MTH 404: Measure and Integration

Mid-Semester Examination | Maximum Duration: 120 minutes
Feb 29, 2016 | Maximum Marks: 75

Note: Clearly state the results used.

1. (10 points)
   (a) Prove that any set with zero outer measure is measurable.
   (b) Let E be such that $m^*E < \infty$. Prove that E is measurable if and only if given there exists a G_δ-set O containing E such that $m^*(O \setminus E) = 0$.

2. (10 points) Let $f : [0,1] \to \mathbb{R}$ be an integrable function.
   (a) For every $n \in \mathbb{N}$, define $g_n(x) = x^n f(x)$, $x \in [0,1]$. Prove that each g_n is an integrable function on $[0,1]$.
   (b) Prove that
   $$\lim_{n\to\infty} \int_0^1 g_n = 0.$$

3. (10 points) Let f be a non-negative integrable function on $\mathbb{R}$. Prove that for any $\epsilon > 0$, there is a measurable set E with $mE < \infty$ and
   $$\int_{\mathbb{R}\setminus E} f < \epsilon.$$
   (Hint: Use the definition of $\int_{\mathbb{R}} f$.)

4. (15 points) Let f be an integrable function on E.
   (a) Prove that f is finite a.e. on E.
   (b) Prove that $\int_A f = 0$ for every measurable subset A of E if and only if $f = 0$ a.e. on E.
   (c) If $f \geq 0$, prove that $\int_E f = 0$ if and only if $f = 0$ a.e. on E.

5. (15 points)
   (a) Let f be a non-negative measurable function on $\mathbb{R}$. Prove that
   $$\lim_{n\to\infty} \int_{-n}^{n} f = \int_{\mathbb{R}} f.$$
   (b) Is part (a) true for a general function f? Prove or give a counterexample.

6. (15 points)
   (a) Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with $\sup_{x\in(a,b)} |f'(x)| \leq M$, for some $M > 0$. Prove that f is a function of bounded variation.
   (b) Let $f : [0,1] \to \mathbb{R}$ be defined as $f(x) = 1$, $x \in \mathbb{Q}$ and $f(x) = -1$, $x \notin \mathbb{Q}$. Is f a function of bounded variation? Justify your answer.
   (c) Is f defined in part (b) an absolutely continuous function? Justify your answer.