

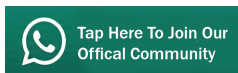


Lecture Handouts

on

VECTORS AND CLASSICAL MECHANICS (MTH-622)

**Virtual University of Pakistan
Pakistan**



About the Handouts

The following books have been mainly followed to prepare the slides and handouts:

1. Spiegel, M.R., *Theory and Problems of Vector Analysis: And an Introduction to Tensor Analysis*. 1959: McGraw-Hill.
2. Spiegel, M.S., *Theory and problems of theoretical mechanics*. 1967: Schaum.
3. Taylor, J.R., *Classical Mechanics*. 2005: University Science Books.
4. DiBenedetto, E., *Classical Mechanics: Theory and Mathematical Modeling*. 2010: Birkhäuser Boston.
5. Fowles, G.R. and G.L. Cassiday, *Analytical Mechanics*. 2005: Thomson Brooks/Cole.

The first two books were considered as main text books. Therefore the students are advised to read the first two books in addition to these handouts. In addition to the above mentioned books, some other reference book and material was used to get these handouts prepared.

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Module No. 1

Introduction to the Course and Mathematics

Introduction to Mathematics

“Mathematics is the branch of science which deals with the study of relations and patterns, and means to represent and communicate them.”

Introduction to the Course

There are two main portions of this course:

- Vectors
- Classical Mechanics

Module No. 2

Scalar and Vector Fields

Scalar Point Function

If to each point (x, y, z) of a region R in space there corresponds a scalar $\varphi(x, y, z)$, then φ is called a scalar point function in R .

Scalar Field

Scalar field is a function define on space whose value at each point is a scalar quantity
The set of all values of scalar point function φ in R together forms a Scalar field.

Examples of Scalar Fields

1. The temperature $T(x, y, z)$ within a body A is a scalar point function because there exist only one temperature at each point of A .
2. The pressure and potential due to gravity of the air in the earth's atmosphere define scalar field.

Vector Point Function

If to each point (x, y, z) of a region R in space there exist a unique vector $\vec{A}(x, y, z)$, then $\vec{A}$ is called a vector point function in R .

Vector Field

A function of a space whose value at each point is a vector quantity is called vector field.
Mathematically, we can write it as

$$\vec{A} = \vec{A}(x, y, z) = A_1(x, y, z) + A_2(x, y, z) + A_3(x, y, z)$$

The set of all values of $\vec{A}$ in R constitute a vector field.

Examples of Vector Field

1. $\vec{A} = \vec{A}(x, y, z) = xy\hat{i} - 2yz^3\hat{j} + y^2z\hat{k}$ defines a vector point function and hence is a vector field.
2. The motion of a moving fluid at any time define vector field.
3. The set of tangent vector of a curve C and the set of normal vectors of a surface S are examples of vector field.

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Module No. 3

The Operator Del and Gradient of Function

The Del Operator

The vector differential operator del, symbolize as ∇ , is defined by

$$\nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$$

The symbol ∇ called **del** or **nabla** is used to symbolize del operator. It is only applied as defined derivative on one-dimensional function, and for more dimensions it may be applied as partial derivative on the function. The del operator is not a particular operator but when we applied it on a scalar point function or a vector point function; this may be known as the gradient, the divergence, and the curl.

Gradient: $\text{grad}(f) = \nabla \phi$

Curl: $\text{curl } \vec{A} = \nabla \times \vec{A}$

Divergence: $\text{div}(\vec{A}) = \nabla \cdot \vec{A}$

We will discuss here the first application of del operator as gradient.

Gradient Function

Let $\phi(x, y, z)$ be a scalar point function defined on a specific region on R and also differentiable on the same domain. Then we can apply del operator on ϕ in order to obtain gradient of scalar function ϕ of $\text{grad}(\phi)$ written as $\nabla \phi$ is defined by

$$\begin{aligned}\nabla\varphi &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\varphi \\ &= \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\end{aligned}$$

it is to be noted that $\nabla\varphi$ defined a vector field. Also $\nabla\varphi = 0$ if and only if φ is constant.

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Module No. 4

Properties of the Gradient

If φ and ψ are Scalar point functions as well as differentiable on a specific domain and C is a constant then we can show that the gradient holds the following properties.

- i. $\nabla(C\varphi) = C\nabla\varphi$
- ii. $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$
- iii. $\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$
- iv. $\nabla\left(\frac{\varphi}{\psi}\right) = \frac{\psi\nabla\varphi - \varphi\nabla\psi}{\psi^2}, \psi \neq 0$

Proof:

i. $\nabla(C\varphi) = C\nabla\varphi$

$$\begin{aligned} \text{L.H.S} &= \nabla(C\varphi) = \frac{\partial(C\varphi)}{\partial x}\hat{i} + \frac{\partial(C\varphi)}{\partial y}\hat{j} + \frac{\partial(C\varphi)}{\partial z}\hat{k} \\ &= C\frac{\partial\varphi}{\partial x}\hat{i} + C\frac{\partial\varphi}{\partial y}\hat{j} + C\frac{\partial\varphi}{\partial z}\hat{k} \\ &= C\left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right) \\ &= C\nabla\varphi = \text{R.H.S} \end{aligned}$$

ii. $\nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$

$$\begin{aligned} \text{L.H.S} &= \nabla(\varphi + \psi) = \frac{\partial(\varphi + \psi)}{\partial x}\hat{i} + \frac{\partial(\varphi + \psi)}{\partial y}\hat{j} + \frac{\partial(\varphi + \psi)}{\partial z}\hat{k} \\ &= \left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}\right) + \left(\frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}\right) \\ &= \nabla\varphi + \nabla\psi = \text{R.H.S} \end{aligned}$$

iii. $\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$

$$\text{L.H.S} = \nabla(\varphi\psi) = \frac{\partial(\varphi\psi)}{\partial x}\hat{i} + \frac{\partial(\varphi\psi)}{\partial y}\hat{j} + \frac{\partial(\varphi\psi)}{\partial z}\hat{k}$$

$$\begin{aligned}
&= \left(\varphi \frac{\partial \psi}{\partial x} + \psi \frac{\partial \varphi}{\partial x} \right) \hat{i} + \left(\varphi \frac{\partial \psi}{\partial y} + \psi \frac{\partial \varphi}{\partial y} \right) \hat{j} + \left(\varphi \frac{\partial \psi}{\partial z} + \psi \frac{\partial \varphi}{\partial z} \right) \hat{k} \\
&= \varphi \left(\frac{\partial \psi}{\partial x} \hat{i} + \frac{\partial \psi}{\partial y} \hat{j} + \frac{\partial \psi}{\partial z} \hat{k} \right) + \psi \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right) \\
&= \varphi \nabla \psi + \psi \nabla \varphi = R.H.S
\end{aligned}$$

iv. $\nabla \left(\frac{\varphi}{\psi} \right) = \frac{\psi \nabla \varphi - \varphi \nabla \psi}{\psi^2}$

$$\begin{aligned}
L.H.S &= \nabla \left(\frac{\varphi}{\psi} \right) = \nabla \left(\varphi \frac{1}{\psi} \right) = \varphi \nabla \left(\frac{1}{\psi} \right) + \psi \nabla \left(\frac{1}{\varphi} \right) \\
&= \varphi \left(\frac{-1}{\psi^2} \right) \nabla \psi + \psi \nabla \left(\frac{1}{\varphi} \right) \\
&= \frac{\psi \nabla \varphi - \varphi \nabla \psi}{\psi^2} = R.H.S
\end{aligned}$$

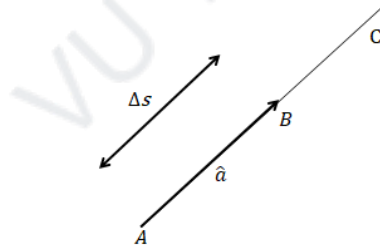
Module No. 5

Directional Derivative

The procedure to determine the derivative in a specific direction, other than the coordinate axes (x, y, z) is called **directional derivative**.

Let $\varphi(x, y, z)$ be a scalar point function defined on a specific region on R and also differentiable on the same domain. The first partial derivatives of $\varphi(x, y, z)$ are the rate of change of φ in the direction of coordinate axes (x, y, z) . It is a restricted way to calculate the rate change is given function. Maybe one ought to need the derivative in a specific direction. Therefore the idea of directional derivative introduced.

To define the directional derivative we choose a point $A(x, y, z)$ in space and a direction at P , given by a unit vector $\hat{a}$. Let C be the ray drawn from P in the direction of $\hat{a}$, and let $A(x + \Delta x, y + \Delta y, z + \Delta z)$ denoted by B be a neighboring point on C , whose distance from P is Δs as shown in figure,



The value of given scalar point function is $\varphi(x, y, z)$ and $\varphi(+\Delta x, y + \Delta y, z + \Delta z)$ at P and P' respectively.

Then the limit

$$\lim_{\Delta s \rightarrow 0} \frac{\Delta \varphi}{\Delta s} = \lim_{\Delta s \rightarrow 0} \frac{\varphi(P') - \varphi(P)}{\Delta s}$$

if it exists, is called the directional derivative of φ at P in the direction of $\hat{a}$ and is denoted by $\frac{\partial \varphi}{\partial s}$.

Obviously,

$$\begin{aligned}
\frac{\partial \varphi}{\partial s} &= \frac{\partial \varphi}{\partial x} \frac{dx}{ds} + \frac{\partial \varphi}{\partial y} \frac{dy}{ds} + \frac{\partial \varphi}{\partial z} \frac{dz}{ds} \\
&= \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right) \left(\frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} + \frac{dz}{ds} \hat{k} \right) \\
&= \nabla \varphi \cdot \frac{d\vec{r}}{ds} = \nabla \varphi \cdot \hat{a}
\end{aligned} \tag{1}$$

Since $\hat{a}$ is a unit vector, directional derivative of φ (i.e. $\frac{\partial \varphi}{\partial s}$) is the component of $\nabla \varphi$ in the direction of this unit vector.

From equation (1), we have the operator equivalence,

$$\frac{\partial}{\partial s} = \nabla \cdot \hat{a}$$

This means that the operator $\nabla \cdot \hat{a}$ applied to the scalar function φ differentiates it w.r.t the distance s in the direction of $\hat{a}$.

Deductions

In particular, if we assume $\hat{a}$ has the direction of the positive x-axis, the $\hat{a} = \hat{i}$ then equation (1) will become $\nabla \varphi \cdot \hat{i}$

$$\begin{aligned}
&= \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right) \cdot \hat{i} \\
\nabla \varphi \cdot \hat{i} &= \frac{\partial \varphi}{\partial x}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\nabla \varphi \cdot \hat{j} &= \frac{\partial \varphi}{\partial y}, \\
\nabla \varphi \cdot \hat{k} &= \frac{\partial \varphi}{\partial z}
\end{aligned}$$

Module No. 6

Theorem related Directional Derivative

Statement

Show that the maximum value of the directional derivative of $\varphi(x, y, z)$ is equal to the magnitude of $\varphi(x, y, z)$ is equal to the magnitude of $\nabla\varphi$ (i.e. $|\nabla\varphi|$) and it takes place in the direction of $\nabla\varphi$.

Proof

We know that

$$\begin{aligned}\frac{\partial\varphi}{\partial s} &= \nabla\varphi \cdot \hat{a} \\ &= |\nabla\varphi| |\hat{a}| \cos\theta\end{aligned}$$

where θ is the angle between $\nabla\varphi$ and $\hat{a}$. Since $-1 \leq \cos\theta \leq 1$, therefore $\frac{\partial\varphi}{\partial s}$ is maximum when $\cos\theta = 1$ or $\theta = 0^\circ$ i.e. when the direction of $\hat{a}$ is the direction of $\nabla\varphi$ and $\max\left(\frac{\partial\varphi}{\partial s}\right) = |\nabla\varphi|$.

Thus the maximum value of directional derivative takes place in the direction of $\nabla\varphi$ and has the magnitude $|\nabla\varphi|$.

It is important to be note that directional derivative $\frac{\partial\varphi}{\partial s}$ is zero, when $\theta = 90^\circ$ i.e when $\nabla\varphi$ and $\hat{a}$ are orthogonal to each other.

Module No. 7

Example of the Directional Derivative

Problem Statement:

Find the directional derivative of $\varphi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\hat{i} - \hat{j} - 2\hat{k}$.

Solution:

As we studied the equation

$$\text{Directional derivative} = \nabla\varphi \cdot \hat{a}$$

$$\nabla\varphi = \nabla(x^2yz + 4xz^2)$$

$$= (2xyz + 4z^2)\hat{i} + (x^2z)\hat{j} + (x^2y + 8xz)\hat{k}$$

$$\nabla\varphi \text{ at } (1, -2, -1) = 8\hat{i} - \hat{j} - 10\hat{k}$$

The unit vector in the given direction can be calculated as

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - \hat{j} - 2\hat{k}}{3} = \frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}$$

then the required directional unit vector is

$$\begin{aligned}\nabla\varphi \cdot \hat{a} &= (8\hat{i} - \hat{j} - 10\hat{k}) \cdot \left(\frac{2}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{2}{3}\hat{k}\right) \\ &= \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}\end{aligned}$$

Since this is positive, φ is increasing in this direction.

Module No. 8

Related Problem 1 of the Directional Derivative

Problem Statement:

Find the directional derivative of $\varphi = 4xz^3 - 3x^2y^2z$ at $(2, -1, 2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$.

Solution:

As we studied the equation

$$\text{Directional derivative} = \nabla\varphi \cdot \hat{a}$$

$$\nabla\varphi = \nabla(4xz^3 - 3x^2y^2z)$$

$$= (4z^3 - 6xy^2z)\hat{i} + (-6x^2yz)\hat{j} + (12xz^2 - 6x^2y^2)\hat{k}$$

$$\nabla\varphi \text{ at } (2, -1, 2) = 8\hat{i} + 24\hat{j} + 84\hat{k}$$

The unit vector in the given direction can be calculated as

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{2\hat{i} - 3\hat{j} + 6\hat{k}}{7} = \frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$$

then the required directional unit vector is

$$\begin{aligned}\nabla\varphi \cdot \hat{a} &= (8\hat{i} + 24\hat{j} + 84\hat{k}) \cdot \left(\frac{2}{7}\hat{i} - \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \\ &= \frac{16}{7} - \frac{144}{7} + \frac{504}{7} = \frac{376}{7}\end{aligned}$$

Since this is positive, φ is increasing in this direction.

Module No. 9

Related problem 2 of Directional Derivative

Problem Statement

- In what direction from the point $(2, 1, -1)$ is the directional derivative of $\varphi = x^2yz^3$ a maximum?
- What is the magnitude of this maximum?

Solution

First of all, we calculate gradient of $\varphi = x^2yz^3$

$$\nabla\varphi = \nabla(x^2yz^3) = (2xyz^3)\hat{i} + (x^2z^3)\hat{j} + (3x^2yz^2)\hat{k}$$

$\nabla\varphi$ at $(2, 1, -1)$ is $-4\hat{i} - 4\hat{j} + 12\hat{k}$

using the result that directional derivative is maximum in the direction of $\nabla\varphi$ and its maximum magnitude is $|\nabla\varphi|$

The directional derivative is a maximum in the direction $\nabla\varphi = -4\hat{i} - 4\hat{j} + 12\hat{k}$

the magnitude of this maximum is $|\nabla\varphi| = |(-4)^2 + (-4)^2 + (12)^2| = \sqrt{176} = 4\sqrt{11}$

Module No. 10

Related Problem 3 of the Directional Derivative

Problem Statement:

Find the values of the constants a, b, c so that the directional derivative of $\varphi = axy^2 + byz + cz^2x^3$ at $(1, 2, -1)$ has a maximum of magnitude 64 in a direction parallel to the z-axis.

Solution:

Since $\varphi = axy^2 + byz + cz^2x^3$, therefore

$$\begin{aligned}\nabla\varphi &= \nabla(axy^2 + byz + cz^2x^3) \\ &= (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)\hat{k}\end{aligned}$$

At the point $(1, 2, -1)$, the value of this gradient is

$$= (4a + 3c)\hat{i} + (4a - b)\hat{j} + (2b - 2c)\hat{k}$$

We know that the maximum directional derivative takes place in the direction of $\nabla\varphi$ and has the magnitude of $\nabla\varphi$. The maximum directional derivative will be parallel to the z-axis if

$$4a + 3c = 0 \tag{1}$$

$$4a - b = 0 \tag{2}$$

Therefore

$$\nabla\varphi = (2b - 2c)\hat{k}$$

since the magnitude of this maximum directional derivative is 64, there fore

$$2b - 2c = 64$$

(3)

Solving equation (1), (2) and (3), we find

$$a = 6, b = 24, c = -8$$

which is the required solution.

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Module No. 11

Geometrical Interpretation of Gradient

Geometrically, gradient of a scalar function represents a normal vector to the surface.

$\nabla\varphi|_{(x,y,z)}$ represents the normal vector of the surface at (x, y, z) .

Example

If $\varphi = x^2yz$

$$\nabla\varphi = 2xyz\hat{i} + x^2z\hat{j} + x^2y\hat{k}$$

At $(1,1,1)$

$$\nabla\varphi|_{(1,1,1)} = 2\hat{i} + \hat{j} + \hat{k}$$

which is normal to the surface $\varphi = x^2yz$

Module No. 12

Theorem Related to Gradient

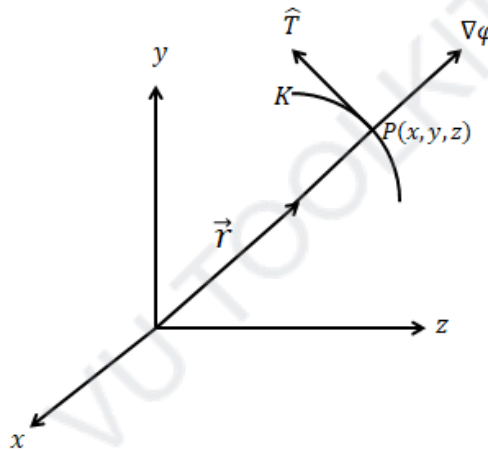
Theorem

Prove that $\nabla\phi$ is a vector perpendicular to the surface $\phi(x, y, z) = C$, where C is a constant.

Proof

Let the parametric equation of the curve K be $x = x(s)$, $y = y(s)$, $z = z(s)$.

let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of the point $P(x, y, z)$, then $\hat{T} = \frac{dr}{ds}$ is unit tangent vector to the curve K at P as shown in figure.



Since the curve K lies on the level surface, together, the condition of any point on the curve must satisfy equation (1), and so $[x(s), y(s), z(s)] = C$.

Differentiating above equation w.r.t 's' using the chain rule,

$$\frac{\partial\phi}{\partial x} \frac{dx}{ds} + \frac{\partial\phi}{\partial y} \frac{dy}{ds} + \frac{\partial\phi}{\partial z} \frac{dz}{ds} = 0$$

or

$$= \left(\frac{\partial\phi}{\partial x} \hat{i} + \frac{\partial\phi}{\partial y} \hat{j} + \frac{\partial\phi}{\partial z} \hat{k} \right) \cdot \left(\frac{dx}{ds} \hat{i} + \frac{dy}{ds} \hat{j} + \frac{dz}{ds} \hat{k} \right) = 0$$

or

$$\nabla\phi \cdot \frac{dr}{ds} = 0 \text{ or } \nabla\phi \cdot \hat{T} = 0$$

Which implies that $\nabla\varphi$ is a vector perpendicular to the unit tangent vector $\hat{T}$ and therefore on the surface $\varphi(x, y, z) = C$.

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Module No. 13

Related Problem 1; Gradient

Problem Statement

If $\vec{A} = 2x^2i - 3yzj + xz^2k$ and $\varphi = 2z - x^3y$

Find

- i. $\vec{A} \cdot \nabla \varphi$
- ii. $\vec{A} \times \nabla \varphi$

Solution

Since $\varphi = 2z - x^3y$ then

$$\begin{aligned}\nabla \varphi &= \frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \\ &= \frac{\partial (2z - x^3y)}{\partial x} \hat{i} + \frac{\partial (2z - x^3y)}{\partial y} \hat{j} + \frac{\partial (2z - x^3y)}{\partial z} \hat{k} \\ \nabla \varphi &= -3x^2yi - x^3j + 2k\end{aligned}$$

$\nabla \varphi$ at $(1, -1, 1)$

$$\nabla \varphi = 3i - j + 2k$$

Given vector $\vec{A} = 2x^2i - 3yzj + xz^2k$ at $(1, -1, 1)$

$$\vec{A} = 2i + 3j + k$$

- i. $\vec{A} \cdot \nabla \varphi = (2i + 3j + k) \cdot (3i - j + 2k)$
 $(2)(3) - (3)(1) + (1)(2) = 5$
- ii. $\vec{A} \times \nabla \varphi = (2i + 3j + k) \times (3i - j + 2k)$

$$\begin{aligned}&= \begin{vmatrix} i & j & k \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix} \\ &= (6 - (-1))i - (4 - 3)j + (-2 - 9)k \\ &= 7i - j - 11k\end{aligned}$$

Module No. 14

Related Problem 2; Gradient

Statement

Show that

$$\nabla f(r) = \frac{f'(r)\vec{r}}{r}$$

Proof

We know that

$$\begin{aligned}\nabla f(r) &= \frac{\partial}{\partial x} f(r)\hat{i} + \frac{\partial}{\partial y} f(r)\hat{j} + \frac{\partial}{\partial z} f(r)\hat{k} \\ &= \frac{\partial f}{\partial x} \frac{\partial r}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \frac{\partial r}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \frac{\partial r}{\partial z} \hat{k} \\ &= \frac{df}{dx} \frac{\partial r}{\partial x} \hat{i} + \frac{df}{dy} \frac{\partial r}{\partial y} \hat{j} + \frac{df}{dz} \frac{\partial r}{\partial z} \hat{k}\end{aligned}\quad (1)$$

since $r = \sqrt{x^2 + y^2 + z^2}$, it follows that

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

putting these values in (1), we get

$$\begin{aligned}&= f'(r) \frac{x}{r} \hat{i} + f'(r) \frac{y}{r} \hat{j} + f'(r) \frac{z}{r} \hat{k} \\ &= \frac{f'(r)}{r} (x\hat{i} + y\hat{j} + z\hat{k}) \\ &= \frac{f'(r)\vec{r}}{r}\end{aligned}$$

Hence Proved

Module No. 15

Related Problem 3; Gradient

Statement

If $\nabla\varphi = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$,

Find

$\varphi(x, y, z)$ if $\varphi(1, -2, 2) = 4$

Solution

Given equation is

$$\nabla\varphi = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k} \quad (1)$$

We know that

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} \quad (2)$$

therefore by comparing equation(1) and (2), we get

$$\frac{\partial\varphi}{\partial x} = 2xyz^3 \quad (3)$$

$$\frac{\partial\varphi}{\partial y} = x^2z^3 \quad (4)$$

$$\frac{\partial\varphi}{\partial z} = 3x^2yz^2 \quad (5)$$

Integrating Equation (3) w.r.t x , keeping y and z constants

$$\varphi = x^2yz^3 + f(y, z) \quad (6)$$

Similarly, from equation (4) and (5), we get

$$\varphi = x^2yz^3 + g(x, z) \quad (7)$$

$$\varphi = x^2yz^3 + h(x, y) \quad (8)$$

Comparison of equation (6), (7) and (8) shows that there will be a common value of φ if we choose

$$f(y, z) = g(x, z) = h(x, y) = C$$

where C is an arbitrary constant.

Thus

$$\varphi = x^2yz^3 + C \quad (9)$$

using $\varphi(1, -2, 2) = 4$ in equation (9) we get

$$C = 20$$

hence from equation (9), we obtained

$$\varphi = x^2yz^3 + 20$$

Which is the required solution of φ .

VU TOOLKIT

Module No. 16

Divergence of a Vector Point Function

Definition

Divergence is a vector operator, when we applied this operator to the quantity of a vector field, it produces a scalar field.

Let $\vec{V}(x, y, z) = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e. V defines a differentiable vector field). Then the divergence of V , written $\nabla \cdot V$ or $\text{div } V$, is defined by

$$\begin{aligned}\nabla \cdot \vec{V} &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (V_1\hat{i} + V_2\hat{j} + V_3\hat{k}) \\ &= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}\end{aligned}$$

Some Deductions

It is important to be noted that $\nabla \cdot \vec{V}$ is a scalar quantity. Also $\nabla \cdot \vec{V} \neq \vec{V} \cdot \nabla$

If $\vec{V}$ is a constant vector, then $\nabla \cdot \vec{V} = 0$.

if $\nabla \cdot \vec{V} = 0$ everywhere in some region of R , Then $\vec{V}$ is called **solenoid vector point function** in the region.

We can consider an example of solenoid to understand the concept.

Example

Determine the constant a so that the vector $\vec{V} = (x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}$ is solenoidal.

Solution

$$\begin{aligned}\nabla \cdot V &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot ((x + 3y)\hat{i} + (y - 2z)\hat{j} + (x + az)\hat{k}) \\ &= \frac{\partial(x + 3y)}{\partial x} + \frac{\partial(y - 2z)}{\partial y} + \frac{\partial(x + az)}{\partial z}\end{aligned}$$

$$= 1 + 1 + a = 2 + a$$

$$a = -2$$

Hence if we substitute $a = -2$ in given vector field then given $\vec{V}$ will become solenoidal.

VU TOOLKIT

Module No. 17

Properties of the Divergence

Statement

If $\vec{A}$ and $\vec{B}$ are differentiable vector point functions, and φ is differentiable scalar point function, then prove that

- i. $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$
- ii. $\nabla \cdot (\varphi \vec{A}) = \varphi (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \varphi)$

Proof

Let

$\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

i. $\nabla \cdot (\vec{A} + \vec{B}) = \nabla \cdot \vec{A} + \nabla \cdot \vec{B}$

$$L.H.S = (\vec{A} + \vec{B}) = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$$

hence

$$\begin{aligned} \nabla \cdot (\vec{A} + \vec{B}) &= \frac{\partial}{\partial x} (A_1 + B_1)\hat{i} + \frac{\partial}{\partial y} (A_2 + B_2)\hat{j} + \frac{\partial}{\partial z} (A_3 + B_3)\hat{k} \\ &= \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) + \left(\frac{\partial B_1}{\partial x} + \frac{\partial B_2}{\partial y} + \frac{\partial B_3}{\partial z} \right) \\ &= \nabla \cdot \vec{A} + \nabla \cdot \vec{B} = R.H.S \end{aligned}$$

ii. $\nabla \cdot (\varphi \vec{A}) = \varphi (\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \varphi)$

$$L.H.S = \nabla \cdot (\varphi \vec{A})$$

$$\text{We have } \varphi \vec{A} = \varphi A_1\hat{i} + \varphi A_2\hat{j} + \varphi A_3\hat{k}$$

Hence

$$\begin{aligned} \nabla \cdot (\varphi \vec{A}) &= \frac{\partial}{\partial x} (\varphi A_1)\hat{i} + \frac{\partial}{\partial y} (\varphi A_2)\hat{j} + \frac{\partial}{\partial z} (\varphi A_3)\hat{k} \\ &= \varphi \frac{\partial A_1}{\partial x} + A_1 \frac{\partial \varphi}{\partial x} + \varphi \frac{\partial A_2}{\partial y} + A_2 \frac{\partial \varphi}{\partial y} + \varphi \frac{\partial A_3}{\partial z} + A_3 \frac{\partial \varphi}{\partial z} \end{aligned}$$

$$\begin{aligned}
&= \varphi \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) + (A_1 \frac{\partial \varphi}{\partial x} + A_2 \frac{\partial \varphi}{\partial y} + A_3 \frac{\partial \varphi}{\partial z}) \\
&= \varphi(\nabla \cdot \vec{A}) + \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right) (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) \\
&\nabla \cdot (\varphi \vec{A}) = \varphi(\nabla \cdot \vec{A}) + (\nabla \varphi) \cdot \vec{A} = \varphi(\nabla \cdot \vec{A}) + \vec{A} \cdot (\nabla \varphi)
\end{aligned}$$

$\nabla \cdot (\varphi \vec{A}) = \varphi(\nabla \cdot \vec{A})$ if φ is constant.

VU TOOLKIT

Module No. 18

Laplacian

The Laplacian or Laplace operator is a second-order differential operator given by the divergence of the gradient of a given function defined over a space R . It is usually denoted by $\nabla \cdot \nabla$, or ∇^2 or Δ .

Thus if $\vec{u}$ is a twice differentiable function, then the Laplacian of $\vec{u}$ is defined by

$$\Delta \vec{u} = \nabla^2 \vec{u} = \nabla \cdot \nabla \vec{u}$$

In cartesian coordinate system, the Laplacian is given by the sum of second order partial derivatives of the function w.r.t each independent variable. Laplace is a Second order differential operator which is obtained by taking the divergence of gradient of any scalar point function.

It is denoted as

$$\begin{aligned} \nabla \cdot \nabla &= \nabla^2 = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \end{aligned}$$

It is a scalar operator.

In one and two dimension, the Laplace operator reduces to

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial x^2} \\ \nabla^2 &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \end{aligned}$$

If $\varphi(x, y, z)$ is a scalar point function, then the divergence of gradient of φ written as $\nabla \cdot \nabla \varphi = \nabla^2 \varphi$ is called the Laplacian of φ and the equation $\nabla^2 \varphi = 0$ is called Laplace's equation.

If a scalar function φ satisfies the Laplace equation $\nabla^2 \varphi = 0$ in a Cartesian region R , then φ is said to be a harmonic function in the region R .

Mathematically, we can write it as

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$$

We can also express the Laplace operator in polar coordinate (r, θ) notation.

Two dimensional Laplace operators can be expressed as

$$\begin{aligned}\nabla^2\varphi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2\varphi}{\partial\theta^2}\right) \\ &= \frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}\end{aligned}$$

The Laplacian occurs in differential equation that describes many physical phenomena, such as Diffusion equation for heat and fluid flow, gravitational potentials and quantum mechanics.

Module No. 19

Example of Divergence

Statement

If $\vec{A} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$, find $\nabla \cdot \vec{A}$ (or $\text{div } A$) at the point $(1, -1, 1)$.

Solution

As we know del operator ∇ is

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

Since given vector point function $\vec{A}$ is

$$\begin{aligned}\vec{A} &= x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k} \\ \nabla \cdot \vec{A} &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}) \\ &= \frac{\partial(x^2z)}{\partial x} + \frac{\partial(-2y^3z^2)}{\partial y} + \frac{\partial(xy^2z)}{\partial z} \\ &= 2xz - 6y^2z^2 + xy^2\end{aligned}$$

As we calculated the expression $\nabla \cdot \vec{A}$, we can easily determine its value at $(1, -1, 1)$.

$$\begin{aligned}\nabla \cdot \vec{A} \text{ at } (1, -1, 1) &= 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 \\ &= 2 - 6 + 1 = -3 \\ \nabla \cdot \vec{A} \text{ at } (1, -1, 1) &= -3\end{aligned}$$

which is the required Result.

Module No. 20

Related Problem 1: Divergence

Statement

If $\vec{A} = 3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}$ and $\varphi = 3x^2 - yz$,

find

- i. $\nabla \cdot \vec{A}$
- ii. $\vec{A} \cdot \nabla \varphi$
- iii. $\nabla \cdot (\varphi \vec{A})$
- iv. $\nabla \cdot (\nabla \varphi)$

at the point $(1, -1, 1)$.

Solution

Since $\vec{A} = 3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}$

also

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

then

- i. $\nabla \cdot \vec{A}$

$$\begin{aligned}\nabla \cdot \vec{A} &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}) \\ &= 3yz^2 + 6xy^2\hat{j} - x^2y\end{aligned}$$

$\nabla \cdot \vec{A}$ at point $(1, -1, 1)$ is

$$\begin{aligned}&= 3(-1)(1)^2 + 6(1)(-1)^2 - (1)^2(-1) \\ &= -3 + 6 - 1 = 2\end{aligned}$$

- ii. $\vec{A} \cdot \nabla \varphi$

Since $\varphi = 3x^2 - yz$

then

$$\begin{aligned}
\nabla\varphi &= \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} \\
&= \frac{\partial(3x^2 - yz)}{\partial x}\hat{i} + \frac{\partial(3x^2 - yz)}{\partial y}\hat{j} + \frac{\partial(3x^2 - yz)}{\partial z}\hat{k} \\
&= 6x\hat{i} - z\hat{j} - y\hat{k}
\end{aligned}$$

Now,

$$\begin{aligned}
\vec{A} \cdot \nabla\varphi &= (3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}) \cdot (6x\hat{i} - z\hat{j} - y\hat{k}) \\
&= (18x^2yz^2 - 2xy^3z + x^2y^2z)
\end{aligned}$$

$\vec{A} \cdot \nabla\varphi$ at point $(1, -1, 1)$ is

$$\begin{aligned}
&= [18(1)^2(-1)(1)^2 - 2(1)(-1)^3(1) + (1)^2(-1)^2(1)] \\
&= -18 + 2 + 1 = -15
\end{aligned}$$

iii. $\nabla \cdot (\varphi\vec{A})$

then

$$\begin{aligned}
\varphi\vec{A} &= (3x^2 - yz)(3xyz^2\hat{i} + 2xy^3\hat{j} - x^2yz\hat{k}) \\
&= (9x^3yz^2 - 3xy^2z^3)\hat{i} + (6x^3y^3 - 2xy^4z)\hat{j} + (3x^4yz - x^2y^2z^2)\hat{k}
\end{aligned}$$

Now

$$\begin{aligned}
\nabla \cdot (\varphi\vec{A}) &= \nabla \cdot ((9x^3yz^2 - 3xy^2z^3)\hat{i} + (6x^3y^3 - 2xy^4z)\hat{j} + (3x^4yz - x^2y^2z^2)\hat{k}) \\
&= \frac{\partial(9x^3yz^2 - 3xy^2z^3)}{\partial x} + \frac{\partial(6x^3y^3 - 2xy^4z)}{\partial y} + \frac{\partial(3x^4yz - x^2y^2z^2)}{\partial z} \\
&= 27x^2yz^2 - 3y^2z^3 + 18x^3y^2 - 8xy^3z + 3x^4y - 2x^4yz
\end{aligned}$$

$\nabla \cdot (\varphi\vec{A})$ at point $(1, -1, 1)$ is

$$\nabla \cdot (\varphi\vec{A}) = -27 - 3 + 18 + 8 - 3 + 2 = -5$$

iv. $\nabla \cdot (\nabla\varphi)$

$$\begin{aligned}
&= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(\frac{\partial(3x^2 - yz)}{\partial x}\hat{i} + \frac{\partial(3x^2 - yz)}{\partial y}\hat{j} + \frac{\partial(3x^2 - yz)}{\partial z}\hat{k} \right) \\
&= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot \left(\frac{\partial(3x^2 - yz)}{\partial x}\hat{i} + \frac{\partial(3x^2 - yz)}{\partial y}\hat{j} + \frac{\partial(3x^2 - yz)}{\partial z}\hat{k} \right)
\end{aligned}$$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (6x\hat{i} - z\hat{j} - y\hat{k}) = 6$$

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Module No. 21

Related Problem 2: Divergence

Problem Statement

Given

$$\varphi = 2x^3y^2z^4.$$

To find

- $\nabla \cdot \nabla \varphi$ (or $\text{div grad} \varphi$).
- Show that $\nabla \cdot \nabla \varphi = \nabla^2 \varphi$, where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ denotes the Laplacian operator.

Solution

i. $\nabla \cdot \nabla \varphi$

As we know the grad of φ

$$\begin{aligned}\nabla \varphi &= \frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \\ &= \frac{\partial(2x^3y^2z^4)}{\partial x} \hat{i} + \frac{\partial(2x^3y^2z^4)}{\partial y} \hat{j} + \frac{\partial(2x^3y^2z^4)}{\partial z} \hat{k} \\ &= 6x^2y^2z^4 \hat{i} + 4x^3yz^4 \hat{j} + 8x^3y^2z^3 \hat{k}\end{aligned}$$

Then the divergence of the grad of φ

$$\begin{aligned}\nabla \cdot \nabla \varphi &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot (6x^2y^2z^4 \hat{i} + 4x^3yz^4 \hat{j} + 8x^3y^2z^3 \hat{k}) \\ &= \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial z} \right) \\ &= \frac{\partial(6x^2y^2z^4)}{\partial x} + \frac{\partial(4x^3yz^4)}{\partial y} + \frac{\partial(8x^3y^2z^3)}{\partial z} \\ &= 12xy^2z^4 + 4x^3yz^4 + 24x^3y^2z^2\end{aligned}$$

ii. $\nabla \cdot \nabla \varphi = \nabla^2 \varphi$

L.H.S = $\nabla \cdot \nabla \varphi$

$$= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right)$$

$$\begin{aligned}
&= \frac{\partial}{\partial x} \left(\frac{\partial \varphi}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{\partial \varphi}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial \varphi}{\partial z} \right) \\
&= \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} \\
&= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \varphi \\
&= \nabla^2 \varphi
\end{aligned}$$

Hence proved.

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Module No. 22

Related Problem 3: Laplacian

Problem Statement

Show that

- i. $\nabla \cdot (r^3 \vec{r})$
- ii. $\nabla \cdot \left[\vec{r} \nabla \left(\frac{1}{r^3} \right) \right] = \frac{3}{r^4}$
- iii. $\nabla \cdot \left[\nabla \cdot \left(\frac{\vec{r}}{r} \right) \right] = -\frac{2\vec{r}}{r^3}$

where $\vec{r}$ is position vector.

Solution

First of all we will derive a general relation for $\nabla \cdot [\vec{r} f(r)]$ then we will substitute our given values to evaluate our required result.

$$\nabla \cdot [\vec{r} f(r)] = f(r)(\nabla \cdot \vec{r}) + \vec{r} \cdot (\nabla f(r))$$

$$\therefore \nabla \cdot \vec{r} = 3$$

We have

$$= 3f(r) + \vec{r} f'(r) \hat{r} = 3f(r) + \vec{r} \cdot \frac{f'(r) \vec{r}}{r}$$

$$\therefore \hat{r} = \frac{\vec{r}}{r}$$

$$= 3f(r) + \frac{f'(r)}{r} \vec{r} \cdot \vec{r}$$

$$\therefore \vec{r} \cdot \vec{r} = r^2$$

$$\nabla \cdot [\vec{r} f(r)] = 3f(r) + r f'(r)$$

Now setting $f(r) = r^n$ in above relation, we obtain

$$\nabla \cdot (\vec{r} r^n) = 3r^n + r(nr^{n-1})$$

$$\nabla \cdot (\vec{r} r^n) = 3r^n + (nr^n)$$

$$\nabla \cdot (r^n \vec{r}) = (3 + n)r^n$$

- i. We have $\nabla \cdot (r^3 \vec{r})$ to evaluate, it is of the form $\nabla \cdot [\vec{r} f(r)]$, where $\square(r) = r^3$

Put $n = 3$ in above relation, we will obtain the required result

$$\nabla \cdot (r^3 \vec{r}) = (3 + 3)r^3 = 6r^3$$

Is the required solution.

$$\text{ii. } L.H.S = \nabla \cdot \left[r \nabla \left(\frac{1}{r^3} \right) \right]$$

We can also write it as

$$= \nabla \cdot [r(\nabla r^{-3})] = \nabla \cdot [r(-3r^{-5}\vec{r})]$$

where we used the result

$$\nabla f(r) = \frac{f'(r)\vec{r}}{r}$$

We have $\nabla \cdot [r(-3r^{-5}\vec{r})]$ to evaluate, it is of the form $\nabla \cdot [\vec{r}f(r)]$, where $f(r) = r^{-4}\vec{r}$ thus,

$$\nabla \cdot \left[\vec{r} \nabla \left(\frac{1}{r^3} \right) \right] = -3 \nabla \cdot (r^{-4}\vec{r}) = -3(-4 + 3)r^{-4} = \frac{3}{r^4} = R.H.S$$

is the required solution.

$$\text{iii. } L.H.S = \nabla \cdot \left[\nabla \cdot \left(\frac{\vec{r}}{r} \right) \right]$$

first we evaluate $\nabla \cdot \left(\frac{\vec{r}}{r} \right)$ using formula $\nabla \cdot [\vec{r}f(r)]$, where $f(r) = r^{-1}\vec{r}$

$$\nabla \cdot (r^n \vec{r}) = (3 + n)r^n$$

put $n = 1$, we have

$$\nabla \cdot (r^{-1}\vec{r}) = (3 - 1)r^{-1} = \frac{2}{r}$$

therefore

$$= \nabla \cdot \left(\frac{2}{r} \right) = -2r^{-3}\vec{r} = -\frac{2\vec{r}}{r^3} = R.H.S$$

Module No. 23

Curl of a vector Point Function

Introduction

The infinitesimal rotation of a vector field is described by the curl. The curl of a specific point is shown by a vector at every point of the vector field. The attributes of this vector (length and direction) characterize the rotation at that point.

The direction of the curl is the axis of rotation, as determined by the right-hand rule, and the magnitude of the curl is the magnitude of rotation.

Definition

If $\vec{V}(x, y, z)$ is a differentiable vector field in a certain region of space, then the curl or rotation of V , written $\nabla \times \vec{V}$, curl V or rot V , is defined by

$$\nabla \times \vec{V} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \times (V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}).$$

Where V_1, V_2, V_3 are the components of vector field along x, y and z-axis.

We can write this expression in matrix form

$$\begin{aligned} \nabla \times \vec{V} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_2 & V_3 \end{vmatrix} \hat{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ V_1 & V_3 \end{vmatrix} \hat{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ V_1 & V_2 \end{vmatrix} \hat{k} \\ &= \left(\frac{\partial V_3}{\partial y} - \frac{\partial V_2}{\partial z} \right) \hat{i} - \left(\frac{\partial V_3}{\partial x} - \frac{\partial V_1}{\partial z} \right) \hat{j} + \left(\frac{\partial V_2}{\partial x} - \frac{\partial V_1}{\partial y} \right) \hat{k} \end{aligned}$$

A vector field whose curl is zero is called irrotational.

