, g\} = \{b, d, f\}$$

### Solution No 2:

$$A = \{x \in \mathbb{Z} \mid 0 < x \leq 2\}$$

$$B = \{x \in \mathbb{Z} \mid 1 \leq x < 4\}$$

$$C = \{x \in \mathbb{Z} \mid 3 \leq x < 9\}$$

$$A \cup B = \{x \in \mathbb{Z} \mid 0 < x \leq 2\} \cup \{x \in \mathbb{Z} \mid 1 \leq x < 4\} = \{x \in \mathbb{Z} \mid 0 < x < 4\}$$

$$(A \cup B)^c = U - A \cup B = \{x \mid x \in \mathbb{Z}^+\} - \{x \in \mathbb{Z} \mid 0 < x < 4\} = \{x \in \mathbb{Z} \mid x \geq 4\}$$

$$A \cap B = \{x \in \mathbb{Z} \mid 0 < x \leq 2\} \cap \{x \in \mathbb{Z} \mid 1 \leq x < 4\} = \{x \in \mathbb{Z} \mid 1 \leq x \leq 2\}$$

$$(A \cap B)^c = U - A \cap B = \{x \mid x \in \mathbb{Z}^+\} - \{x \in \mathbb{Z} \mid 1 \leq x \leq 2\} = \{x \in \mathbb{Z} \mid x \geq 3\}$$

$$A^c = U - A = \{x \mid x \in \mathbb{Z}^+\} - \{x \in \mathbb{Z} \mid 0 < x \leq 2\} = \{x \in \mathbb{Z} \mid x \geq 3\}$$

$$B^c = U - B = \{x \mid x \in \mathbb{Z}^+\} - \{x \in \mathbb{Z} \mid 1 \leq x < 4\} = \{x \in \mathbb{Z} \mid x \geq 4\}$$

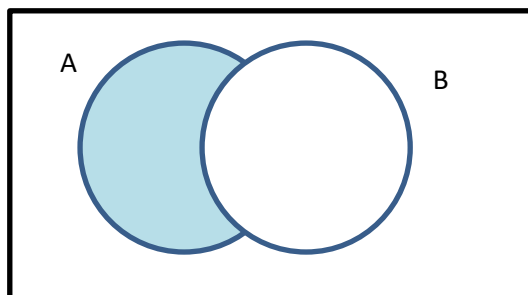
$$A^c \cup B^c = \{x \in \mathbb{Z} \mid x \geq 3\} \cup \{x \in \mathbb{Z} \mid x \geq 4\} = \{x \in \mathbb{Z} \mid x \geq 3\}$$

$$A^c \cap B^c = \{x \in \mathbb{Z} \mid x \geq 3\} \cap \{x \in \mathbb{Z} \mid x \geq 4\} = \{x \in \mathbb{Z} \mid x \geq 4\}$$

$$A \cup C = \{x \in \mathbb{Z} \mid 0 < x \leq 2\} \cup \{x \in \mathbb{Z} \mid 3 \leq x < 9\} = \{x \in \mathbb{Z} \mid 0 < x \leq 8\}$$

### Solution No 3:

y will be in the shaded area



**Solution No 4:**

Do it by yourself.

**Solution No 5:**

$$A = \{4, 8, 12, 16\}$$

$$B = \{2, 4, 6, 8, 10, 12, 14, 16, 18\}$$

$$A \cap B = \{4, 8, 12, 16\}$$

$$B - A = \{2, 6, 10, 14, 18\}$$

## Practice Questions solution Files

### Lecture No (10 – 12)

#### **Solution No 1:**

Let  $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$$A_1 = \{1, 2, 3\}$$

$$A_2 = \{4, 5, 6\}$$

$$A_3 = \{7, 8\}$$

To show that  $A_1, A_2$  and  $A_3$  is the partition of  $A$ , we have to show

$$1) \quad A_1 \cup A_2 \cup A_3 = A$$

$$2) \quad A_1 \cap A_2 \cap A_3 = \emptyset$$

$$A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8\} = A$$

$$A_1 \cap A_2 \cap A_3 = \emptyset$$

Hence  $\{\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8\}\}$  is a partition of  $\{1, 2, 3, 4, 5, 6, 7, 8\}$ .

#### **Solution No 2:**

$$\text{Domain of } R = \{a, b, c\}$$

$$\text{Range of } R = \{8, 9\}$$

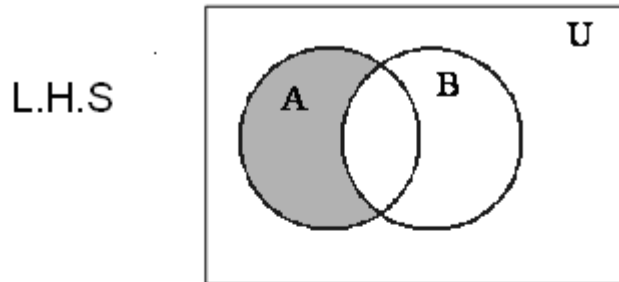
#### **Solution No 3:**

$$A \times B = \{(5, x), (5, y), (9, x), (9, y)\}$$

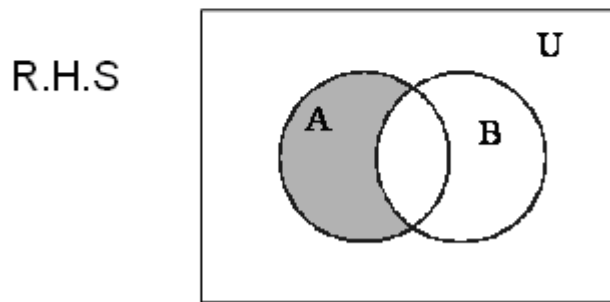
$$B \times A = \{(x, 5), (x, 9), (y, 5), (y, 9)\}$$

$A \times B$  is not equal to  $B \times A$ . Since order matters in Cartesian product.