Identities of Curl

Some identities of curl are stated below

A gradient has zero curl:

$$\nabla \times \nabla \phi = 0.$$

A curl has zero divergence:

$$\nabla \cdot (\nabla \times \vec{V}) = 0$$

Note: $\nabla \cdot (\nabla \times \vec{V}) = (\nabla \times \nabla) \cdot \vec{V} = 0$ because $\nabla \times \nabla = 0$.

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Module No. 24

Properties of the Curl

If $\vec{A}$ and $\vec{B}$ are differentiable vector functions, and φ is differentiable scalar functions of position (x, y, z) , then

- i. $\nabla \times (\vec{A} + \vec{B}) = \nabla \times \vec{A} + \nabla \times \vec{B}$ or $\text{curl } (\vec{A} + \vec{B}) = \text{curl } \vec{A} + \text{curl } \vec{B}$
- ii. $\nabla \times (\varphi \vec{A}) = \varphi (\nabla \times \vec{A}) + (\nabla \varphi) \times \vec{A}$
- iii. $\nabla \times (\nabla \varphi) = 0$ The curl of the gradient of φ is zero.
- iv. $\nabla \cdot (\nabla \times \vec{A}) = 0$ The divergence of the curl of $\vec{A}$ is zero.

Proof

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{B} = B_1\hat{i} + B_2\hat{j} + B_3\hat{k}$, then

i. $\text{curl } (\vec{A} + \vec{B}) = \text{curl } \vec{A} + \text{curl } \vec{B}$

L.H. $S = \text{curl } (\vec{A} + \vec{B})$

$$\vec{A} + \vec{B} = (A_1 + B_1)\hat{i} + (A_2 + B_2)\hat{j} + (A_3 + B_3)\hat{k}$$

Hence by using the definition of curl

$$\begin{aligned} \nabla \times (\vec{A} + \vec{B}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 + B_1 & A_2 + B_2 & A_3 + B_3 \end{vmatrix} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_1 & B_2 & B_3 \end{vmatrix} \\ &= \nabla \times \vec{A} + \nabla \times \vec{B} \end{aligned}$$

Hence proved

ii. $\nabla \times (\varphi \vec{A}) = \varphi (\nabla \times \vec{A}) + (\nabla \varphi) \times \vec{A}$

L.H. $S = \nabla \times (\varphi \vec{A})$

$$\varphi \vec{A} = \varphi (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) = \varphi A_1\hat{i} + \varphi A_2\hat{j} + \varphi A_3\hat{k},$$

Then by using the definition of curl

$$\begin{aligned}
\nabla \times (\varphi \vec{A}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \varphi A_1 & \varphi A_2 & \varphi A_3 \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y} (\varphi A_3) - \frac{\partial}{\partial z} (\varphi A_2) \right) \hat{i} - \left(\frac{\partial}{\partial x} (\varphi A_3) - \frac{\partial}{\partial z} (\varphi A_1) \right) \hat{j} + \left(\frac{\partial}{\partial x} (\varphi A_2) - \frac{\partial}{\partial y} (\varphi A_1) \right) \hat{k} \\
&= \left[\varphi \frac{\partial(A_3)}{\partial y} + A_3 \frac{\partial(\varphi)}{\partial y} - \varphi \frac{\partial(A_2)}{\partial z} - A_2 \frac{\partial(\varphi)}{\partial z} \right] \hat{i} - \left[\varphi \frac{\partial(A_3)}{\partial x} + A_3 \frac{\partial(\varphi)}{\partial x} - \varphi \frac{\partial(A_1)}{\partial z} - A_1 \frac{\partial(\varphi)}{\partial z} \right] \hat{j} \\
&\quad + \left[\varphi \frac{\partial(A_2)}{\partial x} + A_2 \frac{\partial(\varphi)}{\partial x} - \varphi \frac{\partial(A_1)}{\partial y} - A_1 \frac{\partial(\varphi)}{\partial y} \right] \hat{k} \\
&= \varphi \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k} \right] + \left[(A_3 \frac{\partial \varphi}{\partial y} - A_2 \frac{\partial \varphi}{\partial z}) \hat{i} + (A_3 \frac{\partial \varphi}{\partial x} - A_1 \frac{\partial \varphi}{\partial z}) \hat{j} + (A_2 \frac{\partial \varphi}{\partial x} - A_1 \frac{\partial \varphi}{\partial y}) \hat{k} \right] \\
&= \varphi (\nabla \times \vec{A}) + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = \varphi (\nabla \times \vec{A}) + (\nabla \varphi) \times \vec{A}
\end{aligned}$$

Hence Proved

iii. $\nabla \times (\nabla \varphi) = 0$

$$\begin{aligned}
\text{L.H.S.} = \nabla \times (\nabla \varphi) &= \nabla \times \left(\frac{\partial \varphi}{\partial x} \hat{i} + \frac{\partial \varphi}{\partial y} \hat{j} + \frac{\partial \varphi}{\partial z} \hat{k} \right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} \\
&= \left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y} \right) \hat{i} - \left(\frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial^2 \varphi}{\partial x \partial z} \right) \hat{j} + \left(\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \right) \hat{k} \quad (1)
\end{aligned}$$

We assume that φ has continuous second order partial derivative so the order of differentiation can be neglected.

i.e.

$$\frac{\partial^2 \varphi}{\partial y \partial z} = \frac{\partial^2 \varphi}{\partial z \partial y}, \frac{\partial^2 \varphi}{\partial z \partial x} = \frac{\partial^2 \varphi}{\partial x \partial z}, \frac{\partial^2 \varphi}{\partial x \partial y} = \frac{\partial^2 \varphi}{\partial y \partial x}$$

$\Rightarrow$ equation (1) must be equal to zero

Hence the expression.

iv. $\nabla \cdot (\nabla \times \vec{A}) = 0$

$$\text{L.H.S.} = \nabla \cdot (\nabla \times \vec{A})$$

Since

$$\begin{aligned}\nabla \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \\ &= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k}\end{aligned}$$

$$\begin{aligned}\text{Hence } \nabla \cdot (\nabla \times \vec{A}) &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} - \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k} \\ &= \frac{\partial}{\partial x} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \\ &= \left(\frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} \right) - \left(\frac{\partial^2 A_3}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y \partial z} \right) + \left(\frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} \right) \\ &= \frac{\partial^2 A_3}{\partial x \partial y} - \frac{\partial^2 A_2}{\partial x \partial z} + \frac{\partial^2 A_3}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y \partial z} + \frac{\partial^2 A_2}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z \partial y} = 0\end{aligned}$$

Assuming that $\vec{A}$ has continuous second order derivative.

Hence the theorem.

Module No. 25

Example of Curl

Statement

If $\vec{A} = xz^3 \hat{i} - 2x^2yz \hat{j} + 2yz^4 \hat{k}$,

Find

$\nabla \times \vec{A}$ (or curl $\vec{A}$) at the point $(1, -1, 1)$.

Solution

$$\nabla \times \vec{A} = \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \times (xz^3 \hat{i} - 2x^2yz \hat{j} + 2yz^4 \hat{k})$$

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left(\frac{\partial(2yz^4)}{\partial y} - \frac{\partial(-2x^2yz)}{\partial z} \right) \hat{i} + \left(\frac{\partial(2yz^4)}{\partial x} - \frac{\partial(xz^3)}{\partial z} \right) \hat{j} + \left(\frac{\partial(-2x^2yz)}{\partial x} - \frac{\partial(xz^3)}{\partial y} \right) \hat{k}$$

$$\nabla \times \vec{A} = (2z^4 + 2x^2yz) \hat{i} + 3xz^2 \hat{j} - 4xyz \hat{k}$$

Now we calculate the value of $\nabla \times \vec{A}$ at $(1, -1, 1)$

$$\nabla \times \vec{A} = (2 - 2) \hat{i} + 3 \hat{j} + 4 \hat{k}$$

$$\nabla \times \vec{A} = 3 \hat{j} + 4 \hat{k}$$

is the required solution.

Module No. 26

Related Problem 1: Curl

Problem Statement

If $A = 2yz \, i - x^2y \, j + xz^2 \, k$ and $\varphi = 2x^2yz^3$,

Find

- $(A \times \nabla)\varphi$
- $A \times (\nabla\varphi)$

Also show whether they are identical or not.

Solution

Since $A = 2yz \, i - x^2y \, j + xz^2 \, k$ and $\varphi = 2x^2yz^3$

- $(A \times \nabla)\varphi$

$$(A \times \nabla)\varphi = \left[(2yz \, i - x^2y \, j + xz^2 \, k) \times \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \right] \varphi$$

$$(A \times \nabla)\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2yz & -x^2y & xz^2 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{vmatrix} \varphi$$

$$= \left[\left(-x^2y \frac{\partial}{\partial z} - xz^2 \frac{\partial}{\partial y} \right) \hat{i} - \left(2yz \frac{\partial}{\partial z} - xz^2 \frac{\partial}{\partial x} \right) \hat{j} + \left(2yz \frac{\partial}{\partial y} + x^2y \frac{\partial}{\partial x} \right) \hat{k} \right] \varphi$$

Now substituting φ

$$= \left[\left(-x^2y \frac{\partial \varphi}{\partial z} - xz^2 \frac{\partial \varphi}{\partial y} \right) \hat{i} - \left(2yz \frac{\partial \varphi}{\partial z} - xz^2 \frac{\partial \varphi}{\partial x} \right) \hat{j} + \left(2yz \frac{\partial \varphi}{\partial y} + x^2y \frac{\partial \varphi}{\partial x} \right) \hat{k} \right]$$

Since $\varphi = 2x^2yz^3$, substituting value

$$= \left[\left(-x^2y \frac{\partial(2x^2yz^3)}{\partial z} - xz^2 \frac{\partial(2x^2yz^3)}{\partial y} \right) \hat{i} - \left(2yz \frac{\partial(2x^2yz^3)}{\partial z} - xz^2 \frac{\partial(2x^2yz^3)}{\partial x} \right) \hat{j} + \left(2yz \frac{\partial(2x^2yz^3)}{\partial y} + x^2y \frac{\partial(2x^2yz^3)}{\partial x} \right) \hat{k} \right]$$

$$\begin{aligned}
&= \left[\{-x^2y(6x^2yz^2) - xz^2(2x^2z^3)\}\hat{i} - \{2yz(6x^2yz^2) - xz^2(4xyz^3)\}\hat{j} \right. \\
&\quad \left. + \{2yz(2x^2z^3) + x^2y(4xyz^3)\}\hat{k} \right] \\
(A \times \nabla)\varphi &= -(6x^4y^2z^2 + 2x^3z^5)\hat{i} - (12x^2y^2z^3 - 4x^2yz^5)\hat{j} + (4x^2yz^4 + 4x^3y^2z^3)\hat{k} \\
(1)
\end{aligned}$$

ii. $A \times (\nabla\varphi)$

Here $A = 2yz\hat{i} - x^2y\hat{j} + xz^2\hat{k}$ and $\varphi = 2x^2yz^3$

$$\begin{aligned}
\nabla\varphi &= \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} \\
&= \frac{\partial(2x^2yz^3)}{\partial x}\hat{i} + \frac{\partial(2x^2yz^3)}{\partial y}\hat{j} + \frac{\partial(2x^2yz^3)}{\partial z}\hat{k} \\
\nabla\varphi &= 4xyz^3\hat{i} + 2x^2z^3\hat{j} + 6x^2yz^2\hat{k}
\end{aligned}$$

Now

$$\begin{aligned}
A \times (\nabla\varphi) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2yz & -x^2y & xz^2 \\ 4xyz^3 & 2x^2z^3 & 6x^2yz^2 \end{vmatrix} \\
&= [(-x^2y)(6x^2yz^2) - (xz^2)(2x^2z^3)]\hat{i} - [(2yz)(6x^2yz^2) - (xz^2)(4xyz^3)]\hat{j} \\
&\quad + ((2yz)(2x^2z^3) - (-x^2y)(4xyz^3))\hat{k}
\end{aligned}$$

$$\begin{aligned}
A \times (\nabla\varphi) &= -(6x^4y^2z^2 + 2x^3z^5)\hat{i} - (12x^2y^2z^3 - 4x^2yz^5)\hat{j} + (4x^2yz^4 + 4x^3y^2z^3)\hat{k} \\
(2)
\end{aligned}$$

From equation (1) and (2), we conclude that

$$A \times (\nabla\varphi) = (A \times \nabla)\varphi$$

Module No. 27

Related Problem 2: Curl

Problem Statement

A vector $\vec{V}$ is called irrotational if $\text{curl } \vec{V} = 0$.

- i. Find constants a, b, c so that

$$\vec{A} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

is irrotational.

- ii. Show that $\vec{A}$ can be expressed as the gradient of a scalar function.

Solution

Since give vector field is

$$\vec{A} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

- i. Also for a vector to be irrotational, we have expression

$$\text{curl } \vec{A} = \nabla \times \vec{A} = 0$$

$$\begin{aligned} \nabla \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(4x + cy + 2z) - \frac{\partial}{\partial z}(bx - 3y - z) \right) \hat{i} - \left(\frac{\partial}{\partial x}(4x + cy + 2z) - \frac{\partial}{\partial z}(x + 2y + az) \right) \hat{j} \\ &\quad + \left(\frac{\partial}{\partial x}(bx - 3y - z) - \frac{\partial}{\partial y}(x + 2y + az) \right) \hat{k} \\ &= (c - 1)\hat{i} - (4 - a)\hat{j} + (b - 2)\hat{k} \end{aligned}$$

or

$$= (c - 1)\hat{i} + (a - 4)\hat{j} + (b - 2)\hat{k}$$

According to the given condition

$$(c - 1)\hat{i} + (a - 4)\hat{j} + (b - 2)\hat{k} = 0$$

$$\Rightarrow c - 1 = 0 \Rightarrow c = 1$$

$$\Rightarrow a - 4 = 0 \Rightarrow a = 4$$

$$\Rightarrow b - 2 = 0 \Rightarrow b = 2$$

and thus

$$\vec{A} = (x + 2y + 4z)\hat{i} + (2x - 3y - z)\hat{j} + (4x + y + 2z)\hat{k} \quad (1)$$

is irrotational vector field.

ii. Assume that

$$\vec{A} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} \quad (2)$$

By comparing equation (1) and (2), we obtain

$$\frac{\partial\varphi}{\partial x} = x + 2y + 4z \quad (3)$$

$$\frac{\partial\varphi}{\partial y} = 2x - 3y - z \quad (4)$$

$$\frac{\partial\varphi}{\partial z} = 4x + y + 2z \quad (5)$$

Integrating equation (3) w.r.t x , keeping y and z constants,

$$\varphi = \frac{x^2}{2} + 2xy + 4xz + f(y, z) \quad (6)$$

$$\varphi = 2xy - \frac{3y^2}{2} - yz + g(x, z) \quad (7)$$

$$\varphi = 4xz - yz + z^2 + h(x, y) \quad (8)$$

Comparison of equation (6), (7) and (8) shows that there will be a common value of φ if we choose

$$f(y, z) = -\frac{3y^2}{2} + z^2$$

$$g(x, z) = \frac{x^2}{2} + z^2$$

$$h(x, y) = \frac{x^2}{2} - \frac{3y^2}{2}$$

so that

$$\varphi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy + 4xz - yz + \text{constant}$$

Note that we can also add any constant to φ . In general if $\nabla \times \vec{A} = 0$, then we can find φ so that $\vec{A} = \nabla\varphi$. A vector field $\vec{A}$ which can be derived from a scalar field φ so that $\vec{A} = \nabla\varphi$ is called a conservative vector field and φ is called the scalar potential.

Note that conversely if $\vec{A} = \nabla\varphi$, then $\nabla \times \vec{A} = 0$

VU TOOLKIT

Module No. 28

Related Problem 3: Curl

Problem Statement

If $v = \omega \times r$, prove $\omega = \frac{1}{2} \text{curl } v$ where ω is a constant vector.

Solution

$$\begin{aligned}
 \text{curl } v &= \nabla \times v = \nabla \times (\omega \times r) \\
 &= \nabla \times \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = \nabla \times [(\omega_2 z - \omega_3 y)\hat{i} - (\omega_1 z - \omega_3 x)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}] \\
 &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix} \\
 &= \left(\frac{\partial}{\partial y} (\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z} (\omega_3 x - \omega_1 z) \right) \hat{i} - \left(\frac{\partial}{\partial x} (\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z} (\omega_2 z - \omega_3 y) \right) \hat{j} \\
 &\quad + \left(\frac{\partial}{\partial x} (\omega_3 x - \omega_1 z) - \frac{\partial}{\partial y} (\omega_2 z - \omega_3 y) \right) \hat{k} \\
 &= (\omega_1 + \omega_1)\hat{i} - (-\omega_2 - \omega_2)\hat{j} + (\omega_3 - (-\omega_3))\hat{k} \\
 &= 2(\omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k})
 \end{aligned}$$

$$\Rightarrow \nabla \times (\omega \times r) = 2\vec{\omega}$$

$$\Rightarrow \nabla \times v = 2\vec{\omega}$$

$$\Rightarrow \vec{\omega} = \frac{1}{2} \nabla \times v$$

$$\Rightarrow \vec{\omega} = \frac{1}{2} \text{curl } v$$

This problem indicates that the curl of a vector field is linked with rotational properties of the field. We might say that if $\text{curl } \vec{A} = 0$, then there would be no rotation and the field is called irrotational field. A field which is not irrotational is sometimes called a vortex field.

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Module No. 29

Vector Identities

Problem Statement

Evaluate $\nabla \cdot (\vec{A} \times \vec{r})$ if $\nabla \times \vec{A} = 0$.

Solution

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$$\begin{aligned}\vec{A} \times \vec{r} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_1 & A_2 & A_3 \\ x & y & z \end{vmatrix} \\ &= (A_2z - A_3y)\hat{i} - (A_1z - A_3x)\hat{j} + (A_1y - A_2x)\hat{k}\end{aligned}$$

And

$$\begin{aligned}\nabla \cdot (\vec{A} \times \vec{r}) &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot ((A_2z - A_3y)\hat{i} - (A_1z - A_3x)\hat{j} + (A_1y - A_2x)\hat{k}) \\ &= \frac{\partial}{\partial x}(A_2z - A_3y) - \frac{\partial}{\partial y}(A_1z - A_3x) + \frac{\partial}{\partial z}(A_1y - A_2x) \\ &= z \frac{\partial A_2}{\partial x} - y \frac{\partial A_3}{\partial x} - z \frac{\partial A_1}{\partial y} + x \frac{\partial A_3}{\partial y} + y \frac{\partial A_1}{\partial z} - x \frac{\partial A_2}{\partial z} \\ &= x \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) + y \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) + z \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \\ &= (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \left(\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right)\hat{i} + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right)\hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right)\hat{k} \right) \\ &= \vec{r} \cdot (\nabla \times \vec{A}) = \vec{r} \cdot \text{curl } \vec{A}\end{aligned}$$

If $\nabla \times \vec{A} = 0$ this reduces to zero.

Statement

Prove that

$$\nabla \times (\nabla \times \vec{A}) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

Proof

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a vector point function

Then

$$\text{L.H.S} = \nabla \times (\nabla \times \vec{A})$$

Since

$$\begin{aligned} \nabla \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \\ &= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k} \end{aligned}$$

$$\begin{aligned} \text{Hence } \nabla \times (\nabla \times \vec{A}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} & \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \end{vmatrix} \\ &= \left[\frac{\partial}{\partial y} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \right] \hat{i} - \left[\frac{\partial}{\partial x} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) - \frac{\partial}{\partial z} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \right] \hat{j} \\ &\quad + \left[\frac{\partial}{\partial x} \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) - \frac{\partial}{\partial y} \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \right] \hat{k} \\ &= \left(\frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_3}{\partial z \partial x} - \frac{\partial^2 A_1}{\partial z^2} \right) \hat{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} + \frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial y} - \frac{\partial^2 A_2}{\partial z^2} \right) \hat{j} \\ &\quad + \left(\frac{\partial^2 A_1}{\partial x \partial z} - \frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} + \frac{\partial^2 A_2}{\partial y \partial z} \right) \hat{k} \\ &= \left(-\frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right) \hat{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial z^2} \right) \hat{j} + \left(\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} \right) \hat{k} + \left(\frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x} \right) \hat{i} \\ &\quad + \left(\frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial y} \right) \hat{j} + \left(\frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z} \right) \hat{k} \end{aligned}$$

$$\begin{aligned}
&= \left(-\frac{\partial^2 A_1}{\partial x^2} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} \right) \hat{i} + \left(-\frac{\partial^2 A_2}{\partial x^2} - \frac{\partial^2 A_2}{\partial y^2} - \frac{\partial^2 A_2}{\partial z^2} \right) \hat{j} + \left(\frac{\partial^2 A_3}{\partial x^2} - \frac{\partial^2 A_3}{\partial y^2} - \frac{\partial^2 A_3}{\partial z^2} \right) \hat{k} \\
&\quad + \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x} \right) \hat{i} + \left(\frac{\partial^2 A_2}{\partial y^2} + \frac{\partial^2 A_1}{\partial x \partial y} + \frac{\partial^2 A_3}{\partial z \partial y} \right) \hat{j} \\
&\quad + \left(\frac{\partial^2 A_3}{\partial z^2} + \frac{\partial^2 A_1}{\partial x \partial z} + \frac{\partial^2 A_2}{\partial y \partial z} \right) \hat{k} \\
&= - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) + \frac{\partial}{\partial x} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \\
&\quad + \frac{\partial}{\partial y} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) + \frac{\partial}{\partial z} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \\
&= -\nabla^2 \vec{A} + \nabla \cdot \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) \\
&= -\nabla^2 \vec{A} + \nabla \cdot (\nabla \cdot \vec{A}) = \nabla \cdot (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}
\end{aligned}$$

Hence the result.

Problem Statement

Find $\text{curl}(\vec{r}f(r))$ where $f(r)$ is differentiable.

Solution

Let $\vec{r} = xi + yj + zk$ be the given vector
then

$$\begin{aligned}
\text{curl}(\vec{r}f(r)) &= \nabla \times (\vec{r}f(r)) \\
&= \nabla \times (xf(r)\hat{i} + yf(r)\hat{j} + zf(r)\hat{k}) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} \\
&= \left[\frac{\partial}{\partial y} zf(r) - \frac{\partial}{\partial z} yf(r) \right] \hat{i} + \left[\frac{\partial}{\partial z} xf(r) - \frac{\partial}{\partial x} zf(r) \right] \hat{j} + \left[\frac{\partial}{\partial x} yf(r) - \frac{\partial}{\partial y} xf(r) \right] \hat{k} \\
&= \left[z \frac{\partial f(r)}{\partial y} - y \frac{\partial f(r)}{\partial z} \right] \hat{i} + \left[x \frac{\partial f(r)}{\partial z} - z \frac{\partial f(r)}{\partial x} \right] \hat{j} \\
&\quad + \left[y \frac{\partial f(r)}{\partial x} - x \frac{\partial f(r)}{\partial y} \right] \hat{k} \tag{1}
\end{aligned}$$

As we know $r = \sqrt{x^2 + y^2 + z^2}$, therefore

$$\frac{\partial f(r)}{\partial x} = \frac{\partial f(r)}{\partial r} \frac{\partial r}{\partial x} = \frac{\partial f(r)}{\partial r} \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} = f'(r) \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{f'(r)x}{r}$$

Similarly,

$$\frac{\partial f(r)}{\partial y} = \frac{f'(r)y}{r}$$

And

$$\frac{\partial f(r)}{\partial z} = \frac{f'(r)z}{r}$$

By substituting these values in equation (1), we obtain

$$= \left[z \frac{f'(r)y}{r} - y \frac{f'(r)z}{r} \right] i + \left[x \frac{f'(r)z}{r} - z \frac{f'(r)x}{r} \right] j + \left[y \frac{f'(r)x}{r} - x \frac{f'(r)y}{r} \right] k = 0$$

Hence the result.

Module No. 30

Line Integral

An integral where the function is evaluated along a curve is called line integral. It is also named as path integral, curve integral, and curvilinear integral; contour integral as well. The function to be integrated may be a scalar field or a vector field. The value of the line integral is the sum of values of the field at all points on the curve, weighted by some scalar function on the curve. A line integral is a natural generalization of definite integral. Line integral can be transformed into double integral and surface integrals and vice versa.

The symbolic form of line integral is

$$\int_C \vec{A} \cdot d\vec{r}$$

Where $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$ is called the differential displacement vector. The integrals that involves differential displacement vector $d\vec{r}$ are called line integrals.

Line integral can also be expressed as

$$\begin{aligned} \int_C \vec{A} \cdot d\vec{r} &= \int_C (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \int_C (A_1dx + A_2dy + A_3dz) \end{aligned}$$

The line integral $\int_C \vec{A} \cdot d\vec{r}$ is sometimes called a scalar line integral of a vector field $\vec{A}$.

If C is a closed curve which we shall assume a simple closed curve (a curve which does not intersect itself anywhere), the line integral around C is denoted by

$$\oint_C \vec{A} \cdot d\vec{r} = \oint_C (A_1dx + A_2dy + A_3dz)$$

Applications

Some applications of line integrals are:

- i. If the vector field to be integrated $\vec{A}$ is the force $\vec{F}$ on a particle move along C, then this line integral show the work done by the force.

- ii. In fluid mechanics, if the vector field to be integrated $\vec{A}$ represents the velocity of some fluid then the line integral is called the circulation of $\vec{A}$ about C .
- iii. In general, we can say that any integral which is to be calculated along a curve is called a line integral.

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Module No. 31

Other forms and General Properties of line Integrals

Other forms of Line integrals

Generally, we define line integral by the vector function $\vec{A}$ by

$$\int_C \vec{A} \cdot d\vec{r}$$

As we studied previously, that the function to be integrated can be either scalar point function or vector point function.

Here, we will define line integral using scalar function,

$$\begin{aligned} \int_C \phi d\vec{r} &= \int_C \phi (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \int_C (\phi dx\hat{i} + \phi dy\hat{j} + \phi dz\hat{k}) \\ \int_C \phi \cdot d\vec{r} &= \hat{i} \int_C \phi dx + \hat{j} \int_C \phi dy + \hat{k} \int_C \phi dz \end{aligned}$$

Also the general line integral indicates the dot product (“.”) between the given vector field and the differential displacement vector. However we can also express it as a cross product (×) of both the vector according to our requirement.

$$\int_C \vec{A} \times d\vec{r} = \hat{i} \int_C (A_2 dz - A_3 dy) + \hat{j} \int_C (A_3 dx - A_1 dz) + \hat{k} \int_C (A_1 dy - A_2 dx)$$

Thus any integral that involves differential displacement vector $d\vec{r}$ are called line integrals.

General Properties of Line Integral

The following are the properties of the line integrals that are useful in computational subjects and application

- i. $\int_C K \vec{A} \cdot d\vec{r} = K \int_C \vec{A} \cdot d\vec{r}$ (K is any real constant)
- ii. $\int_C (\vec{A} + \vec{B}) \cdot d\vec{r} = \int_C \vec{A} \cdot d\vec{r} + \int_C \vec{B} \cdot d\vec{r}$

$$\text{iii. } \int_C \vec{A} \cdot d\vec{r} = \int_{C_1} \vec{A} \cdot d\vec{r} + \int_{C_2} \vec{A} \cdot d\vec{r}$$

Where the path C is subdivided into two arcs C_1 and C_2 that has the same orientations as C . If the sense of orientation along C is reversed, the value of the integral is multiplied by ‘-1’.

iv. If C is piece-wise smooth, consisting of smooth curves $C_1, C_2, \dots, C_n$ then the line integral of $\vec{A}$ over C is defined as the sum of the line integrals of $\vec{A}$ over each of the smooth curve making up C :

$$\int_C \vec{A} \cdot d\vec{r} = \int_{C_1} \vec{A} \cdot d\vec{r} + \int_{C_2} \vec{A} \cdot d\vec{r} + \dots + \int_{C_n} \vec{A} \cdot d\vec{r}$$

In the sum, the orientation along C must be maintained over curves $C_1, C_2, \dots, C_n$. That is the initial point of C_j is the terminal point of C_{j-1} .

Module No. 32

Example of Line Integral

Problem Statement

If $\varphi = 2xyz^2$, $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$ and C is the curve $x = t^2, y = 2t, z = t^3$ from $t = 0$ to $t = 1$.

Evaluate the line integrals

- i. $\int_C \varphi \cdot d\vec{r}$
- ii. $\int_C \vec{F} \times d\vec{r}$

Solution

- i. $\int_C \varphi \cdot d\vec{r}$

By substituting the given values of x, y, z, we obtain

$$\varphi = 2xyz^2 = 2(t^2)(2t)(t^3)^2 = 4t^9$$

and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\Rightarrow \vec{r} = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$$

Also $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\Rightarrow d\vec{r} = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$$

As we studied the relation

$$\begin{aligned} \int_C \varphi \cdot d\vec{r} &= \hat{i} \int_C \varphi dx + \hat{j} \int_C \varphi dy + \hat{k} \int_C \varphi dz \\ \Rightarrow \int_C \varphi d\vec{r} &= \int_0^1 4t^9(2t\hat{i} + 2\hat{j} + 3t^2\hat{k}) dt = \hat{i} \int_0^1 8t^{10} dt + \hat{j} \int_0^1 8t^9 dt + \hat{k} \int_0^1 12t^{11} dt \\ &= \frac{8}{11}\hat{i} + \frac{8}{10}\hat{j} + \hat{k} \end{aligned}$$

Hence the required solution.

- ii. $\int_C \vec{F} \times d\vec{r}$

Since $\vec{F} = xy\hat{i} - z\hat{j} + x^2\hat{k}$, first of all we will calculate the value of F using parametric equations of x,y,z

$$\vec{F} = 2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}$$

Also we calculated, $d\vec{r} = (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})dt$

Then

$$\vec{F} \times d\vec{r} = (2t^3\hat{i} - t^3\hat{j} + t^4\hat{k}) \times (2t\hat{i} + 2\hat{j} + 3t^2\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t^3 & -t^3 & t^4 \\ 2t & 2 & 3t^2 \end{vmatrix} = (-3t^5 - 2t^4)\hat{i} - (6t^5 - 2t^5)\hat{j} + (4t^3 + 2t^4)\hat{k}$$

Using the expression

$$\int_C \vec{A} \times d\vec{r} = \hat{i} \int_C (A_2 dz - A_3 dy) + \hat{j} \int_C (A_3 dx - A_1 dz) + \hat{k} \int_C (A_1 dy - A_2 dx)$$

Substituting values, we get

$$\begin{aligned} \int_C \vec{F} \times d\vec{r} &= \hat{i} \int_0^1 (-3t^5 - 2t^4)dt - \hat{j} \int_0^1 (6t^5 - 2t^5)dt + \hat{k} \int_0^1 (4t^3 + 2t^4)dt \\ &= -\frac{9}{10}\hat{i} - \frac{2}{3}\hat{j} + \frac{7}{5}\hat{k} \end{aligned}$$

Module No. 33

Example of Line Integral

Problem statement

Find the total work done in moving a particle in a force field given by $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1, y = 2t^2, z = t^3$ from $t = 1$ to $t = 2$.

Solution

Since $\vec{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$,

Substituting values $x = t^2 + 1, y = 2t^2, z = t^3$, in $\vec{F}$.

$$\vec{F} = 3(t^2 + 1)(2t^2)\hat{i} - 5(t^3)\hat{j} + 10(t^2 + 1)\hat{k}$$

$$\vec{F} = (6t^4 + 6t^2)\hat{i} - 5(t^3)\hat{j} + (10t^2 + 10)\hat{k}$$

Also, $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\Rightarrow d\vec{r} = (2t\hat{i} + 4t\hat{j} + 3t^2\hat{k})dt$$

Total work done $= \int_C \vec{F} \cdot d\vec{r} =$

$$= \int_1^2 [(6t^4 + 6t^2)\hat{i} - 5(t^3)\hat{j} + (10t^2 + 10)\hat{k}] \cdot (2t\hat{i} + 4t\hat{j} + 3t^2\hat{k})dt$$

$$= \int_1^2 2t(6t^4 + 6t^2) - 4t(5t^3) + 3t^2(10t^2 + 10)dt$$

$$= \int_1^2 (12t^5 + 12t^3) - (20t^4) + (30t^4 + 30t^2)dt$$

$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)dt$$

$$= \int_1^2 12t^5 dt + \int_1^2 10t^4 dt + \int_1^2 12t^3 dt + \int_1^2 30t^2 dt$$

$$= \left| \frac{12t^6}{6} \right|_1^2 + \left| \frac{10t^5}{5} \right|_1^2 + \left| \frac{12t^4}{4} \right|_1^2 + \left| \frac{30t^3}{3} \right|_1^2$$

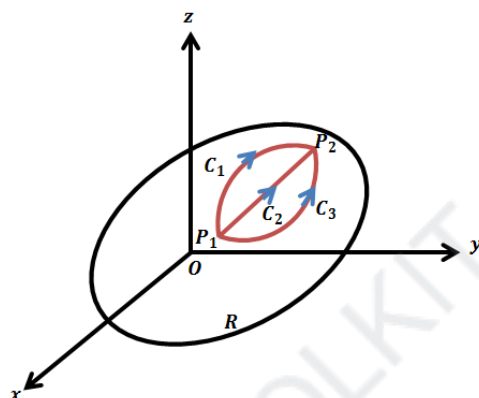
$$\begin{aligned} &= 2[(2)^6 - (1)^6] + 2[(2)^5 - (1)^5] + 3[(2)^4 - (1)^4] + 10[(2)^3 - (1)^3] \\ &= 303 \end{aligned}$$

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Module No. 34

Line Integral Dependent on Path

The value of line integrals $\int_C \vec{A} \cdot d\vec{r}$ generally depends not only on the end points P_1 and P_2 of the curve C but also the geometric shape of the path C i.e. if we integrate from point P_1 to P_2 along different paths, we in general, obtain different values of the integral.



We will illustrate this concept by the following example

If $A = (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k}$, evaluate $\int_C \vec{A} \cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths C

- The straight lines from $(0,0,0)$ to $(1,0,0)$, then to $(1,1,0)$, and then to $(1,1,1)$.
- The straight line joining $(0,0,0)$ and $(1,1,1)$.

Solution

- Along the straight line from $(0,0,0)$ to $(1,0,0)$ $y = 0, z = 0, dy = 0, dz = 0$ while x varies from 0 to 1. Then

the integral over this part of the path is

$$\begin{aligned} \int_C \vec{A} \cdot d\vec{r} &= \int_C (A_1 dx + A_2 dy + A_3 dz) \\ &= \int_0^1 3x^2 + 6(0) dx - 14y(0)dy + 20x(0)^2 dz = \int_0^1 3x^2 dx = 1 \end{aligned}$$

Along the straight line from $(1,0,0)$ to $(1,1,0)$ $x = 1, z = 0, dx = 0, dz = 0$ while y varies from 0 to 1.

Then the integral over this part of the path is

$$\int_0^1 3(1)^2 + 6y \, dx - 14y(0)dy + 20x(0)^2 = \int_0^1 3 + 6y \, dx = 0$$

Along the straight line from $(1,1,0)$ to $(1,1,1)$ $x = 1, y = 1, dx = 0, dy = 0$ while z varies from 0 to 1.

Then the integral over this part of the path is

$$\int_0^1 3(1)^2 + 6(1)(0) - 14(1)z(0) + 20(1)z^2 \, dz = \int_0^1 20z^2 \, dz = \frac{20}{3}$$

By adding all the results, we have

$$\int_0^1 \vec{A} \cdot d\vec{r} = 1 + 0 + \frac{20}{3} = \frac{23}{3}$$

- ii. The straight line joining $(0,0,0)$ and $(1,1,1)$ is given in parametric form by $x = t, y = t, z = t$. Then

$$\begin{aligned} \int_C \vec{A} \cdot d\vec{r} &= \int_C (3x^2 + 6y)\hat{i} - 14yz\hat{j} + 20xz^2\hat{k} \cdot d\vec{r} \\ &= \int_0^1 (3t^2 + 6t) - 14t^2 + 20t^3 \, dt = \int_0^1 6t - 11t^2 + 20t^3 \, dt = \frac{13}{3} \end{aligned}$$

In example, we discuss two cases here. In each case the end points of the curve were same but we chose different path to obtain results. The results obtained by both the cases are different. Hence we conclude that the line integral not only depends on the end points of the curve but also on the path taken by the particle.

Module No. 35

Independence of Path

Definition

The line integral $\int_C \vec{V} \cdot d\vec{r}$ is said to be independent of the path C in a given region R , if the value of the line integral $\int_{P_1}^{P_2} \vec{V} \cdot d\vec{r}$ is the same for all paths C joining any two points P_1 and P_2 in R .

Theorem

Prove that a necessary and sufficient condition for $\int_{P_1}^{P_2} \vec{V} \cdot d\vec{r}$ to be independent of the path joining any two points P_1 and P_2 (i.e. $\vec{V}$ to be conservative) in a given region is that $\oint_C \vec{V} \cdot d\vec{r} = 0$ for all closed path C in the region.

Proof

Let C be any simple closed curve, and let P_1 and P_2 be any two points on C as shown.

Then since by the supposition, the integral is independent of path (i.e. $\vec{V}$ to be conservative), we have

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} = \int_{P_1 B P_2} \vec{V} \cdot d\vec{r}$$

Reversing the direction of integration in the integral on the right, we have

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} = - \int_{P_2 B P_1} \vec{V} \cdot d\vec{r}$$

or

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} + \int_{P_2 B P_1} \vec{V} \cdot d\vec{r} = 0$$

or

$$\oint_C \vec{V} \cdot d\vec{r} = 0$$

Conversely, if $\oint_C \vec{V} \cdot d\vec{r} = 0$,

then

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} + \int_{P_2 B P_1} \vec{V} \cdot d\vec{r} = 0$$

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} = - \int_{P_2 B P_1} \vec{V} \cdot d\vec{r}$$

Again changing the direction in the integral, we have

$$\int_{P_1 A P_2} \vec{V} \cdot d\vec{r} = \int_{P_1 B P_2} \vec{V} \cdot d\vec{r}$$

Which shows the line integral is independent of the path joining P_1 and P_2 as required.

Module No. 36

Theorems on line Integral

Scalar Potential Function

A scalar potential function φ is a single-valued function for which there exists a continuous vector field $\vec{V}$ in a simply connected region R that satisfies the relation $\vec{V} = \nabla\varphi$.

Theorem 1 Statement

Prove that a necessary and sufficient condition for $\int_{P_1}^{P_2} \vec{V} \cdot d\vec{r}$ to be independent of the path joining any two points $P_1(x, y, z)$ and $P_2(x, y, z)$ (i.e. $\vec{V}$ to be conservative) is that there exist a scalar function φ such that $\vec{V} = \nabla\varphi$, where φ is single valued and has continuous partial derivatives.

Proof

Let $\vec{V} = \nabla\varphi$, then

$$\begin{aligned} \int_C \vec{V} \cdot d\vec{r} &= \int_{P_1}^{P_2} \vec{V} \cdot d\vec{r} = \int_{P_1}^{P_2} \nabla\varphi \cdot d\vec{r} \\ &= \int_{P_1}^{P_2} \left(\frac{\partial\varphi}{\partial x} \hat{i} + \frac{\partial\varphi}{\partial y} \hat{j} + \frac{\partial\varphi}{\partial z} \hat{k} \right) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \int_{P_1}^{P_2} \left(\frac{\partial\varphi}{\partial x} dx + \frac{\partial\varphi}{\partial y} dy + \frac{\partial\varphi}{\partial z} dz \right) \\ &= \int_{P_1}^{P_2} d\varphi = \varphi(P_2) - \varphi(P_1) = \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1) \end{aligned}$$

Thus the line integrals only depends only on points P_1 and P_2 and not on the path joining them i.e. $\vec{V}$ is conservative.

Conversely, let $\int_C \vec{V} \cdot d\vec{r}$ be the independent of the path C joining any two points, We choose these points as a fixed point $P_1 \equiv (x_1, y_1, z_1)$ and a variable point $P_2 \equiv (x, y, z)$, so that the result is a function of only of the coordinates (x, y, z) of the variable end points. Then

$$\varphi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x, y, z)} \vec{V} \cdot d\vec{r} = \int_{(x_1, y_1, z_1)}^{(x, y, z)} \vec{V} \cdot \frac{d\vec{r}}{ds} ds$$

By taking derivative, we have

$$\frac{d\varphi}{ds} = \vec{V} \cdot \frac{d\vec{r}}{ds} \quad (1)$$

But

$$\frac{d\varphi}{ds} = \frac{\partial \varphi}{\partial s} = \nabla \varphi \cdot \frac{d\vec{r}}{ds} \quad (2)$$

From (1) and (2) we have

$$\vec{V} \cdot \frac{d\vec{r}}{ds} = \nabla \varphi \cdot \frac{d\vec{r}}{ds} = (\vec{V} - \nabla \varphi) \cdot \frac{d\vec{r}}{ds} = 0$$

Since $\frac{d\vec{r}}{ds}$ is a unit tangent vector, therefore $\frac{d\vec{r}}{ds} \neq 0$

$$\Rightarrow \vec{V} - \nabla \varphi = 0$$

Or

$$\vec{V} = \nabla \varphi$$

Hence Proved.

Theorem 2 Statement

Prove that a necessary and sufficient condition that a vector field $\vec{V}$ be conservative is that $\nabla \times \vec{V} = 0$ (i.e. $\vec{V}$ is irrotational).