**Solution No 4:**



**$A - B$  is shaded**



**$A \cap B^c$  is shaded**

**Solution No 5:**

- (a)  $R$  is not transitive because  $(d, b) \in R$  and  $(b, c) \in R$  but  $(d, c) \notin R$
- (b)  $R$  is not anti-symmetric because  $(b, c)$  and  $(c, b) \in R$  but  $b \neq c$

## Lecture 13-15

### Question No 1:

Let R and S be the following relations on  $A = \{1, 2, 3\}$

$$R = \{(1, 1), (1, 2), (2, 3), (3, 1), (3, 3), (1, 3), (2, 2)\}$$

$$S = \{(1, 2), (1, 3), (2, 1), (3, 3), (2, 2)\}$$

Find the following

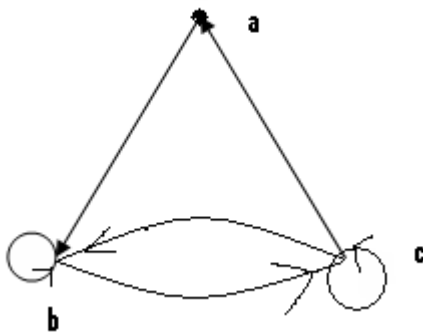
- a)  $R \cap S$
- b)  $R \cup S$
- c)  $R \circ S$
- d)  $S^2 = S \circ S$

### Solution:

- a)  $R \cap S = \{(1, 2), (3, 3), (1, 3), (2, 2)\}$
- b)  $R \cup S = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 3)\}$
- c)  $R \circ S = \{(1, 2), (1, 1), (2, 3), (3, 2), (1, 3), (2, 1), (2, 2), (3, 1), (3, 3)\}$
- d)  $S^2 = S \circ S = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 1), (2, 3), (3, 3)\}$

### Question No 2:

Determine whether the relation shown by the directed graph is reflexive, symmetric, anti-symmetric or transitive?



### Solution:

#### **Reflexive:**

There is not a loop at every point of the directed graph so it is not reflexive.

**Symmetric & Anti-symmetric:**

It is neither symmetric nor anti-symmetric, since there is an edge from a to b but not one from b to a. but there are edges in both directions b and c.

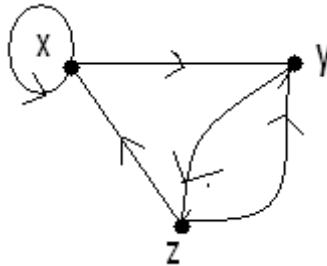
**Transitive:**

It is not transitive, since there is an edge from a to b and an edge from b to c, but no edge from a to c.

**Question No 3:**

In the following diagram

- (a) Write the relation R as set of ordered pairs then write inverse of R.
- (b) Whether it is symmetric or anti-symmetric relation?



**Solution:**

- (a)  $R = \{(x, x), (x, y), (y, z), (z, y), (z, x)\}$   
 $R^{-1} = \{(x, x), (y, x), (z, y), (y, z), (x, z)\}$

(b) It is neither symmetric nor anti-symmetric, since there is an edge from z to x but not one from x to z. but there are edges in both directions z and y.

**Question No 4:**

Let  $X = \{1, 2, 3, 4\}$  be a set and  $R = \{(x, y) | x \leq y\}$  is a relation on X. Find all the ordered pairs of R and  $R^{-1}$ . Also draw the directed graph of R.

**Solution:**

$$X = \{1, 2, 3, 4\}$$

$$X \times X = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$$

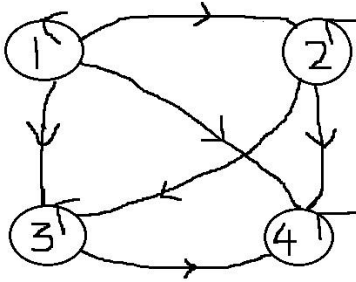
So

$$= \left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), \\ (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4) \end{array} \right\}$$

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,3), (2,4), (3,3), (3,4), (4,4)\}$$

$$R^{-1} = \{(1,1), (2,1), (3,1), (4,1), (2,2), (3,2), (4,2), (3,3), (4,3), (4,4)\}$$

The directed graph of R is



**Question No 5:**

Let  $R_1$  and  $R_2$  be the relations on a set  $A = \{1, 2, 3, 4\}$  given by

$$R_1 = \{(1,1), (1,2), (3,2), (3,4), (4,2)\}$$

$$R_2 = \{(1,2), (2,1), (3,1), (4,4), (2,2)\}$$

Find the ordered pairs of  $R_1 \circ R_2$  and  $R_2 \circ R_1$ .

**Solution:**

$$R_2 \circ R_1 = \{(1,2), (1,1), (4,1), (3,1), (3,2), (3,4), (4,2)\}$$

$$R_1 \circ R_2 = \{(2,1), (2,2), (3,1), (3,2), (4,2)\}$$

**Question No 6:**

Find the matrix representing the relations  $S \circ R$  where the matrices representing R and S are as follows

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$M_S = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

**Solution:**

$$\begin{aligned} M_{SOR} = M_R O M_S &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

**Question No 7:**

If  $f(x) = \frac{x}{2} - 3$  and  $g(x) = \frac{3}{4}x + 2$  then find the value of  $5f(-2) - 7g(-4)$ .

**Solution:**

$$f(-2) = -1 - 3 = -4$$

$$g(-4) = -3 + 2 = -1$$

Then

$$\begin{aligned} 5f(-2) - 7g(-4) &= 5(-4) - 7(-1) \\ &= -20 + 7 = -13 \end{aligned}$$

**Question No 8:**

Determine whether  $f : Z \rightarrow R$  such that  $f(n) = 2\sqrt{n}$  is a function or not.

**Solution:**

$f$  is not defined for  $n < 0$ , since  $f$  results in imaginary values (not real). So  $f$  is not a function.

## Solution to Practice questions for Lecture No. 16 – 18

### Solution 1:

Given function is

$$y = \frac{3x+2}{x+2}$$

We know that a function will be injective if for any two values  $x_1, x_2$  from the domain of  $f$ ,

$$f(x_1) = f(x_2)$$

Then

$$x_1 = x_2$$

Lets assume that for  $x_1, x_2$ , we have

$$\frac{3x_1+2}{x_1+2} = \frac{3x_2+2}{x_2+2}$$

$$(3x_1+2)(x_2+2) = (3x_2+2)(x_1+2)$$

$$3x_1x_2 + 6x_1 + 2x_2 + 4 = 3x_1x_2 + 6x_2 + 2x_1 + 4$$

$$6x_1 + 2x_2 = 6x_2 + 2x_1$$

$$4x_1 = 4x_2$$

$$x_1 = x_2$$

Thus, the given function is injective or one to one.

We know that, a function is surjective if for every value of  $y \in Y$ , there exist a  $x \in X$  such that  $f(x) = y$ .

For that let's take

$$y = \frac{3x+2}{x+2}$$

And try to find  $x$  such that  $f(x) = y$  for all  $y$  belongs to  $Y$ .

$$y = \frac{3x+2}{x+2}$$

$$y(x+2) = 3x+2$$

$$xy + 2y = 3x+2$$

$$xy + 2y - 3x = 2$$

$$x(y-3) = 2-2y$$

$$x = \frac{2-2y}{y-3}$$

So, apparently, we see that for every  $y$ , there is an  $x$  as given above, but if you watch carefully, the value of  $x$  is undefined for  $y = 3$ . Saying so, in other words, we can very conveniently claim that for  $y = 3$ , there exist no  $x$ , Therefore, the given function is not surjective.

### **Solution 2:**

Given that

$$f(x) = x^2 - 1 \text{ and } g(x) = 3x + 5$$

Then

$$f \circ g(x) = f(g(x))$$

$$= f(3x+5)$$

$$= (3x+5)^2 - 1$$

$$= 9x^2 + 30x + 24$$

$$g \circ f(x) = g(f(x))$$

$$= g(x^2 - 1)$$

$$= 3(x^2 - 1) + 5$$

$$= 3x^2 + 2$$

### **Solution 3:**

Given function is

$$y = \frac{-2}{x-5}$$

$$y(x-5) = -2$$

$$xy - 5y = -2$$

$$xy = -2 + 5y$$

$$x = \frac{5y-2}{y}$$

Thus,

$$f^{-1}(y) = \frac{5y-2}{y}$$

**Solution 4:**

Given that

$$y = g(x) = (2x - 1)$$

$$y+1 = 2x$$

$$x = \frac{y+1}{2}$$

Which is the required inverse function.

Now, replacing x and y , we get

$$g^{-1}(x) = \frac{x+1}{2}$$

$$g^{-1}(5) = \frac{5+1}{2} = 3$$

**Solution 5:**

Given that

$f(x) = ax + b$  and  $g(x) = cx + d$  , where a, b, c and d are constants.

$$f \circ g(x) = f(g(x))$$

$$= f(cx + d)$$

$$= a(cx + d) + b$$

$$= acx + ad + b$$

$$g \circ f(x) = g(f(x))$$

$$= g(ax + b)$$

$$= c(ax + b) + d$$

$$= cax + cb + d$$

$$f \circ g(x) = g \circ f(x)$$

$$acx + ad + b = cax + cb + d$$

If

$$acx + ad + b = cax + cb + d$$

$$ad + b = cb + d$$

$$d(a - 1) = b(c - 1)$$

$$(a - 1) / (c - 1) = b / d$$



**Solution File**  
**Sequence, Series and Recursion**  
**(Lecture # 19-22)**

**Question No 1:**

Write the geometric sequence with positive terms whose third term is 3 and fourth term is 1.

**Solution:**

$$3^{\text{rd}} \text{ term} = ar^2 = 3 \dots\dots 1$$

$$4^{\text{th}} \text{ term} = ar^3 = 1 \dots\dots 2$$

To find r and a, (2)/(1)

$$ar^3 / ar^2 = 1/3$$

$$r = 1/3 \text{ and } a = 27$$

So geometric sequence is, 27, 9, 3, 1,...

**Question No 2:**

Find the 15<sup>th</sup> term of the Arithmetic sequence, if its 6<sup>th</sup> term is 19 and 9<sup>th</sup> term is 31.

**Solution:**

$$6^{\text{th}} \text{ term} = a + 5d = 19 \dots\dots (1)$$

$$9^{\text{th}} \text{ term} = a + 8d = 31 \dots\dots (2)$$

By subtracting (2) and (1), we get

$$3d = 12, d = 4 \text{ and } a = -1$$

$$15^{\text{th}} \text{ term} = a + 14d = -1 + 14 \times 4 = 55$$

**Question No 3:**

The sixth term of an arithmetic sequence is 22 and eighth term is 32 .Find the first four terms of this sequence.