Proof

If $\vec{V}$ is conservative field then by the previous theorem, we have $\vec{V} = \nabla \varphi$.

Thus

$$\nabla \times \vec{V} = \nabla \times \nabla \varphi = 0$$

Conversely, if $\nabla \times \vec{V} = 0$, then

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = 0$$

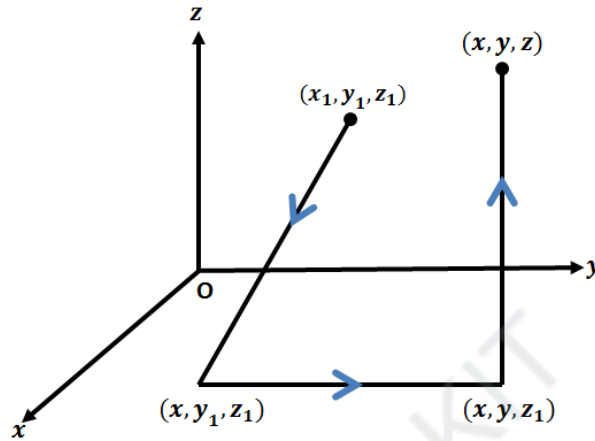
and thus

$$\frac{\partial V_3}{\partial y} = \frac{\partial V_2}{\partial z}, \frac{\partial V_1}{\partial z} = \frac{\partial V_3}{\partial x}, \frac{\partial V_2}{\partial x} = \frac{\partial V_1}{\partial y}$$

we must prove that $\vec{V} = \nabla\phi$ follows as a consequence of this,

Now,

$$\int_C \vec{V} \cdot d\vec{r} = \int_C (V_1(x, y, z)dx + V_2(x, y, z)dy + V_3(x, y, z)dz)$$



where C is path joining (x_1, y_1, z_1) and (x_2, y_2, z_3) . Let us choose as a particular path, the straight line segment from (x_1, y_1, z_1) to (x, y_1, z_1) to (x, y, z) and $\phi(x, y, z)$ the value of the integral along this particular path. Then omitting the integrand, we have

$$\phi(x, y, z) = \int_{(x_1, y_1, z_1)}^{(x, y_1, z_1)} [] + \int_{(x, y_1, z_1)}^{(x, y, z_1)} [] + \int_{(x, y, z_1)}^{(x, y, z)} [] \quad (1)$$

- i. Along the straight line (x_1, y_1, z_1) to (x, y_1, z_1) , $y = \text{constant} = y_1, z = \text{constant} = z_1$, so that $dy = 0, dz = 0$, while x varies from x_1 to x .
- ii. Along the straight line (x, y_1, z_1) to (x, y, z_1) , $x = \text{constant}, z = \text{constant}$, so that $dx = 0, dz = 0$, while y varies from y_1 to y .
- iii. Along the straight line (x, y, z_1) to (x, y, z) , $x = \text{constant}, y = \text{constant}$, so that $dx = 0, dy = 0$, while z varies from z_1 to z .

Thus we can write equation (1) as

$$\phi(x, y, z) = \int_{x_1}^x V_1(x, y_1, z_1)dx + \int_{y_1}^y V_2(x, y, z_1)dy + \int_{z_1}^z V_3(x, y, z)dz$$

it follows that

$$\begin{aligned}
 \frac{\partial \phi}{\partial z} &= V_3(x, y, z) \\
 \frac{\partial \phi}{\partial y} &= V_2(x, y, z_1) + \int_{z_1}^z \frac{\partial V_3}{\partial y}(x, y, z) dz \\
 &= V_2(x, y, z_1) + \int_{z_1}^z \frac{\partial V_2}{\partial z}(x, y, z) dz = V_2(x, y, z_1) + |V_2(x, y, z)|_{z_1}^z = V_2(x, y, z) \\
 \frac{\partial \phi}{\partial x} &= V_1(x, y_1, z_1) + \int_{y_1}^y \frac{\partial V_2}{\partial x}(x, y, z_1) dy + \int_{z_1}^z \frac{\partial V_3}{\partial x}(x, y, z) dz \\
 &= V_1(x, y_1, z_1) + \int_{y_1}^y \frac{\partial V_1}{\partial y}(x, y, z_1) dy + \int_{z_1}^z \frac{\partial V_1}{\partial z}(x, y, z) dz \\
 &= V_1(x, y_1, z_1) + |V_1(x, y, z_1)|_{y_1}^y + |V_1(x, y, z_1)|_{z_1}^z = V_1(x, y, z) \\
 &= V_1(x, y_1, z_1) + V_1(x, y, z_1) - V_1(x, y_1, z_1) + V_1(x, y, z) - V_1(x, y, z_1) \\
 \frac{\partial \phi}{\partial x} &= V_1(x, y, z)
 \end{aligned}$$

Then we have,

$$\vec{V} = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k} = \frac{\partial \phi}{\partial x} \hat{i} + \frac{\partial \phi}{\partial y} \hat{j} + \frac{\partial \phi}{\partial z} \hat{k} = \nabla \phi$$

hence $\vec{V}$ is conservative i.e $\vec{V} = \nabla \phi$.

Module No. 37

Selected Example/Problem 1

Problem Statement

If $\vec{A}(t) = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{B}(t) = 2t^2\hat{i} + 6t\hat{k}$,

Evaluate

- i. $\int_0^2 \vec{A} \cdot \vec{B} dt$
- ii. $\int_0^2 \vec{A} \times \vec{B} dt$

Solution

i. $\int_0^2 \vec{A} \cdot \vec{B} dt$

Since $\vec{A}(t) = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{B}(t) = 2t^2\hat{i} + 6t\hat{k}$

So,

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (t\hat{i} - t^2\hat{j} + (t-1)\hat{k}) \cdot (2t^2\hat{i} + 0\hat{j} + 6t\hat{k}) \\ &= 2t^3 + 6t(t-1)\end{aligned}$$

Now

$$\begin{aligned}\int_0^2 \vec{A} \cdot \vec{B} dt &= \int_0^2 [2t^3 + 6t(t-1)] dt \\ &= \int_0^2 2t^3 dt + \int_0^2 6t^2 dt - \int_0^2 6t dt \\ &= \left[\frac{t^4}{2} + 2t^3 - 3t^2 \right]_0^2 \\ &= 8 + 16 - 12 = 12\end{aligned}$$

Hence

$$\int_0^2 \vec{A} \cdot \vec{B} dt = 12$$

is required solution.

ii. $\int_0^2 \vec{A} \times \vec{B} dt$

$$\begin{aligned}\vec{A} \times \vec{B} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix} \\ &= [(-t^2)(6t) - (t-1)(0)]\hat{i} - [(t)(6t) - (t-1)(2t^2)]\hat{j} + [(t)(0) - (-t^2)(2t^2)]\hat{k} \\ &= -6t^3\hat{i} - (6t^2 + 2t^2 - 2t^3)\hat{j} + 2t^4\hat{k} \\ &= -6t^3\hat{i} - (8t^2 - 2t^3)\hat{j} + 2t^4\hat{k}\end{aligned}$$

Now,

$$\begin{aligned}\int_0^2 \vec{A} \times \vec{B} dt &= \int_0^2 (-6t^3\hat{i} - (8t^2 - 2t^3)\hat{j} + 2t^4\hat{k}) dt \\ &= -\hat{i} \int_0^2 6t^3 dt - \hat{j} \int_0^2 (8t^2 - 2t^3) dt + \hat{k} \int_0^2 2t^4 dt \\ &= \left| -\hat{i} \frac{3}{2} t^4 - \hat{j} \left(\frac{8}{3} t^3 - \frac{t^4}{2} \right) + \hat{k} \frac{2}{5} t^5 \right|_0^2 \\ &= -\hat{i} \frac{3(16)}{2} - \hat{j} \left(\frac{8(8)}{3} - \frac{16}{2} \right) + \hat{k} \frac{2(32)}{5} \\ &= -\hat{i} 24 - \hat{j} \frac{80}{6} + \hat{k} \frac{64}{5} \\ \int_0^2 \vec{A} \times \vec{B} dt &= -24\hat{i} - \frac{40}{3}\hat{j} + \frac{64}{5}\hat{k}\end{aligned}$$

is the required result.

Module No. 38

Selected Example/Problem 2: Line Integrals

Problem Statement

If $F = 3xy\hat{i} - y^2\hat{j}$, evaluate work done by the force on the curve C in the xy plane, $y = 2x^2$, from (0,0) to (1,2).

Solution

As we know total work done on a curve is

$$\int_C \vec{F} \cdot d\vec{r}$$

Also,

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

By substituting

$$\begin{aligned} &= \int_0^1 (3xy\hat{i} - y^2\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) \\ &= \int_0^1 (3xy \, dx - y^2 \, dy) \end{aligned}$$

As $y = 2x^2$ is given, this implies $dy = 4x \, dx$, now substituting these values in above integral, we obtain

$$\begin{aligned} &= \int_0^1 (3x(2x^2) \, dx - (2x^2)^2 4x \, dx) \\ &= \int_0^1 6x^3 \, dx - \int_0^1 16x^5 \, dx \\ &= \left[\frac{3}{2}x^4 - \frac{8}{3}x^6 \right]_0^1 \\ &= \frac{3}{2} - \frac{8}{3} = \frac{-7}{6} \end{aligned}$$

Note that if the curve were traversed in the opposite sense, i.e. from (1,2) to (0,0), the value of the integral would have been $7/6$ instead of $-7/6$.

Module No. 39

Selected Example/Problem 3: Line Integrals

Problem Statement

Find the work done in moving a particle once around a circle C in the xy plane, if the circle has center at the origin and radius 3 and if the force field is given by

$$\vec{F} = (2x - y + z)\hat{i} + (x + y - z^2)\hat{j} + (3x - 2y + 4z)\hat{k}$$

Solution

In the plane $z = 0$, $\vec{F} = (2x - y)\hat{i} + (x + y)\hat{j}$ and $d\vec{r} = dx\hat{i} + dy\hat{j}$

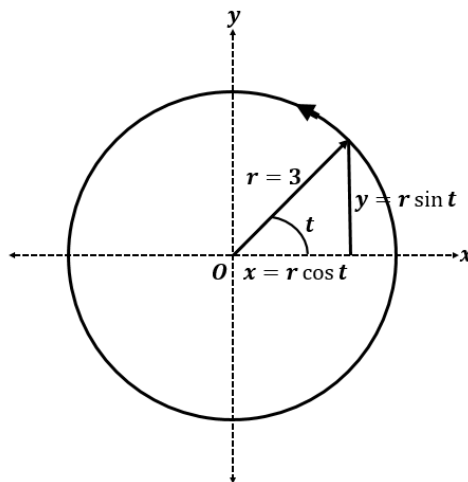
So, that the work done on the curve is

$$\begin{aligned} \text{Total work done} &= \int_C \vec{F} \cdot d\vec{r} \\ &= \int_C ((2x - y)\hat{i} + (x + y)\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) \\ &= \int_C ((2x - y)dx + (x + y)dy) \end{aligned}$$

As radius of the circle is $r = 3$, we choose the parametric equations of the circle as

$$x = 3 \cos t \Rightarrow dx = -3 \sin t \, dt$$

$$y = 3 \sin t \Rightarrow dy = 3 \cos t \, dt$$



where t varies from 0 to 2π . Then the line integral

$$\begin{aligned}
 &= \int_0^{2\pi} ((2(3 \cos t) - 3 \sin t)(-3 \sin t) dt) + (3 \cos t + 3 \sin t)(3 \cos t) dt \\
 &= \int_0^{2\pi} (6 \cos t - 3 \sin t)(-3 \sin t) + 9(\cos t)^2 + 9 \sin t \cos t \, dt \\
 &= \int_0^{2\pi} -18 \cos t \sin t + 9 \sin^2 t + 9 \cos^2 t + 9 \sin t \cos t \, dt \\
 &= \int_0^{2\pi} 9 - 9 \cos t \sin t \, dt \\
 &= \left| 9t - \frac{9}{2} \sin^2 t \right|_0^{2\pi} \\
 &= 18\pi
 \end{aligned}$$

In traversing C we have chosen the counterclockwise direction indicated in the adjoining figure. We call this the positive direction, or say that C has been traversed in the positive sense. If C were traversed in the clockwise (negative) direction the value of the integral would be -18π .

Module No. 40

Selected Example/Problem 4: Line Integral

Problem Statement

Given a force vector $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$.

- Show that $\vec{F}$ a conservative force field
- Find the scalar potential
- Find the work done in moving an object in this field from $(1, -2, 1)$ to $(3, 1, 4)$.

Solution

- We derived a necessary and sufficient condition that a force will be conservative is that

$$\text{curl}\vec{F} = \nabla \times \vec{F} = 0.$$

Now

$$\begin{aligned}\text{curl}\vec{F} &= \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} \\ &= \left(\frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(x^2) \right) \hat{i} - \left(\frac{\partial}{\partial x}(3xz^2) - \frac{\partial}{\partial z}(2xy + z^3) \right) \hat{j} \\ &\quad + \left(\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + z^3) \right) \hat{k} \\ &= 0\hat{i} - (3z^2 - 3z^2)\hat{j} + (2x - 2x)\hat{k} = 0 \\ \text{curl}\vec{F} &= 0\end{aligned}$$

Hence $\vec{F}$ is conservative.

- As $\vec{F}$ is conservative, so $\vec{F} = \nabla\varphi$

and

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k}$$

this implies=

$$\vec{F} = \nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}\hat{k} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$$

By comparing, we get

$$\frac{\partial\varphi}{\partial x} = 2xy + z^3 \quad (1)$$

$$\frac{\partial\varphi}{\partial y} = x^2 \quad (2)$$

$$\frac{\partial\varphi}{\partial z} = 3xz^2 \quad (3)$$

By integrating equation (1), (2) and (3), we get

$$\varphi = x^2y + xz^3 + f(y, z)$$

$$\varphi = x^2y + g(x, z)$$

$$\varphi = xz^3 + h(x, y)$$

These equation will agree if we choose

$$f(y, z) = 0, g(x, z) = xz^3 \text{ and } h(x, y) = x^2y$$

so that $\varphi = x^2y + xz^3$ to which may be added any constant.

iii. From part (ii), we have

$$\varphi = x^2y + xz^3 + K$$

Where K is any constant,

$$\text{Then work done} = \varphi(P_2) - \varphi(P_1)$$

$$= \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1) \quad (4)$$

$$\varphi(x_2, y_2, z_2) = \varphi(3, 1, 4) = (3)^2(1) + 3(4)^3 = 9 + 192 = 201$$

$$\varphi(x_1, y_1, z_1) = \varphi(1, -2, 1) = (1)^2(-2) + 1(1)^3 = -2 + 1 = -1$$

Now substituting these values in equation (4)

$$\text{Work done} = \varphi(P_2) - \varphi(P_1) = \varphi(x_2, y_2, z_2) - \varphi(x_1, y_1, z_1) = 201 - (-1) = 202$$

202 is the required work done.

Module No. 41

Surface Integral

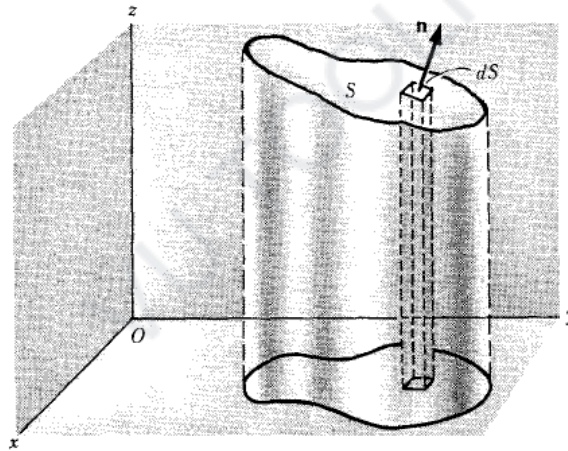
Definition

Integrals that involves the differential surface elements $d\vec{S}$ are called surface integrals.

Mathematically, we can represents it as

$$\iint_S \vec{A} \cdot d\vec{S}$$

Let S be a two-sided surface (open or closed), and $\vec{A}(x, y, z) = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ be a defined and continuous vector function over the surface S . Let one side of S be considered arbitrarily as the positive side (if S is a closed surface this is taken as the outer side). A unit normal $\hat{n}$ to any point of the positive side of S is called a positive or outward drawn unit normal.



[1]

Associate with the differential of surface area dS a vector $d\vec{S}$ whose magnitude is dS and whose direction is that of n . Then $d\vec{S} = \hat{n} dS$.

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_S \vec{A} \cdot d\vec{S}$$

The surface integral of a vector field $\vec{A}$ is called the flux of $\vec{A}$ through S .

The other forms of surface integrals are

$$\iint_S \varphi d\vec{S} \text{ and } \iint_S \vec{A} \times d\vec{S}$$

Where φ is a scalar point function.

If S is a closed surface, then the surface integrals may be written as

$$\oint \vec{A} \cdot d\vec{S},$$

$$\oint \vec{A} \times d\vec{S},$$

Or

$$\oint \varphi d\vec{S}$$

Properties of Surface Integrals

Like the double integrals, surface integrals have the following properties:

- i. $\iint_S K \vec{A} \cdot d\vec{S} = K \iint_S \vec{A} \cdot d\vec{S}$ (K is any real constant)
- ii. $\iint_S (\vec{A} + \vec{B}) \cdot d\vec{S} = \iint_S \vec{A} \cdot d\vec{S} + \iint_S \vec{B} \cdot d\vec{S}$
- iii. $\iint_S \vec{A} \cdot d\vec{S} = \iint_{S_1} \vec{A} \cdot d\vec{S} + \iint_{S_2} \vec{A} \cdot d\vec{S}$

Where the surface S is subdivided into two smooth surfaces S_1 and S_2 having atmost a curve in common.

- iv. If the surface S is partitioned by the smooth curves into a finite number of non-overlapping smooth patches $S_1, S_2, \dots, S_n$ (i.e. S is piece-wise smooth), then the normal surface integral of $\vec{A}$ over S is the sum of the normal surface integrals of $\vec{A}$ over all the smooth patches, i.e.

$$\iint_S \vec{A} \cdot d\vec{S} = \iint_{S_1} \vec{A} \cdot d\vec{S} + \iint_{S_2} \vec{A} \cdot d\vec{S} + \dots + \iint_{S_n} \vec{A} \cdot d\vec{S}$$

Module No. 42

Evaluation of the Surface Integral

To evaluate the surface integral, it is convenient to express them as double integrals taken over the projected area of the surface S on one of the coordinate planes.

Evaluation of surface integrals can be made by done by the following result

Theorem Statement

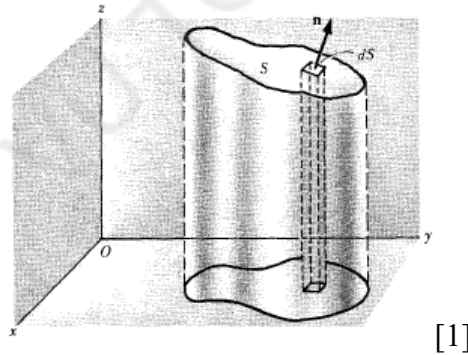
Let R be the projection of the surface S on the xy -plane, then prove that

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \hat{k}|}$$

Proof

Let the surface S and its projection R on the xy -plane be as shown in figure.

Divide R into a rectangles of area ΔA_k , $k = 1, 2, \dots, n$ and erect a vertical column on each of these sub-regions to intersect S in an element of surface area ΔS_k .



Choose a point (x_k, y_k, z_k) on each surface element ΔS_k and draw the unit normal $\hat{n}_k$ to this element at this point. Let γ_k be the acute angle between this unit normal $\hat{n}_k$ and the positive z -axis.

If this surface element is sufficiently small, it can be regarded as a plane.

We know from geometry, that if two planes intersect at an acute angle, an area in one plane may be projected into the other by multiplying the cosine of the included angle as shown in figure.

Since the angle between two planes is the angle between their normal, therefore

$$\Delta S_k \cos \gamma_k \approx \Delta A_k$$

or

$$\Delta S_k \approx \sec \gamma_k \Delta A_k = \frac{\Delta x_k \Delta y_k}{|\hat{n} \cdot \hat{k}|}$$

Thus the sum (1) in the definition of unit normal surface integrals

$$\sum_{k=1}^n A_k \cdot \hat{n}_k \Delta S_k \approx \sum_{k=1}^n A_k \cdot \hat{n}_k \frac{\Delta x_k \Delta y_k}{|\hat{n} \cdot \hat{k}|}$$

and the limit of this sum can be written as

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \hat{k}|}$$

Similarly, we can prove that if R is the projection of the surface S on the yz-plane, then

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dydz}{|\hat{n} \cdot \hat{i}|}$$

And if R is the projection of S on the xz-plane, then

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dzdx}{|\hat{n} \cdot \hat{j}|}$$

Module No. 43

Example to the previous topic: Surface Integrals

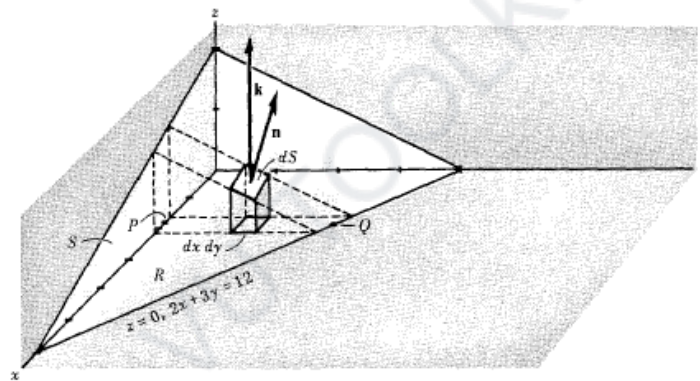
Problem Statement

Evaluate

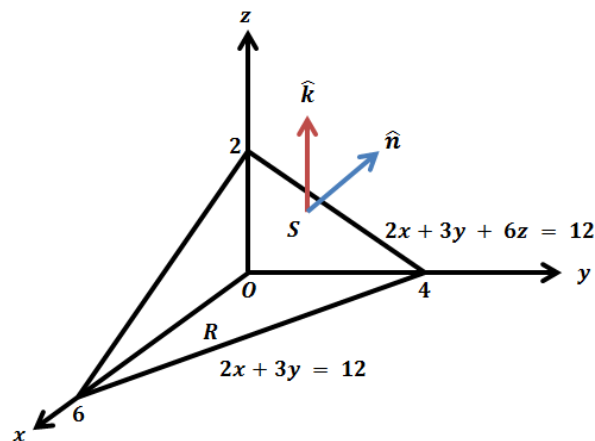
$\iint_S \vec{A} \cdot \hat{n} dS$ where $A = 18z\hat{i} - 12\hat{j} + 3y\hat{k}$ and S is that part of the plane $2x + 3y + 6z = 12$ which is located in the first octant.

Solution

The surface S and its projection R on the xy plane are shown in the figure below.



[1]



We know that

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \hat{k}|}$$

Also we know that a normal vector to the surface $2x + 3y + 6z = 12$ is given by

$$\nabla(2x + 3y + 6z) = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Then a unit normal $\hat{n}$ to any point of S is $\hat{n} = \frac{2\hat{i}+3\hat{j}+6\hat{k}}{7} = \frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}$

Thus $\hat{n} \cdot \hat{k} = \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \cdot \hat{k} = \frac{6}{7}$

and so

$$\frac{dxdy}{|\hat{n} \cdot \hat{k}|} = \frac{7}{6} dxdy$$

Also

$$\begin{aligned} \vec{A} \cdot \hat{n} &= (18z\hat{i} - 12\hat{j} + 3y\hat{k}) \cdot \left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} + \frac{6}{7}\hat{k}\right) \\ &= \frac{36z - 36 + 18y}{7} \end{aligned}$$

Now from the equation of the surface S $2x + 3y + 6z = 12$,

$$z = \frac{12 - 2x - 3y}{6}$$

Therefore

$$\vec{A} \cdot \hat{n} = \frac{6(12 - 2x - 3y) - 36 + 18y}{7} = \frac{36 - 12x}{7}$$

Then

$$\begin{aligned} \iint_S \vec{A} \cdot \hat{n} dS &= \iint_R \vec{A} \cdot \hat{n} \frac{dxdy}{|\hat{n} \cdot \hat{k}|} \\ &= \iint_R \frac{36 - 12x}{7} \frac{7}{6} dxdy \\ &= \int_{x=0}^6 \int_{y=0}^{\frac{12-2x}{3}} (6 - 2x) dy dx \\ &= \int_0^6 (6 - 2x) \left(\frac{12 - 2x}{3}\right) dx \end{aligned}$$

$$\begin{aligned} &= \int_0^6 \left(24 - 12x + \frac{4}{3}x^2 \right) dx \\ &= \left| 24x - 6x^2 + \frac{4}{9}x^3 \right|_0^6 \\ &= 144 - 216 + 96 = 24 \end{aligned}$$

If we had chosen the positive unit normal $\hat{n}$ opposite to that in the figure above, we would have obtained the result — 24 .

Module No. 44

Further Example on Surface Integral

Problem Statement

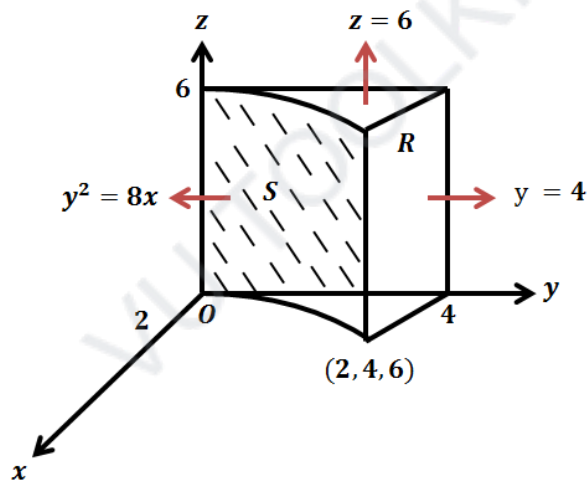
If $\vec{F} = 2y\hat{i} - z\hat{j} + x^2\hat{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$,

Evaluate

$$\iint_S \vec{F} \cdot \hat{n} dS$$

Solution

The surface S and its projection R on the yz -plane are shown in figure.



[1]

A vector normal to S is

$$\nabla(8x - y^2) = 8\hat{i} - 2y\hat{j}$$

and therefore

$$\hat{n} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{8^2 + (2y)^2}} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{64 + 4y^2}} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{4(16 + y^2)}} = \frac{4\hat{i} - y\hat{j}}{\sqrt{16 + y^2}} = \frac{4\hat{i}}{\sqrt{16 + y^2}} + \frac{-y\hat{j}}{\sqrt{16 + y^2}}$$

Also,

$$\hat{n} \cdot \hat{i} = \frac{4}{\sqrt{16 + y^2}}$$

and

$$\begin{aligned}\vec{F} \cdot \hat{n} &= (2y\hat{i} - z\hat{j} + x^2\hat{k}) \cdot \left(\frac{4\hat{i}}{\sqrt{16+y^2}} + \frac{-y\hat{j}}{\sqrt{16+y^2}} \right) \\ &= \frac{8y + zy}{\sqrt{16+y^2}}\end{aligned}$$

Thus

$$\begin{aligned}\iint_S \vec{A} \cdot \hat{n} dS &= \iint_R \vec{A} \cdot \hat{n} \frac{dzdy}{|\hat{n} \cdot \hat{i}|} \\ &= \int_0^4 \int_0^6 \frac{8y + zy}{\sqrt{16+y^2}} \frac{\sqrt{16+y^2}}{4} dzdy \\ &= \frac{1}{4} \int_0^4 \int_0^6 (8y + zy) dzdy = \frac{1}{4} \int_0^4 66y dy \\ &= \left| \frac{33}{4} y^2 \right|_0^4 = \frac{33}{4} 4^2 = 132 \\ \iint_S \vec{F} \cdot \hat{n} dS &= 132\end{aligned}$$

is the required result.

Module No. 45

Selected Example/Problem 1: Surface Integrals

Problem Statement

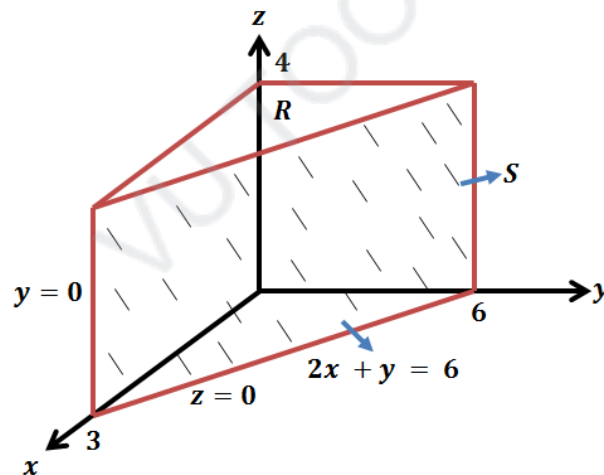
Evaluate

$$\iint_S \vec{A} \cdot \hat{n} dS$$

for $\vec{A} = y\hat{i} + 2x\hat{j} - z\hat{k}$ and S is the surface of the plane $2x + y = 6$ in the first octant cut off by the plane $z = 4$

Solution

The Surface S and its projection R in the yz -plane are shown in figure.



[1]

A vector normal to S is given by

$$\nabla(2x + y) = 2\hat{i} + \hat{j}$$

Therefore,

$$\hat{n} = \frac{2\hat{i} + \hat{j}}{\sqrt{2^2 + (1)^2}} = \frac{2\hat{i} + \hat{j}}{\sqrt{4 + 1}} = \frac{2\hat{i} + \hat{j}}{\sqrt{5}}$$

Also,

$$\hat{n} \cdot \hat{i} = \frac{2}{\sqrt{5}}$$

and

$$\begin{aligned}\vec{A} \cdot \hat{n} &= (y\hat{i} + 2x\hat{j} - z\hat{k}) \cdot \left(\frac{2\hat{i}}{\sqrt{5}} + \frac{\hat{j}}{\sqrt{5}} \right) \\ &= \frac{2y}{\sqrt{5}} + \frac{2x}{\sqrt{5}} = \frac{2}{\sqrt{5}}(x + y)\end{aligned}$$

and

$$\begin{aligned}\iint_S \vec{A} \cdot \hat{n} dS &= \iint_R \vec{A} \cdot \hat{n} \frac{dzdy}{|\hat{n} \cdot \hat{i}|} \\ &= \int_{y=0}^6 \int_{z=0}^4 \frac{2}{\sqrt{5}}(x + y) \frac{\sqrt{5}}{2} dzdy \\ 2x + y &= 6 \Rightarrow x = \frac{6 - y}{2} \\ &= \int_{y=0}^6 \int_{z=0}^4 y + \left(\frac{6 - y}{2} \right) dzdy = \int_{y=0}^6 \int_{z=0}^4 \left(3 + \frac{y}{2} \right) dzdy \\ &= \int_{y=0}^6 \left(3 + \frac{y}{2} \right) |z|_0^4 dy \\ &= 4 \int_{y=0}^6 \left(3 + \frac{y}{2} \right) dy \\ &= \left| 3y + \frac{y^2}{4} \right|_0^6 \\ &= 4(18 + 9) = 108\end{aligned}$$

Module No. 46

Selected Example/Problem 2: Surface Integral

Problem Statement

Evaluate

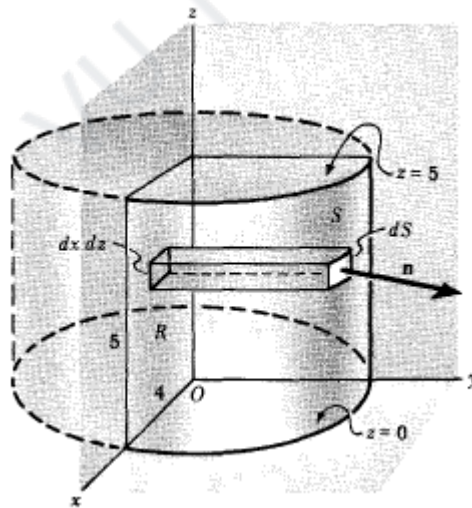
i. $\iint_S \vec{A} \cdot \hat{n} dS$

ii. $\iint_S \varphi \hat{n} dS$

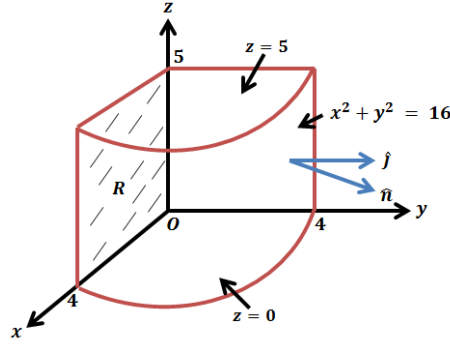
where $\vec{A} = z\hat{i} + x\hat{j} - 3y^2z\hat{k}$ and $\varphi = \frac{3}{8}xyz$ S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.

Solution

- i. The surface S and its projection R on the xz -plane are shown in figure.



[1]



[1]

Then

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_R \vec{A} \cdot \hat{n} \frac{dx dz}{|\hat{n} \cdot \hat{j}|} \quad (1)$$

A normal vector to $x^2 + y^2 = 16$ is $\nabla(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$

Thus the unit normal $\hat{n}$ to S is

$$\begin{aligned} \hat{n} &= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4(x^2 + y^2)}} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4(16)}} \\ &= \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{64}} = \frac{2(x\hat{i} + y\hat{j})}{8} = \frac{(x\hat{i} + y\hat{j})}{4} \end{aligned}$$

Using $x^2 + y^2 = 16$ on S

Now,

$$\begin{aligned} \vec{A} \cdot \hat{n} &= (z\hat{i} + x\hat{j} - 3y^2z\hat{k}) \cdot \left(\frac{x\hat{i} + y\hat{j}}{4} \right) \\ &= \frac{1}{4}(zx + xy) \end{aligned}$$

And

$$\hat{n} \cdot \hat{j} = \left(\frac{x\hat{i} + y\hat{j}}{4} \right) \cdot \hat{j} = \frac{y}{4}$$

Thus putting obtained values in equation (1)

$$\begin{aligned} \iint_S \vec{A} \cdot \hat{n} dS &= \iint_R \frac{1}{4}(zx + xy) \frac{4}{y} dx dz \\ &= \int_{z=0}^5 \int_{x=0}^4 \frac{zx + xy}{y} dx dz \\ &= \int_{z=0}^5 \int_{x=0}^4 \frac{zx}{y} + x dx dz \end{aligned} \quad (2)$$

Utilizing the equation of the surface of the cylinder

$$x^2 + y^2 = 16 \Rightarrow y^2 = 16 - x^2 \Rightarrow y = \sqrt{16 - x^2}$$

Substituting the value of y in equation (2)

$$\begin{aligned} &= \int_{z=0}^5 \int_{x=0}^4 \frac{zx}{\sqrt{16-x^2}} + x \, dx \, dz \\ &= \int_0^5 \left[-z\sqrt{16-x^2} + \frac{x^2}{2} \right]_0^4 dz \\ &= \int_0^5 (4z + 8) dz = [2z^2 + 8z]_0^5 = 90 \end{aligned}$$

ii. $\iint_S \varphi \hat{n} dS$

$$\iint_S \varphi \cdot \hat{n} dS = \iint_R \varphi \hat{n} \frac{dx dz}{|\hat{n} \cdot \hat{j}|}$$

Substituting values, we get

$$\begin{aligned} \iint_S \varphi \cdot \hat{n} dS &= \iint_R \left(\frac{3}{8} xyz \right) \left(\frac{x\hat{i} + y\hat{j}}{4} \right) \frac{4}{y} dx dz \\ &= \frac{3}{8} \int_{z=0}^5 \int_{x=0}^4 xz (x\hat{i} + y\hat{j}) dx dz \end{aligned}$$

Using the value of $y = \sqrt{16 - x^2}$ in above integral,

$$\begin{aligned} &= \frac{3}{8} \int_{z=0}^5 \int_{x=0}^4 (x^2 z \hat{i} + xz \sqrt{16 - x^2} \hat{j}) dx dz \\ &= \int_0^5 \left[z \hat{i} \left| \frac{x^3}{3} \right|_0^4 + z \hat{j} \left| \frac{-1}{3} (16 - x^2)^{3/2} \right|_0^4 \right] dz \\ &= \int_0^5 \left(\frac{64}{3} z \hat{i} + \frac{64}{3} z \hat{j} \right) dz = 8 \left[\frac{z^2}{2} \hat{i} + \frac{z^2}{2} \hat{j} \right]_0^5 \\ &= 8 \left(\frac{25}{2} \hat{i} + \frac{25}{2} \hat{j} \right) = 100\hat{i} + 100\hat{j} \end{aligned}$$

is required solution.

Module No. 47

Selected Example/Problem 3: Surface Integral

Problem Statement

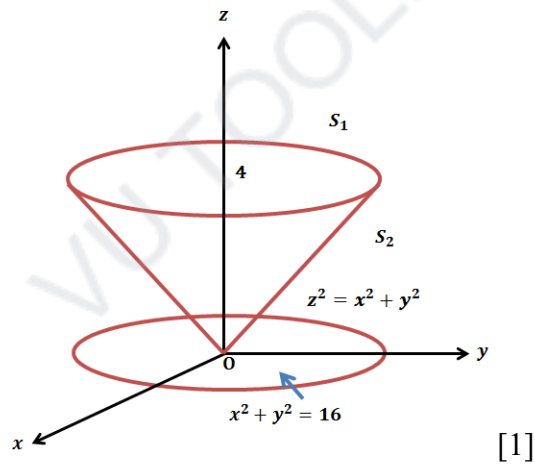
Evaluate $\iint_S \vec{A} \cdot \hat{n} dS$

over the entire surface of the region above the xy plane bounded by the cone S

$z^2 = x^2 + y^2$ and the plane $z = 4$, if $\vec{A} = 4xz \hat{i} + xyz^2 \hat{j} + 3z \hat{k}$.

Solution

The surface S and its projection R on the xy-plane are shown in figure.



Then

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_{S_1} \vec{A} \cdot \hat{n} dS_1 + \iint_{S_2} \vec{A} \cdot \hat{n} dS_2$$

For S_1 , $z = 4$, $\hat{n} = \hat{k}$ and $\hat{n} \cdot \hat{k} = 1$

Then

$$\begin{aligned} \iint_{S_1} \vec{A} \cdot \hat{n} dS &= \iint_{S_1} 12 dS_1 = 12 \iint_R dx dy \\ &= 12 \times \pi(4)^2 = 192\pi \end{aligned}$$

For S_2 , the normal vector is given by

$$\nabla(x^2 + y^2 - z^2) = 2x\hat{i} + 2y\hat{j} - 2z\hat{k}$$

Therefore,

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j} - 2z\hat{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{2(x\hat{i} + y\hat{j} - z\hat{k})}{2\sqrt{x^2 + y^2 + z^2}} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{x^2 + y^2 + z^2}} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})}{\sqrt{2z^2}}$$

$$\hat{n} = \frac{(x\hat{i} + y\hat{j} - z\hat{k})}{z\sqrt{2}}$$

Also,

$$\vec{A} \cdot \hat{n} = \frac{1}{\sqrt{2}}(4x^2 + xy^2z - 3z) \text{ and } \hat{n} \cdot \hat{k} = \frac{-1}{\sqrt{2}}$$

Therefore

$$\iint_{S_2} \vec{A} \cdot \hat{n} dS_2 = \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \frac{1}{\sqrt{2}}(4x^2 + xy^2z - 3z) \cdot \sqrt{2} dy dx$$

Using $z^2 = x^2 + y^2$ in the above integral, we obtain

$$= \int_{-4}^4 \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} (4x^2 + xy^2\sqrt{x^2 + y^2} - 3\sqrt{x^2 + y^2}) dy dx$$

Now we will further solve it by using polar coordinates

Let $x = r \cos \theta$, $y = r \sin \theta$, therefore $0 \leq r \leq 4$ and $0 \leq \theta \leq 2\pi$

and

$$\iint_{S_2} \vec{A} \cdot \hat{n} dS_2 = \int_0^{2\pi} \int_0^4 (4r^2 \cos^2 \theta + r^4 \cos \theta \sin^2 \theta - 3r) r dr d\theta$$

$$= \int_0^{2\pi} \int_0^4 (4r^3 \cos^2 \theta + r^5 \cos \theta \sin^2 \theta - 3r^2) dr d\theta$$

integrating w.r.t r

$$= \int_0^{2\pi} \left[r^4 \cos^2 \theta + \frac{r^6}{6} \cos \theta \sin^2 \theta - r^3 \right]_0^4 d\theta$$

$$= \int_0^{2\pi} \left(256 \cos^2 \theta + \frac{4^6}{6} \cos \theta \sin^2 \theta - 4^3 \right) d\theta$$

$$\begin{aligned}
&= \int_0^{2\pi} \left(256 \cos^2 \theta + \frac{4^6}{6} \cos \theta \sin^2 \theta - 4^3 \right) d\theta \\
&= \int_0^{2\pi} \left(128(1 + \cos 2\theta) + \frac{2048}{3} \cos \theta \sin^2 \theta - 64 \right) d\theta
\end{aligned}$$

Now integrating w.r.t θ , we get

$$= \left| 128\theta + 64 \sin 2\theta + \frac{2048}{3} \sin^3 \theta - 64\theta \right|_0^{2\pi}$$

$$\iint_{S_2} \vec{A} \cdot \hat{n} dS_2 = 128\pi$$

Thus,

$$\iint_S \vec{A} \cdot \hat{n} dS = \iint_{S_1} \vec{A} \cdot \hat{n} dS_1 + \iint_{S_2} \vec{A} \cdot \hat{n} dS_2$$

$$\iint_S \vec{A} \cdot \hat{n} dS = 192\pi + 128\pi = 320\pi$$

is the required solution.