**Solution:**

Do it by yourself.

**Question No 4:**

A man deposited Rs.20 in a bank in the first month; Rs.25 in the second month; Rs.30 in the third month and so on. Find how much he will have deposited in the bank by the 7th month.

**Solution:**

1<sup>st</sup> term =20, 2<sup>nd</sup> term =25, 3<sup>rd</sup> term= 30,  $d=2^{\text{nd}} \text{ term} - 1^{\text{st}} \text{ term} = 5$

In 7<sup>st</sup> month =  $a+6d = 20+5*6 = 50$ .

**Question No 5:**

Find a common fraction for the recurring decimal  $1.\overline{73}$ .

**Solution:**

$$1.\overline{73} = 1.73737373\dots$$

$$= 1 + .73737373\dots = 1 + .73 + 0.0073 + 0.000073$$

Now using the geometric sequence on  $0.73 + 0.0073 + 0.000073$ , here  $a = 0.73$  and  $r =$

$$0.0073/0.73 = 0.01$$

$$\text{Formula} = a / (1-r) = 0.73 / (1-0.01)$$

$$1.\overline{73} = 1 + 0.73 / (1-0.01)$$

**Question No 6:**

Sum the following series upto n terms.

$$(1 \times 1) + (2 \times 3) + (3 \times 5) + \dots$$

**Solution:**

It can be written as

$$(1 \times 1) + (2 \times 3) + (3 \times 5) + \dots = (1, 2, 3, \dots) * (1, 3, 5, \dots)$$

Use Arithmetic Progression on (1,2,3..) and (1,3,5..), and then multiply both answers in order to get the sum of series upto n terms.

**Question No 7:**

Find the sum of first n terms of an arithmetic series.

**Solution:**

Do it by yourself

**Question No 8:**

How many terms are there in  $-9 - 6 - 3 + 0 + \dots$  amount to 45?

**Solution:**

$$a = -9 \text{ and } d = -6 + 9 = 3 \text{ and } a_n = 45$$

$$a_n = 45 = a + (n-1)d = -9 + (n-1)(3)$$

$$54 = (n-1)(3)$$

$$n-1 = 18$$

$$n = 19$$

**Question No 9:**

Let a and b be integers. Suppose a function Q is defined recursively as follows:

$$Q(a, b) = \begin{cases} 5 & \text{if } a < b \\ Q(a - b, b) - a & \text{if } b \leq a \end{cases}$$

Find Q (25, 5)

**Solution:**

Now

$$\begin{aligned} Q(25, 5) &= Q(25 - 5, 5) - 25 \\ &= Q(20, 5) - 25 \\ &= [ Q(20 - 5, 5) - 20 ] - 25 \\ &= Q(15, 5) - 20 - 25 \end{aligned}$$

$$\begin{aligned}
&= Q(15, 5) - 45 \\
&= [Q(15 - 5, 5) - 15] - 45 \\
&= Q(10, 5) - 60 \\
&= [Q(10 - 5, 5) - 10] - 60 \\
&= [Q(5, 5) - 70] = [Q(5 - 5, 5) - 5] - 70 \\
&= Q(0, 5) - 75 \\
&= 5 - 75 \\
&= -70
\end{aligned}$$

**Question No 10:**

Suppose that  $f$  is defined recursively by  $f(0) = 2, f(n+1) = 4f(n) + 2$  then find  $f(3)$ .

**Solution:**

For  $n = 0, f(0+1) = 4f(0) + 2 = 4(2) + 2 = 10$

$f(1) = 10$

For  $n = 1, f(1+1) = 4f(1) + 2$  or  $f(2) = 4f(1) + 2 = 4(10) + 2 = 42$

Similarly find the value for  $f(3) = 170$

**Question No 12:**

Show that the sequence  $0, 1, 3, 7, \dots, 2^n - 1, \dots$ , for  $n \geq 0$  satisfies the recurrence relation

$$d_k = 3d_{k-1} - 2d_{k-2}, \text{ for all integers } k \geq 2.$$

**Solution:**

The sequence is given by the formula

$$d_n = 2^n - 1 \quad \text{for } n \geq 0$$

Substituting  $k - 1$  for  $n$  we get  $d_{k-1} = 2^{k-1} - 1$

Substituting  $k - 2$  for  $n$  we get  $d_{k-2} = 2^{k-2} - 1$

We want to prove that

$$d_k = 3d_{k-1} - 2d_{k-2}$$

$$\begin{aligned} \text{R.H.S.} &= 3(2^{k-1} - 1) - 2(2^{k-2} - 1) \\ &= 3 \cdot 2^{k-1} - 3 - 2 \cdot 2^{k-2} + 2 \\ &= 3 \cdot 2^{k-1} - 2^{k-1} - 1 \\ &= (3 - 1) \cdot 2^{k-1} - 1 \\ &= 2 \cdot 2^{k-1} - 1 = 2^k - 1 = d_k = \text{L.H.S.} \end{aligned}$$

### **Question No 11:**

Define a sequence  $a_0, a_1, a_2, \dots$  by the formula  $a_n = 2n + 1$ , for all integers  $n \geq 0$ . Show that this sequence satisfies the recurrence relation  $a_k = a_{k-1} + 2$ , for all integers  $k \geq 1$ .

### **Solution:**

Do it by yourself

## Solution File Lecture No 23-25

### Question No.1:

Use mathematical induction to prove that  $4^n > 3^n + 4$  is true for integral values of  $n \geq 2$

### Solution:

Let  $P(n) = 4^n > 3^n + 4$

Let  $P(n) = 4^n > 3^n + 4$

**BASIS STEP:**

$$n = 2$$

$$4^2 > 3^2 + 4 = 16 > 13$$

So  $P(2)$  is true

**INDUCTIVE STEP:**

Suppose  $P(k)$  is true for all  $k \geq 2$

$$4^k > 3^k + 4 \quad (1)$$

we are to prove  $P(k+1)$  is true i.e.

$$4^{k+1} > 3^{k+1} + 4 \quad (2)$$

Now

$$\text{LHS} = 4^{k+1} = 4^k \cdot 4 > 4 \cdot (3^k + 4) = 4 \cdot 3^k + 16 \text{ by (1) for } k \geq 2.$$

$$4^{k+1} > 3^k \cdot (3+1) + 16$$

$$4^{k+1} > 3^k \cdot 3 + 3^k + 16$$

$$4^{k+1} > 3^{k+1} + 3^k + 4 + 12$$

$$4^{k+1} > 3^{k+1} + 4 + 12 + 3^k$$

$$4^{k+1} > 3^{k+1} + 4 \quad \text{ignoring } 12 + 3^k \text{ as if } a > b + c \text{ then } a > b$$

Hence it is proved by mathematical induction that  $4^n > 3^n + 4$  is true for integral values of  $n \geq 2$ .

## **Question No.2**

Use mathematical induction to prove that  $3+6+9+\dots+3n = \frac{3n(n+1)}{2}$  for all positive integers  $n$ .

### **Solution:**

$$\text{Let } P(n): \quad 3+6+9+\dots+3n = \frac{3n(n+1)}{2}$$

*BASIS STEP:*

$$LHS = 3(1) = 3$$

$$RHS = \frac{3(1+1)}{2} = 3$$

So  $P(1)$  is true

*INDUCTIVE STEP:*

Suppose  $P(k)$  is true for all  $k \geq 1$

$$3+6+9+\dots+3k = \frac{3k(k+1)}{2} \quad (1)$$

we are to prove  $P(k+1)$  is true i.e.

$$P(k+1): \quad 3+6+9+\dots+3(k+1) = \frac{3(k+1)((k+1)+1)}{2} \quad (2)$$

Now

$$\begin{aligned} LHS &= 3+6+9+\dots+3k+3(k+1) = (3+6+9+\dots+3k)+3(k+1) \\ &= \frac{3k(k+1)}{2} + 3(k+1) \\ &= 3(k+1) \left( \frac{k}{2} + 1 \right) \\ &= 3(k+1) \frac{k+2}{2} \\ &= \frac{3(k+1)[(k+1)+1]}{2} = RHS \end{aligned}$$

### **Question No.3**

Suppose  $n^3 - n$  is divisible by 6 is true for any positive integer. Justify the statement for  $n = k+1$ .

### **Solution:**

Inductive Step:

Suppose for  $n=k$ ,  $k^3 - k$  is dividible by 6

then  $k^3 - k = 6.q$

we need to prove  $(k+1)^3 - (k+1)$  is dividible by 6

$$\begin{aligned}(k+1)^3 - (k+1) &= (k^3 + 3k^2 + 3k + 1) - (k+1) \\ &= k^3 - k + 3k(k+1) \\ &= 6.q + 3k(k+1)\end{aligned}$$

As the product  $k(k+1)$  of integers is even so we can take as  $2r$

$$\begin{aligned}&= 6.q + 3k(k+1) \\ &= 6.q + 3.2r \\ &= 6(q+r)\end{aligned}$$

So  $(k+1)^3 - (k+1)$  is dividible by 6

Hence by induction method,  $n^3 - n$  is dividible 6

### **Question No 4:**

Suppose  $5^n - 1$  is dividible by 4 is true for  $n = k$ . Justify the statement for  $n = k + 1$ .

### **Solution:**

Inductive Step:

Let for  $n = k$ ,  $5^k - 1$  is divisible by 4 then

$$5^k - 1 = 4.q$$

we need to prove that  $5^{k+1} - 1$  is divisible by 4

$$\begin{aligned}5^{k+1} - 1 &= 5 \cdot 5^k - 1 \\ &= 5^k (4 + 1) - 1 \\ &= 5^k \cdot 4 + 5^k - 1 \\ &= 5^k \cdot 4 + 4.q \\ &= 4(5^k + q)\end{aligned}$$

Hence  $5^{k+1} - 1$  is divisible by 4

### **Question No 5**

Prove that if  $x$  divides  $y$  and  $y$  divides  $z$  then  $x$  divides  $z$ .

### **Solution:**

By our assumptions, and the definition of divisibility, there are natural numbers  $k_1$  and  $k_2$  such that

$$y = xk_1 \text{ and } z = yk_2.$$

Consequently,

$$z = yk_2 = xk_1k_2.$$

Let  $k = k_1k_2$ . Now  $k$  is a natural number and  $z = xk$ , so by the definition of divisibility,  $x$  divides  $z$ .

### **Question No 6:**

Show that for all integers  $a$  and  $b$ , if  $a + b$  is even then  $a - b$  is also even.

### **Solution:**

Take  $a + b = 2k$

$$a = 2k - b$$

Subtracting 'b' from both sides

$$a - b = 2k - b - b$$

$$a - b = 2k - 2b$$

$$a - b = 2(k - b)$$

Hence  $a - b$  is also even.