Module No. 48

Volume Integral

A volume integral refers to an integral over a 3-dimensional domain, that is, it is a special case of multiple integrals.

Definition

Let $\vec{A}$ be a given vector point function which is defined and continuous in a closed region R. Then

$$\iiint_R \vec{A} dV$$

If $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then the above integral may be written as

$$\iiint_R \vec{A} dV = \hat{i} \iiint_R A_1 dV + \hat{j} \iiint_R A_2 dV + \hat{k} \iiint_R A_3 dV$$

If we have a scalar point function $\varphi(x, y, z)$ defined as continuous over the region R, then the volume integral becomes

$$\iiint_R \varphi dV$$

In rectangular coordinate system, $dV = dxdydz$ so the volume integral becomes

$$\iiint_R \varphi dV = \iiint_R \varphi(x, y, z) dxdydz$$

Which is ordinary triple integral of $\varphi(x, y, z)$ over the region R.

If $\varphi(x, y, z) = 1$, the volume V of the region R is given by

$$\iiint_R dxdydz = \iiint_R dV$$

Notations for other than Cartesian system

Volume integral can be expressed also in cylindrical and spherical coordinates as

- i. Volume integral in cylindrical coordinates:

$$\iiint_R g(\rho, \varphi, z) \rho d\rho d\varphi dz$$

ii. Volume integral in cylindrical coordinates:

$$\iiint_R g(r, \theta, \varphi) r^2 \sin \varphi dr d\theta d\varphi$$

Which are equivalent to ordinary triple integrals in cylindrical and spherical coordinates.

VU TOOLKIT

Module No. 49

Example on Volume Integral

Problem Statement

Let $\vec{F} = 2xzi - xj + y^2k$.

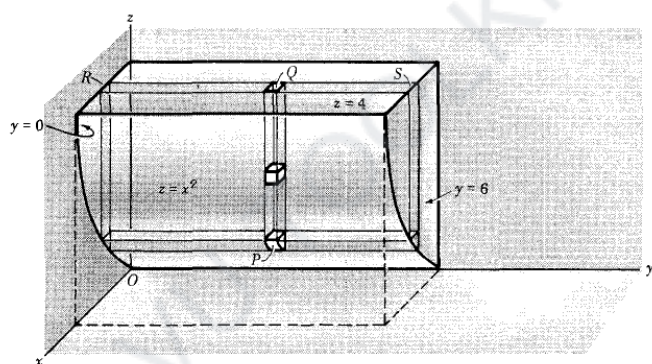
Evaluate

$$\iiint_R \vec{F} dV$$

where V is the region bounded by the surfaces $x = 0, y = 0, y = 6, z = x^2, z = 4$.

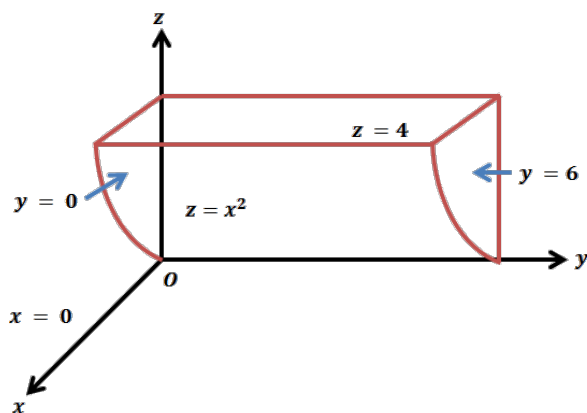
Solution

We will evaluate the given problem through following scheme.



[1]

The region V is covered (i) by keeping x and y fixed and integrating from $z = x^2$ to $z = 4$ (base to top of column PQ),



[1]

- (ii) Then by keeping x fixed and integrating from $y = 0$ to $y = 6$ (R to S in the slab),
 (iii) Finally integrating from $x = 0$ to $x = 2$ (where $z = x^2$ meets $z = 4$). Then the required integral is

$$\begin{aligned}\iiint_R \vec{F} dV &= \int_0^2 \int_0^6 \int_{x^2}^4 (2xz\hat{i} - x\hat{j} + y^2\hat{k}) dz dy dx \\ &= \hat{i} \int_0^2 \int_0^6 \int_{x^2}^4 2xz dz dy dx - \hat{j} \int_0^2 \int_0^6 \int_{x^2}^4 x dz dy dx + \hat{k} \int_0^2 \int_0^6 \int_{x^2}^4 y^2 dz dy dx\end{aligned}$$

Solving integrals turn by turn

$$\begin{aligned}\int_0^2 \int_0^6 \int_{x^2}^4 2xz dz dy dx &= \int_0^2 \int_0^6 x(16 - x^4) dy dx = \int_0^2 x(16 - x^4) |y|_0^6 dx = 6 \int_0^2 x(16 - x^4) dx \\ &= 6 \int_0^2 (16x - x^5) dx = 6 \left| \frac{16x^2}{2} - \frac{x^6}{6} \right|_0^2 = 6 \left| \frac{64}{2} - \frac{64}{6} \right| = 128\end{aligned}$$

$$\begin{aligned}\int_0^2 \int_0^6 \int_{x^2}^4 x dz dy dx &= \int_0^2 \int_0^6 x |z|_{x^2}^4 dy dx = \int_0^2 \int_0^6 x(4 - x^2) dy dx = \int_0^2 x(4 - x^2) |y|_0^6 dx \\ &= 6 \int_0^2 x(4 - x^2) dx = 6 \left| 2x^2 - \frac{x^4}{4} \right|_0^2 = 6 \left| 8 - \frac{16}{4} \right| = 24\end{aligned}$$

$$\begin{aligned}\int_0^2 \int_0^6 \int_{x^2}^4 y^2 dz dy dx &= \int_0^2 \int_0^6 y^2 |z|_{x^2}^4 dy dx = \int_0^2 \int_0^6 y^2(4 - x^2) dy dx = \int_0^2 \left| \frac{y^3}{3} \right|_0^6 (4 - x^2) dx \\ &= 72 \int_0^2 (4 - x^2) dx = 72 \left| 4x - \frac{x^3}{3} \right|_0^2 = 72 \left(8 - \frac{8}{3} \right) = 384\end{aligned}$$

Hence

$$\iiint_R \vec{F} dV = \hat{i} \int_0^2 \int_0^6 \int_{x^2}^4 2xz dz dy dx - \hat{j} \int_0^2 \int_0^6 \int_{x^2}^4 x dz dy dx + \hat{k} \int_0^2 \int_0^6 \int_{x^2}^4 y^2 dz dy dx$$

$$\iiint_R \vec{F} dV = 128\hat{i} - 24\hat{j} + 384\hat{k}$$

is the required solution.

Module No. 50

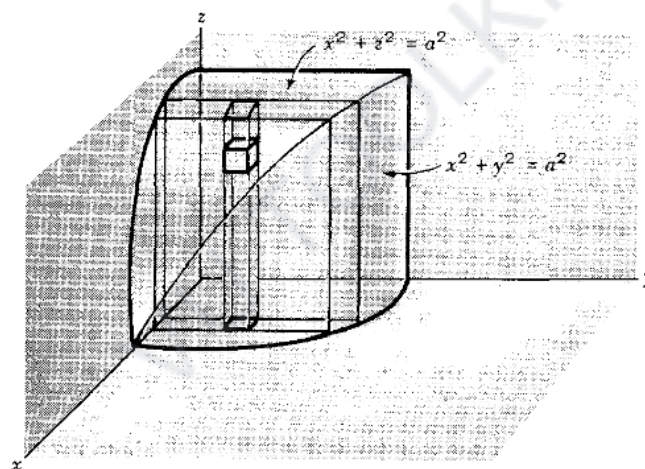
Selected Example/Problem 1: Volume Integral

Problem Statement

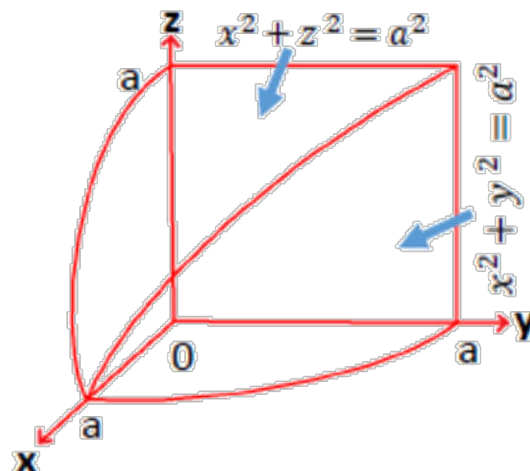
Find the volume of the region common to the intersecting cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$.

Solution

Let M be the required common region then



[1]



[1]

Required volume = 8 times volume of region shown in above figure.

$$\begin{aligned}
 \iiint_V M dV &= 8 \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2}} dz dy dx \\
 &= \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} z dy dx = \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2} dy dx = \int_{x=0}^a (a^2-x^2) dx \\
 &= \frac{16a^3}{3}
 \end{aligned}$$

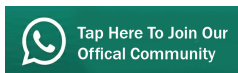


Lecture Handouts

on

VECTORS AND CLASSICAL MECHANICS (MTH-622)

**Virtual University of Pakistan
Pakistan**



About the Handouts

The following books have been mainly followed to prepare the slides and handouts:

1. Spiegel, M.R., *Theory and Problems of Vector Analysis: And an Introduction to Tensor Analysis*. 1959: McGraw-Hill.
2. Spiegel, M.S., *Theory and problems of theoretical mechanics*. 1967: Schaum.
3. Taylor, J.R., *Classical Mechanics*. 2005: University Science Books.
4. DiBenedetto, E., *Classical Mechanics: Theory and Mathematical Modeling*. 2010: Birkhäuser Boston.
5. Fowles, G.R. and G.L. Cassiday, *Analytical Mechanics*. 2005: Thomson Brooks/Cole.

The first two books were considered as main text books. Therefore the students are advised to read the first two books in addition to these handouts. In addition to the above mentioned books, some other reference book and material was used to get these handouts prepared.

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Module No. 51

Selected Example/Problem 2: Volume Integral

Problem Statement

If $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$, evaluate

$$\iiint_R \nabla \cdot \vec{F} dV$$

the closed region bounded by the planes $x = 0, y = 0, z = 0$ and $2x + 2y + z = 4$.

Solution

Since $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$,

Thus

$$\begin{aligned} \nabla \cdot \vec{F} &= \left(\frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \right) \cdot ((2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}) \\ &= \frac{\partial(2x^2 - 3z)}{\partial x} - \frac{\partial 2xy}{\partial y} - \frac{\partial 4x}{\partial z} \\ &= 4x - 2x = 2x \end{aligned}$$

$$\iiint_R \nabla \cdot \vec{F} dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} 2x dz dy dx$$

Using equation $2x + 2y + z = 4$ for limit

Integrating w.r.t z we obtain,

$$\begin{aligned} &= 2 \int_0^2 \int_0^{2-x} x |z|_0^{4-2x-2y} dy dx = 2 \int_0^2 \int_0^{2-x} x(4 - 2x - 2y) dy dx = \int_0^2 \int_0^{2-x} (8x - 4x^2 - 4xy) dy dx \\ &= \int_0^2 (8x|y|_0^{2-x} - 4x^2|y|_0^{2-x} - 4x \left| \frac{y^2}{2} \right|_0^{2-x}) dx = \int_0^2 8x(2-x) - 4x^2(2-x) - 2x(2-x)^2 dx \end{aligned}$$

$$\begin{aligned} &= \int_0^2 8x - 8x^2 + 2x^3 dx \\ &= \left[4x^2 - \frac{8}{3}x^3 + \frac{x^4}{2} \right]_0^2 \\ &= 16 - \frac{64}{3} + 8 = -\frac{8}{3} \end{aligned}$$

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Module No. 52

Divergence Theorem

Divergence theorem is also called Gauss's divergence theorem. Gauss's divergence theorem has wide applications in physics and engineering and is used to derive equation governing the flow of fluids, heat conduction, wave propagation and electrical fields.

In words the divergence theorem may states that the surface integral of the normal component of a vector function $\vec{A}$ taken over a closed surface S is equal to the integral of the divergence of $\vec{A}$ taken over the region R enclosed by the surface.

We can write it mathematically as

Statement

It states that if R is the region bounded by a closed surface S and $\vec{A}$ is a vector point function with continuous first partial derivatives, then

$$\iint_S \vec{A} \cdot \hat{n} dS = \iiint_R \nabla \cdot \vec{A} dV$$

Where $\hat{n}$ is outward drawn unit normal to S.

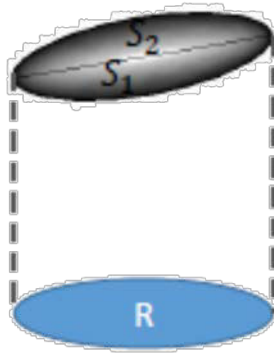
Proof

If $\vec{A}$ is expressed as $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then the divergence theorem can be written component wise as

$$\iint_S (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) \cdot \hat{n} dS = \iiint_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV$$

To establish this relation, we will prove that the respective integrals on each sides are equal.

We prove this for a closed surface S, which has the property that any line parallel to the coordinate axes cuts S in at most two points. Under this assumption, it follows that S is doubled valued surface over its projection on each of the coordinate planes.



Let R' be the projection of S on the xy -plane. Divide the surface S into the lower and upper parts S_1 and S_2 and assume the equations of S_1 and S_2 to be $z = f_1(x, y)$ and $z = f_2(x, y)$ respectively. Consider

$$\begin{aligned}
 \iiint_R \frac{\partial A_3}{\partial z} dV &= \iiint_R \frac{\partial A_3}{\partial z} dz dy dx \\
 &= \iint_{R'} \left[\int_{z=f_1(x,y)}^{z=f_2(x,y)} \frac{\partial A_3}{\partial z} dz \right] dy dx = \iint_{R'} [A_3(x, y, z)]_{z=f_1(x,y)}^{z=f_2(x,y)} dy dx \\
 &= \iint_{R'} \{A_3[(x, y, f_2(x, y))] - A_3[(x, y, f_1(x, y))]\} dy dx \quad (1)
 \end{aligned}$$

For the upper part S_2 , $dy dx = \cos \gamma_2 dS_2 = \hat{k} \cdot \hat{n}_2 dS_2$, since the normal $\hat{n}$ to S_2 makes an acute angle with $\hat{k}$. For the lower part S_1 , $dy dx = -\cos \gamma_1 dS_1 = \hat{k} \cdot \hat{n}_1 dS_1$, since the normal $\hat{n}_1$ to S_1 , makes an angle γ_1 with $-\hat{k}$.

Then

$$\iint_{R'} A_3[(x, y, f_2(x, y))] dy dx = \iint_{S_2} A_3 \hat{k} \cdot \hat{n}_2 dS_2$$

And

$$\iint_{R'} A_3[(x, y, f_1(x, y))] dy dx = - \iint_{S_1} A_3 \hat{k} \cdot \hat{n}_1 dS_1$$

and therefore the equation (1) becomes

$$\iiint_R \frac{\partial A_3}{\partial z} dV = \iint_{S_2} A_3 \hat{k} \cdot \hat{n}_2 dS_2 + \iint_{S_1} A_3 \hat{k} \cdot \hat{n}_1 dS_1$$

$$\iiint_R \frac{\partial A_3}{\partial z} dV = \iint_S A_3 \hat{k} \cdot \hat{n} dS \quad (2)$$

Similarly, by projecting S on the yz and zx coordinate plane, we obtain respectively,

$$\iiint_R \frac{\partial A_1}{\partial z} dV = \iint_S A_1 \hat{i} \cdot \hat{n} dS \quad (3)$$

$$\iiint_R \frac{\partial A_2}{\partial z} dV = \iint_S A_2 \hat{j} \cdot \hat{n} dS \quad (4)$$

By adding equation (2),(3) and (4), we obtain

$$\iiint_R \frac{\partial A_1}{\partial z} dV + \iiint_R \frac{\partial A_2}{\partial z} dV + \iiint_R \frac{\partial A_3}{\partial z} dV = \iint_S A_1 \hat{i} \cdot \hat{n} dS + \iint_S A_2 \hat{j} \cdot \hat{n} dS + \iint_S A_3 \hat{k} \cdot \hat{n} dS$$

Which is equal to

$$\iiint_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV = \iint_S (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) \cdot \hat{n} dS$$

or

$$\iint_S (A_1 \hat{i} + A_2 \hat{j} + A_3 \hat{k}) \cdot \hat{n} dS = \iiint_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV$$

Hence the theorem.

Module No. 53

Divergence theorem in Rectangular Form

As we studied earlier about the rectangular coordinate system and rectangular coordinates. The Cartesian coordinate system (x, y, z) is also called rectangular system. In this article, we will express the Gauss's divergence theorem in the form of Cartesian coordinates.

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, and $\hat{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$

Then

$$\begin{aligned}\nabla \cdot \vec{A} &= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \right) \cdot (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) \\ &= \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\end{aligned}$$

and

$$\vec{A} \cdot \hat{n} = (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) \cdot (n_1\hat{i} + n_2\hat{j} + n_3\hat{k})$$

The unit normal to S is $\hat{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$. Then $\hat{n} \cdot \hat{i} = n_1 = \cos \alpha$, $\hat{n} \cdot \hat{j} = n_2 = \cos \beta$ and $\hat{n} \cdot \hat{k} = n_3 = \cos \gamma$, where α, β, γ are the angles which $\hat{n}$ makes with the positive x, y, z -axes or $\hat{i}, \hat{j}, \hat{k}$ directions respectively. The quantities $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines of $\hat{n}$.

Hence

$$\vec{A} \cdot \hat{n} = A_1 \cos \alpha + A_2 \cos \beta + A_3 \cos \gamma$$

Using these values, Gauss's divergence theorem can be written as

$$\iiint_R \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV = \iint_S (A_1 \cos \alpha + A_2 \cos \beta + A_3 \cos \gamma) dS$$

is the required form of divergence theorem.

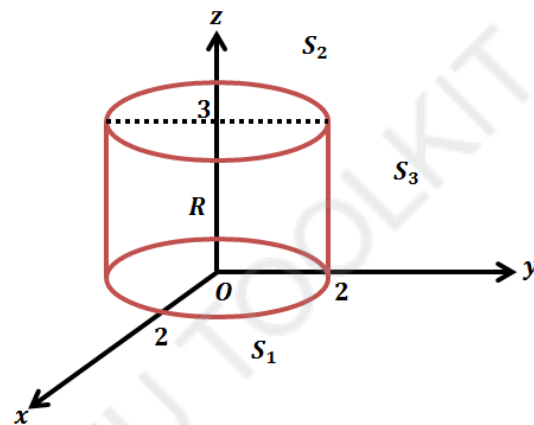
Module No. 54

Verification of Divergence Theorem by an Example

Problem Statement

Verify the divergence theorem for $\vec{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2 + y^2 = 4, z = 0$ and $z = 3$.

Solution



[1]

As we know the divergence theorem is

$$\begin{aligned} \iint_S \vec{A} \cdot \hat{n} dS &= \iiint_V \nabla \cdot \vec{A} dV \\ \iiint_V \nabla \cdot \vec{A} dV &= \iiint_V \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) dV \\ &= \iiint_V \left(\frac{\partial 4x}{\partial x} - \frac{\partial 2y^2}{\partial y} + \frac{\partial z^2}{\partial z} \right) dV = \iiint_V (4 - 4y + 2z) dV \\ &= \int_{x=-2}^2 \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^3 (4 - 4y + 2z) dz dy dx \end{aligned}$$

Here we use the given equation $x^2 + y^2 = 4$, for limits

The surface S of the cylinder consists of a base S_1 ($r = 0$), the top S_2 ($r = 3$) and the convex portion S_3 ($x^2 + y^2 = 4$). Then

$$\text{Surface Integral} = \iint_S \vec{A} \cdot \hat{n} dS = \iint_{S_1} \vec{A} \cdot \hat{n} dS_1 + \iint_{S_2} \vec{A} \cdot \hat{n} dS_2 + \iint_{S_3} \vec{A} \cdot \hat{n} dS_3$$

On Surface S_1 ($z = 0$), $\hat{n} = -\hat{k}$, $\vec{A} = 4x\hat{i} - 2y^2\hat{j}$

Therefore $\vec{A} \cdot \hat{n} = (4x\hat{i} - 2y^2\hat{j}) \cdot (-\hat{k}) = 0$

$$\Rightarrow \iint_{S_1} \vec{A} \cdot \hat{n} dS_1 = 0$$

On Surface S_2 ($z = 3$), $\hat{n} = \hat{k}$, $\vec{A} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

Therefore $\vec{A} \cdot \hat{n} = (4x\hat{i} - 2y^2\hat{j} + 9\hat{k}) \cdot (\hat{k}) = 9$

$$\Rightarrow \iint_{S_2} \vec{A} \cdot \hat{n} dS_2 = 9 \iint_{S_2} dS_2 = 9(4\pi) = 36\pi$$

Since we have $x^2 + y^2 = 4 \Rightarrow \sqrt{x^2 + y^2} = 2 = r$, radius of the base of the cylinder.

Therefore the area of the base of cylinder is $\pi r^2 = 4\pi$

On S_3 ($x^2 + y^2 = 4$). A perpendicular to $x^2 + y^2 = 4$ has the direction

$$\nabla(x^2 + y^2) = 2x\hat{i} + 2y\hat{j}$$

Then a unit normal is

$$\hat{n} = \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{4x^2 + 4y^2}} = \frac{2(x\hat{i} + y\hat{j})}{2\sqrt{x^2 + y^2}} = \frac{(x\hat{i} + y\hat{j})}{\sqrt{4}} = \frac{(x\hat{i} + y\hat{j})}{2}$$

Since $x^2 + y^2 = 4$

$$\vec{A} \cdot \hat{n} = (4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}) \cdot \frac{(x\hat{i} + y\hat{j})}{2} = 2x^2 - y^3$$

$$\iint_{S_3} (2x^2 - y^3) dS_3$$

Since $r = 2$ is the radius of the base of the cylinder

So, using cylindrical coordinates, $x = 2 \cos \theta$, $y = 2 \sin \theta$, $dS_3 = 2d\theta dz$

We have

$$\begin{aligned}
 \iint_{S_3} (2x^2 - y^3) dS_3 &= \int_{\theta=0}^{2\pi} \int_{z=0}^3 [2(2 \cos \theta)^2 - (2 \sin \theta)^3] 2 dz d\theta \\
 &= \int_0^{2\pi} 48 \cos^2 \theta - 48 \sin^2 \theta d\theta = 48\pi
 \end{aligned}$$

Then the surface integral = $0 + 36\pi + 48\pi = 84\pi$, agreeing with the volume integral and verifying the divergence theorem.

Module No. 55

Another Example: Divergence Theorem

Problem Statement

Evaluate

$$\iiint_R \vec{F} \cdot \hat{n} dV$$

where $F = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded S by $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

Solution

By the divergence theorem, the required integral is equal to

$$\begin{aligned} \iint_S \vec{A} \cdot \hat{n} dS &= \iiint_R \nabla \cdot \vec{A} dV \\ \nabla \cdot \vec{A} &= \nabla \cdot (4xz\hat{i} - y^2\hat{j} + yz\hat{k}) \\ &= \frac{\partial 4xz}{\partial x} - \frac{\partial y^2}{\partial y} + \frac{\partial yz}{\partial z} \\ \iiint_R \nabla \cdot \vec{A} dV &= \iiint_R \left(\frac{\partial 4xz}{\partial x} - \frac{\partial y^2}{\partial y} + \frac{\partial yz}{\partial z} \right) dV \end{aligned}$$

Now

$$\begin{aligned} &= \iiint_R (4z - 2y + y) dV = \iiint_R (4z - y) dV \\ &= \int_0^1 \int_0^1 \int_0^1 (4z - y) dz dy dx \\ &= \int_0^1 \int_0^1 2z^2 - yz dy dx = \int_0^1 \int_0^1 2 - y dy dx \end{aligned}$$

$$= \int_0^1 2y - \frac{y^2}{2} dx = \int_0^1 2 - \frac{1}{2} dx = \frac{3}{2} \int_0^1 dx = \frac{3}{2} |x|_0^1 = \frac{3}{2}$$

Hence

$$\iiint_R \vec{F} \cdot \hat{n} dV = \frac{3}{2}$$

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Module No. 56

Further Example 1 of Divergence Theorem

Problem Statement

Prove the identity

$$\iiint_V \nabla \phi dV = \iint_S \phi \hat{n} dS$$

Proof

In the divergence theorem, let $\vec{A} = \phi \vec{C}$ where $\vec{C}$ a constant vector is. Then

$$\iiint_V \nabla \cdot (\phi \vec{C}) dV = \iint_S \phi \vec{C} \cdot \hat{n} dS$$

Since $\nabla \cdot (\phi \vec{C}) = (\nabla \phi) \cdot \vec{C} = \vec{C} \cdot \nabla \phi$ and $\phi \vec{C} \cdot \hat{n} = \vec{C} \cdot (\phi \hat{n})$

Substituting these values in above integral, we get

$$\iiint_V \vec{C} \cdot \nabla \phi dV = \iint_S \vec{C} \cdot (\phi \hat{n}) dS$$

Taking $\vec{C}$ outside the integrals,

$$\vec{C} \cdot \iiint_V \nabla \phi dV = \vec{C} \cdot \iint_S (\phi \hat{n}) dS$$

and since C is an arbitrary constant vector,

$$\iiint_V \nabla \phi dV = \iint_S (\phi \hat{n}) dS$$

Hence the result.

Module No. 57

Further Example 2 of Divergence Theorem

A fluid of density $\rho(x, y, z, t)$ moves with velocity $v(x, y, z, t)$. If there are no sources or sinks, prove that

$$\nabla \cdot J + \frac{\partial \rho}{\partial t} = 0$$

Solution

Consider an arbitrary surface enclosing a volume V of the fluid. At any time the mass of fluid within V is

$$M = \iiint_V \rho dV$$

The time rate of increase of this mass is

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \iiint_V \rho dV = \iiint_V \frac{\partial \rho}{\partial t} dV$$

Let $A =$ velocity v at any point of a moving fluid

Volume of fluid crossing dS in Δt seconds = volume contained in cylinder of base dS and slant height $v\Delta t$

$$= (v\Delta t) \cdot \hat{n} dS = v \cdot \hat{n} dS \Delta t$$

Then, volume per second of fluid crossing $dS = v \cdot \hat{n} dS$

The relation of mass of fluid per unit time leaving V is

$$\iint_S \rho v \cdot \hat{n} dS$$

and the time rate of increase in mass is

$$- \iint_S \rho v \cdot \hat{n} dS = \iiint_V \nabla \cdot \rho v dV$$

by the divergence theorem. Then

$$\iiint_V \frac{\partial \rho}{\partial t} dV = - \iiint_V \nabla \cdot (\rho \mathbf{v}) dV$$

$$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0$$

Suppose that $\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV = 0$ for all the region V . If we suppose that $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) > 0$ at a point P , then from the continuity of the derivatives it follows that $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) > 0$ in some region A surrounding P . If Γ is the boundary of A then

$\iiint_V \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) dV > 0$ which contradicts the assumption that the line integral is zero around every closed curve. Similarly the assumption $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) < 0$ leads to a contradiction. Hence the integrand $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})$ must be equal to zero.

Hence from the continuity of the derivatives it follows that

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

Where $\mathbf{J} = \rho \mathbf{v}$.

The equation is called the continuity equation. If ρ is a constant, the fluid is incompressible and $\nabla \cdot \mathbf{v} = 0$, i.e. $\mathbf{v}$ is solenoidal.

The continuity equation also arises in electromagnetic theory, where ρ is the charge density and $\mathbf{J} = \rho \mathbf{v}$ is the current density.

Module No. 58

Further Example 3 of Divergence Theorem

Problem Statement

Prove the relation:

$$\iiint_V \nabla \times \vec{B} dV = \iint_S \hat{n} \times \vec{B} dS$$

where $\vec{B}$ is any vector field.

Proof

Since the divergence theorem,

$$\iiint_V \nabla \cdot \vec{A} dV = \iint_S \vec{A} \cdot \hat{n} dS$$

let $\vec{A} = \vec{B} \times \vec{C}$ where $\vec{C}$ is a constant vector. Then

$$\iiint_V \nabla \cdot (\vec{B} \times \vec{C}) dV = \iint_S (\vec{B} \times \vec{C}) \cdot \hat{n} dS$$

Since

$$\nabla \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\nabla \times \vec{B}) \text{ and } (\vec{B} \times \vec{C}) \cdot \hat{n} = \vec{B} \cdot (\vec{C} \times \hat{n}) = (\vec{C} \times \hat{n}) \cdot \vec{B} = \vec{C} \cdot (\hat{n} \times \vec{B}),$$

Then the above integral will become

$$\iiint_V \vec{C} \cdot (\nabla \times \vec{B}) dV = \iint_S \vec{C} \cdot (\hat{n} \times \vec{B}) dS$$

Taking $\vec{C}$ outside the integral,

$$\vec{C} \cdot \iiint_V (\nabla \times \vec{B}) dV = \vec{C} \cdot \iint_S (\hat{n} \times \vec{B}) dS$$

and since $\vec{C}$ is an arbitrary constant vector,

$$\iiint_V \nabla \times \vec{B} dV = \iint_S \hat{n} \times \vec{B} dS$$

Hence the result.

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Module No. 59

Stokes' Theorem

In words we can state Stokes' theorem as the line integral of the tangential component of a vector function $\vec{A}$ taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of $\vec{A}$ taken over any surface S having C as its boundary.

Statement

It states that if S is an open, two sided surface bounded by a simple closed curve C, then if $\vec{A}$ has continuous first partial derivatives

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS$$

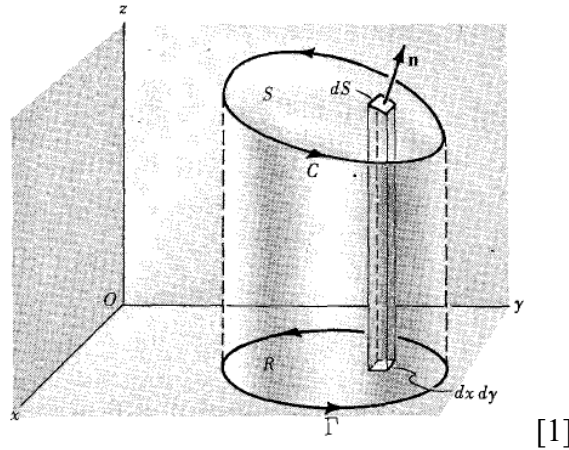
Where C is traversed in the positive direction.

Proof

If $\vec{A}$ is expressed as $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$, then the divergence theorem can be written as

$$\iint_S \nabla \times (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) dS = \oint_C A_1 dx + A_2 dy + A_3 dz$$

We will prove this theorem for a surface S which has the property that its projection on the xy, yz and zx planes are regions bounded by simple closed curves as shown in figure.



Assume S to have representation $z = f(x, y)$ or $x = g(y, z)$ or $y = h(x, z)$, where f, g, h are continuous and differentiable functions.

Consider first

$$\iint_S [\nabla \times (A_1 \hat{i})] \cdot \hat{n} dS$$

Since,

$$\nabla \times (A_1 \hat{i}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & 0 & 0 \end{vmatrix} = \left(\frac{\partial A_1}{\partial y} \hat{j} - \frac{\partial A_1}{\partial z} \hat{k} \right)$$

Therefore,

$$[\nabla \times (A_1 \hat{i})] \cdot \hat{n} dS = \left(\frac{\partial A_1}{\partial z} \hat{n} \cdot \hat{j} - \frac{\partial A_1}{\partial y} \hat{n} \cdot \hat{k} \right) dS \quad (1)$$

If $z = f(x, y)$ is taken the equation of S , then the position vector to any point of S is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = x\hat{i} + y\hat{j} + f(x, y)\hat{k}$$

so that

$$\frac{\partial \vec{r}}{\partial y} = \hat{j} + \frac{\partial z}{\partial y} \hat{k} = \hat{j} + \frac{\partial f}{\partial y} \hat{k}$$

But $\frac{\partial \vec{r}}{\partial y}$ is a vector tangent to S and thus perpendicular to $\hat{n}$, so that

$$\hat{n} \cdot \frac{\partial \vec{r}}{\partial y} = \hat{n} \cdot \hat{j} + \frac{\partial z}{\partial y} \hat{n} \cdot \hat{k} = 0$$

or we can write

$$\hat{n} \cdot \hat{j} = -\frac{\partial z}{\partial y} \hat{n} \cdot \hat{k}$$

Substituting this value in equation (1), we obtain

$$\begin{aligned} [\nabla \times (A_1 \hat{i})] \cdot \hat{n} dS &= \left(-\frac{\partial A_1}{\partial z} \frac{\partial z}{\partial y} \hat{n} \cdot \hat{k} - \frac{\partial A_1}{\partial y} \hat{n} \cdot \hat{k} \right) dS \\ &= -\left(\frac{\partial A_1}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial A_1}{\partial y} \right) \hat{n} \cdot \hat{k} dS \end{aligned} \quad (2)$$

$$\text{Now on } S, A_1(x, y, z) = (x, y, f(x, y)) = F(x, y) \quad (3)$$

Hence equation $\frac{\partial A_1}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial A_1}{\partial y} = \frac{\partial F}{\partial y}$ and (2) becomes

$$[\nabla \times (A_1 \hat{i})] \cdot \hat{n} dS = -\frac{\partial F}{\partial y} \hat{n} \cdot \hat{k} dS = -\frac{\partial F}{\partial y} dx dy$$

Then

$$\iint_S [\nabla \times (A_1 \hat{i})] \cdot \hat{n} dS = \iint_R -\frac{\partial F}{\partial y} dx dy$$

where R is the projection of S on the xy-plane.

By the Green's theorem in the plane, the last integral equals $\oint_{\Gamma} F dx$ where Γ is the boundary of R. From equation (3), since at each point (x, y) of Γ the value of F is the same as the value of A_1 at each point (x, y, z) of C, and since dx is the same for both curves, we must have

$$\oint_{\Gamma} F dx = \oint_C A_1 dx$$

or

$$\iint_S [\nabla \times (A_1 \hat{i})] \cdot \hat{n} dS = \oint_C A_1 dx \quad (4)$$

Similarly, by projections on the other coordinate planes, we have

$$\iint_S [\nabla \times (A_2 \hat{j})] \cdot \hat{n} dS = \oint_C A_2 dy \quad (5)$$

$$\iint_S [\nabla \times (A_3 \hat{k})] \cdot \hat{n} dS = \oint_C A_3 dz \quad (6)$$

Addition of equation (4), (5) and (6) gives us the required results and completes the theorem.

The theorem is also valid for surfaces S which may not satisfy the restrictions imposed above. For assume that S can be subdivided into surfaces $S_1, S_2, S_3, \dots, S_k$ with boundaries $C_1, C_2, C_3, \dots, C_k$ which do satisfy the restrictions. Then Stokes' theorem holds for each such surface. Adding these surface integrals, the total surface integral over S is obtained. Adding the corresponding line integrals over $C_1, C_2, C_3, \dots, C_k$, the line integral over C is obtained.

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Module No. 60

Stokes' Theorem in Rectangular Form

Let $\vec{A} = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ and $\hat{n} = n_1\hat{i} + n_2\hat{j} + n_3\hat{k}$ be the outward drawn unit normal to the surface S. If α, β and γ are the angles which the unit normal $\hat{n}$ makes with the positive directions of x, y and z axes respectively, then

$$n_1 = \hat{n} \cdot \hat{i} = \cos \alpha, n_2 = \hat{n} \cdot \hat{j} = \cos \beta \text{ and } n_3 = \hat{n} \cdot \hat{k} = \cos \gamma.$$

The quantities $\cos \alpha, \cos \beta$, and $\cos \gamma$ are the direction cosine of $\hat{n}$. Then

$$\hat{n} = \cos \alpha \hat{i} + \cos \beta \hat{j} + \cos \gamma \hat{k}$$

Thus

$$\begin{aligned} \nabla \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} \\ &= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k} \end{aligned}$$

and

$$\begin{aligned} (\nabla \times \vec{A}) \cdot \hat{n} &= \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \hat{i} + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \hat{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \hat{k} \right] \cdot (\cos \alpha \hat{i} \\ &\quad + \cos \beta \hat{j} + \cos \gamma \hat{k}) \\ &= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \cos \alpha + \left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z} \right) \cos \beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \cos \gamma \end{aligned}$$

Also

$$\begin{aligned} \vec{A} \cdot d\vec{r} &= (A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= A_1dx + A_2dy + A_3dz \end{aligned}$$

and the stokes theorem becomes

$$\oint_C A_1 dx + A_2 dy + A_3 dz \\
 = \iint_S \left[\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \right) \cos \alpha + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right) \cos \beta + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right) \cos \gamma \right] dS$$

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Module No. 61

Verification of Stokes' Theorem by an Example

Problem Statement

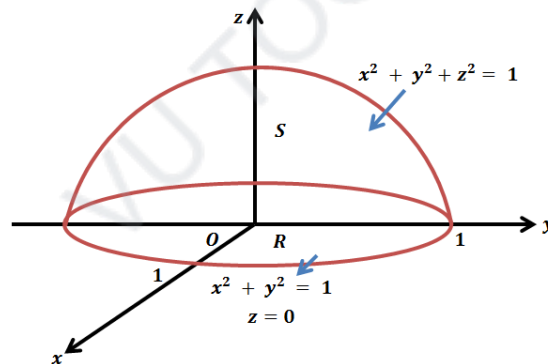
Verify Stokes' theorem for $\vec{A} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary.

Solution

The Stokes' theorem is

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS$$

we will verify the above statement using given vector function $\vec{A}$.



[1]

Let we solve first $\oint_C \vec{A} \cdot d\vec{r}$

The boundary C of S is a circle in the xy plane of radius one and center at the origin. Let $x = \cos t, y = \sin t, z = 0, 0 < t < 2\pi$ be parametric equations of C. (since $r = 1$)

$$\oint_C \vec{A} \cdot d\vec{r} = \oint_C (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k} \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C (2x - y)dx - yz^2dy - y^2zdz$$

Substitute $x = \cos t, y = \sin t, z = 0, 0 < t < 2\pi$, we get

$$\begin{aligned} & \int_{\theta=0}^{2\pi} (2\cos t - \sin t)(-\sin t)dt \\ &= \int_{\theta=0}^{2\pi} (-2\sin t \cos t + \sin^2 t)dt = \int_{\theta=0}^{2\pi} (-\sin 2t) + \left(\frac{1 + \cos 2t}{2}\right)dt \\ &= \left[\frac{\cos 2t}{2} + \frac{t}{2} - \frac{\sin 2t}{4}\right]_0^{2\pi} = \frac{1}{2} + \pi - \frac{1}{2} = \pi \end{aligned}$$

Now, $\iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS$

$$\begin{aligned} \nabla \times \vec{A} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2z \end{vmatrix} \\ &= \left(\frac{\partial(-y^2z)}{\partial y} - \frac{\partial(-yz^2)}{\partial z}\right)\hat{i} - \left(\frac{\partial(-y^2z)}{\partial x} - \frac{\partial(2x - y)}{\partial z}\right)\hat{j} + \left(\frac{\partial(-yz^2)}{\partial x} - \frac{\partial(2x - y)}{\partial y}\right)\hat{k} \\ &= (-2yz + 2yz)\hat{i} - (0 - 0)\hat{j} + (0 + 1)\hat{k} = \hat{k} \end{aligned}$$

Then

$$\iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS = \iint_S \hat{k} \cdot \hat{n} dS = \iint_R dx dy$$

(Since $\hat{k} \cdot \hat{n} dS$)

$$= 4 \int_0^1 \int_0^{\sqrt{1-x^2}} dy dx = 4 \int_0^1 \sqrt{1-x^2} dx$$

Let $x = \sin t \Rightarrow dx = \cos t dt, 0 \leq t \leq \pi/2$. Then

$$\begin{aligned} \iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS &= 4 \int_0^{\pi/2} \sqrt{1 - \sin^2 t} \cos t dt \\ &= 4 \int_0^{\pi/2} \cos^2 t dt = \frac{4}{2} \int_0^{\pi/2} (1 + \cos 2t) dt \end{aligned}$$

$$\begin{aligned} &= \left| t + \frac{\sin 2t}{2} \right|_0^{\pi/2} \\ &= 2(\pi/2) = \pi \end{aligned}$$

and stokes' theorem is verified.

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Module No. 62

Another Example: Stokes' Theorem

Problem Statement

Prove

$$\oint_C d\vec{r} \times \vec{B} = \iint_S (\hat{n} \times \nabla) \times \vec{B} dS$$

Proof

Stokes' theorem is

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS$$

In Stokes' theorem, let $\vec{A} = \vec{B} \times \vec{C}$ where $\vec{C}$ a constant vector is. Then

$$\oint_C (\vec{B} \times \vec{C}) \cdot d\vec{r} = \iint_S (\nabla \times (\vec{B} \times \vec{C})) \cdot \hat{n} dS$$

$$\oint_C d\vec{r} \cdot (\vec{B} \times \vec{C}) = \iint_S [(\vec{C} \cdot \nabla) \vec{B} - \vec{C} (\nabla \cdot \vec{B})] \cdot \hat{n} dS$$

$$\oint_C \vec{C} \cdot (d\vec{r} \times \vec{B}) = \iint_S [(\vec{C} \cdot \nabla) \vec{B}] \cdot \hat{n} dS - \iint_S [\vec{C} (\nabla \cdot \vec{B})] \cdot \hat{n} dS$$

$$\vec{C} \cdot \oint_C (d\vec{r} \times \vec{B}) = \iint_S \vec{C} \cdot [\nabla(\vec{B} \cdot \hat{n})] dS - \iint_S \vec{C} \cdot [\hat{n} (\nabla \cdot \vec{B})] dS$$

$$\vec{C} \cdot \oint_C (d\vec{r} \times \vec{B}) = \vec{C} \cdot \iint_S [\nabla(\vec{B} \cdot \hat{n}) - \hat{n} (\nabla \cdot \vec{B})] dS$$

$$\vec{C} \cdot \oint_C (d\vec{r} \times \vec{B}) = \vec{C} \cdot \iint_S (\hat{n} \times \nabla) \times \vec{B} dS$$

Since C is an arbitrary constant vector, therefore

$$\oint_C d\vec{r} \times \vec{B} = \iint_S (\hat{n} \times \nabla) \times \vec{B} dS$$

Hence the result.