### **Question No 7:**

Assume that  $m$  and  $n$  are particular integers. Justify your answer to each of the following:

- 1) Is  $4m + 6n$  even?
- 2) Is  $8mn + 5$  odd?

### **Solution:**

- 1) Yes:  $4m + 6n = 2 \cdot (2m + 3n)$  [by definition of even]  
( $2m + 3n$ ) is an integer because 2, 3,  $m$ ,  $n$  are integers and products and sums of integers are integers.
- 2) Yes:  $8mn + 5 = 2 \cdot (4mn + 2) + 1$  [by definition of odd]  
( $4mn + 2$ ) is an integer because 4, 2,  $m$ ,  $n$  are integers and products and sums of integers are integers.

## Solution to Practice questions Lecture 26-28

Solution 1:

We are to prove by contradiction that  $4+3\sqrt{2}$  is an irrational number. So, let's assume that  $4+3\sqrt{2}$  is a rational number. Then by definition of rational number, it can be written in the following form

$$4+3\sqrt{2} = \frac{a}{b} \text{ for some integers } a \text{ and } b \text{ with } b \neq 0$$

$$4+3\sqrt{2} = \frac{a}{b}$$

$$3\sqrt{2} = \frac{a}{b} - 4$$

$$\sqrt{2} = \frac{a-4b}{3b}$$

Since  $a$  and  $b$  are integers, so are  $a-4b$  and  $3b$  with  $3b \neq 0$ . This shows that  $\sqrt{2}$  is a rational number. Which is a contradiction to the fact that  $\sqrt{2}$  is an irrational number. This contradiction shows that our initial supposition was incorrect and thus,  $4+3\sqrt{2}$  is not an irrational number that is  $4+3\sqrt{2}$  is a rational number.

Solution 2:

We are to prove by contradiction that if  $5n+1$  is odd then  $n$  is even. Suppose for some integer  $n$ ,  $5n+1$  is odd but  $n$  is not even. That is  $n$  is odd. So it can be written as

$$n = 2m+1 \text{ for some integer } m.$$

$$5n+1 = 5(2m+1)+1$$

$$= 10m+6$$

$$= 2(5m+3)$$

This shows that  $5n+1$  is equal to some multiple of 2 and so it is even which is a contradiction to our initial supposition. So, the correct supposition will be that if  $5n+1$  is odd then  $n$  is even.

Solution 3:

We are to prove by contraposition that if  $n^2$  is odd then  $n$  is odd. That is we are to prove that if  $n$  is even then  $n^2$  is even.

Lets assume  $n$  is even. Then it can be written as a multiple of 2. That is

$$\begin{aligned}
n &= 2k \text{ for some integer } k \\
n^2 &= (2k)^2 \\
&= 4k^2 \\
&= 2(2k^2)
\end{aligned}$$

Clearly,  $n^2$  is a multiple of 2 so  $n^2$  is also even.

This Proves that if  $n^2$  is odd then  $n$  is odd.

Solution 4:

The following are pre and post conditions.

Pre-condition: The input variables  $a$  and  $b$  are positive integers.

Post-condition: The output variable  $q$  and  $r$  are positive integers such that

$$a = b \cdot q + r \text{ and } 0 \leq r < b.$$

Solution 5:

We are to find the GCD (61114, 94).

Divide 61114 by 94

$$61114 = 94 \times 650 + 14$$

Divide 94 by 14

$$94 = 14 \times 6 + 10$$

Divide 14 by 10

$$14 = 10 \times 1 + 4$$

Divide 10 by 4

$$10 = 4 \times 2 + 2$$

Divide 4 by 2

$$4 = 2 \times 2 + 0$$

Thus, the GCD of 61114 and 94 is 2.

## Solution to practice questions for Lecture No. 29-31

### Solution 1:

For a round trip, the person will travel from city A to B, B to C then C to B and then B to A i.e.,

$$A \rightarrow B \rightarrow C \rightarrow B \rightarrow A$$

The person can travel 3 ways from A to B and 5 ways from B to C and back that is

$$A \xrightarrow{3} B \xrightarrow{5} C \xrightarrow{5} B \xrightarrow{3} A$$

Thus, there are

$$3 \times 5 \times 5 \times 3 = 225$$

ways to have a round trip.

### Solution 2:

Each bit (binary digit) is either 0 or 1.

Hence, there are 2 ways to choose each bit. Since we have to choose seven bits therefore, the product rule shows, there are a total of  $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^7 = 128$

### Solution 3:

Given that

$$P(n,2) = 42$$

That is

$$\frac{n!}{(n-2)!} = 42$$

$$\frac{n(n-1)(n-2)!}{(n-2)!} = 42$$

$$n(n-1) = 42$$

$$n^2 - n - 42 = 0$$

$$n^2 - 7n + 6n - 42 = 0$$

$$n(n-7) + 6(n-7) = 0$$

$$(n+6)(n-7) = 0$$

$$n = -6, \quad n = 7$$

Ignoring the negative value, the desired value of n is 7.

**Solution 4:**

5 boys can be selected from 10 in following number of ways

$${}^{10}C_5 = 252$$

5 girls can be selected from 15 in following number of ways

$${}^{15}C_5 = 3003$$

Total number of ways a team of 10 students be selected =  $252 \times 3003 = 756756$

**Solution 5:**

There are 13 Spade cards in a deck of cards so  $C(13,2) = 78$

There are 13 Heart cards in a deck of cards so  $C(13,2) = 78$

$$C(13,2) \times C(13,2) = 78 \times 78 = 6084$$

## **Practice questions Lectures 32-34**

### **Question No 1:**

How many distinguishable ways can the letter of the word HULLABALOO be arranged?

### **Question No 2:**

How many signals can be given by 7 flags of different colors using 3 flags at a time?

### **Question No 3:**

How many 4-digit numbers can be formed by using each of the digits 1, 2, 3, 4, 5, 6 only once?

### **Question No 4:**

Suppose that there are 2000 students at your university. Of these 500 are taking a course in computer science, 670 are taking a course in mathematics and 300 are taking course in both computer science and mathematics. How many are taking a course either in computer science or in mathematics?

### **Question No 5:**

There are 24 students in a class, only 13 are willing to go to a tour for Murree. In how many ways can these be chosen?

### **Question No 6:**

In an office, there are 75 faculty members can speak Malay (M) and 25 can speak Chinese (C), while only 10 can speak both Malay and Chinese. Using Inclusion-exclusion Principle find how many faculty members can speak either Malay (M) or Chinese (C)?

## Practice Questions for Lecture No. 35 to 37

### Question 1:

A box contains seven discs numbered 1 to 7. Find for each integer  $k$  ( $k$  is from 3 to 11), the probability that the numbers on two discs drawn without replacement have a sum equal to  $k$ .

### Question 2:

A die is weighted so that the outcomes produce the following probability distribution:

Outcome	1	2	3	4	5	6
Probability	0.1	0.3	0.2	0.1	0.1	0.2

Consider the event  $A = \{\text{odd number occur on die}\}$  then find  $P(A)$  and  $P(A^c)$ .

### Question 3:

There are 20 girls and 40 boys in a class. Half of the boys and half of the girls have blue eyes. Find the probability that one student chosen as monitor is either a girl or a student having blue eyes.

### Question 4:

A pair of fair dice is thrown. Find the probability  $P$  that the sum is 9 or greater if

- (i) 5 appear on first die
- (ii) 5 appear on at least one die

## Prctice Question Lecture No 39-41

### Question No 1:

Find the maximum number of edges for a complete graph  $K_n$  by using handshaking theorem.

### Solution

The total degrees in graph  $K_n = 2$  (number of edges in  $K_n$ )

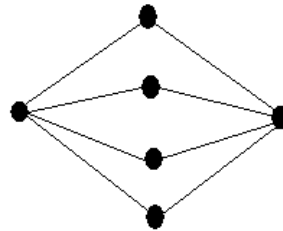
$$\Rightarrow n(n-1) = 2(\text{number of edges in } K_n)$$

$\Rightarrow$

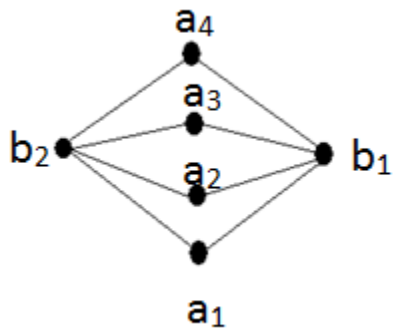
$$\text{Number of edges in } K_n = \frac{n(n-1)}{2}$$

### Question No 2:

Is the graph below is Complete Bipartite or not? Redraw the bipartite graph so that its bipartite nature is evident?



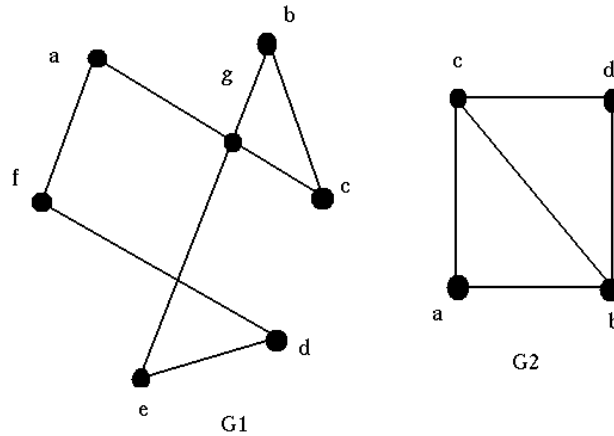
### Solution:



Yes it is a complete bipartite graph.

**Question No 3:**

Check whether the given graphs have an Euler circuit? If it does, find such a circuit, if it does not, give an argument to show why no such circuit exists.

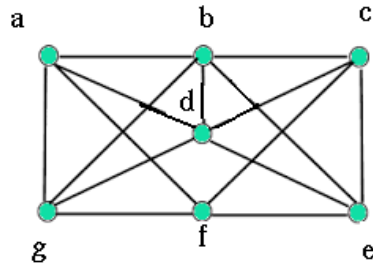


**Solution:**

In G1: each vertex has even degree so it an Euler circuit. In G2: vertices “c” and “b” has odd degree so it is not an Euler circuit

**Question No 4:**

Find two Hamiltonian circuits of the following graph



**Solution:**

(i) abcdefga ii) abcefgda

**Question No 5:**

Find the product AB and BA of the matrices (if not possible then give reason).

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 4 \end{bmatrix}$$

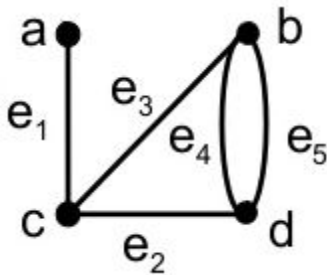
**Solution:**

$$AB = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 2+0 & 4+0 & 6+0 \\ 1-3 & 2-2 & 3-4 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ -2 & 0 & -1 \end{bmatrix}$$

BA is not possible as B has 3 columns and A has two rows, so multiplication is not possible.