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Module No. 63

Related Theorem: Stokes' Theorem

Theorem Statement

Prove that a necessary and sufficient condition that $\oint_C \vec{A} \cdot d\vec{r} = 0$ for every closed curve C is that $\nabla \times \vec{A} = 0$ identically.

Proof

Sufficiently. Suppose $\nabla \times \vec{A} = 0$. Then by the Stokes' theorem

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS$$

Since $\nabla \times \vec{A} = 0$, therefore $\iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS = 0$

Hence

$$\oint_C \vec{A} \cdot d\vec{r} = 0$$

Necessity. Suppose $\oint_C \vec{A} \cdot d\vec{r} = 0$ around every closed path C, and assume $\nabla \times \vec{A} = 0$ at some point P. Then assuming $\nabla \times \vec{A}$ is continuous there will be a region with P as an interior point, where $\nabla \times \vec{A} = 0$. Let S be a surface contained in this region whose normal $\hat{n}$ at each point has the same direction as $\nabla \times \vec{A}$, i.e. $\nabla \times \vec{A} = \alpha \hat{n}$ where α is a positive constant. Let C be the boundary of S. Then by Stokes' theorem

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS = \iint_S \alpha \hat{n} \cdot \hat{n} dS > 0$$

which contradicts the hypothesis that $\oint_C \vec{A} \cdot d\vec{r} = 0$ and shows that $\nabla \times \vec{A} = 0$.

Hence the theorem.

Module No. 64

Related Theorem: Stokes' Theorem

Problem Statement

Prove that

$$\oint_C \varphi d\vec{r} = \iint_S d\vec{S} \times \nabla \varphi$$

Solution

By Stokes' theorem, we have

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS$$

Let $\vec{A} = \varphi \vec{C}$ where $\vec{C}$ is a constant non-zero vector, then

$$\oint_C \varphi \vec{C} \cdot d\vec{r} = \iint_S (\nabla \times (\varphi \vec{C})) \cdot \hat{n} dS$$

or

$$\oint_C \vec{C} \cdot \varphi d\vec{r} = \iint_S (\nabla \varphi \times \vec{C}) \cdot d\vec{S}$$

since $\nabla \times \vec{C} = \vec{0}$

$$\oint_C \vec{C} \cdot \varphi d\vec{r} = \iint_S \nabla \varphi \cdot \vec{C} \times d\vec{S} = \iint_S \vec{C} \times d\vec{S} \cdot \nabla \varphi = \iint_S \vec{C} \cdot d\vec{S} \times \nabla \varphi$$

or

$$\vec{C} \cdot \oint_C \varphi d\vec{r} = \vec{C} \cdot \iint_S d\vec{S} \times \nabla \varphi$$

Since $\vec{C}$ is an arbitrary constant vector, therefore

$$\oint_C \varphi d\vec{r} = \iint_S d\vec{S} \times \nabla \varphi$$

Hence the result.

Module No. 65

Further Example 2 of Stokes' Theorem

Problem Statement

If

$$\vec{A} = 2yz\hat{i} - (x + 3y - 2)\hat{j} + (x^2 + z)\hat{k}$$

then using Stokes' theorem, evaluate

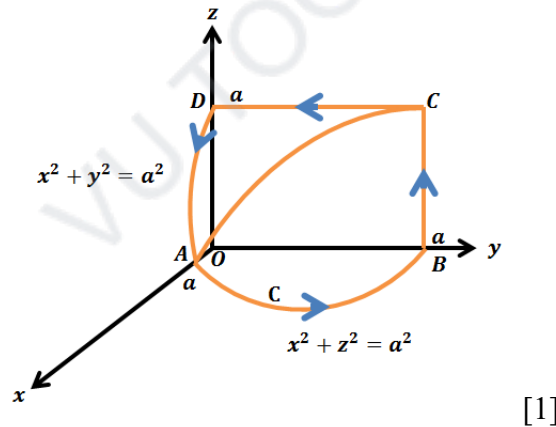
$$\iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS$$

over the surface of intersection of the cylinders $x^2 + y^2 = a^2$, $x^2 + z^2 = a^2$ which is included in first octant.

Solution

By Stokes' theorem, we have

$$\iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS = \oint_C \vec{A} \cdot d\vec{r}$$



$$= \oint_{ABCD} (2yz\hat{i} - (x + 3y - 2)\hat{j} + (x^2 + z)\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_{ABCD} (2yzdx - (x + 3y - 2)dy + (x^2 + z)dz) \quad (1)$$

For AB, $z = 0$ therefore $dz = 0$ and the integral (1) over the part of the curves becomes

$$\int_{AB} -(x + 3y - 2)dy = \int_0^a -\sqrt{a^2 - y^2} + 3y - 2)dy$$

Let $x = a \cos \theta$, $y = a \sin \theta$, $dy = a \cos \theta d\theta$, $0 \leq \theta \leq \pi/2$, then

$$\begin{aligned}
\int_{AB} -(x + 3y - 2)dy &= \int_0^{\pi/2} -(a \cos \theta + 3a \sin \theta - 2)a \cos \theta d\theta \\
&= \int_0^{\pi/2} -\left[\frac{a^2}{2}(1 + \cos 2\theta) + 3a^2 \sin \theta \cos \theta - 2a \cos \theta\right] d\theta \\
&= -\left[\frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + \frac{3}{2}a^2 \sin^2 \theta - 2a \sin \theta\right]_0^{\pi/2} \\
&= -\left[\frac{a^2}{2}\left(\frac{\pi}{2}\right) + \frac{3}{2}a^2 - 2a\right] = \frac{-a^2\pi}{4} - \frac{3}{2}a^2 + 2a
\end{aligned}$$

For BC , $x = 0, y = a$ therefore $dx = dy = 0$ and integral (1) over this part of the curve becomes

$$\int_{BC} z dz = \int_0^a z dz = \frac{a^2}{2}$$

For CD , $x = 0, z = a$ therefore $dx = dz = 0$ and the integral (1) over this part of the curve becomes

$$\int_{CD} -(3y - 2)dy = \int_a^0 -(3y - 2)dy = \frac{3a^2}{2} - 2a$$

For DA , $y = 0$ therefore $dy = 0$ and the integral (1) over this part of the curve becomes

$$\begin{aligned}
\int_{DA} (x^2 + 2)dz &= \int_a^0 (a^2 - z^2 + z)dz \\
&= \left[a^2z - \frac{z^3}{3} + \frac{z^2}{2}\right]_a^0 \\
&= -\frac{2}{3}a^3 - \frac{a^2}{2}
\end{aligned}$$

Thus from equation (1), we get

$$\begin{aligned}
\iint (\nabla \times \vec{A}) \cdot \hat{n} dS &= \frac{-a^2\pi}{4} - \frac{3}{2}a^2 + 2a + \frac{a^2}{2} + \frac{3a^2}{2} - 2a - \frac{2}{3}a^3 - \frac{a^2}{2} \\
&= -\frac{a^2}{12}(3\pi + 8a)
\end{aligned}$$

required solution.

Module No. 66

Further Example 3 of Stokes' Theorem

Problem Statement

If

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{1}{C} \frac{\partial}{\partial t} \iint_S H \cdot d\vec{S}$$

where S is any surface bounded by the curve C, show that

$$\nabla \times \vec{E} = -\frac{1}{C} \frac{\partial \vec{H}}{\partial t}.$$

Solution

As we know the Stokes' theorem

$$\oint_C \vec{A} \cdot d\vec{r} = \iint_S (\nabla \times \vec{A}) \cdot \hat{n} dS$$

by the Stokes' theorem

$$\oint_C \vec{E} \cdot d\vec{r} = \iint_S (\nabla \times \vec{E}) \cdot d\vec{S}$$

therefore

$$\iint_S (\nabla \times \vec{E}) \cdot d\vec{S} = - \iint_S \frac{1}{C} \frac{\partial \vec{H}}{\partial t} \cdot d\vec{S}$$

or

$$\iint_S \left(\nabla \times \vec{E} + \frac{1}{C} \frac{\partial \vec{H}}{\partial t} \right) \cdot d\vec{S} = 0$$

This implies

$$\nabla \times \vec{E} + \frac{1}{C} \frac{\partial \vec{H}}{\partial t} = 0$$

or

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}$$

is required result.

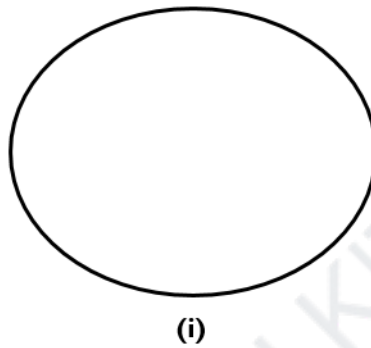
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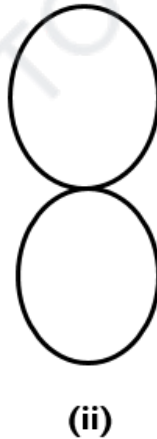
Simply and Multiply Connected Regions

Simply Connected Region

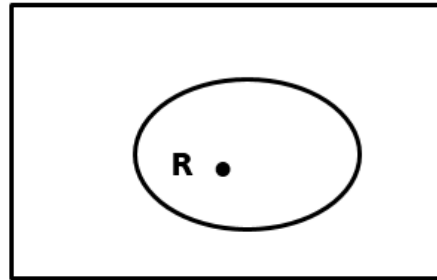
A simple closed curve is a closed curve which does not intersect itself anywhere. For example the curve in the figure (i) is a simple closed curve



while the curve the curve in figure (ii) is not a simple closed curve.



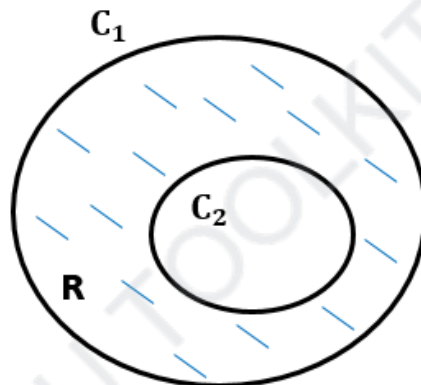
A region R which is said to be simply connected if any simple closed curve lying in R can be continuously shrunk to a point. For example, the interior of a rectangle as shown in figure (iii) is an example of simply connected region.



(iii)

Multiply Connected Regions

A region R which is not simply connected is called multiply connected. For example, the region R exterior to C_2 and interior to C_1 is not simply connected because a circle drawn within R and enclosing C_2 cannot shrunk to a point without crossing C_2 as shown in figure (iv).



(iv)

In other words, we can say that the regions which have holes are called multiply connected.

Module No. 68

Green's Theorem in the Plane

We will consider the vector function of just x and y and derive a relationship between a line integral around a closed curve and a double integral over the part of the plane enclosed by the curve.

Theorem Statement

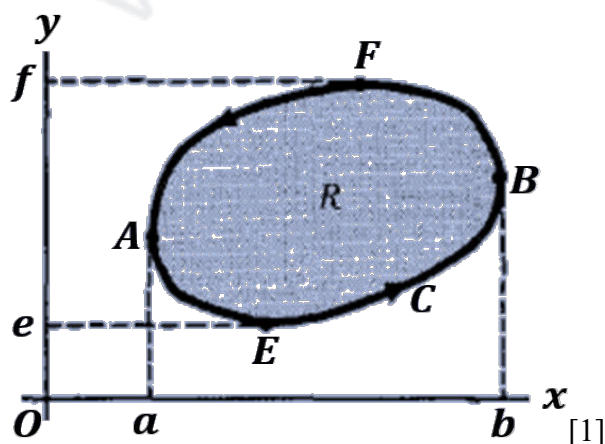
If R is simply-connected region of the xy -plane bounded by a closed curve C and if M and N are continuous functions of x and y having continuous derivatives in R , then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

where C is described in the positive (counter-clockwise) direction.

Proof

We prove the theorem for a closed curve C which has the property that any straight line parallel to the coordinate axes cuts C in at most two points as shown in figure.



Let the equation of the curves AEB and AFB be $y = f_1(x)$ and $y = f_2(x)$ respectively. If R is the region bounded by C , we have

$$\begin{aligned}
\iint_R \frac{\partial M}{\partial y} dx dy &= \int_{x=a}^b \left[\int_{y=f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y} dy \right] dx \\
&= \int_a^b |M(x, y)|_{y=f_1(x)}^{y=f_2(x)} dx \\
&= \int_a^b [M(x, f_2(x)) - M(x, f_1(x))] dx \\
&= - \int_a^b M(x, f_1) dx - \int_a^b M(x, f_2) dx \\
&= - \oint_C M dx
\end{aligned}$$

Then,

$$\oint_C M dx = - \iint_R \frac{\partial M}{\partial y} dx dy \quad (1)$$

Similarly let the equation of the curves EAF and EBF be $x = g_1(y)$ and $x = g_2(y)$ respectively.

Then

$$\begin{aligned}
\iint_R \frac{\partial N}{\partial x} dx dy &= \int_{y=e}^f \left[\int_{x=g_1(y)}^{g_2(y)} \frac{\partial N}{\partial x} dx \right] dy \\
&= \int_e^f |N(x, y)|_{x=g_1(y)}^{x=g_2(y)} dy \\
&= \int_a^b [N(x, g_2(x)) - N(x, g_1(x))] dy \\
&= \int_a^b N(x, g_1) dy + \int_a^b N(x, g_2) dy \\
&= \oint_C N dy
\end{aligned}$$

Then,

$$\iint_R \frac{\partial N}{\partial x} dx dy = \oint_C N dy \quad (2)$$

Adding equation (1) and (2), we get

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

Hence the theorem.

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Module No. 69

Related Example: Green's Theorem

Problem Statement

Verify Green's theorem in the plane for

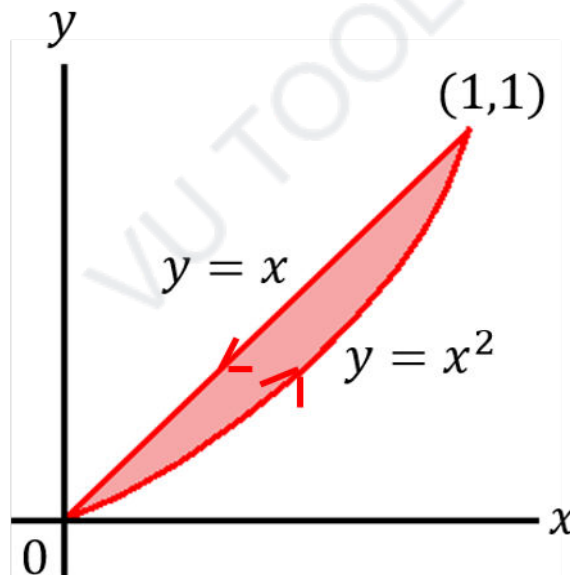
$$\oint_C (xy + y^2) dx + x^2 dy$$

where C is the closed curve of the region bounded by $y = x$ and $y = x^2$.

Solution

The plane curves $y = x$ and $y = x^2$ intersect at $(0,0)$ and $(1,1)$. Let C_1 be the curve $y = x^2$ and C_2 the curve $y = x$ and let the closed curve C be formed from C_1 and C_2 .

The positive direction in traversing C is as shown in the adjacent diagram.



[1]

As we know the Green's theorem

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

By comparing the given relation with Green's theorem, we get

$M = xy + y^2$ and $N = x^2$, using these values, we must show Green's theorem.

Now

$$\oint_C M dx + N dy = \oint_C (xy + y^2) dx + x^2 dy$$

Along the curve $C_1: y = x^2, dy = 2dx$, while x varies from 0 to 1. The line Integral (1) equals to

$$\begin{aligned} \int_{C_1} M dx + N dy &= \int_0^1 (3x^3 + x^4) dx \\ &= \left[\frac{3}{4}x^4 + \frac{x^5}{5} \right]_0^1 = \frac{3}{4} + \frac{1}{5} = \frac{19}{20} \end{aligned} \quad (2)$$

Along the curve $C_2: y = x, dy = dx$, while x varies from 1 to 0. The line Integral (1) equals to

$$\begin{aligned} \int_{C_2} M dx + N dy &= \int_1^0 2x^2 dx + x^2 dx = \int_1^0 3x^2 dx \\ &= |x^3|_1^0 = -1 \end{aligned}$$

Thus from equation (1) and (2), we have

$$\oint_C (xy + y^2) dx + x^2 dy = \frac{19}{20} - 1 = -\frac{1}{20}$$

Now we will calculate the relation

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy$$

since

$$\frac{\partial M}{\partial y} = x + 2y, \text{ and } \frac{\partial N}{\partial x} = 2x$$

then

$$\begin{aligned} \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy &= \iint_R (2x - x - 2y) dx dy \\ &= \int_{x=0}^1 \int_{y=x^2}^x (x - 2y) dy dx = \int_0^1 (x^4 - x^3) dx \end{aligned}$$

integrating and applying limit, we get

$$= \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

so the theorem is verified.

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Module No. 70

Green's Theorem in the Plane in Vector Notation

First Vector Form (or tangential form) of Green's Theorem

We have Green's theorem

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy \quad (1)$$

Now

$$Mdx + Ndy = (M\hat{i} + N\hat{j}) \cdot (dx\hat{i} + dy\hat{j}) = \vec{A} \cdot d\vec{r}$$

where

$$\vec{A} = M\hat{i} + N\hat{j} \text{ and } d\vec{r} = dx\hat{i} + dy\hat{j}.$$

Also, if $\vec{A} = M\hat{i} + N\hat{j}$, then

$$\nabla \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = -\frac{\partial N}{\partial z}\hat{i} + \frac{\partial M}{\partial z}\hat{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\hat{k}$$

$$\text{so that } (\nabla \times \vec{A}) \cdot \hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Then from equation (1) Green's theorem in the plane can be written

$$\oint_C \vec{A} \cdot d\vec{r} = \iint (\nabla \times \vec{A}) \cdot \hat{k} dR$$

where $dR = dxdy$

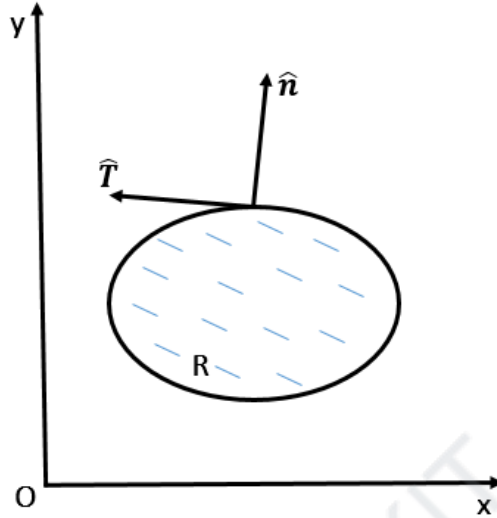
A generalization of this surface to S in space having C as boundary leads quite naturally to Stokes' theorem. This form of Green's theorem is sometimes called Stokes' theorem in the plane.

The Green's theorem in plane is a special case of Stokes theorem.

Second Vector Form (or normal form) of Green's Theorem

As we derive in first vector form of Green's theorem

$$Mdx + Ndy = \vec{A} \cdot d\vec{r} = \vec{A} \cdot \hat{T} ds$$



where $\hat{T} = \frac{d\vec{r}}{ds}$ = unit tangent vector to C as shown in figure.

If $\hat{n}$ is the outward drawn unit normal to C, then $\hat{T} = \hat{k} \times \hat{n}$. so that

$$Mdx + Ndy = \vec{A} \cdot \hat{T} ds = \vec{A} \cdot (\hat{k} \times \hat{n}) ds = (\vec{A} \times \hat{k}) \cdot \hat{n} ds$$

Since $\vec{A} = M\hat{i} + N\hat{j}$, therefore

$$\vec{B} = \vec{A} \times \hat{k} = (M\hat{i} + N\hat{j}) \times \hat{k} = N\hat{i} - M\hat{j}$$

and

$$\nabla \cdot \vec{B} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

then the equation (1) becomes

$$\oint_C \vec{B} \cdot \hat{n} ds = \iint_S \nabla \cdot \vec{B} dR$$

where $dR = dxdy$

these are the required vector notations of Green's theorem.

Module No. 71

Green's Theorem in the Plane as Special case of Stokes' Theorem

Green's theorem can be expressed in the plane vector notations which are also named the tangential form or normal forms of Green's theorem.

The tangential form of Green's theorem is also called the first vector form of Green's theorem. This generalize form of Green's theorem in plane also called Stokes' theorem in the plane. Thus we can say that Green's theorem is a special case of Stokes' theorem when applied to a region in the xy-plane.

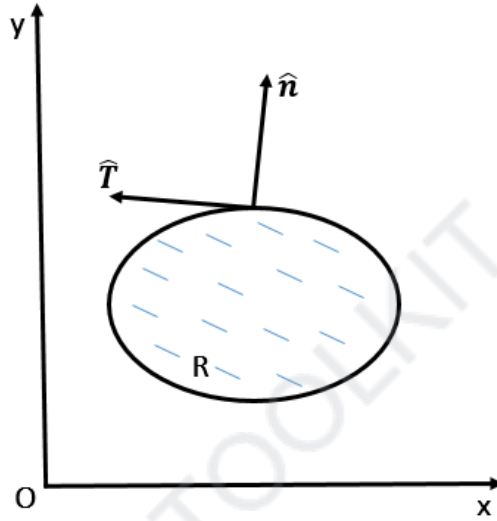
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Module No. 72

Gauss' Divergence Theorem as Generalization of Green's Theorem

As we derive in first vector form of Green's theorem

$$Mdx + Ndy = \vec{A} \cdot d\vec{r} = \vec{A} \cdot \hat{T} ds$$



where $\hat{T} = \frac{d\vec{r}}{ds}$ = unit tangent vector to C as shown in figure.

If $\hat{n}$ is the outward drawn unit normal to C, then $\hat{T} = \hat{k} \times \hat{n}$. so that

$$Mdx + Ndy = \vec{A} \cdot \hat{T} ds = \vec{A} \cdot (\hat{k} \times \hat{n}) ds = (\vec{A} \times \hat{k}) \cdot \hat{n} ds$$

Since $\vec{A} = M\hat{i} + N\hat{j}$, therefore

$$\vec{B} = \vec{A} \times \hat{k} = (M\hat{i} + N\hat{j}) \times \hat{k} = N\hat{i} - M\hat{j}$$

and

$$\nabla \cdot \vec{B} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

then the equation (1) becomes

$$\oint_C \vec{B} \cdot \hat{n} ds = \iint_S \nabla \cdot \vec{B} dR$$

where $dR = dxdy$

these are the required vector notations of Green's theorem.

The generalization of Green's theorem as Gauss's divergence theorem is also called the second vector form (normal form) of Green's theorem.

Generalization of this to the case where the differential arc length ds of a closed curve C is replaced by the differential of surface area dS of a closed surface S , and the corresponding plane region R enclosed by C is replaced by the volume V enclosed by S , leads to Gauss' divergence theorem or Green's theorem in space.

$$\iint_S \vec{B} \cdot \hat{n} dS = \iiint_R \nabla \cdot \vec{B} dV$$

Module No. 73

Green's First Identity

Theorem Statement

If φ and ψ are scalar point functions with continuous second order derivatives in a region R bounded by a closed surface S , then

$$\iiint_R [\varphi \nabla^2 \psi + (\nabla \varphi)(\nabla \psi)] dV = \iint_S (\varphi \nabla \psi) \cdot d\vec{S}$$

Proof

Since divergence theorem is

$$\iint_S \vec{A} \cdot \hat{n} dS = \iiint_R \nabla \cdot \vec{A} dV$$

Now substitute $\vec{A} = \varphi \nabla \psi$ in the divergence theorem, we obtain

$$\iiint_R \nabla \cdot (\varphi \nabla \psi) dV = \iint_S (\varphi \nabla \psi) \cdot \hat{n} dS = \iint_S (\varphi \nabla \psi) \cdot d\vec{S} \quad (1)$$

But

$$\nabla \cdot (\varphi \nabla \psi) = (\nabla \varphi) \cdot (\nabla \psi) + \varphi (\nabla \cdot \nabla \psi) = \varphi \nabla^2 \psi + (\nabla \varphi)(\nabla \psi)$$

thus the equation (1) becomes

$$\iiint_R [\varphi \nabla^2 \psi + (\nabla \varphi)(\nabla \psi)] dV = \iint_S (\varphi \nabla \psi) \cdot d\vec{S}$$

Hence the theorem.

Alternative Forms of Green's first Identity

We know that

$$\nabla \psi \cdot \hat{n} = \frac{\partial \psi}{\partial n} \text{ and } \nabla \varphi \cdot \hat{n} = \frac{\partial \varphi}{\partial n}$$

Thus

$$\nabla\psi \cdot d\vec{S} = \nabla\psi \cdot \hat{n}dS = \frac{\partial\psi}{\partial n}dS$$

and

$$\nabla\varphi \cdot d\vec{S} = \nabla\varphi \cdot \hat{n}dS = \frac{\partial\varphi}{\partial n}dS$$

Hence the Green's first Identity can be written as

$$\iiint_R [\varphi \nabla^2 \psi + (\nabla \varphi)(\nabla \psi)] dV = \iint_S \varphi \frac{\partial \psi}{\partial n} dS$$

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Module No. 74

Green's Second Identity

If φ and ψ are scalar point functions with continuous second order derivatives in a region R bounded by a closed surface S , then

$$\iiint_R [\varphi \nabla^2 \psi - \psi \nabla^2 \varphi] dV = \iint_S (\varphi \nabla \psi - \psi \nabla \varphi) \cdot d\vec{S}$$

Proof

We have Green's first identity

$$\iiint_R [\varphi \nabla^2 \psi + (\nabla \varphi)(\nabla \psi)] dV = \iint_S (\varphi \nabla \psi) \cdot d\vec{S} \quad (1)$$

Interchanging φ and ψ in equation (1), we obtain

$$\iiint_R [\psi \nabla^2 \varphi + (\nabla \psi)(\nabla \varphi)] dV = \iint_S (\psi \nabla \varphi) \cdot d\vec{S} \quad (2)$$

Subtracting equation (2) from (1), we have

$$\begin{aligned} \iiint_R [\psi \nabla^2 \varphi + (\nabla \psi)(\nabla \varphi)] - [\varphi \nabla^2 \psi + (\nabla \varphi)(\nabla \psi)] dV &= \iint_S (\psi \nabla \varphi) - (\varphi \nabla \psi) \cdot d\vec{S} \\ \iiint_R [\varphi \nabla^2 \psi - \psi \nabla^2 \varphi] dV &= \iint_S (\varphi \nabla \psi - \psi \nabla \varphi) \cdot d\vec{S} \end{aligned}$$

which is called Green's second identity or symmetrical theorem.

Alternative Forms of Green's Second Identity

We know that

$$\nabla \psi \cdot \hat{n} = \frac{\partial \psi}{\partial n} \text{ and } \nabla \varphi \cdot \hat{n} = \frac{\partial \varphi}{\partial n}$$

Thus

$$\nabla \psi \cdot d\vec{S} = \nabla \psi \cdot \hat{n} dS = \frac{\partial \psi}{\partial n} dS$$

and

$$\nabla\varphi \cdot d\vec{S} = \nabla\varphi \cdot \hat{n}dS = \frac{\partial\varphi}{\partial n}dS$$

Hence the Green's Second Identity can be written as

$$\iiint_R [\varphi \nabla^2 \psi - \psi \nabla^2 \varphi] dV = \iint_S \left(\varphi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \varphi}{\partial n} \right) dS$$

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Module No. 75

Related Example: Green's Theorem

Problem Statement

Evaluate

$$\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2dy$$

along the path $x^4 - 6xy^3 = 4y^2$.

Solution

A direct evaluation is difficult. By comparing it with Green's Theorem, we get

$$\oint_C Mdx + Ndy = \int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2dy$$

here $M = 10x^4 - 2xy^3$ and $N = -3x^2y^2$ and

$$\frac{\partial M}{\partial y} = -6x^2y = \frac{\partial N}{\partial x}$$

it follows that the integral is independent of the path. Then we can use any path, for example the path consisting of straight line segments from (0,0) to (2,0) and then from (2,0) to (2,1).

Along the straight line path from (0,0) to (2,0), $y = 0$, $dy = 0$ and the integral equals

$$\int_{x=0}^2 10x^4 dx = \frac{10}{5} x^5 = 2(32) = 64$$

Along the straight line path from (2,0) to (2,1), $x = 2$, $dx = 0$ and the integral equals

$$\int_{y=0}^1 -12y^2 dy = -12 \left(\frac{y^3}{3} \right) = -4(1) = -4$$

Then the value of the line integral

$$\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2dy = 64 - 4 = 60$$

is the required solution.

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Module No. 76

Selected Problem 1: Green's Theorem

Problem Statement

Prove that

$$\oint Mdx + Ndy = 0$$

$Mdx + Ndy = 0$ around every closed curve C in a simply-connected region if and only

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

everywhere in the region.

Proof

Assume that M and N are continuous and have continuous partial derivatives everywhere in the region R bounded by C , so that Green's theorem is applicable.

Then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$

Sufficient: If

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

in R , then Clearly

$$\oint Mdx + Ndy = 0$$

Necessity: suppose

$$\oint Mdx + Ndy = 0$$

for all curves C . If $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} > 0$ at a point P , then from the continuity of the derivatives it follows that $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} > 0$ in some region A surrounding P . If Γ is the boundary of A then

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy > 0$$

which contradicts the assumption that the line integral is zero around every closed curve.

Similarly the assumption $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} < 0$ leads to a contradiction. Thus $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ at all points on R .

Module No. 77

Selected Problem 2: Green's Theorem

Problem Statement

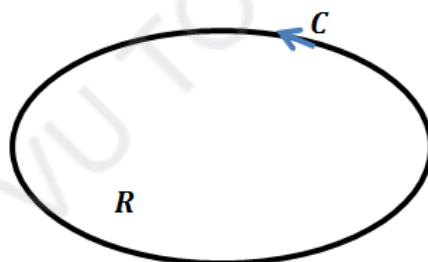
Show that the area bounded by a simple closed curve C is given by

$$\frac{1}{2} \oint_C xdy - ydx$$

Proof

Since Green's theorem is

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy$$



Put $M = -y$ and $N = x$ in Green's theorem, we get

$$\oint_C xdt - ydx = \iint_R \left(\frac{\partial(x)}{\partial x} - \frac{\partial(-y)}{\partial y} \right) dxdy = 2 \iint_R dxdy = 2A$$

$$\oint_C xdt - ydx = 2A$$

or

$$\text{Area} = A = \frac{1}{2} \oint_C xdy - ydx$$

Hence the result.

Now we will illustrate this formula through an example

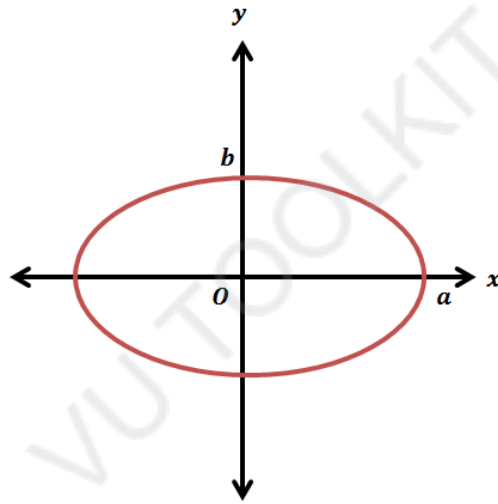
Example Statement

Find the area of the ellipse $x = a \cos \theta, y = b \sin \theta$

Solution

By making use of above result,

$$\begin{aligned} \text{Area} = A &= \frac{1}{2} \oint_C x dy - y dx \\ &= \frac{1}{2} \int_0^{2\pi} (a \cos \theta)(b \cos \theta) d\theta - (b \sin \theta)(-a \sin \theta) d\theta \end{aligned}$$



$$\begin{aligned} &= \frac{1}{2} \int_0^{2\pi} ab \cos^2 \theta d\theta + ab \sin^2 \theta d\theta = \frac{1}{2} \int_0^{2\pi} ab (\cos^2 \theta + \sin^2 \theta) d\theta \\ &= ab \cdot \frac{1}{2} \int_0^{2\pi} d\theta = ab\pi \end{aligned}$$

is the required area.

Module No. 78

Selected Problem 3: Green's Theorem

Problem Statement

Evaluate

$$\oint_C (y - \sin x)dx + \cos x dy$$

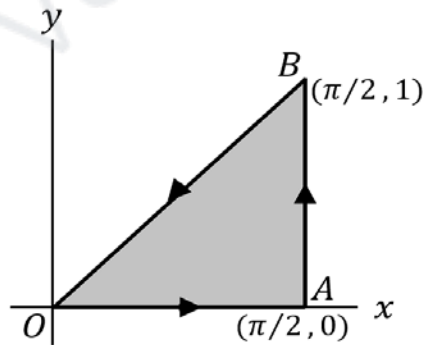
where C is the triangle of the adjoining figure:

- Directly,
- By using Green's theorem in the plane.

Solution

- Along OA, $y = 0, dy = 0$ and the integral equals

$$\int_0^{\pi/2} (0 - \sin x)dx + \cos x (0) = \int_0^{\pi/2} -\sin x dx = \cos x \Big|_0^{\pi/2} = -1$$



Along AB, $x = \pi/2, dx = 0$ and the integral equals to

$$\int_0^1 (y - 1)(0) + 0dy = 0$$

Along BO , $y = \frac{2x}{\pi}$, $dy = \frac{2}{\pi} dx$ and the integral equals

$$\begin{aligned} \int_{\pi/2}^0 \left(\frac{2x}{\pi} - \sin x \right) dx + \frac{2}{\pi} \cos x dx &= \left| \frac{x^2}{\pi} + \cos x + \frac{2}{\pi} \sin x \right|_{x=\pi/2}^0 \\ &= 1 - \frac{\pi}{4} - \frac{2}{\pi} \end{aligned}$$

Then the integral along $C = -1 + 0 + 1 - \frac{\pi}{4} - \frac{2}{\pi} = -\frac{\pi}{4} - \frac{2}{\pi}$

Hence

$$\oint_C (y - \sin x) dx + \cos x dy = -\frac{\pi}{4} - \frac{2}{\pi}$$

ii. Since the green's theorem is

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx dy \quad (1)$$

by comparing with the given integral, we get

$$\oint_C M dx + N dy = \oint_C (y - \sin x) dx + \cos x dy$$

$$M = y - \sin x, \quad N = \cos x, \quad \frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -\sin x$$

then the equation (1) becomes

$$\begin{aligned} &= \oint_C (y - \sin x) dx + \cos x dy = \iint_R (-\sin x - 1) dx dy \\ &= \int_{x=0}^{\pi/2} \left[\int_{y=0}^{\frac{2x}{\pi}} (-\sin x - 1) dy \right] dx \\ &= \int_{x=0}^{\pi/2} \left[-y \sin x - y \right]_0^{\frac{2x}{\pi}} dx \\ &= \int_{x=0}^{\pi/2} \left(-\frac{2x}{\pi} \sin x - \frac{2x}{\pi} \right) dx \end{aligned}$$

$$\left| -\frac{2}{\pi}(-x \cos x + \sin x) - \frac{\pi^2}{2} \right|_{x=0}^{x=\pi/2}$$

$$= -\frac{\pi}{4} - \frac{2}{\pi}$$

in agreement with part (i).

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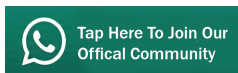


Lecture Handouts

on

VECTORS AND CLASSICAL MECHANICS (MTH-622)

**Virtual University of Pakistan
Pakistan**



About the Handouts

The following books have been mainly followed to prepare the slides and handouts:

1. Spiegel, M.R., *Theory and Problems of Vector Analysis: And an Introduction to Tensor Analysis*. 1959: McGraw-Hill.
2. Spiegel, M.S., *Theory and problems of theoretical mechanics*. 1967: Schaum.
3. Taylor, J.R., *Classical Mechanics*. 2005: University Science Books.
4. DiBenedetto, E., *Classical Mechanics: Theory and Mathematical Modeling*. 2010: Birkhäuser Boston.
5. Fowles, G.R. and G.L. Cassiday, *Analytical Mechanics*. 2005: Thomson Brooks/Cole.

The first two books were considered as main text books. Therefore the students are advised to read the first two books in addition to these handouts. In addition to the above mentioned books, some other reference book and material was used to get these handouts prepared.

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Module No. 79

Introduction to Classical Mechanics

Mechanics is the science of motion. It studies the states of rest and motion and the laws governing rest, equilibrium and motion.

A broad division of mechanics is made through the terms classical mechanics and quantum mechanics.

Classical Mechanics

The science of classical mechanics deals with the motion of objects through absolute space and time in the Newtonian sense. Classical mechanics deals with the macroscopic object i.e. those bodies which we encounter in everyday life. Such bodies may be those bodies in the form of billiard balls, parts of machinery and astronomical bodies.

Classical mechanics itself has two broad divisions in the form of non-relativistic and relativistic mechanics. Non-relativistic mechanics, based on the laws of Newton is concerned with bodies moving at speeds and velocity negligibly small as compared to the speed of light $c = 3 \times 10^5$ per kilometer per second. Relativistic mechanics is dispensable for particles and bodies moving with speed comparable to the speed of light.

Quantum mechanics

The quantum mechanics is the mechanics of microscopic object. Its basic concepts, principals and laws are of entirely different from those of classical mechanics.

Division of Classical Mechanics

Three major divisions of classical mechanics are the following:

Mechanics of particles and rigid bodies: It is based on newton's law. Basic concepts and terms are space, time and mass; particle and body; velocity, momentum and acceleration; force and energy.

Mechanics of fluid: it is also based on newton's law and their extensions and deal with the behavior of the fluid (liquid and gases) in motion. Its two well-known branches are hydrodynamics (for fluid) and Aerodynamics (for gases).

Mechanics of elastic solids: it deals with the behavior of solids when they undergo deformation under forces.

Module No. 80

Introductions to the Basics of Newton's Law

Newton's three laws of motion are formulated in terms of crucial underlying concepts: the notations of space, time, mass and force.

In this section we will review all of the factors of formulation of Newton's law.

Space and time:

We shall assume that space and time are described strictly in the Newtonian sense.

- **Space.** This is closely related to the concepts of point, position, direction and displacement sense. Three-dimensional space is Euclidian, and positions of points in that space are specified by a set of three numbers (x,y,z) relative to the origin (0,0,0) of a rectangular Cartesian coordinate system. Measurement in space involves the concepts of length or distance, with which we assume familiarity. Units of length are feet, meters, miles, etc.
- **Time.** This concept is derived from our experience of having one event taking place after, before or simultaneous with another event. Measurement of time is achieved, for example, by use of clocks. Units of time are seconds, hours, years, etc. the basic unit of time in SI, began as an arbitrary fraction ($1/86,400$) of a mean solar day ($24 \times 60 \times 60 = 86,400$).
- **Mass** Physical objects are composed of "small bits of matter" such as atoms and molecules. From this we arrive at the concept of a material object called a particle which can be considered as occupying a point in space and perhaps moving as time goes by. A measure of the "quantity of matter" associated with a particle is called its mass. The mass of an object is characterized by its inertia. It's the resistance to be accelerated. Units of mass are grams, kilograms, etc. Unless otherwise stated we shall assume that the mass of a particle does not change with time. Length, mass and time are often called dimensions from which other physical quantities are constructed.

- **Force** The informal notion of force as a push or pull. It is a vector quantity. As the unit of force we naturally adopt the newton (abbreviated N) defined as the magnitude of any single force that accelerates a standard kilogram mass with an acceleration of 1 m/s^2 . If we apply a given force F (and no other forces) to any object at rest, the direction of F is defined as the direction of the resulting acceleration, that is, the direction in which the body moves off.

Now that we know, at least in principle, what we mean by positions, times, masses, and forces, we can proceed to discuss the cornerstone of our subject — Newton's three laws of motion.

Module No. 81

Introductions to Rectangular Components of Velocity & Acceleration

Rectangular Components

The process of splitting a vector into various parts or components is called “Resolution of vector” and these parts are called components of vector. If we split a vector in a rectangular plane OXY, such components are called rectangular components of a vector.

Component Along x-axis is called horizontal component of vector.

Component Along y-axis is called vertical component of vector.

Position vector

It is often convenient to describe the motion of a particle in terms of its x, y or rectangular components, relative to a fixed frame of reference. In a given reference system, the position of a particle can be specified by a single vector, namely, the displacement of the particle relative to the origin of the coordinate system. This vector is called the position vector of the particle. In rectangular coordinates (see Figure), the position vector is simply

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$$

The components of the position vector of a moving particle are functions of the time, namely,

$$x = x(t), y = y(t)$$

RECTANGULAR COMPONENTS: VELOCITY

In above Equation we gave the formal definition of the derivative of any vector with respect to some parameter. In particular, if the vector is the position vector $\mathbf{r}$ of a moving particle and the parameter is the time t , the derivative of $\mathbf{r}$ with respect to t is called the velocity, which we shall denote by

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k}$$

where the dots indicate differentiation with respect to t .

where

$$v_x = \dot{x}, v_y = \dot{y}, v_z = \dot{z}$$

The magnitude of the velocity is called the speed. In rectangular components the speed is just

The magnitude of the velocity vector is

$$v = (v_x^2 + v_y^2 + v_z^2)^{1/2}$$

The direction of $\mathbf{v}$ is tangent to the path of motion.