**Question No 6:**

Find the adjacency matrix of the graph shown below.



Solution:

$$\begin{matrix}
 & \begin{matrix} a & b & c & d \end{matrix} \\
 \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}
 \end{matrix}$$

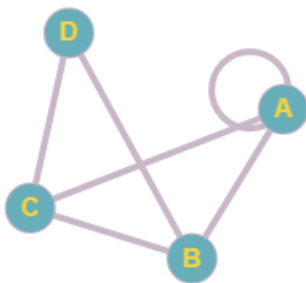
**Question No 7:**

Draw a graph with the adjacency matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices a, b, c, d.

**Solution:**



## Lecture 41-43

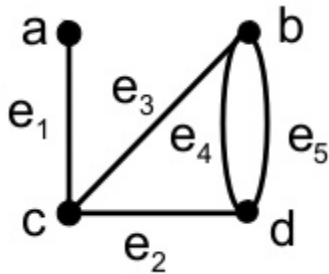
### Question No 1:

Find the product  $AB$  and  $BA$  of the matrices (if not possible then give reason).

$$A = \begin{bmatrix} 2 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 4 \end{bmatrix}$$

### Question No 2:

Find the adjacency matrix of the graph shown below.



### Question No 3:

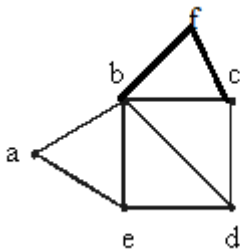
Draw a graph with the adjacency matrix.

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

with respect to the ordering of vertices  $a, b, c, d$ .

### Question No 4 :

Find the degree sequence of the following graph.

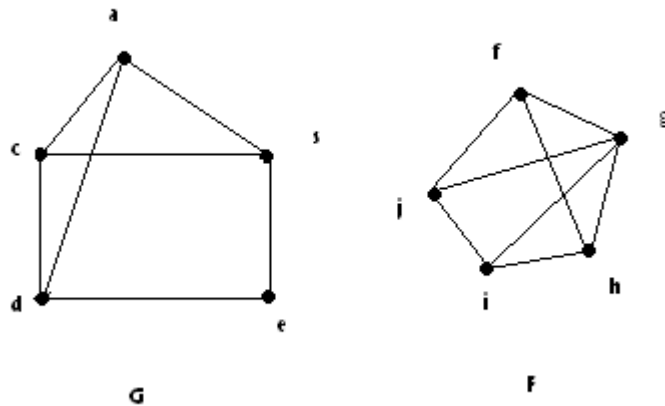


**Question No 5:**

Draw all possible simple graphs with 2 vertices, which are non-isomorphic to each other.

**Question No 6:**

Determine whether the given graphs are isomorphic? (justify your answer)

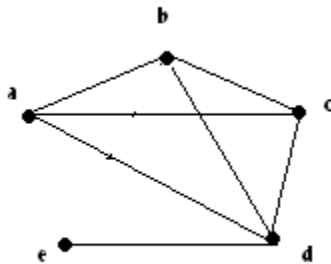


**Question No 7:**

Suppose that a connected planar simple graph has 22 edges. If you want to draw a graph having 11 faces, how many vertices does this graph have?

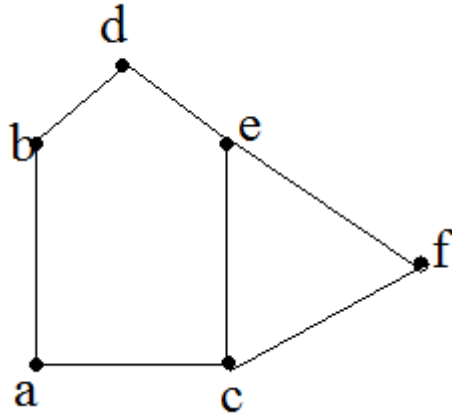
**Question No 8:**

Re-draw the given graph to prove it planar.



**Question No 9:**

Determine the chromatic number of the given graph.



## Practice Questions for Lecture No. 44 and 45

### Question 1:

Find the total number of internal and terminal vertices of a full binary tree with nine vertices

### Question 2:

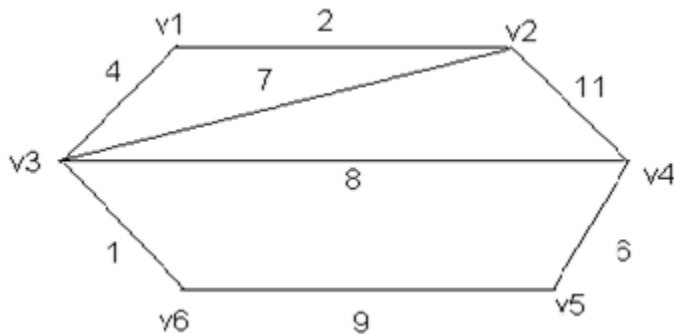
Draw a binary tree with height 3 and having seven terminal vertices.

### Question 3:

Draw any two non-isomorphic trees of five vertices.

### Question 4:

Use Kruskal's Algorithm to draw the minimal spanning tree for the graph given below. Indicate the order in which edges are added to form a tree.



### Question 5:

Find a spanning tree for the graph  $K_{1,5}$ ?