RECTANGULAR COMPONENTS: ACCELERATION

Acceleration represents the rate of change in the velocity of a particle.

If a particle's velocity changes from $\mathbf{v}$ to $\mathbf{v}'$ over a time increment Δt , the average acceleration during that increment is: avg $\mathbf{a} = \Delta \mathbf{v} / \Delta t = (\mathbf{v} - \mathbf{v}') / \Delta t$ the instantaneous accelerations the time derivative of velocity:

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2\mathbf{r}}{dt^2} = i\mathbf{a}_x + j\mathbf{a}_y + k\mathbf{a}_z$$

In rectangular components,

$$\mathbf{a} = v_x \mathbf{i} + v_y \mathbf{j} + v_z \mathbf{k}$$

$$\mathbf{a} = i\ddot{x} + j\ddot{y} + k\ddot{z}$$

$$a_x = \ddot{x}, a_y = \ddot{y}, a_z = \ddot{z}$$

Thus, acceleration is a vector quantity whose components, in rectangular coordinates, are the second derivatives of the positional coordinates of a moving particle.

The direction of $\mathbf{a}$ is usually not tangent to the path of the particle.

Module No. 82

Introduction to Tangential and Normal Components of Velocity & Acceleration

Introduction To Tangential And Normal Components Of Vector

In mathematics, given a vector at a point on a curve, that vector can be decomposed uniquely as a sum of two vectors, one tangent to the curve, called the tangential component of the vector, and another one perpendicular to the curve, called the normal component of the vector.

Introduction to tangential and normal components of velocity

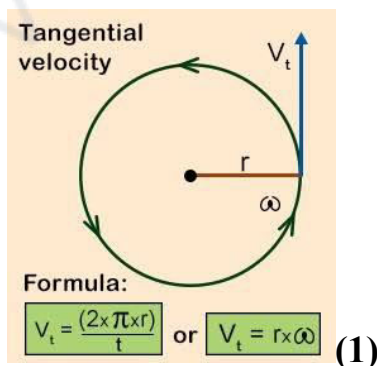
The tangential velocity is the velocity measured at any point tangent to a circular path. Thus tangential velocity, V_t is related to the angular velocity of the wheel, ω , and the radius of the wheel, r .

$$V_t = \omega r$$

V_t = tangential velocity

ω = angular velocity

r = radius of wheel



Total velocity of a particle at a point on a curve can be write as

$$v = v_t + v_N$$

We can determine normal component of velocity by

$$v_N = v - v_t$$

Tangential and normal components of Acceleration

The rate of change of tangential velocity of an object traveling in a circular orbit or path is known as Tangential acceleration. It is directed towards tangent to the path of a body.

Total acceleration of a particle at a point on a curve can be write as

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_N$$

where

$$a_t = \frac{dv_t}{dt}, a_N = \frac{v^2}{r}$$

where v is total velocity of the object and r is radius of circular path.

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Module No. 83

Example of Tangential and Normal Acceleration

Example

Consider the space curve C with the position vector $\mathbf{r}$ given to be at time t

$$\mathbf{r} = 5 \cos 3t \mathbf{i} + 3 \sin 3t \mathbf{j} + 10t \mathbf{k}$$

(a) Find a unit tangent vector $\mathbf{T}$ to the curve.

(b) Verify that $\mathbf{v} = v \mathbf{T}$.

Solution:

(a) Tangent vector to the space curve C is

$$\frac{d\mathbf{r}}{dt} = -15 \sin 3t \mathbf{i} + 9 \cos 3t \mathbf{j} + 10 \mathbf{k}$$

The magnitude of this vector is

$$\begin{aligned} v = \left| \frac{d\mathbf{r}}{dt} \right| &= \frac{ds}{dt} = \sqrt{(-15 \sin 3t)^2 + (9 \cos 3t)^2 + (10)^2} \\ &= \sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100} \end{aligned}$$

The unit tangent vector to C is

$$\mathbf{T} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|} = \frac{-15 \sin 3t \mathbf{i} + 9 \cos 3t \mathbf{j} + 10 \mathbf{k}}{\sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100}}$$

(b) From the first part, it follows that This follows directly from (a) since

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = -15 \sin 3t \mathbf{i} + 9 \cos 3t \mathbf{j} + 10 \mathbf{k}$$

$$\begin{aligned}
 v &= -15 \sin 3t \, i + 9 \cos 3t \, j + 10k \frac{\sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100}}{\sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100}} \\
 v &= \sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100} \frac{(-15 \sin 3t \, i + 9 \cos 3t \, j + 10k)}{\sqrt{225 \sin^2 3t + 81 \cos^2 3t + 100}} \\
 &= v \, T
 \end{aligned}$$

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Module No. 84

Curvature and Radius of Curvature

Curvature is the amount by which a geometric object such as a surface differs from being a flat plane, or a curve from being straight as in the case of a line.

Statement:

If T is a unit tangent vector to a space curve C , then show that $\frac{dT}{ds}$ is normal to T .

Proof:

Since T is a unit vector, we have $T \cdot T = 1$. Then differentiating with respect to s , we obtain

$$T \cdot \frac{dT}{ds} + \frac{dT}{ds} \cdot T = 0$$

$$2 \frac{dT}{ds} \cdot T = 0$$

$$\frac{dT}{ds} \cdot T = 0$$

Which states that $\frac{dT}{ds}$ is normal, i.e. perpendicular to T .

If N is a unit vector in the direction of $\frac{dT}{ds}$, we have

$$\frac{dT}{ds} = kN$$

and we call N the unit principal normal to C . The scalar $k = \left| \frac{dT}{ds} \right|$ is called the curvature,

while $R = 1/k$ is called the radius of curvature.

Example:

Find the

- Curvature,
- Radius of curvature
- Unit principal normal N to any point of the space curve of given unit tangent vector is

$$T = -\frac{3}{5} \sin 2t \mathbf{i} + \frac{3}{5} \cos 2t \mathbf{j} + \frac{4}{5} \mathbf{k}$$

moving with 10 ms^{-1} .

Solution:

Since given tangential vector is

$$T = -\frac{3}{5} \sin 2t \mathbf{i} + \frac{3}{5} \cos 2t \mathbf{j} + \frac{4}{5} \mathbf{k}$$

Then,

$$\begin{aligned} \frac{dT}{ds} &= \frac{\frac{dT}{dt}}{\frac{ds}{dt}} = \frac{-\left(\frac{6}{5}\right) \cos 2t \mathbf{i} - \left(\frac{6}{5}\right) \sin 2t \mathbf{j}}{10} \\ &= -\frac{3}{25} \cos 2t \mathbf{i} - \frac{3}{25} \sin 2t \mathbf{j} \end{aligned}$$

Thus the curvature is $k = \left| \frac{dT}{ds} \right| = \sqrt{\left(-\frac{3}{25} \cos 2t\right)^2 + \left(-\frac{3}{25} \sin 2t\right)^2} = \frac{3}{25}$

Radius of curvature $= \frac{1}{k} = \frac{25}{3}$

We have unit principal normal $N = \frac{1}{k} \cdot \frac{dT}{ds} = \frac{25}{3} \left(-\frac{3}{25} \cos 2t \mathbf{i} - \frac{3}{25} \sin 2t \mathbf{j}\right)$
 $= -\cos 2t \mathbf{i} - \sin 2t \mathbf{j}$

Module No. 85

Introduction to Radial and Transverse Components of Velocity & Acceleration

It is often convenient to employ polar coordinates r, θ to express the position of a particle moving in a plane. Vectorially, the position of the particle can be written as the product of the radial distance r by a unit radial vector:

$$\mathbf{r} = r\mathbf{e}_r$$

As the particle moves, both r and $\mathbf{e}_r$ vary; thus, they are both functions of the time.

Hence, if we differentiate with respect to t , we have

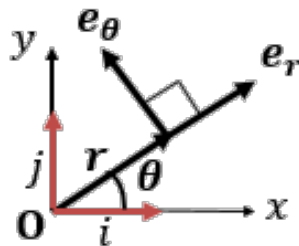
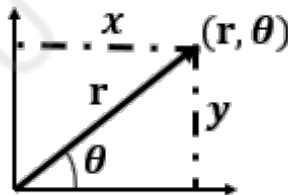
$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{r}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt}$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r$$

From fig $\mathbf{e}_r = \cos \theta \mathbf{i} + \sin \theta \mathbf{j}$ and $\mathbf{e}_\theta = -\sin \theta \mathbf{i} + \cos \theta \mathbf{j}$

So, $\dot{\mathbf{e}}_r = -\sin \theta \dot{\theta} \mathbf{i} + \cos \theta \dot{\theta} \mathbf{j}$ or $\dot{\mathbf{e}}_r = (-\sin \theta \mathbf{i} + \cos \theta \mathbf{j})\dot{\theta}$

This implies $\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{e}_\theta$



$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$$\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

Thus, $\dot{r}$ is the radial component of the velocity vector, and $r\dot{\theta}$ is the transverse component.

To find the acceleration vector, we take the derivative of the velocity with respect to time. This gives

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{r}\mathbf{e}_r + \dot{r}\frac{d\mathbf{e}_r}{dt} + (\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta + r\dot{\theta}\frac{d\mathbf{e}_\theta}{dt}$$

The values of $\frac{d\mathbf{e}_r}{dt}$ and $\frac{d\mathbf{e}_\theta}{dt}$ are given by Equations

$$\dot{\mathbf{e}}_r = \dot{\theta}\mathbf{k} \times \mathbf{e}_r = \dot{\theta}\mathbf{e}_\theta \text{ and } \dot{\mathbf{e}}_\theta = \dot{\theta}\mathbf{k} \times \mathbf{e}_\theta = -\dot{\theta}\mathbf{e}_r$$

and yield the following equation for the acceleration vector in plane polar coordinates:

$$\underline{a} = (\ddot{r} - r\dot{\theta}^2)\underline{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\underline{e}_\theta$$

Thus, the radial component of the acceleration vector is

$$a_r = \ddot{r} - r\dot{\theta}^2$$

and the transverse component is

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r} \frac{d}{dt}(r^2\dot{\theta})$$



Module No. 86

Example of Radial and Transverse Components of Velocity and Acceleration

Statement:

On a horizontal turntable that is rotating at constant angular speed, a bug is crawling outward on a radial line such that its distance from the center increases quadratic ally with time: $r = bt^2$, $\theta = \omega t$, where b and ω are constants. Find the acceleration of the bug.

Solution:

The equation of acceleration in plane polar coordinates is given by

$$\mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta \quad (1)$$

As we have $r = bt^2$ and $\theta = \omega t$, therefore $\dot{r} = 2bt$, $\ddot{r} = 2b$, $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$.

By putting these values in equation (1), we have

$$\begin{aligned} \mathbf{a} &= (2b - bt^2\omega^2)\mathbf{e}_r + (bt^2(0) + 2(2bt)\omega)\mathbf{e}_\theta \\ \mathbf{a} &= b(2 - t^2\omega^2)\mathbf{e}_r + 4bt\omega\mathbf{e}_\theta \end{aligned}$$

Which is required acceleration of bug.

Module No. 87

Reference System or Inertial Frame

Framework

A framework that is used for the observation and mathematical description of physical phenomena and the formulation of physical laws, usually consisting of an observer, a coordinate system, and a clock or clocks assigning times at positions with respect to the coordinate system. a system of geometric axes in relation to which measurements of size, position, or motion can be made.

Inertial frames

A frame of reference in which a body remains at rest or moves with constant linear velocity unless acted upon by forces: any frame of reference that moves with constant velocity relative to an inertial system is itself an inertial system. These are also called inertial frame.

Mechanics is based on Newton's laws of motion. These laws are usually stated without explicitly discussing the role of frames of references. It is to be emphasized that Newton's laws are not valid w.r.t every frame of references e.g. they are not true in a rotating frame or coordinate system. They hold only in restricted class of frames, so called the inertial frames (Newtonian frames). Such frames are assumed to be un-accelerated i.e. neither moving in straight line with variable velocity nor rotating. In other words, it is supposed to be absolutely at rest (or moving with uniform velocity w.r.t absolutely-at-rest frames). We can easily see that if S is an inertial frame, then any other frame S' which is in uniform motion relative to S is also an inertial frame. To all observers in inertial frames, the force acting on a particle will be the same i.e. the law of motion will be invariant under transformation connecting inertial frames. In fact all laws of mechanics are same in all frames of reference. This statement is called Newtonian principal of relativity.

Module No. 88

Example of inertial Frame

In order to illustrate the concept of inertia, we will consider the example of earth.

Statement:

Calculate the centripetal acceleration relative to the acceleration due to gravity g , of

- a) A point on the surface of the Earth's equator (the radius of the Earth is $R_e = 6.4 \times 10^6 \text{ km}$)
- b) The Earth in its orbit about the Sun (the radius of the Earth's orbit is
 - a. $a_e = 150 \times 10^6 \text{ km}$)
- c) The Sun in its rotation about the center of the galaxy (the radius of the Sun's orbit about the center of the galaxy is $R_G = 2.8 \times 10^4 \text{ LY}$. its orbital speed is $v_G = 220 \text{ km/s}$)

Solution:

The centripetal acceleration of a point rotating in a circle of radius R is given by

$$a_c = \omega^2 R = \left(\frac{2\pi}{T}\right)^2 R = \frac{4\pi R^2}{T^2}$$

where T is period of one complete rotation ($T = 3.16 \times 10^7$). Thus, relative to g (gravitational acceleration), we have

$$\frac{a_c}{g} = \frac{4\pi^2 R^2}{gT^2} \quad (1)$$

- a) By putting value from (a) in (1), we get

$$\begin{aligned} \frac{a_c}{g} &= \frac{4(3.14)^2(6.4 \times 10^6)^2}{(9.8)(3.16 \times 10^7)^2} \\ \frac{a_c}{g} &= 3.4 \times 10^{-3} \end{aligned}$$

- b) By putting values from (b) in (1), here

$$R = a_e = 150 \times 10^6 \text{ km}$$

$$\frac{a_c}{g} = \frac{4(3.14)^2(150 \times 10^6)^2}{(9.8)(3.16 \times 10^7)^2}$$

$$\frac{a_c}{g} = 6 \times 10^4$$

c) By putting values from (c),

$$R = R_G = 2.8 \times 10^4 \text{ LY}$$

$$\frac{a_c}{g} = \frac{4(3.14)^2(150 \times 10^6)^2}{(9.8)(2.8 \times 10^4)^2}$$

$$\frac{a_c}{g} = 1.5 \times 10^{-12}$$

Module No. 89

Newton's First and Second Laws

Sir Issac Newton derives the Newton's laws of motion which are the three physical laws which together forms the foundation of classical mechanics.

Newton's 1st law of Motion

Newton's 1st law of motion is also called the law of inertia. This law measures the force of an object qualitatively.

We can state it as

“In the absence of an external force, an object continues its state of rest or motion with constant velocity.”

The symbolic form of first law of motion is

$$\sum F = 0 \Rightarrow ma = 0$$

as the mass of the object is non-zero, therefore the acceleration of the concerned object must be zero

$$a = 0 \Rightarrow \frac{dv}{dt} = 0 \Rightarrow v = \text{constant}$$

As the consequence of first law of motion

- An object that is at rest will stay at rest unless a force acts upon it.
- An object continues its constant motion until an external force abrupt its state of motion.

The first law of motion says that if the net force acting on a body is zero then the velocity of object is said to be constant. The velocity of an object is vector quantity so if we say that the velocity of an object is constant we conclude that the speed and direction of the object is fixed.

Examples

Some of the examples of first law of motion are

- i. A ball kicked in a ground.
- ii. A car moving with constant velocity
- iii. A book lying in a book shelf.

Newton's 2nd law of Motion

Newton's 2nd law of motion describes the relationship among the force, mass and acceleration of the given object.

We can state the 2nd law of motion as

For any particle of mass m , the net force F on the particle of mass m times the particle's acceleration.

The symbolic form of first law of motion is

$$F = ma$$

The second law can be rephrased in terms of the particle's momentum, defined as

$$p = mv$$

by differentiating we get

$$\dot{p} = m\dot{v} = ma$$

hence we can state it as

$$F = \dot{p}$$

Consequently we can say the net force applied on a body is equals to the rate of change of momentum of the particle.

Examples

- i. When we apply same force to move a truck and a bicycle, the bicycle will have more acceleration than the truck, because the mass of bicycle is less than the truck.
- ii. An empty shopping cart is much easier to move than a full one, because the empty one has less mass.

Module No. 90

The Third Law and Law of Conservation of Momentum

Newton's first two laws concern the response of a single object to applied forces. The third law addresses a quite different issue: Every force on an object inevitably involves a second object the object that exerts the force. The nail is hit by the hammer; the cart is pulled by the horse, and so on. Newton realized that if an object 1 exerts a force on another object 2, then object 2 always exerts a force (the "reaction" force) back on object 1.

Newton's third law can be stated very compactly:

To every action there is an equal and opposite reaction.

The third law states that all forces between two objects exist in equal magnitude and opposite direction: if one object 1 exerts a force F_{12} on a second object 2, then Simultaneously exerts a force F_{21} on 1, and the two forces are equal in magnitude and opposite in direction: $F_{12} = -F_{21}$. This law is sometimes referred to as the *action-reaction law*, with F_{12} called the "action" and F_{21} the "reaction".

In other situations the magnitude and directions of the forces are determined jointly by both bodies and it isn't necessary to identify one force as the "action" and the other as the "reaction". The action and the reaction are simultaneous, and it does not matter which is called the *action* and which is called *reaction*; both forces are part of a single interaction, and neither force exists without the other.

Examples

A variety of action-reaction force pairs are evident in nature.

- i. A fish's thrust through the water.
- ii. A bird's fly in the air.
- iii. A rocket's launch.
- iv. The car moving on a road.
- v. The nail hit by hammer.

Law of Conservation of Momentum

Definition of Linear Momentum

If a body of mass m is moving with velocity, then the momentum of that body is equals to the product of mass and velocity of the specific body.

Mathematically, we can write is as

$$\vec{p} = m\vec{v}$$

it is a vector quantity.

It S.I unit is $kg \cdot s^{-1}$.

Momentum of system of particles

The total momentum of a system of particles is the vector sum of the momenta of the individual particles:

$$\vec{p}_{system} = \vec{p}_1 + \vec{p}_2 + \cdots \cdots \cdots + \vec{p}_n = \sum_{i=1}^n \vec{p}_i$$

where the system consists of n particles. we can also express it as

$$\vec{p}_{system} = \sum_{i=1}^n \vec{p}_i = \sum_{i=1}^n m_i \vec{v}_i$$

where m_i is the mass of i th particle and $\vec{v}_i$ is corresponding velocity.

Statement “Law of conservation of momentum”

Law of conservation of momentum can be stated as:

If the sum of the external forces on a system is zero, the total momentum of the system does not change.

Mathematically, we can write is as

if

$$F_{ext} = 0 \Rightarrow \vec{p} = 0 \Rightarrow \sum_i m_i v_i = Mv = \text{constant}$$

Momentum is always conserved (even if forces are non-conservative).

As we stated earlier the second law of motion,

$$F_{ext} = ma$$

We can deduct the law of conservation of momentum from the last expression as well.

$$F_{ext} = 0 \Rightarrow ma = 0 \Rightarrow m \frac{dv}{dt} = 0$$

as mass is non-zero constant, therefore

$$\frac{dv}{dt} = 0 \text{ or } v = \text{constant} \Rightarrow mv = \text{constant.}$$

Explanation

In simple terms, momentum is considered to be a quantity of motion. This quantity is measurable because if an object is moving and has mass, then it has momentum. Something that has a large mass has a large momentum or something that is moving very fast has a large momentum. The momentum of individual component may change but the total momentum of system remains conserved.

Example

- i. A 3000 kg vehicle moving at 30 m/sec has a momentum of 90,000 kgm/sec as a result of product of the mass and the velocity.
- ii. Two hockey players of equal mass are traveling towards each other, one is moving at 9 m/sec and the other at 5 m/sec. The one moving with the faster velocity has a greater momentum and will knock the other one backwards.
- iii. A bullet fired from a gun, although small in mass, has a large momentum because of an extremely large velocity.

Module No. 91

Validity of Newton's Law

- It is known that the Newton's laws are not valid universally after the quantum mechanics and theory of relativity.
- Initially Newton's laws were considered as fundamental.
- But now quantum laws are fundamental and Newton's laws can be observed as derived from the quantum world.
- So, we can say that

Quantum Mechanics

Classical Mechanics

- In the Classical phenomena, Newton's first two laws are always valid.

Examples

- If the speed is extremely increased, even than first law remains valid.
- Newton's second law $F = ma$ and its another form in terms of the particle's momentum $p = mv$ are no longer equivalent in relativity.

Module No. 92

Example of Newton's Law 1, 2

NEWTON'S 1st LAW

Example:

Because of the force field, a particle of mass 4 moves along a space curve C whose position vector is given as a function of time t by

$$r = (3t^3 + t^2)\hat{i} + (t^4 + 8t)\hat{j} - 5t\hat{k}$$

Find

- i. The velocity,
- ii. The acceleration
- iii. The momentum,
- iv. The force field at any time t .

Solution:

- i. Velocity = $v = \frac{dr}{dt}$

$$= \frac{d}{dt}((3t^3 + t^2)\hat{i} + (t^4 + 8t)\hat{j} - 5t\hat{k})$$

$$= (9t^2 + 2t)\hat{i} + (4t^3 + 8)\hat{j} - 5\hat{k}$$
- ii. Acceleration = $a = \frac{dv}{dt} = \frac{d}{dt}((9t^2 + 2t)\hat{i} + (4t^3 + 8)\hat{j} - 5\hat{k})$

$$= (18t + 2)\hat{i} + 12t^2\hat{j}$$
- iii. Momentum = $p = mv = 4((9t^2 + 2t)\hat{i} + (4t^3 + 8)\hat{j} - 5\hat{k})$

$$= (36t^2 + 8t)\hat{i} + (16t^3 + 32)\hat{j} - 20\hat{k}$$
- iv. Force = $F = \frac{dp}{dt} = \frac{d}{dt}((36t^2 + 8t)\hat{i} + (16t^3 + 32)\hat{j} - 20\hat{k})$

$$= (72t + 8)\hat{i} + 48t^2\hat{j}$$

Newton's 2nd Law

Example:

A particle of mass 4 units moves in a force field given by

$$F = 4t^2\hat{i} + (16t - 4)\hat{j} - 12t\hat{k}$$

Assume that initially, i.e at $t = 0$ the particle is located at $r_0 = i + 2j - 3k$ and has velocity $v_0 = i + 4j - 9k$, find

- The velocity
- The position at any time t .

Solution:

- Since we know that

$$F = ma$$

As we have $m = 4$, therefore

$$F = 4a = 4 \frac{dv}{dt} = 4t^2\hat{i} + (16t - 4)\hat{j} - 12t\hat{k}$$

$$\frac{dv}{dt} = t^2\hat{i} + (4t - 1)\hat{j} - 3t\hat{k}$$

Integrating the above equation gives the velocity

$$v = \frac{1}{3}t^3\hat{i} + (2t^2 - t)\hat{j} + \frac{3}{2}t^2\hat{k} + C$$

where C is the constant of integration.

It is given that at $t = 0$, $v_0 = i + 4j - 9k$ at $t = 0$, therefore

$$i + 4j - 9k = C$$

So,

$$v = \frac{1}{3}t^3\hat{i} + (2t^2 - t)\hat{j} + \frac{3}{2}t^2\hat{k} + i + 4j - 9k$$

$$v = \left(\frac{1}{3}t^3 + 1\right)\hat{i} + (2t^2 - t + 4)\hat{j} + \left(\frac{3}{2}t^2 - 9\right)\hat{k}$$

- Since we know that

$$v = \frac{d\vec{r}}{dt} = \left(\frac{1}{3}t^3 + 1\right)\hat{i} + (2t^2 - t + 4)\hat{j} + \left(\frac{3}{2}t^2 - 9\right)\hat{k}$$

There integrating w.r.to t gives the position

$$\vec{r} = \left(\frac{1}{12}t^4 + t\right)\hat{i} + \left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 4t\right)\hat{j} + \left(\frac{1}{2}t^3 - 9t\right)\hat{k} + C'$$

As $r_0 = i + 2j - 3k$ at $t = 0$, we get $C' = i + 2j - 3k$

$$\vec{r} = \left(\frac{1}{12}t^4 + t\right)\hat{i} + \left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 4t\right)\hat{j} + \left(\frac{1}{2}t^3 - 9t\right)\hat{k} + i + 2j - 3k$$

$$\vec{r} = \left(\frac{1}{12}t^4 + t + 1\right)\hat{i} + \left(\frac{2}{3}t^3 - \frac{1}{2}t^2 + 4t + 2\right)\hat{j} + \left(\frac{1}{2}t^3 - 9t - 3\right)\hat{k}$$

VU TOOLKIT

Module No. 93

Example of Newton's Law: 3

Example 1:

Determine the constant force needed to bring a 4000 lb mass moving at a speed of 90 ft/sec to rest in 3 sec?

Solution:

We know that the motion can be in any direction but for simplicity, we assume that the mass is moving in a straight axis. We have the information that initially the mass is moving, and finally it has to be at rest, therefore we can write as

$$V_1 = 90 \text{ ft./sec,}$$

$$V_2 = 0 \text{ ft./sec,}$$

where

$$t = 3 \text{ sec}$$

Then using newton's law $F = ma$

$$\text{Where } a = \frac{V_2 - V_1}{t} \quad (\text{by using first equation of motion})$$

$$\text{Now, } F = m \left(\frac{V_2 - V_1}{t} \right) = (4000) \left(\frac{0 - 90}{3} \right) = -120000 \text{ ft lb/sec}^2$$

This is important to note that there is a negative sign in the expression for F . This negative sign is because of the fact that the force is opposite to the direction of motion.

Example

A person of mass 85 kg is standing in a lift which is accelerating downwards at 0.45 ms^{-2} .

Draw a diagram to show the forces acting on the person and calculate the force the person exerts on the floor of the lift. (© mathcentre 2009)

Solution

The weight of the person is $85\,g$, where g is the acceleration due to gravity.
where

$$W = 85\,g = 85 \times 9.81\,N$$

$$a = 0.45\,\text{ms}^{-2}$$

The resultant force is $W - R$ and using Newton's second law gives:

$$F = ma$$

$$W - R = F$$

$$W - R = 85 \times 0.45$$

$$R = W - 85 \times 0.45$$

$$R = 85 \times 9.81 - 85 \times 0.45$$

$$R = 795.6\,N = 800\,N$$

Therefore, using Newton's third law the force the person exerts on the floor of the lift is equal to the force of the floor acting on the person, i.e. R , which equals $800\,N$.

Module No. 94

Introduction to Energy: Kinetic Energy

Energy:

Energy is defined as the ability to do the work by the object. It is a measurable characteristic of a system which may be in the form of kinetic energy or potential.

There exist many forms of energy. The energy neither can be created nor be destroyed but can be converted from one form to another.

In mechanics, energy is the characteristic that transferred from one particle to another. The SI unit of energy is the joule; 1 joule can be defined as the energy transferred to an object by the work done of moving it a distance of 1 meter against a force of 1 newton.

The forms of energy include kinetic energy, potential energy, elastic energy, chemical energy, thermal energy and many others.

Kinetic energy is the energy stored in a body due to its motion. It can be transferred from one objects to another and transformed into other kinds of energy.

In classical mechanics, the kinetic energy is equal to 1/2 the product of the mass and the square of the speed. In formula form:

$$K.E = T = \frac{1}{2}mv^2$$

The measuring unit of kinetic energy is the joule.

It is denoted by T .

The kinetic energy increases with the square of the velocity. If a car is moving with double velocity then we can say that it has four times as much kinetic energy. As a consequence of this quadrupling, it takes four times the work to double the velocity.

if p denotes momentum of the object and m is the mass then we can symbolize the kinetic energy in the form of momentum as

$$T = \frac{p^2}{m}$$

Module No. 95

Introduction to Work – Theorem

When some external force is applied on an object, work is done by this force in the direction of force. Also when some work is done by the applied force, energy transferred from one place to another.

The work done can be defined as a product of force and the displacement in the direction of applied force. The amount of work done can be expressed as the following equation:

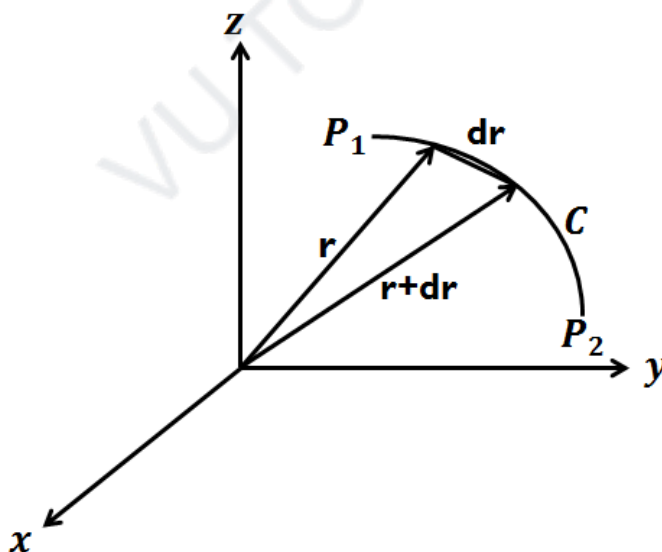
$$\text{Work} = \text{Applied Force} \times \text{Distance}$$

The SI unit of work is the joule (J), which is defined as the work done by a force of one newton through a displacement of one meter.

If a force $\vec{F}$ acting on a particle gives it a displacement $d\vec{r}$, then the work done by the force on the particle is defined as

$$dW = \vec{F} \cdot d\vec{r}$$

Since only the component of $\vec{F}$ in the direction of $d\vec{r}$ is effective in producing the motion.



The total work done by a force field (vector field) F in moving the particle from point P_1 to point P_2 along the curve C of Fig. is given by the line integral

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r}$$

Where r_1 and r_2 are the position vectors of P_1 and P_2 respectively.

Theorem Statement

A particle of constant mass m moves in space under the influence of a force field F . Assuming that at times t_1 and t_2 the velocity is $\vec{v}_1$ and $\vec{v}_2$ respectively, prove that the work done is the change in kinetic energy, i.e.,

$$\text{Work done} = \int_{t_1}^{t_2} \vec{F} \cdot d\vec{r} = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2$$

Proof

$$\text{L. H. S} = \text{Work done} = \int_{t_1}^{t_2} \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{F} \cdot \frac{d\vec{r}}{dt} dt$$

$$= \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt$$

$$\therefore \frac{d\vec{r}}{dt} = \vec{v}$$

$$= \int_{t_1}^{t_2} m\vec{a} \cdot \vec{v} dt$$

$$\therefore F = ma$$

$$= m \int_{t_1}^{t_2} \frac{d\vec{v}}{dt} \cdot \vec{v} dt = m \int_{t_1}^{t_2} \vec{v} \cdot d\vec{v}$$

$$= \frac{1}{2}m \int_{t_1}^{t_2} d(\vec{v} \cdot \vec{v}) = \frac{1}{2}m|\vec{v}^2|_{t_1}^{t_2}$$

$$\text{Work done} = \frac{1}{2}m\vec{v}_2^2 - \frac{1}{2}m\vec{v}_1^2$$

Hence the required equation.

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Module No. 96

Example of Work Done

Example 1:

Find the work done in moving an object along a vector $r = 8\hat{i} + 2\hat{j} - 5\hat{k}$ if the applied force is $F = 2\hat{i} - \hat{j} - \hat{k}$.

Solution:

As we know the formula for the work done is

$$\begin{aligned} W &= \mathbf{F} \cdot \mathbf{r} \\ &= (2\hat{i} - \hat{j} - \hat{k}) \cdot (8\hat{i} + 2\hat{j} - 5\hat{k}) \\ &= 16 - 2 + 5 = 19 \text{ joule} \end{aligned}$$

Work done = 19 joule

Example 2:

Find the work done by the force of a body of mass $m = 10^4 \text{ kg}$ with initial velocity $v_1 = 1.5 \times 10^3 \text{ ms}^{-1}$ to the final velocity $v_2 = 3.0 \times 10^3 \text{ ms}^{-1}$ at any time t .

Solution:

As we studied the result that the work done is equals to the difference of kinetic energy of the initial and final points

$$\begin{aligned} \text{Work done} &= \frac{1}{2} m \vec{v}_2^2 - \frac{1}{2} m \vec{v}_1^2 \\ &= \frac{1}{2} m (\vec{v}_2^2 - \vec{v}_1^2) \\ &= \frac{1}{2} (10^4) [(3.0 \times 10^3)^2 - (1.5 \times 10^3)^2] \\ &= 3.38 \times 10^{10} \text{ kgms}^{-1} \\ &= 3.38 \times 10^{10} \text{ N} \end{aligned}$$

Hence the required work done is $3.38 \times 10^{10} \text{ N}$.

Module No. 97

Conservative Force Field

In order to understand the concept of conservative force field, we must understand the phenomena of path independence.

Path Independence

Independence of path is defined as that the work done by an object remains same between initial point and the final destination of the particle, it does not matter which path is taken by the object. The example of path independence is work done by the gravitation force. When an object freely falls from same height, the work done remains same no matter what path chose, when the object hits the ground due to the fixed gravitational force and the perpendicular distance between the object and the ground. The symbolic statement of path independence is

If a particle is moving along a curve C from P_1 to P_2 is, then the total work done can be expressed as

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} -(\nabla V) \cdot d\vec{r} = -|dV|_{P_1}^{P_2} = -[V(P_2) - V(P_1)] = V(P_1) - V(P_2)$$

In such case we can observe that the work done by the object depend only on the end points of the curve P_1 and P_2 .

Now we can define the conservative force field in terms of path independence of a particle moving along a curve that:

“A force field is said to conservative if the total work done by the particle moving along a curve is independent of the path taken by the particle and depend upon the end points of the curve only.”

Necessary and sufficient conditions for a Conservative Force Field

Conservative force fields conserve the following properties:

- i. A force field F is conservative if and only if there exists a continuously differentiable scalar field V such that $\vec{F} = -\nabla V$ or, equivalently, if and only if

$$\text{curl } \vec{F} = \nabla \times \vec{F} = 0$$

identically

- ii. A continuously differentiable force field F is conservative if and only if for any closed non-intersecting curve C (simple closed curve)

$$W = \oint_C \vec{F} \cdot d\vec{r} = 0$$

i.e. the total work done in moving a particle around any closed path is zero.

Examples of Conservative Forces

- Gravitational force is an example of a conservative force.
- Elastic spring force is example of conservative force.
- The work done of a particle moving along a closed path is zero and the force which causes such motion is conservative.

Module No. 98

Example of Conservative Field

Problem:

Prove that the force field

$$\vec{F} = (y^2 - 2xyz^3)\hat{i} + (3 + 2xy - x^2z^3)\hat{j} + (6z^3 - 3x^2yz^2)\hat{k}$$

is conservative.

Solution:

We know that the force field is conservative iff $\text{curl } \vec{F}$ is zero;

i.e. $\nabla \times \vec{F} = 0$

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - 2xyz^3 & 3 + 2xy - x^2z^3 & 6z^3 - 3x^2yz^2 \end{vmatrix} \\ &= \hat{i} \left[\frac{\partial}{\partial y} (6z^3 - 3x^2yz^2) - \frac{\partial}{\partial z} (3 + 2xy - x^2z^3) \right] - \hat{j} \left[\frac{\partial}{\partial x} (6z^3 - 3x^2yz^2) - \frac{\partial}{\partial z} (y^2 - 2xyz^3) \right] \\ &\quad + \hat{k} \left[\frac{\partial}{\partial x} (3 + 2xy - x^2z^3) - \frac{\partial}{\partial y} (y^2 - 2xyz^3) \right] \\ &= (-3x^2z^2 + 3x^2z^2)\hat{i} - (-6xyz^2 + -6xyz^2)\hat{j} + (2y - 2xz^2 - 2y + 2xz^2)\hat{k} \\ &= 0\end{aligned}$$

As we obtain $\nabla \times \vec{F} = 0$

We conclude that $\vec{F}$ is conservative.

Module No. 99

Related Topic of Conservative Force Fields

Conservation of Energy Theorem in Case of Conservative Force Field

The concept of energy is a useful alternative to the concept of force. We will derive the law of conservation of energy using the concept of conservative force field.

If P_1 and P_2 are any two points on the trajectory of a particle, the total work done by the total force acting on the particle is given by

$$\begin{aligned}
 W &= \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{t_1}^{t_2} \frac{d}{dt} mv \cdot \frac{d\vec{r}}{dt} dt = \int_{t_1}^{t_2} m \frac{dv}{dt} \cdot v dt \\
 &= \frac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt} (v^2) dt = \frac{1}{2} m |v^2|_{t_1}^{t_2} = \frac{1}{2} mv_2^2 - \frac{1}{2} mv_1^2 = T_2 - T_1
 \end{aligned} \tag{1}$$

where $v(t_1) = v_1$ & $v(t_2) = v_2$

Now we suppose that the applied force F on the particle is conservative. Then we can express it as $F = -\nabla V$, and the integral for W can be written as

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = \int_{P_1}^{P_2} -(\nabla V) \cdot d\vec{r} = -|dV|_{P_1}^{P_2} = -[V(P_2) - V(P_1)] = V_1 - V_2 \tag{2}$$

where V_1 and V_2 corresponds to P.E. at initial and final positions of the particle. By comparing expression (1) and (2), we have

$$T_2 - T_1 = V_1 - V_2 \quad \text{or} \quad T_1 + V_1 = T_2 + V_2 \tag{3}$$

Expression (3) shows that the quantity $T_1 + V_1$ is a constant of motion. It is called total energy of the particle. Denoting the total energy by E , we have

$$E = T + V = \frac{1}{2} mv^2 + V$$

Conservative Systems and Orbits of Particles

A single particle moving in a conservative field of forces may perform an important type of motion. Suppose the total energy E of the system is a constant of motion i.e.

$$\frac{1}{2}m\dot{r}^2 + V(r) = E$$

where E is some constant denoting total energy of the system.

Suppose the particle's motion is such that it returns to the same position, represented by the position vector r_0 , at a later time. Then it must have the same K.E. and therefore the same speed. It follows that in a conservative system it is possible for closed trajectories to occur. This fact is very relevant in the study of Earth's motion about the Sun.

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Module No. 100

Non-Conservative Force Field

Forces that cannot be expressed in the term of a potential energy function are called non-conservative forces. We can also state that forces that do not store energy are called non-conservative or dissipative forces. If there is no scalar function V such that $F = -\Delta V$ [or, equivalently, if $\nabla \times F = 0$], then F is called a non-conservative force field.

Friction is a non-conservative force, and there are others. It is always opposed to the direction of motion and is not a single valued function of position alone.

Similarly the impulse (time dependent force) is also non-conservative and cannot be derived from a scalar point function.

An example of non-conservative force, we have $F = kv$, where v is the velocity of the particle, then,

$$\begin{aligned}\oint \vec{F} \cdot d\vec{r} &= \int \vec{F} \cdot \frac{d\vec{r}}{dt} dt \\ &= \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt = k \int_{t_1}^{t_2} v^2 dt > 0\end{aligned}$$

which shows the integral is not equals to zero. Hence the force is non-conservative.

Work-Energy relation and Non-conservative Forces

We have already shown that for any general force F

$$W = \int_{P_1}^{P_2} \vec{F} \cdot d\vec{r} = T_2 - T_1$$

When the force F can be broken into conservative and non-conservative parts

$$\vec{F} = \vec{F}^{(c)} + \vec{F}^{(nc)}$$

we have

$$\int_{P_1}^{P_2} \vec{F}^{(c)} \cdot d\vec{r} + \int_{P_1}^{P_2} \vec{F}^{(nc)} \cdot d\vec{r} = T_2 - T_1 \quad (1)$$

But $\vec{F}^{(c)} = -\nabla V$ and equation (1) can be written as

$$V_1 - V_2 + \int_{P_1}^{P_2} \vec{F}^{(nc)} \cdot d\vec{r} = T_2 - T_1$$

which can also be written as

$$T_1 + V_1 + \int_{P_1}^{P_2} \vec{F}^{(nc)} \cdot d\vec{r} = T_2 + V_2 \quad (2)$$

The work done in overcoming friction is always negative, because $\vec{F}^{(nc)}$ is opposite to the displacement relation (2) proves that the influence of friction is dissipative and therefore decrease the total mechanical energy of the system.

Alternatively (2) can be expressed as

$$(V_2 - V_1) + (T_2 - T_1) = \int_{P_1}^{P_2} \vec{F}^{(nc)} \cdot d\vec{r}$$

or

$$\Delta(V + T) = \int_{P_1}^{P_2} \vec{F}^{(nc)} \cdot d\vec{r}$$

It is interesting to remember that the process in which work is converted into internal energy (due to friction) are irreversible.

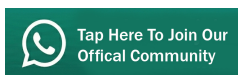


Lecture Handouts

on

VECTORS AND CLASSICAL MECHANICS (MTH-622)

**Virtual University of Pakistan
Pakistan**



About the Handouts

The following books have been mainly followed to prepare the slides and handouts:

1. Spiegel, M.R., *Theory and Problems of Vector Analysis: And an Introduction to Tensor Analysis*. 1959: McGraw-Hill.
2. Spiegel, M.S., *Theory and problems of theoretical mechanics*. 1967: Schaum.
3. Taylor, J.R., *Classical Mechanics*. 2005: University Science Books.
4. DiBenedetto, E., *Classical Mechanics: Theory and Mathematical Modeling*. 2010: Birkhäuser Boston.
5. Fowles, G.R. and G.L. Cassiday, *Analytical Mechanics*. 2005: Thomson Brooks/Cole.

The first two books were considered as main text books. Therefore the students are advised to read the first two books in addition to these handouts. In addition to the above mentioned books, some other reference book and material was used to get these handouts prepared.

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Module No. 101

Example of Non- Conservative Field

Problem:

Show that the force field given by

$$\vec{F} = x^2yz\hat{i} - xyz^2\hat{j} + 2xz\hat{k}$$

is non-conservative.

Solution:

We have

$$\vec{F} = x^2yz\hat{i} - xyz^2\hat{j} + 2xz\hat{k}$$

To verify whether the force field $\vec{F}$ is conservative or not, we will check whether $\nabla \times \vec{F} = 0$ or not.

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2yz & -xyz^2 & 2xz \end{vmatrix} \\ &= \hat{i} \left[\frac{\partial}{\partial y}(2xz) - \frac{\partial}{\partial z}(-xyz^2) \right] - \hat{j} \left[\frac{\partial}{\partial z}(x^2yz) - \frac{\partial}{\partial x}(2xz) \right] + \hat{k} \left[\frac{\partial}{\partial x}(-xyz^2) - \frac{\partial}{\partial y}(x^2yz) \right] \\ &= (2xyz)\hat{i} - (x^2y - 2z)\hat{j} + (yz^2 + x^2z)\hat{k} \\ &\neq 0\end{aligned}$$

As we obtain $\nabla \times \vec{F} \neq 0$

We conclude that $\vec{F}$ is non-conservative.

Module No. 102

Introduction to Simple Harmonic Motion and Oscillator

Simple Harmonic Motion (SHM) is a particular type of oscillation and periodic motion in which restoring force of an object is directly proportional to the displacement of the object acting in opposite direction of displacement.

Mathematically, the restoring force F is given by

$$F_R = -kx$$

where the subscript R represents the restoring force and k is the constant of proportionality often called the spring constant or modulus of elasticity.

In Newtonian mechanics, by Newton's second law we have eq. of S.H.M

$$F = ma = m \frac{d^2x}{dt^2} = -kx$$

or

$$m\ddot{x} + kx = 0$$

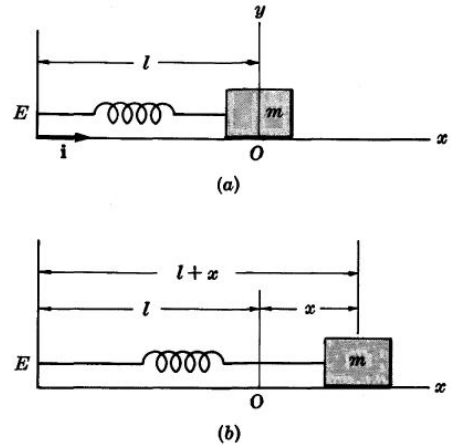
This vibrating system is called a simple harmonic oscillator or linear harmonic oscillator.

This type of motion is often called simple harmonic motion.

The following physical systems are some examples of simple harmonic oscillator

Mass on a spring

An object of mass m linked to a spring of spring constant k represents the simple harmonic motion in closed region.



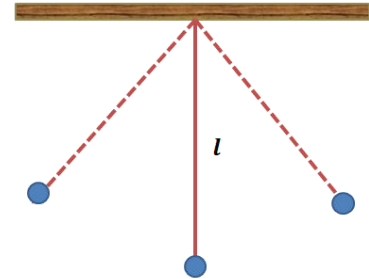
The equation representing the period for the mass attached on a spring

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The above equation expresses that the time period of oscillation is independent of amplitude as well as the acceleration.

Simple pendulum

The movement of the mass attached to a simple pendulum is considered as simple harmonic motion.



The time period of a mass m attached to a pendulum of length l with gravitational acceleration g is given by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Module No. 103

Amplitude, Time period, Frequency and Energy of S.H.M

Amplitude

Amplitude is defined as the maximum distance covered by the oscillating body in one oscillation or length of a wave measured from its mean position.

The amplitude of a pendulum is one-half the distance that the mass covered in moving from one terminal to the other. The vibrating sources generate waves, whose amplitude is proportional to the amplitude of the vibrating source.

Time Period

Time period is minimum time required by a oscillation system to complete its one cycle of oscillation of the specific system.

It is denoted by T and measured in seconds.

Frequency

The frequency (f) of an oscillatory system is the number of oscillations pass through a specific point in one second.

It is measure in hertz (Hz).

The frequency of S.H.M can be calculate by using the following relation

$$f = \frac{1}{T}$$

Energy of S.H.M

If T is the kinetic energy, V the potential energy and $E = T + V$ the total energy of a simple harmonic oscillator, then we have

$$K.E = T = \frac{1}{2}mv^2$$

and

$$V = \frac{1}{2}kx^2$$

Then the total energy of S.H.M will be

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

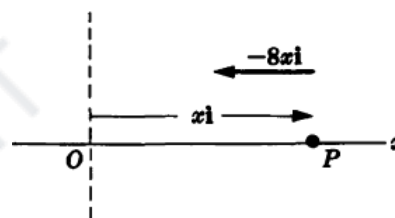
Module No. 104

Example of S.H.M

Problem Statement

A particle of mass 4 moves along y axis attracted towards the origin by a force whose magnitude is numerically equal to $6y$. If the particle is initially at rest at $y = 10$,

- The differential equation and initial conditions describing the motion,
- The position of the particle at any time,
- The speed and velocity of the particle at any time,
- The amplitude, time period and frequency of the vibration.



Solution:

- Let $r = y$ be the position vector of the particle. The acceleration of the particle is

$$\frac{d^2 r}{dt^2} = \frac{d^2 y}{dt^2}$$

The net force acting on the particle is $-6y$. Then by using the Newton's second law,

$$-6y = 4 \frac{d^2 y}{dt^2}$$

or
$$\frac{d^2 y}{dt^2} + \frac{3}{2}y = 0$$

which is the required differential equation. The required initial conditions (I. Cs) are

$$y = 10, \quad dy/dt = 0 \text{ at } t = 0$$

- Since the general solution the diff. eq. is

When $t = 0, x = 20$ so that $A = 20$. Thus

$$y = A \cos \sqrt{3/2}t + B \sin \sqrt{3/2}t \quad (1)$$

Using I. Cs

At $t = 0, y = 10$. Thus $A = 10$. So,

$$y = 10 \cos \sqrt{3/2}t + B \sin \sqrt{3/2}t \quad (2)$$

$$\frac{dy}{dt} = -10\sqrt{3/2} \sin \sqrt{3/2} t + \sqrt{3/2} B \cos \sqrt{3/2} t$$

(3)

Since at $t = 0, \Rightarrow \frac{dy}{dt} = 0$

Thus, we get $B = 0$. Hence (2) becomes

$$y = 10 \cos \sqrt{3/2} t \quad (4)$$

This gives the position at any time.

iii. Speed at any time?

From (6) $\frac{dy}{dt} = -10\sqrt{3/2} \sin \sqrt{3/2} t$

This gives the speed at any time.

iv. Amplitude, T, F ?

General form for the position of a particle moving to and fro is

$$y(t) = A \cos(2\pi t/T)$$

Thus, $y = 10 \cos \sqrt{3/2} t$

$$\text{Amplitude} = 10$$

$$\text{Period} = T = 2\sqrt{\frac{2}{3}}\pi$$

$$\text{Frequency} = 1/T$$

Module No. 105

Example of Energy of S.H.M

ENERGY OF A SIMPLE HARMONIC OSCILLATOR

Example:

Find the total energy of the force $\vec{F} = -3xi$ acting on a simple harmonic oscillator, where i represents the direction.

Solution:

Since the total energy of SHM

$$E = T + V$$

By Newton's second law,

$$F = ma$$

therefore

$$F = m \frac{dv}{dt} = -3x$$

$$\frac{dv}{dt} = -\frac{3}{m}x$$

Integrating with respect to t , we have

$$v = -\frac{3}{2m}x^2 + c$$

c can be calculated if the initial condition is given. Assuming $v = 0$ initially, which gives $c = 0$.

Thus
$$v = -\frac{3}{2m}x^2$$

So,

$$T = \frac{9}{8m}x^4$$

Now, as the potential energy is given by V where $\vec{F} = -\nabla V$

or
$$F = -3xi = -\left(\frac{\partial V}{\partial x}i + \frac{\partial V}{\partial y}j + \frac{\partial V}{\partial z}k\right)$$

Then

$$\frac{\partial V}{\partial x} = 3x, \quad \frac{\partial V}{\partial y} = 0, \quad \frac{\partial V}{\partial z} = 0$$

By integrating, we obtain

$$V = \frac{3}{2}x^2 + c_1$$

Assuming $V = 0$, corresponding to $x = 0$, we get $c_1 = 0$.

So the potential energy is $V = \frac{1}{2}kx^2$

Hence

$$E = \frac{9}{8m}x^4 + \frac{1}{2}kx^2$$

Module No. 106

Damped Harmonic Oscillator

The forces acting on a harmonic oscillator are called damping forces which tend to decrease the amplitude of the successive oscillations or simply force apposing the motion. Since the damping force is proportional to the velocity. Thus, mathematically,

$$\begin{aligned} F_d &= -bv \\ &= -b\dot{x} \end{aligned}$$

where d represents the damping force and b is the damping coefficient.

The negative sign shows that that direction of F_d is opposite to the velocity v .

As we know that for the restoring force:

$$m\ddot{x} + kx = 0$$

Adding the restoring force with the damping force, the equation of motion of the damped harmonic oscillator will be,

$$m\ddot{x} + kx = -b\dot{x}$$

or

$$m\ddot{x} + b\dot{x} + kx = 0$$

Since the mass is not equal to zero, therefore $\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = 0$

let

$$\frac{b}{m} = 2\zeta, \quad \sqrt{\frac{k}{m}} = \omega_0$$

then the equation can be written as

$$\ddot{x} + 2\zeta\dot{x} + \omega_0^2x = 0$$

Module No. 107

Example of Damped Harmonic Oscillator

Problem Statement

A particle of mass 4 moves along y axis attracted towards the origin by a force whose magnitude is numerically equal to $6y$. If the particle is initially at rest at $y = 10$. Consider that the particle has also a damping force whose magnitude is numerically equal to 6 times the instantaneous speed.

Find

- Position of the particle
- Velocity of the particle at any time

Solution

- i. Since we have by Newton's second law

$$F = ma = 4\ddot{y}$$

According to the given information, the damping force is $-6\dot{y}$.

So the net force is

$$-6y - 6\dot{y}$$

Hence,

$$4\ddot{y} = -6y - 6\dot{y}$$

or

$$\ddot{y} + \frac{2}{3}\dot{y} + \frac{2}{3}y = 0$$

The solution of the above equation is

$$y = e^{-\sqrt{2/3}t}(A + Bt)$$

and

$$\dot{y} = Be^{-\sqrt{2/3}t} - \sqrt{2/3}(A + Bt)e^{-\sqrt{2/3}t}$$

To find the values of constants, we use I. Cs

At $t = 0$, $y = 10$ and $dy/dt = 0$; thus,

$$A = 10$$

$$\text{and } 0 = B(1) - \sqrt{2/3}(10)$$

$$B = 10\sqrt{2/3}$$

and the solution gives

$$\begin{aligned} y &= e^{-\sqrt{2/3}t}(A + Bt) \\ &= 10e^{-\sqrt{2/3}t}(1 + \sqrt{2/3}t) \end{aligned}$$

the position at any time t .

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Module No. 108

Euler's Theorem – Derivation

The following theorem, called Euler's theorem, is fundamental in the motion of rigid bodies.

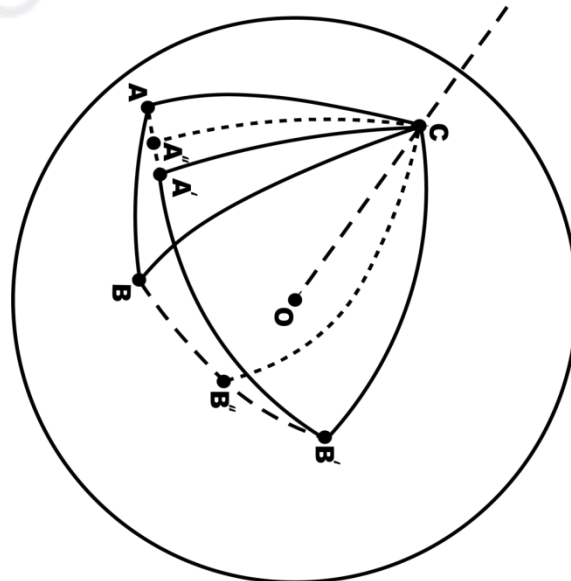
Theorem Statement

“A rotation of a rigid body about a fixed point of the body is equivalent to a rotation about a line which passes through the point.”

The line referred to is called the instantaneous axis of rotation. Rotations can be considered as finite or infinitesimal. Finite rotations cannot be represented by vectors since the commutative law fails. However, infinitesimal rotations can be represented by vectors.

Proof:

Let O be the fixed point in the body, which we take as a sphere S . Further, we take O at the center of the sphere. Let A, B be two distinct points on the sphere. As the body moves, the point O (on the axis) remains fixed and A and B suffer displacement.



Let A' and B' be the new locations of the points A and B after an infinitesimal time interval δt respectively. We join (A, B) and (A', B') by great circular arcs.

Also we join (A, A') and (B, B') by means of great circular arcs. Let A'' and B'' draw axes at right angles, which meet at the point C on the sphere.

We join C with A, B, A', B' by means of great circular arcs.

Consider the spherical triangles $\triangle CA'A''$ and $\triangle CAA''$. Obviously

- i. $m \angle CAA'' \cong \angle CA'A''$
- ii. $AA'' = A''A'$
- iii. CA'' is common to triangle

Therefore $\triangle CAA'' \cong \triangle CA'A''$ and it follows that $CA = CA'$

Similarly for the triangle $\triangle CAB$ and $\triangle CA'B'$, we have

$$CB = CB', \quad CA' = CA \text{ and } AB = A'B'$$

The last relation is due to the fact that by the definition of rigid body, the distance between a pair of particles remains unchanged. Hence

$$\triangle CAB \cong \triangle CA'B'$$

The portion of rigid body lying in $\triangle CAB$ has moved to $\triangle CA'B'$.

In this process the point O and C have remained fixed, Although the later was at rest only instantaneously. Therefore the body has under gone a rotation about the axis OC.

Hence the theorem.

Module No. 109

Chasles' Theorem

Theorem Statement:

Chasle's theorem states that the most general rigid body displacement can be produced by a translation along a line (called its screw axis) followed (or preceded) by a rotation about that line.

Explanation:

- A rigid body has six degrees of freedom.
- By Euler's theorem, three of these are associated with pure rotation.
- The remaining three must be associated with translation.
- To describe the general motion of a rigid body, think of the general motion as translation of a fixed point O in the body to a point O' followed by the rotation about an axis through O' .

Module No. 110

Kinematics of a System of Particles (Space, time & matter)

Kinematics is the branch of mechanics deals with the moving objects without reference to the forces which cause the motion.

In other words we can say those kinematics are the features or properties of motion of concerned with system of particles (rigid bodies).

Here some features of rigid body motion are

- Displacement
- Position
- Velocity
- Linear Velocity & Angular Velocity
- Linear Acceleration & Angular Acceleration
- Motion of a Rigid Body (Translation & Rotation)

From everyday experience, we all have some idea as to the meaning of each of the following terms or concepts. However, we would certainly find it difficult to formulate completely satisfactory definitions. We take them as undefined concepts.

- I. **Space.** This is closely related to the concepts of point, position, direction and displacement. Measurement in space involves the concepts of length or distance, with which we assume familiarity. Units of length are feet, meters, miles, etc.
- II. **Time.** This concept is derived from our experience of having one event taking place after, before or simultaneous with another event. Measurement of time is achieved, for example, by use of clocks. Units of time are seconds, hours, years, etc.
- III. **Matter.** Physical objects are composed of "small bits of matter" such as atoms and molecules. From this we arrive at the concept of a material object called a particle which can be considered as occupying a point in space and perhaps moving as time goes by. A measure of the "quantity of matter" associated with a particle is called its mass. Units of mass are grams, kilograms, etc. Unless otherwise stated we shall assume that the mass of a particle does not change with time.

Module No. 111

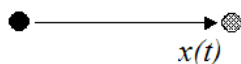
The Concept of Rectilinear Motion of Particles, Uniform Rectilinear Motion, Uniformly Accelerated Rectilinear Motion

When a moving particle remains on a single straight line, the motion is said to be rectilinear. In this case, without loss of generality we can choose the x -axis as the line of motion.

The general equation of motion is then

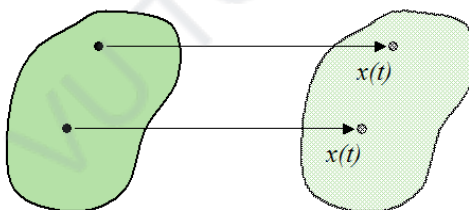
$$F = ma \Rightarrow F(x, \dot{x}, \ddot{x}) = m\ddot{x}$$

Rectilinear motion for a particle:



Rectilinear motion of a body is defined by considering the two point of a body covered the same distance in the parallel direction. The figures below illustrate rectilinear motion for a particle and body.

Rectilinear motion for a body:



In the above figures, $x(t)$ represents the position of the particles along the direction of motion, as a function of time t .

An example of linear motion is an athlete running along a straight track.

The rectilinear motion can be of two types:

- i. Uniform rectilinear motion
- ii. Non uniform rectilinear motion

Uniform Rectilinear Motion:

Uniform rectilinear motion is a type of motion in which the body moves with uniform velocity or zero acceleration.

In contrast, Non uniform rectilinear motion is such type of motion with variable velocity or non-zero acceleration.

Uniformly accelerated rectilinear motion:

Uniformly accelerated rectilinear motion is a special case of non-uniform rectilinear motion along a line is that which arises when an object is subjected to constant acceleration. This kind of motion is called uniformly accelerated motion.

Uniformly accelerated motion is a type of motion in which the velocity of an object changes by an equal amount in every equal intervals of time.

An example of uniformly accelerated body is freely falling object in which the amount of gravitational acceleration remains same.

$$F = mg$$

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Module No. 112

The Concept of Curvilinear Motion of Particles

The motion of a particle moving in a curved path is called curvilinear motion. Example: A stone thrown into the air at an angle.

Curvilinear motion describes the motion of a moving particle that conforms to a known or fixed curve. The study of such motion involves the use of two co-ordinate systems, the first being planar motion and the latter being cylindrical motion.

Tangential and normal unit vectors are usually denoted by $\vec{e}_t$ and $\vec{e}_n$ respectively.

Velocity of Curvilinear motion

If the tangential and normal unit vectors are $\vec{e}_t$ and $\vec{e}_n$ respectively, then the velocity will be

$$\vec{v} = \frac{ds}{dt} \vec{e}_t$$

You have already learnt that

$$v = vT$$

Acceleration of Curvilinear motion

If the tangential and normal unit vectors are $\vec{e}_t$ and $\vec{e}_n$ respectively, then the acceleration will be

$$\vec{a} = \frac{d^2s}{dt^2} \vec{e}_t + \frac{\left(\frac{ds}{dt}\right)^2}{\rho} \vec{e}_n$$

You have already learnt that

$$a = T \frac{dv}{dt} + \frac{v^2}{r} N$$

Example

- A stone thrown into the air at an angle.
- A car driving along a curved road.
- Throwing paper airplanes or paper darts is an example of curvilinear motion.

Module No. 113

Example Related to Curvilinear Coordinates

Example 1

Problem Statement

Prove that a cylindrical coordinate system is orthogonal.

Solution

The position vector of any point in cylindrical coordinates is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

We studied that the in cylindrical coordinates

$$x = \rho \cos \varphi, \quad y = \rho \sin \varphi, \quad z = z$$

this implies

$$\vec{r} = \rho \cos \varphi \hat{i} + \rho \sin \varphi \hat{j} + z\hat{k}$$

The tangent vectors to the ρ , φ and r curves are given respectively by $\frac{\partial r}{\partial \rho}$, $\frac{\partial r}{\partial \varphi}$, $\frac{\partial r}{\partial z}$ and where

$$\frac{\partial r}{\partial \rho} = \cos \varphi \hat{i} + \sin \varphi \hat{j}$$

$$\frac{\partial r}{\partial \varphi} = -\rho \sin \varphi \hat{i} + \rho \cos \varphi \hat{j}$$

$$\frac{\partial r}{\partial z} = \hat{k}$$

The unit vectors in these directions are

$$\begin{aligned} e_1 = e_\rho &= \frac{\partial r / \partial \rho}{|\partial r / \partial \rho|} = \frac{\cos \varphi \hat{i} + \sin \varphi \hat{j}}{|\cos \varphi \hat{i} + \sin \varphi \hat{j}|} \\ &= \frac{\cos \varphi \hat{i} + \sin \varphi \hat{j}}{\sqrt{\cos^2 \varphi + \sin^2 \varphi}} = \cos \varphi \hat{i} + \sin \varphi \hat{j} \\ e_2 = e_\varphi &= \frac{\partial r / \partial \varphi}{|\partial r / \partial \varphi|} = \frac{-\rho \sin \varphi \hat{i} + \rho \cos \varphi \hat{j}}{|-\rho \sin \varphi \hat{i} + \rho \cos \varphi \hat{j}|} \\ &= \frac{\rho(-\sin \varphi \hat{i} + \cos \varphi \hat{j})}{\sqrt{\rho^2 \cos^2 \varphi + \rho^2 \sin^2 \varphi}} = -\sin \varphi \hat{i} + \cos \varphi \hat{j} \\ e_3 = e_z &= \frac{\partial r / \partial z}{|\partial r / \partial z|} = \hat{k} \end{aligned}$$

Then

$$\begin{aligned}
e_1 \cdot e_2 &= (\cos \varphi \hat{i} + \sin \varphi \hat{j}) \cdot (-\sin \varphi \hat{i} + \cos \varphi \hat{j}) \\
&= -\sin \varphi \cos \varphi + \sin \varphi \cos \varphi = 0 \\
e_2 \cdot e_3 &= (-\sin \varphi \hat{i} + \cos \varphi \hat{j}) \cdot \hat{k} = 0 \\
e_3 \cdot e_1 &= \hat{k} \cdot (\cos \varphi \hat{i} + \sin \varphi \hat{j}) = 0
\end{aligned}$$

and so e_1, e_2 and e_3 are mutually perpendicular and the coordinate system is orthogonal.

Example 2

Problem Statement

Prove

$$\begin{aligned}
\frac{de_\rho}{dt} &= \dot{\varphi} e_\varphi \\
\frac{de_\varphi}{dt} &= -\dot{\varphi} e_\rho
\end{aligned}$$

where dots denote differentiation with respect to time t .

Solution

We have

$$e_\rho = \cos \varphi \hat{i} + \sin \varphi \hat{j}$$

and

$$e_\varphi = -\sin \varphi \hat{i} + \cos \varphi \hat{j}$$

Then

$$\begin{aligned}
\frac{de_\rho}{dt} &= \frac{d}{dt} (\cos \varphi \hat{i} + \sin \varphi \hat{j}) \\
&= -\sin \varphi \dot{\varphi} \hat{i} + \cos \varphi \dot{\varphi} \hat{j} = \dot{\varphi} (-\sin \varphi \hat{i} + \cos \varphi \hat{j}) = \dot{\varphi} e_\varphi \\
\frac{de_\varphi}{dt} &= \frac{d}{dt} (-\sin \varphi \hat{i} + \cos \varphi \hat{j}) \\
&= -\cos \varphi \dot{\varphi} \hat{i} - \sin \varphi \dot{\varphi} \hat{j} = -\dot{\varphi} (\cos \varphi \hat{i} + \sin \varphi \hat{j}) = -\dot{\varphi} e_\rho
\end{aligned}$$

Module No. 114

Introduction to Projectile, Motion of a Projectile

Introduction to Projectile

If a ball is thrown from one person to another or an object is dropped from a moving plane, then their path of traveling/motion is often called a projectile. If air resistance is negligible, a projectile can be considered as a freely falling body so the Eq of motion will be

$$m \frac{d^2 r}{dt^2} = -mg$$

or

$$\frac{d^2 r}{dt^2} = -g$$

with appropriate I.Cs

Motion of a Projectile with Resistance

Earlier, we studied the motion of a projectile under the assumption that air resistance is negligible. If we further assume that the motion takes place in the vertical plane, (XY-plane), and the only force acting on the projectile is the gravity, then the equation of motion will be

$$\frac{d^2 \mathbf{r}}{dt^2} = \vec{g} = -g\hat{j}$$

Here we will discuss the problem of projectile motion when air resistance is taken into account. Assuming the model of retarding force in which

$$F \propto v$$

or

$$F = -k_1 v$$

Let the initial velocity of the projectile be v_0 and angle of projection be θ

The initial conditions can be taken as

$$x(t = 0) = y(t = 0) = 0$$

$$\dot{x}(t = 0) = v_0 \cos \theta = u_1$$

$$\dot{y}(t = 0) = v_0 \sin \theta = v_1$$

The equation of motion in the horizontal and vertical direction can be written as

$$m\ddot{x} = -k_1 \dot{x}$$

$$\text{or } \ddot{x} = -\frac{k_1}{m} \dot{x} = -k\dot{x} \quad (1)$$

$$m\ddot{y} = -k\dot{y} - mg \quad (2)$$

By the substitution $\dot{x} = z$ we can solve eq (1) as

$$z = \dot{x} = Ae^{-kt}$$

With the initial condition $\dot{x} = u_1$, when $t = 0$, we obtain the constant of integration $A = u_1$ so that

$$\dot{x} = u_1 e^{-kt}$$

Another integration gives

$$x = -\frac{u_1 e^{-kt}}{k} + B$$

The initial condition $x(0) = 0$ gives $B = u_1/k$. Hence the solution of equation (1) can be written as

$$x = \frac{u_1}{k} (1 - e^{-kt}) \quad (3)$$

We can solve equation (2) in the same manner and obtain

$$y = -\frac{gt}{k} + \frac{kv_1 + g}{k^2} (1 - e^{-kt}) \quad (4)$$

The path of motion can be obtained from (3) and (4) by eliminating t

$$y = \frac{g}{k^2} \ln \left(1 - \frac{kx}{u_1} \right) + \frac{kv_1 + g}{k^2} \frac{kx}{u_1}$$

which is no longer a parabola.

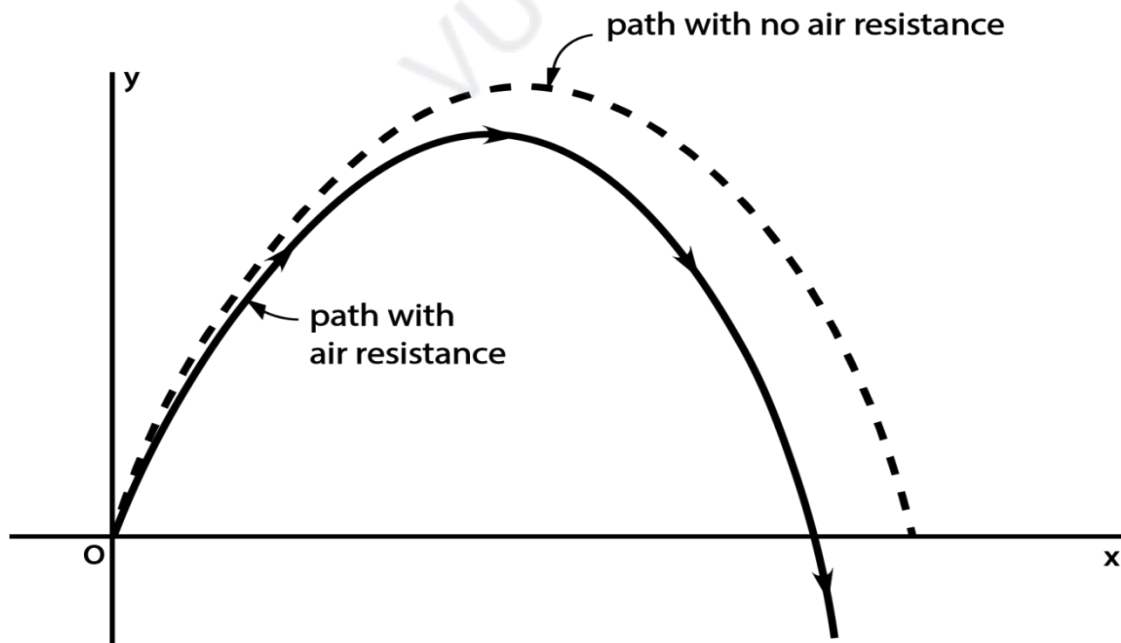
We can solve equation (2) in the same manner and obtain

$$y = -\frac{gt}{k} + \frac{kv_1 + g}{k^2} (1 - e^{-kt}) \quad (4)$$

The path of motion can be obtained from (3) and (4) by eliminating t

$$y = \frac{g}{k^2} \ln \left(1 - \frac{kx}{u_1} \right) + \frac{kv_1 + g}{k^2} \frac{kx}{u_1}$$

which is no longer a parabola.



Time of Flight τ

Time of flight can be found by putting in equation (4) $y = 0$ and $t = \tau$

$$0 = -\frac{g\tau}{k} + \frac{kv_1 + g}{k^2}(1 - e^{-k\tau})$$

$$\tau = \frac{kv_1 + g}{gk}(1 - e^{-k\tau}) \quad (5)$$

This equation can be solved exactly for τ . To find an approximate solution, we expand the exponential factor in equation (5), using

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \dots \dots$$

Form equation (5)

$$\tau = \frac{kv_1 + g}{gk} \left(k\tau - \frac{k^2\tau^2}{k\tau} + \frac{k^3\tau^3}{3!} + \dots \dots \right)$$

$$1 = \frac{kv_1 + g}{gk} \left(k - \frac{k^2\tau}{k\tau} + \frac{k^3\tau^2}{3!} + \dots \dots \right)$$

which on simplification gives

$$\tau = \frac{2v_1}{kv_1 + g} + \frac{1}{3}k\tau^2 + \dots \dots$$

In the limit of small air resistance (i.e. $k \rightarrow 0$), the above expression reduces to

$$\tau = \frac{2v_1}{kv_1 + g} + \frac{k}{3}\tau^2 \quad (6)$$

When there is no resistance, ($k = 0$), obtain from (6)

$$\tau_0 = \frac{2v_1}{g} = \frac{2v_0 \sin \theta}{g}$$

If k is non-zero small number, then the time of flight τ will be very close to τ_0 . Substituting τ_0 for τ on the R.H.S of (6), and simplifying we get

$$\tau = \frac{2v_1}{g} \left(1 - \frac{kv_1}{3g} \right)$$

The above equation gives a formula for approximate time of flight in the presence of a weak retarding force.

Module No. 115

Conservation of Energy for a System of Particles

Statement

“The law of conservation of energy describes that the net energy of an isolated system remains conserved. Energy can neither be created nor destroyed; rather, it transforms from one form to another.”

Theorem Statement

(Principle of conservation of energy)

In case of conservative force field, the total energy is a constant. i.e.,

If T is for kinetic energy and V is for potential energy, then the total energy E is

$$E = T + V = \text{constant}$$

Explanation

For a conservative force field, we have already learned that the work done by the system of particles is

$$\text{Work done} = \text{change in kinetic energy} = W = T_2 - T_1$$

$$\text{Also Work done} = \text{change in potential energy} = W = V_1 - V_2$$

By comparing, we get

$$T_2 - T_1 = V_1 - V_2$$

or

$$T_1 + V_1 = T_2 + V_2$$

which can be written as

$$\frac{1}{2}mv_1^2 + V_1 = \frac{1}{2}mv_2^2 + V_2$$

Module No. 116

Conservation of Energy

Example

Consider the force field $F = -kr^3\vec{r}$

- Find whether the given field is conservative or not.
- Find the potential energy of the given force field of part (i).
- For the particle moving in xy – plane, find the work done by the force in moving the particle from A to B, where A is the point where $r = a$, and B where $r = b$.
- If the particle of mass m moves with velocity $v = \frac{dr}{dt}$ in this field, show that if E is the constant, total energy then
- $\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{1}{5} kr^5 = E$

Solution

- As $\vec{F} = -kr^3\vec{r} = -k(x^2 + y^2)^{3/2}(xi + yj)$

$$\begin{aligned}\nabla \times \vec{F} &= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -k(x^2 + y^2)^{3/2}x & -k(x^2 + y^2)^{3/2}y & 0 \end{vmatrix} \\ &= i \left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(-k(x^2 + y^2)^{3/2}y) \right] + j \left[\frac{\partial}{\partial z}(-k(x^2 + y^2)^{3/2}x) - \frac{\partial}{\partial x}(0) \right] \\ &\quad + k \left[\frac{\partial}{\partial x}(-k(x^2 + y^2)^{3/2}y) - \frac{\partial}{\partial y}(-k(x^2 + y^2)^{3/2}x) \right] \\ &= -3k(x^2 + y^2)^{1/2}xy + 3k(x^2 + y^2)^{1/2}xy = 0\end{aligned}$$

Thus the given field is conservative.

- Since the field is conservative there exists a potential V such that

$$F = -\nabla V$$

$$\begin{aligned}F &= -kr^3\vec{r} = -k(x^2 + y^2)^{3/2}(xi + yj) \\ &= -k(x^2 + y^2)^{3/2}xi - k(x^2 + y^2)^{3/2}yj = -\nabla V \\ &= -\frac{\partial V}{\partial x}i - \frac{\partial V}{\partial y}j - \frac{\partial V}{\partial z}k\end{aligned}$$

By comparing, we get

$$\frac{\partial V}{\partial x} = k(x^2 + y^2)^{3/2}x, \quad \frac{\partial V}{\partial y} = k(x^2 + y^2)^{3/2}y \quad \text{and} \quad \frac{\partial V}{\partial z} = 0$$

From which, by omitting the constants, we get

$$V = \frac{1}{5} k(x^2 + y^2)^{5/2} = \frac{1}{5} kr^5$$

is required potential.

iii. The velocity potential at point A

$$= \frac{1}{5} ka^5$$

and the velocity potential at point B

$$= \frac{1}{5} kb^5$$

So, the work done from A to B is

= Potential at A - Potential at B

$$= \frac{1}{5} ka^5 - \frac{1}{5} kb^5 = \frac{1}{5} k(a^5 - b^5)$$

which is required work done by the given force field.

iv. Since the kinetic energy of a particle of mass m moving with velocity $v = \frac{dr}{dt}$ is

$$T = \frac{1}{2} mv^2 = \frac{1}{2} m \left(\frac{dr}{dt} \right)^2$$

From part (ii), we have the potential energy

$$V = \frac{1}{5} kr^5$$

Thus the total energy E will be

$$\begin{aligned} E &= T + V \\ &= \frac{1}{2} m \left(\frac{dr}{dt} \right)^2 + \frac{1}{5} kr^5 \end{aligned}$$

Hence proved.

Module No. 117

Introduction to Impulse – Derivation

Impulse is a special type of force defined by applying the integral of a force F , over the time interval, t , for which it acts on the body.

Impulse is a directional (vector) quantity in the same direction of force as force is also a directional quantity. When Impulse is applied to a rigid body, it results a corresponding vector change in its linear momentum along the same direction. The SI unit of impulse is the newton second ($N \cdot s$), and the dimensionally equivalent unit of momentum is the kilogram meter per second ($kgms^{-1}$). The particle is located at P_1 and P_2 at times t_1 and t_2 where it has velocities v_1 and v_2 respectively. The time integral of the force F given by

$$\int_{t_1}^{t_2} F dt$$

is called the impulse of the force F .

Theorem Statement

The impulse is equal to the change in momentum; or, in symbols,

$$\int_{t_1}^{t_2} F dt = mv_2 - mv_1 = p_2 - p_1$$

Proof

We have to prove that the impulse of a force is equal to the change in momentum.

By definition of impulse and Newton's second law, we have

$$\int_{t_1}^{t_2} F dt = \int_{t_1}^{t_2} m \frac{dv}{dt} dt = \int_{t_1}^{t_2} m dv = m|v|_{t_1}^{t_2} = mv_2 - mv_1 = p_2 - p_1$$

where we use the conditions

$$v(t_1) = v_1 \text{ and } v(t_2) = v_2$$

The theorem is true even when the mass is variable and the force is non-conservative.

Module No. 118

Example of Impulse

Example:

What is the magnitude of the impulse developed by a mass of 200 gm which changes its velocity from $5i - 3j + 7k$ m/sec to $2i + 3j + k$ m/sec?

Solution:

Since we have the following information:

$$\begin{aligned} m &= 200 \text{ gm} \\ v_1 &= 5i - 3j + 7k \\ v_2 &= 2i + 3j + k \end{aligned}$$

As we know that

$$\begin{aligned} \text{Impulse} &= p_2 - p_1 \\ &= mv_2 - mv_1 \\ &= m(v_2 - v_1) \end{aligned}$$

Substituting the values, we get

$$\begin{aligned} \text{Impulse} &= 200(2i + 3j + k - (5i - 3j + 7k)) \\ &= 200(-3i + 6j - 6k) \end{aligned}$$

The magnitude of the Impulse will be

$$\begin{aligned} \text{Impulse magnitude} &= 200\sqrt{9 + 36 + 36} \\ &= 200\sqrt{81} \\ &= 200(9) \\ &= 1800 \text{ mgm/sec} \\ &= 1.8 \text{ N sec} \end{aligned}$$

Module No. 119

Torque

Definition

Torque is defined as the turning effect of a body. It is trend of an acting force due to which the rotational motion of a body changes. It is also called twist and rotational force on an object. Mathematically, torque is defined as the cross product of the force vector to the distance vector, which causes rotational motion of the body.

$$\boldsymbol{\tau} = \vec{r} \times \vec{F}$$

The magnitude of torque depends upon the applied force, the length of the lever arm connecting the axis to the point where the force applied, and the angle between the force vector and the length of lever arm. Symbolically we can write it as:

$$\tau = |r||F| \sin \theta$$

Torque is a vector quantity implies that it has direction as well as magnitude.

The SI unit for torque is the newton meter (Nm).

The direction of torque can be approximate using **Right Hand Rule**.

Theorem:

The torque acting on a particle equals the time rate of change in its angular momentum, i.e.,

$$\tau = \frac{d\Omega}{dt}$$

where $\Omega = r \times p$ is defined angular momentum of the system.

Proof:

As we know torque is defined as

$$\tau = r \times F = r \times ma$$

as m is constant, therefore

$$\begin{aligned} &= m(r \times a) \\ &= m\left(r \times \frac{dv}{dt}\right) \\ &= m \frac{d}{dt}(r \times v) \\ &= \frac{d}{dt}(r \times mv) \\ &= \frac{d}{dt}(r \times p) \end{aligned}$$

where $p = mv$ is defined as linear momentum and hence the theorem.

Module No. 120

Example of Torque

Example

A particle of mass m moves Along a space curve defined by $r = a \cos \omega t \, i + b \sin \omega t \, j$.

➤ Find

(i) The Torque

(ii) The angular momentum about the origin.

Solution

(i) As we know that the torque acting on a particle is equal to the time rate of change of its angular momentum, i.e.

$$\tau = \frac{d\Omega}{dt} = \frac{d}{dt}(r \times p)$$

where $p = mv$.

As $r = a \cos \omega t \, i + b \sin \omega t \, j$. Therefore

$$v = \frac{dr}{dt}$$

$$v = -a\omega \sin \omega t \, i + b\omega \cos \omega t \, j$$

So,

$$p = -am\omega \sin \omega t \, i + bm\omega \cos \omega t \, j$$

Now

$$\begin{aligned} & r \times p \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a \cos \omega t & b \sin \omega t & 0 \\ -am\omega \sin \omega t & bm\omega \cos \omega t & 0 \end{vmatrix} \\ &= [abm\omega \cos^2 \omega t + abm\omega \sin^2 \omega t] \hat{k} \\ &= abm\omega \hat{k} \end{aligned}$$

$$\tau = \frac{d\Omega}{dt} = \frac{d}{dt}(r \times p) = 0$$

(ii) From part (i), we already have

$$\Omega = r \times p$$

$$= abm\omega\hat{k}$$

is the required angular momentum.

VU TOOLKIT

Module No. 121

Introduction to Rigid Bodies and Elastic Bodies

Definition of Rigid Bodies

- When a force is applied to an object/ system of particles, and if the object maintains its overall shape, then the object is called a rigid body.
- Gap between two fixed points on the rigid body remains same regardless of external forces exerted on it.
- We can neglect the deformation of such bodies.
- A rigid body usually has continuous distribution of mass.

Definition of Elastic Bodies

- When a force is applied to a system of particles, it changes the distance between individual particles. Such systems are often called deformable or elastic bodies.

Examples

- A spring and rubber band are some common examples of elastic bodies.
- A wheel is a common example of rigid body.

Module No. 122

Properties of Rigid Bodies

Following are some of the properties of the rigid bodies.

Degree of freedom

The number of coordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.

For example a particle moving freely in space requires 3 coordinates, e.g. (x, y, z), to specify its position. Thus the number of degrees of freedom is 3.

Similarly, a system consisting of N particles moving freely in space requires 3N coordinates to specify its position. Thus the number of degrees of freedom is 3N.

Translations

A displacement of a rigid body is a direct change of position of its particles. Translational motion is the displacement of all particles of the body by the same amount and the line segment joining the initial and the final position of the particles represented by parallel vectors.

Examples of translational motion are particles freely falling down to earth and the motion of a bullet fired from a gun.

Rotations

Circular motion of a body about a fixed point or axis is called rotation. If during a displacement the points of the rigid body on some line remains fixed and all other are displaced through the same angle, then this displacement is called rotation. A rigid performs rotations around an imaginary line called a rotation axis.

If the axis of rotation passes through the center of mass of the rigid body then body is said to be spin or rotate upon itself. If a body rotates about some external fixed point is called revolution or orbital motion of the rigid body. The example of revolution is the rotation of earth around sun and motion of moon around sun.

Rotational motion concerns only with rigid bodies. The reverse rotation of a body (inverse rotation) is also a rotation.

A wheel is common examples of rotation.

Module No. 123

Instantaneous Axis and Center of Rotation

Introduction to General Plane Motion

The general plane motion of a rigid body can be considered as:

- Translational motion along the given fixed plane and rotational motion about a suitable axis perpendicular to the plane.
- This fixed axis is specifically chosen to pass through the center of mass of the rigid body.

Instantaneous Axis of Rotation

- The axis about which the rigid body rotates is called instantaneous axis of rotation, where this axis is perpendicular to the plane.

Instantaneous Center of Rotation

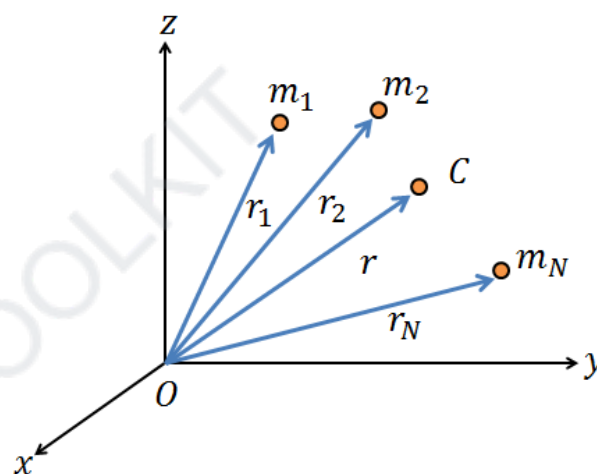
- The point where instantaneous axis meets the fixed plane along which the body performs translation motion is described as the instantaneous center of rotation.

Module No. 124

Centre of Mass & Motion of the Center of Mass

The center of mass (c.m.) or centroid of system of particles is a hypothetical particle such that if the entire mass of the system were concentrated there, the mechanical properties would remain the same. In particular expression of linear momentum, angular momentum and kinetic energy assume simpler or more convenient forms when referred to the coordinated of this hypothetical particle and the equation of motion can be reduced to simpler equation of a single particle.

Let $r_1, r_2, r_3, \dots, r_N$ be the position vectors of a system of N particles of masses $m_1, m_2, m_3, \dots, m_N$ respectively [see Fig.].



The center of mass or centroid of the system of particles is defined as that point C having position vector

$$r_c = \frac{\sum_i m_i r_i}{\sum_i m_i}$$

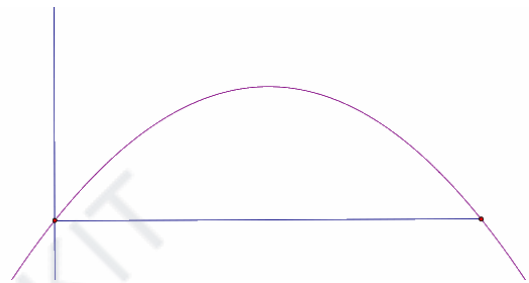
When the system is moving the position vector will depend on time t and therefore r_c will also be a function of t . the velocity and acceleration of center of mass will be then given by

$$\begin{aligned} \dot{r}_c &= v_c = \frac{\sum_i m_i \dot{r}_i}{\sum_i m_i} \\ \ddot{r}_c &= a_c = \frac{\sum_i m_i \ddot{r}_i}{\sum_i m_i} \end{aligned}$$

Motion of Center of Mass

Motion of center of mass can be examined by considering the following points:

1. If a system experiences no external force, the center-of-mass of the system will remain at rest, or will move at constant velocity if it is already moving.
2. If there is an external force, the center of mass accelerates according to $F = ma$.
3. Basically, the center-of-mass of a system can be treated as a point mass, following Newton's Laws.
4. If an object is thrown into the air, different parts of the object can follow quite complicated paths, but the center-of-mass will follow a parabola.



5. If an object explodes, the different pieces of the object will follow seemingly independent paths after the explosion. The center of mass, however, will keep doing what it was doing before the explosion. This is because an explosion involves only internal forces.

Module No. 125

Centre of Mass & Motion of the Center of Mass

The moment of inertia of a rigid body is a property which depends upon its mass and shape, (i.e. the mass distribution of the body) and determines its behavior in rotational motion.

In rotational motion, the moment of inertia plays the same role as the mass in linear motion.

Definition of Moment of Inertia

Formally the moment of inertia I of the particle of mass m about a line is defined by

$$I = md^2$$

where d is the perpendicular distance between the particle and the line (called the axis).

Moment of Inertia of System of particles

The moment of inertia of a system of particles, with masses $m_1, m_2, m_3, \dots, m_N$ about the line or axis AB is defined as

$$\begin{aligned} I &= \sum_{i=1}^N m_i d_i^2 \\ &= m_1 d_1^2 + m_2 d_2^2 + \dots + m_N d_N^2 \end{aligned}$$

In dimensions, the moment of inertia can be expressed as

$$[I] = [M][L^2]$$

Moment of Inertia in Coordinate System

The moment of inertia of a particle of mass m with coordinates (x, y, z) relative to the orthogonal Cartesian coordinate system $OXYZ$ about X, Y, Z axes will be

$$\begin{aligned} I_{xx} &= m(y^2 + z^2) \\ I_{yy} &= m(z^2 + x^2) \\ I_{zz} &= m(x^2 + y^2) \end{aligned}$$

Product of Inertia

The product of inertia for the same particle w.r.to the pair of coordinate axes are defined as

$$I_{xy} = -mxy$$

$$I_{yz} = -myz$$

$$I_{zx} = -mzx$$

These definitions can be easily generalized to a system of particle and a rigid body.

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Module No. 126

Example of moment of Inertia and Product of Inertia

Problem Statement:

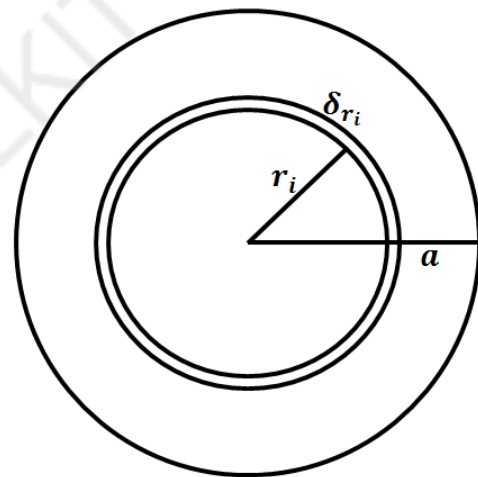
Find the moments of inertia of a ring of radius a about an axis through center.

Solution:

Let M be the mass and a the radius of the ring. Then the mass per unit length will be $M/2\pi a$.

We regard this ring to be composed of small elements of mass (δm) each of length δs , we can write it as

$$\frac{\delta m}{\delta s} = \frac{M}{2\pi a} \Rightarrow \delta m = \frac{M}{2\pi a} \delta s$$



Moment of inertia of the element about an axis through center O and the perpendicular to the plane of the ring equals $\frac{M}{2\pi a} \delta s a^2$.

Therefore the M.I of the whole ring will be

$$I_{ring} = \frac{M}{2\pi a} a^2 \sum_{elements} \delta s = \frac{Ma}{2\pi} \int ds = \frac{Ma}{2\pi} \times 2\pi a = Ma^2$$

Hence Ma^2 is the required moment of inertia of the ring.

Module No. 127

Radius of Gyration

Radius of gyration specifies the distribution of the elements of body around the axis in terms of the mass moment of inertia. As it is the perpendicular distance from the axis of rotation to a point mass m that gives an equivalent inertia to the original object m . The nature of the object does not affect the concept, which applies equally to a surface bulk mass.

Mathematically the radius of gyration is the root mean square distance of the object's parts from either its center of mass or the given axis, depending on the relevant application.

Let $I = \sum_{i=1}^N m_i d_i^2$ be the moment of inertia of a system of particles about AB, and

$M = \sum_{i=1}^N m_i$ be the total mass of the system.

Then the quantity K such that

$$K^2 = \frac{I}{M} = \frac{\sum_{i=1}^N m_i d_i^2}{\sum_{i=1}^N m_i}$$

or

$$K = \sqrt{\frac{I}{M}}$$

is called the radius of gyration of the system AB.

Example:

Find the radius of gyration, K , of the triangular lamina of mass M and moment of inertia $I = \frac{1}{6} M h^2$.

Solution:

Since formula for radius of gyration is given by

$$K^2 = \frac{I}{M}$$

by substituting value in above, we get

$$K^2 = \frac{\frac{1}{6} M h^2}{M} = \frac{1}{6} h^2$$

or

$$K = \frac{h}{\sqrt{6}}$$

Module No. 128

Principal Axes for the Inertia Matrix

Principal Axes

When a rigid body is rotating about a fixed point O, the angular velocity vector ω and the angular momentum vector L (about O) are not in general in the same direction. However it can be proved that at each point in the body there exists distinct directions, which are fixed relative to the body, along which the two vectors are aligned i.e. coincident. Such directions are called **principal directions** and the axes along them are referred to as **principal axes of inertia**. The corresponding moments of inertia are called principal moments of inertia.

Orthogonality of Principal Axes

If the principal axes at each point of the body exist, then their orthogonality can be proved by stating that axes relative to which product of inertia are zero are the principal axes.

Module No. 129

Introduction to the Dynamics of a System of Particles

Dynamics

Dynamics is the branch of mechanics deals with forces and relationship fundamentally to the motion but sometimes also to the equilibrium of bodies.

The Dynamics of a System of Particles

Suppose we have a system of N particles in motion under forces. These will be two types of forces, internal and external.

Internal forces act between particles of the system; all other are external. We suppose further that the internal forces satisfy Newton's third law of motion. i.e. Internal forces between a pair of particles are equal and opposite. If $F_{ij}^{(int)}$ denotes the internal force on the i th particle due to the j th particle, then in the view of this assumption we can write

$$F_{ij}^{(int)} = -F_{ji}^{(int)} \quad (1)$$

The equation of motion for the i th particle of system can therefore be written as

$$F_i^{(ext)} + \sum_{j=1}^N F_{ij}^{(int)} = m_i \vec{a}_i \quad (2)$$

where $F_i^{(ext)}$ denotes the external force on the i th particle, and $\vec{a}_i$, the acceleration of the i th particle. It is clear from equation (1) that the internal force on a given particle due to itself is zero and therefore the term $i = j$ does not contribute to the sum.

To obtain an equation of motion for the whole system we sum over i , (from 1 to N), on the both sides of equation (2), and obtain

$$\sum_i F_i^{(ext)} + \sum_i \sum_j F_{ij}^{(int)} = \sum_i m_i \vec{a}_i$$

or

$$F^{(ext)} + \sum_{i,j} F_{ij}^{(int)} = \sum_i m_i \vec{a}_i \quad (3)$$

Where $F^{(ext)}$ denotes the total external force on the system. Now we will show that because of condition (1), the second term on L.H.S of (3) is zero.

$$\sum_{i,j} F_{ij}^{(int)} = \frac{1}{2} \sum_{i,j} (F_{ij}^{(int)} + F_{ji}^{(int)})$$

Since the indices i and j are dummy and vary over the same set of integers, we can interchange them in the second sum on the R.H.S of the last equation. Therefore we have

$$\begin{aligned}\sum_{i,j} F_{ij}^{(int)} &= \frac{1}{2} \sum_{i,j} (F_{ij}^{(int)} + F_{ji}^{(int)}) \\ &= \frac{1}{2} \sum_{i,j} (F_{ij}^{(int)} - F_{ij}^{(int)}) = 0\end{aligned}$$

Therefore the equation (3) becomes

$$F^{(ext)} = \sum_i m_i \vec{a}_i \quad (4)$$

In case of centroid, the relation becomes

$$\sum_i m_i \vec{a}_i = M \vec{a}_c$$

where $\vec{r}_c = \vec{a}_c$ is the acceleration of center of mass (c.m.) and M is the total mass of the system.

Hence equation (4) can be written as

$$F^{(ext)} = M \vec{a}_c = \frac{d}{dt} (M \vec{v}_c) \quad (5)$$

If $F^{(ext)} = 0$ then $\frac{d}{dt} (M \vec{v}_c) = 0$, which implies that $M \vec{v}_c = \text{constant}$.

We conclude that if external forces on a system of particles are zero, then its momentum will be a constant of motions or a conserved quantity. Equation (5) describes the translational motion of the system and may be referred to as the translational equation of motion which describes the rotational motion of the system, we take the cross product of both sides of equation (2) with $\vec{r}_i$, and obtain

$$\vec{r}_i \times F_i^{(ext)} + \sum_{j=1}^N \vec{r}_i \times F_{ij}^{(int)} = m_i \vec{r}_i \times \vec{a}_i$$

summing over i

$$\sum_i \vec{r}_i \times F_i^{(ext)} + \sum_{i,j=1}^N \vec{r}_i \times F_{ij}^{(int)} = \sum_i m_i \vec{r}_i \times \vec{a}_i \quad (6)$$

Now we will show that in view of our assumption (1) about internal forces, the second term on L.H.S of equation (6) is zero. On interchanging the dummy indices and using (1) we have

$$\begin{aligned}\sum_{i,j} \vec{r}_i \times F_{ij}^{(int)} &= \frac{1}{2} \sum_{i,j} (\vec{r}_i \times F_{ij}^{(int)} + \vec{r}_j \times F_{ji}^{(int)}) \\ &= \frac{1}{2} \sum_{i,j} (\vec{r}_i \times F_{ij}^{(int)} - \vec{r}_j \times F_{ij}^{(int)}) \\ &= \frac{1}{2} \sum_{i,j} (\vec{r}_i - \vec{r}_j) \times F_{ij}^{(int)}\end{aligned} \quad (7)$$

The vector $\vec{r}_i - \vec{r}_j$ is in the direction of the line segment joining the particles i and j and therefore it is parallel to F_{ij} . Therefore the last term on R.H.S of equation (7) is zero.

Hence we finally obtain

$$\sum_{i,j} \vec{r}_i \times F_{ij}^{(ext)} = \sum_i m_i \vec{r}_i \times \vec{a}_i$$

or

$$\begin{aligned}
\sum_i G_i &= G^{ext} = \sum_i m_i r_i \times \vec{a}_i = \frac{d}{dt} \left(\sum_i m_i r_i \times \vec{v}_i \right) \\
&= \frac{d}{dt} \left(\sum_i r_i \times m_i \vec{v}_i \right) \\
&= \frac{d}{dt} \sum_i L_i = \frac{dL}{dt}
\end{aligned}$$

which is usually written as

$$\frac{dL}{dt} = G^{(ext)}$$

where $G_i^{(ext)}$ is the torque or momentum due to external force on the i th particle, and $G_i^{(ext)}$ is the torque due to all external forces on the system.

Similarly L is the total angular momentum of the system.

$$\sum_{i,j} r_i \times F_{ij}^{(ext)} = \sum_i G_i = G$$

Thus we have obtained the following two equations of motion for a system of particles.

$$\frac{d}{dt} (M \vec{v}_c) = F^{(ext)} \quad (8)$$

and

$$\frac{dL}{dt} = G^{(ext)} \quad (9)$$

It follows from equation (8) that if the total external torque on the system is zero, then its angular momentum will be a constant of motion or a conserved quantity.

Module No. 130

Introduction to Center of Mass and Linear Momentum

The rigid bodies are a system of particles in which the position of particles is relatively fixed. We consider such a system of particles in this article and will discuss its center of mass and linear momentum.

Center of Mass

Our general system consists of n particles of masses $m_1, m_2, \dots, m_n$ whose position vectors are, respectively, $r_1, r_2, \dots, r_n$. We define the center of mass of the system as the point whose position vector r_{cm} is given by

$$r_{cm} = \frac{m_1 r_1 + m_2 r_2 + m_3 r_3 + \dots + m_n r_n}{m_1 + m_2 + \dots + m_n} = \frac{\sum_{i=1}^n m_i r_i}{m} \quad (1)$$

where $m = \sum_{i=1}^n m_i$ is the total mass of the system.

The above definition of center of mass leads us to three equations

$$x_{cm} = \frac{\sum_{i=1}^n m_i x_i}{m}, y_{cm} = \frac{\sum_{i=1}^n m_i y_i}{m}, z_{cm} = \frac{\sum_{i=1}^n m_i z_i}{m}$$

which are the rectangular coordinates of the center of mass of the system.

Linear Momentum

We define the linear momentum p of the system as the vector sum of the linear momentum of the individual particles, namely,

$$p = \sum_i p_i = \sum_i m_i v_i$$

From equation (1)

$$r_{cm} = \frac{\sum_{i=1}^n m_i r_i}{m} \quad (1)$$

$$\dot{r}_{cm} = v_{cm} = \frac{\sum_{i=1}^n m_i v_i}{m}$$

it follows that

$$p = m v_{cm}$$

that is, the linear momentum of a system of particles is the product of the velocity of the center of mass and the total mass of the system.

Module No. 131

Law of Conservation of Momentum for Multiple Particles

Statement

In the absence of the external force, the momentum of the system of particles will be conserved.

Proof

Suppose we have a system of N particles in motion under forces. These will be two types of forces, internal and external.

Suppose now that there are external forces $F_1, F_2, \dots, F_i, \dots, F_n$ acting on the respective particles. In addition, there may be internal forces of interaction between any two particles of the system.

We denote these internal forces by $F_{ij}^{(int)}$, meaning the force exerted on particle i by particle j .

Internal Forces act between particles of the system; all over the external. We suppose further that the internal force satisfy Newton's third law of motion. i.e. Internal forces between a pair of particles are equal and opposite.

If $F_{ij}^{(int)}$ denotes the internal force on the i th particle due to the j th particle, then in the view of this assumption we can write

$$F_{ji}^{(int)} = -F_{ij}^{(int)} \quad (1)$$

The equation of motion for the i th particle of the system can therefore be written as

$$F_i^{(ext)} + \sum_{j=1}^N F_{ij}^{(int)} = m_i a_i = \dot{p}_i \quad (2)$$

where $F_i^{(ext)}$ denotes the total external force on the i th particle, and a_i the acceleration of the i th particle.

To obtain an equation of motion for the whole system, we sum over i (from 1 to N), on both sides of the equation (2).

$$\sum_i F_i^{(ext)} + \sum_i \sum_{j=1}^N F_{ij}^{(int)} = \sum_i m_i a_i = \dot{p}_i$$

$$F^{(ext)} + \sum_{i,j} F_{ij}^{(int)} = \sum_i m_i a_i = \dot{p}_i \quad (3)$$

where $F^{(ext)}$ denotes the total force of the system.

now we will show that because of equation 1, L.H.S of equation 3 is zero.

$$\dot{p}_i = \sum_{i,j} F_{ij}^{(int)} = \left(\frac{1}{2} \sum_{i,j} F_{ij}^{(int)} + \frac{1}{2} \sum_{i,j} F_{ij}^{(int)} \right)$$

since the indices I and j are dummy and vary over the same set of integers, we can interchange them in the second sum on R.H.S of the last equation. Therefore we have

$$\dot{p}_i = \sum_{i,j} F_{ij}^{(int)} = \left(\frac{1}{2} \sum_{i,j} F_{ij}^{(int)} + \frac{1}{2} \sum_{i,j} F_{ji}^{(int)} \right)$$

using equation (1), we have

$$\sum_{i,j} F_{ij}^{(int)} = \left(\frac{1}{2} \sum_{i,j} F_{ij}^{(int)} - \frac{1}{2} \sum_{i,j} F_{ji}^{(int)} \right) = 0$$

therefore equation (3) becomes

$$\dot{p}_i = F^{(ext)} = \sum_i m_i a_i$$

for centroid, we know that $\sum_i m_i a_i = m a_c$ where a_c is the acceleration of centroid.

Hence equation can be written as

$$\dot{p}_i = F^{(ext)} = M a_c = \frac{d}{dt} (M v_c)$$

If $F^{(ext)} = 0$ then $\dot{p}_i = \frac{d}{dt} (M v_c) = 0$ or $p = M v_c = \text{constant}$

We conclude that if external forces on a system of particles are zero, then its momentum will be a constant of motion or a conserved quantity.

Module No. 132

Example of Conservation of Momentum

Problem Statement

A particle A , of mass 6 kg , travelling in a straight line at 5 ms^{-1} collides with a particle B , of mass 4 kg , travelling in the same straight line, but in the opposite direction, with a speed of 3 ms^{-1} . Given that after the collision particle A continues to move in the same direction at 1.5 ms^{-1} , what speed does particle B move with after the collision? (© mathcentre 2009)

Given Data

Mass of particle $A = m_1 = 6 \text{ kg}$

Velocity of A before collision $= u_1 = 5 \text{ ms}^{-1}$

Velocity of A after collision $= v_1 = 1.5 \text{ ms}^{-1}$

Mass of Particle $B = m_2 = 4 \text{ kg}$

Velocity of B before collision $= u_2 = 3 \text{ ms}^{-1}$

Required

Velocity of B after collision $= v_2 = ?$

Solution

It is always useful to depict the collision with the velocities both before and after.
Using the principle of conservation of momentum:

$$\begin{aligned} m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2 \\ 6 \times 5 + 4 \times (-3) &= 6 \times 1.5 + 4 \times v_2 \\ 9 &= 4 \times v_2 \\ v_2 &= \frac{9}{4} = 2.3 \text{ m s}^{-1} \end{aligned}$$

So after the collision particle B moves with a speed of 2.3 m s^{-1} , in the same direction as A .

Example 2

Problem Statement

A particle A, of mass 8 kg, collides with a particle B, of mass m_2 kg. The velocity of particle A before the collision was $(-1i + 4j) \text{ m s}^{-1}$ and the velocity of particle B before the collision was $(-0.8i + 1.4j) \text{ m s}^{-1}$. Given the velocity of particle A after the collision was $(-2i + 2j) \text{ m s}^{-1}$, and the velocity of particle B was $3j \text{ m s}^{-1}$, what was the mass of particle B?

Given Data

Mass of particle A = $m_1 = 8 \text{ kg}$

Velocity of A before collision = $u_1 = (-1i + 4j) \text{ m s}^{-1}$

Velocity of A after collision = $v_1 = (-2i + 2j) \text{ m s}^{-1}$

Velocity of B before collision = $u_2 = (-0.8i + 1.4j) \text{ m s}^{-1}$

Velocity of B after collision = $v_2 = ? 3j \text{ m s}^{-1}$

Required

Mass of Particle B = $m_2 = ?$

Solution

It is always useful to depict the collision with the velocities both before and after.

Using the principle of conservation of momentum:

$$\begin{aligned}
 m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2 \\
 6 \times 5 + 4 \times (-3) &= 6 \times 1.5 + 4 \times v_2 \\
 9 &= 4 \times v_2 \\
 v_2 &= \frac{9}{4} = 2.3 \text{ ms}^{-1}
 \end{aligned}$$

So after the collision particle B moves with a speed of 2.3 ms^{-1} , in the same direction as A.

Example 2

Problem Statement

A particle A, of mass 8 kg, collides with a particle B, of mass 2 kg. The velocity of particle A before the collision was $(-1i + 4j) \text{ ms}^{-1}$ and the velocity of particle B before the collision was $(-0.8i + 1.4j) \text{ ms}^{-1}$. Given the velocity of particle A after the collision was $(-2i + 2j) \text{ ms}^{-1}$, and the velocity of particle B was $3j \text{ ms}^{-1}$, what was the mass of particle B?

Given Data

Mass of particle $A = m_1 = 8\text{kg}$

Velocity of A before collision $= u_1 = (-1i + 4j)\text{ms}^{-1}$

Velocity of A after collision $= v_1 = (-2i + 2j)\text{ms}^{-1}$

Velocity of B before collision $= u_2 = (-0.8i + 1.4j)\text{ms}^{-1}$

Velocity of B after collision $= v_2 = 3j\text{ms}^{-1}$

Required

Mass of Particle $B = m_2 = ?$

Solution

By making use of principle of conservation of momentum:

$$\begin{aligned}
 m_1 u_1 + m_2 u_2 &= m_1 v_1 + m_2 v_2 \\
 8(-1i + 4j) + m_2(-0.8i + 1.4j) &= 8(-2i + 2j) + m_2(3j) \\
 -8i + 32j + 16i - 16j + m_2(-0.8i + 1.4j) - m_2(3j) &= 0 \\
 8i + 16j + m_2(-0.8i - 1.6j) &= 0 \\
 8i + 16j - m_2(+0.8i + 1.6j) &= 0 \\
 8i + 16j &= m_2(0.8i + 1.6j) \\
 m_2 &= \frac{0.8i + 1.6j}{8i + 16j}
 \end{aligned}$$

By rationalizing and solving the above fraction, we obtain

$$m_2 = 10\text{kg}$$

Module No. 133

Angular Momentum – Derivation

Angular momentum of a particle of mass m , position vector r and linear momentum p is defined as $r \times p$. It is also called moment of momentum.

Let there be a system consists of N particles with position vector r_i and the momentum p_i ($i = 1, 2, 3, \dots, N$) then the total angular momentum L is given by

$$L = \sum_i r_i \times p_i = \sum_i r_i \times (mv_i)$$

To find the relation between L and $\vec{\omega}$, we proceed as follows:

$$\begin{aligned} L &= \sum_i r_i \times (m_i v_i) \\ &= \sum_i r_i \times m_i (\vec{\omega} \times r_i) \\ &= \sum_i m_i [r_i \times (\vec{\omega} \times r_i)] \\ &= \sum_i m_i [(r_i \cdot r_i) \vec{\omega} - (r_i \cdot \vec{\omega}) r_i] \end{aligned} \quad (1)$$

Here $(r_i \cdot r_i) \vec{\omega}$ and $(r_i \cdot \vec{\omega}) r_i$ are not parallel vectors. If they were parallel then we could easily say that L is parallel to $\vec{\omega}$.

We conclude that in general L and $\vec{\omega}$ are not parallel to each other. To find the relationship between the vectors L and $\vec{\omega}$ we proceed as follows:

Since

$$r_i = x_i \hat{i} + y_i \hat{j} + z_i \hat{k}$$

therefore

$$r_i \cdot r_i = x_i^2 + y_i^2 + z_i^2$$

and

$$\vec{\omega} \cdot r_i = \omega_x x_i + \omega_y y_i + \omega_z z_i$$

Therefore substituting these values in equation (1), we have

$$L = \sum_i m_i (x_i^2 + y_i^2 + z_i^2) (\omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}) - (\omega_x x_i + \omega_y y_i + \omega_z z_i) (x_i \hat{i} + y_i \hat{j} + z_i \hat{k}) \quad (2)$$

Writing

$$L = L_x \hat{i} + L_y \hat{j} + L_z \hat{k} \equiv L_1 \hat{i} + L_2 \hat{j} + L_3 \hat{k}$$

and

$$\omega = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k} \equiv \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

and comparing the coefficient of $\hat{i}$ on both sides of equation (2), we obtain

$$\begin{aligned}
L_x &= \sum_i m_i (x_i^2 + y_i^2 + z_i^2) \omega_x - (\omega_x x_i + \omega_y y_i + \omega_z z_i) x_i \\
&= \sum_i m_i [x_i^2 \omega_x + (y_i^2 + z_i^2) \omega_x - \omega_x x_i^2 - \omega_y x_i y_i - \omega_z x_i z_i] \\
&= \omega_x \sum_i m_i (y_i^2 + z_i^2) - \omega_y \sum_i m_i x_i y_i - \omega_z \sum_i m_i x_i z_i
\end{aligned} \tag{3}$$

As we studied the definitions of moment of inertia and product of inertia be defined as

$$\begin{aligned}
I_{xx} &= \sum_i m_i (y_i^2 + z_i^2) \\
I_{yy} &= \sum_i m_i (z_i^2 + x_i^2) \\
I_{zz} &= \sum_i m_i (x_i^2 + y_i^2)
\end{aligned}$$

and

$$\begin{aligned}
I_{xy} &= - \sum_i m_i x_i y_i \\
I_{yz} &= - \sum_i m_i y_i z_i \\
I_{zx} &= - \sum_i m_i x_i z_i
\end{aligned}$$

Using these definitions and noting that $I_{xy} = I_{yx}$ etc. we can write

$$L_x = I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z$$

Similarly we obtain

$$\begin{aligned}
L_y &= I_{yx} \omega_x + I_{yy} \omega_y + I_{yz} \omega_z \\
L_z &= I_{zx} \omega_x + I_{yz} \omega_y + I_{zz} \omega_z
\end{aligned}$$

Module No. 134

Angular Momentum in Case of Continuous Distribution of Mass

An actual rigid body consists of very large number of particles and therefore we may suppose that there is a continuous distribution of mass. If $\rho(r)$ denotes density of a volume element dV , surrounding the point r , then the mass of this element will be $\rho(r)dV$.

Hence we can write the expression of angular momentum in case of continuous distribution of mass by

$$I_{11} \equiv I_{xx} = \int \rho(r)(y^2 + z^2)dV$$

where

$$r = (x, y, z) \equiv (x_1, x_2, x_3), \quad dV = dxdydz$$

Similarly the other two moment of inertia are defined as

$$I_{22} \equiv I_{yy} = \int \rho(r)(z^2 + x^2)dV$$

and

$$I_{33} \equiv I_{zz} = \int \rho(r)(x^2 + y^2)dV$$

Similarly the product of inertia are defined as

$$I_{12} \equiv I_{xy} = - \int \rho(r)xydV,$$

$$I_{23} \equiv I_{yz} = - \int \rho(r)yzdV,$$

$$I_{31} \equiv I_{zx} = - \int \rho(r)zxdV$$

By using these definitions of moment of inertia, we can find the expression of angular momentum by using the following result

$$L = L_x \hat{i} + L_y \hat{j} + L_z \hat{k}$$

$$L = I_{xx}\omega_x + I_{yy}\omega_y + I_{zz}\omega_z + I_{xy}(\omega_x + \omega_y) + I_{yz}(\omega_y + \omega_z) + I_{zx}(\omega_z + \omega_x)$$

where

$$L_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$$

$$L_y = I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z$$

$$L_z = I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z$$

and $\omega_x, \omega_y, \omega_z$ are the components of angular velocity along (x, y, z) .

or

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

where we have used the property of products of inertia that

$I_{xy} = I_{yx}$, $I_{yz} = I_{zy}$ and

$$I_{xz} = I_{zx}$$

The above matrix form can be written as

$$L = I\omega$$

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Module No. 135

Law of Conservation of Angular Momentum

Angular Momentum

The angular momentum of a single particle is defined as the cross product of linear momentum and position vector of concerned particle.

Mathematically $L = r \times mv$.

Angular Momentum of System of Particles

The angular momentum L of a system of particles is defined accordingly, as the vector sum of the individual angular momentum, namely,

$$L = \sum_{i=1}^n r_i \times m_i v_i \quad (1)$$

Law of Conservation of Angular Momentum

The time rate change of angular momentum in the absence of some external forces is zero.

Mathematically, we can write

$$\frac{dL}{dt} = 0 \Rightarrow L = \text{constant}$$

Let us calculate the time derivative of the angular momentum. Using the rule for differentiating the cross product, we find

$$\frac{dL}{dt} = \frac{d}{dt} \left(\sum_{i=1}^n r_i \times m_i v_i \right) = \sum_{i=1}^n (v_i \times m_i v_i) + \sum_{i=1}^n (r_i \times m_i a_i)$$

Now the first term on the right vanishes, because, $v_i \times v_i = 0$ and, because $m_i a_i$ is equal to the total force acting on particle i , we can write

$$\frac{dL}{dt} = \sum_{i=1}^n (r_i \times m_i a_i)$$

$$\frac{dL}{dt} = \sum_{i=1}^n [r_i \times (\sum_i F_i^{(ext)} + \sum_{j=1}^n \sum_{j=1}^n F_{ij}^{(int)})] \quad (2)$$

$$\frac{dL}{dt} = \sum_{i=1}^n r_i \times F_i^{(ext)} + \sum_{i=1}^n \sum_{j=1}^n F_{ij}^{(int)} \quad (3)$$

where F_i denotes the total external force on particle i , and F_{ij} denotes the (internal) force exerted on particle i by any other particle j . Now the double summation on the right consists of pairs of terms of the form

$$(r_i \times F_j) + (r_j \times F_{ji}) \quad (4)$$

Denoting the vector displacement of particle j relative to particle i by r_{ij} , we see from the triangle shown in Figure that

$$r_{ij} = r_j - r_i \quad (5)$$

Therefore, because $F_{ji} = -F_{ij}$, expression (3) reduces to

$$-r_{ij} \times F_{ij} \quad (6)$$

which clearly vanishes if the internal forces are central, that is, if they act along the lines connecting pairs of particles. Hence, the double sum in Equation (3) vanishes. Now the cross product $(r_i \times F_i)$ is the moment of the external force F_i . The sum $\sum (r_i \times F_i)$ is, therefore, the total moment of all the external forces acting on the system. If we denote the total external torque, or moment of force, by N , Equation (3) takes the form

$$\frac{dL}{dt} = N$$

That is, the time rate of change of the angular momentum of a system is equal to the total moment of all the external forces acting on the system.

If a system is isolated, then $N = 0$, and the angular momentum remains constant in both magnitude and direction:

$$L = \sum_{i=1}^n r_i \times m_i v_i = \text{Constant vector} \quad (8)$$

This is a statement of the principle of conservation of angular momentum. It is a generalization for a single particle in a central field.

Module No. 136

Example Related to Angular Momentum

Problem Statement

The moon revolves around the earth so that we always see the same face of the moon.

- Find how the spin angular momentum and orbital angular momentum of the moon w.r.t. the earth are related?
- Find the change in spin angular momentum of moon so that we could see the entire moon's surface during a month.

Solution

- Let M, R_m denotes the mass and the radius of the moon respectively.

Then its spin angular momentum i.e. angular momentum about its axis of rotation is given by

$$L_s = I\omega_s$$

where I is the moment of inertia and ω_s angular velocity of the moon about its own axis

The moment of inertia of the sphere is

$$I = \frac{2}{5}mr^2$$

by generalizing it for moon, we obtain the moment of inertia for the moon

$$I = \frac{2}{5}MR_m^2$$

then the angular momentum of moon will be

$$L_s = \frac{2}{5}MR_m^2\omega_s \quad (1)$$

In addition to spinning about its own axis, the moon is also performing orbital motion about the earth. If we denote the orbital angular momentum by L_0 , then

$$\begin{aligned} L_0 &= R \times (MV) = R \times (MR\omega_0) \\ &= MR^2\omega_0 \end{aligned} \quad (2)$$

$$\therefore v = \omega r$$

where we have treated the moon as a particle and R is the distance between the moon and the earth. Since we always see the same face of the moon, the moon makes one rotation about its axis in the same time as it makes one revolution around the earth i.e. $\omega_s = \omega_0$. From equation (1) and (2) we have

$$\frac{L_s}{L_0} = \frac{\frac{2}{5}MR_m^2\omega_s}{MR^2\omega_0}$$

$$= \frac{2}{5} \left(\frac{R_m}{R} \right)^2$$

- ii. If we have to see the entire moon's surface during a month, then L_s must either increase or decrease by one-half of its present value.

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Module No. 137

Kinetic Energy of a System about Principal Axes – Derivation

Kinetic Energy

The kinetic energy for a particle is given by the following scalar equation:

$$T = \frac{1}{2}mv^2$$

Where:

- T is the kinetic energy of the particle with respect to ground (an inertial reference frame)
- m is the mass of the particle
- v is the velocity of the particle, with respect to ground

Kinetic Energy of a Rigid Body

For a rigid body experiencing planar (two-dimensional) motion, the kinetic energy is given by the following general scalar equation:

$$T = \frac{1}{2}mv_c^2 + \frac{1}{2}I_c\omega^2 \quad (1)$$

where the first term in equation (1) shows the kinetic energy due to the motion of center of mass and second term shows the rotational kinetic energy and subscript c denotes the center of mass and v_c denotes the velocity of c.m and I_c denotes the inertia matrix w.r.t c.m.

Kinetic Energy of a Rigid Body w.r.t Origin

If the rigid body is rotating about a fixed point O that is attached to ground, we can express the kinetic energy as:

$$T = \frac{1}{2}I_0\omega^2$$

Where:

I_0 is the moment of inertia of the rigid body about an axis passing through the fixed point O , and perpendicular to the plane of motion and ω is the angular velocity.

Kinetic Energy of a Rigid Body about Principal Axes

If the center of mass is oriented along with origin then the coordinate axes (xyz axes) and the principal axes of the rigid body are aligned. In this case, the inertia matrix reduces, i.e. products of inertia are zero along the principal axes, ($I_{xy} = I_{yz} = I_{zx} = 0$) .

For general three-dimensional motion, the kinetic energy of a rigid body about principal axes is given by the following general scalar equation:

$$T = \frac{1}{2}mv_0^2 + \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2 \quad (3)$$

where I_x, I_y, I_z are the moment of inertia along principal axes (xyz axes) also called principal moment of inertia and products of inertia I_{xy}, I_{yz}, I_{zx} are zero along the axes

Equation (3) is the required expression of Kinetic Energy along principal axes.

If the rigid body has a fixed point O that is attached to ground, we can give an alternate scalar equation for the kinetic energy of the rigid body:

$$T = \frac{1}{2}I_x\omega_x^2 + \frac{1}{2}I_y\omega_y^2 + \frac{1}{2}I_z\omega_z^2$$

in this case, the kinetic energy due to the motion of c.m. vanishes.

Module No. 138

Moment of Inertia of a Rigid Body about a Given Line

Let M be the mass of the system and $\hat{e}$ a unit vector along the line l . Then $\hat{e} = \lambda\hat{i} + \mu\hat{j} + \nu\hat{k}$, where (λ, μ, ν) are direction cosines of the line.

If I_l denotes the record moment of inertia, then

$$I_l = \sum_i m_i d_i^2$$

Form the figure

$$d_i = |OP| \sin \theta_i = |r_i| \sin \theta_i = r_i \sin \theta_i = |e_i \times r_i|$$

Therefore

$$I_l = \sum_i m_i |e_i \times r_i|^2 \quad (1)$$

Now

$$\begin{aligned} |e_i \times r_i| &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \lambda & \mu & \nu \\ x_i & y_i & z_i \end{vmatrix} \\ &= (\mu z_i - \nu y_i)\hat{i} + (\nu x_i - \lambda z_i)\hat{j} + (\lambda y_i - \mu x_i)\hat{k} \\ |e_i \times r_i|^2 &= (\mu z_i - \nu y_i)^2 + (\nu x_i - \lambda z_i)^2 + (\lambda y_i - \mu x_i)^2 \end{aligned}$$

Hence on substitution in equation (1), we have

$$\begin{aligned} I_l &= \sum_i m_i [(\mu z_i - \nu y_i)^2 + (\nu x_i - \lambda z_i)^2 + (\lambda y_i - \mu x_i)^2] \\ I_l &= \sum_i m_i [\mu^2 z_i^2 + \nu^2 y_i^2 - 2\mu\nu y_i z_i + \nu^2 x_i^2 + \lambda^2 z_i^2 - 2\nu\lambda x_i z_i + \lambda y_i + \mu x_i - 2\mu\lambda x_i y_i] \\ &= \sum_i m_i [\mu^2(x_i^2 + z_i^2) + \nu^2(x_i^2 + y_i^2) + \lambda^2(y_i^2 + z_i^2) - 2\mu\lambda x_i y_i - 2\mu\nu y_i z_i - 2\nu\lambda x_i z_i] \\ &= \sum_i m_i [\lambda^2(y_i^2 + z_i^2) + \mu^2(x_i^2 + z_i^2) + \nu^2(x_i^2 + y_i^2) - 2\mu\lambda x_i y_i - 2\mu\nu y_i z_i - 2\nu\lambda x_i z_i] \\ &= \lambda^2 \sum_i m_i (y_i^2 + z_i^2) + \mu^2 \sum_i m_i (x_i^2 + z_i^2) + \nu^2 \sum_i m_i (x_i^2 + y_i^2) + 2\mu\lambda \left(- \sum_i m_i x_i y_i \right) \\ &\quad + 2\mu\nu \left(- \sum_i m_i y_i z_i \right) + 2\nu\lambda \left(- \sum_i m_i x_i z_i \right) \end{aligned}$$

or finally,

$$I_l = \lambda^2 I_{xx} + \mu^2 I_{yy} + v^2 I_{zz} + 2\mu\lambda I_{xy} + 2\mu v I_{yz} + 2v\lambda I_{zx}$$

which may also be written as

$$I_l = \lambda^2 I_{11} + \mu^2 I_{22} + v^2 I_{33} + 2\mu\lambda I_{12} + 2\mu v I_{23} + 2v\lambda I_{31}$$

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Module No. 139

Example of M.I of a Rigid Body About Given Line

Problem Statement

Calculate the moment of inertia of a right circular cone about its axis of symmetry.

Solution

Let M be the mass, a the radius and h the height of right circular cone. We regard the cone as composed of elementary circular discs of small thickness each parallel to the base of the cone. We choose the z -axis along the axis of symmetry, and consider a typical disc of radius r and width δz at a distance z from the base.

Mass of the disc is given by

$$\delta m = \rho \pi r^2 \delta z$$

We regard the disk to be composed of concentric elementary circular rings of varying radii say r' then the mass $\delta m'$ of one circular ring with height δz will be

$$\delta m' = \rho 2\pi r' \delta z$$

Then the M.I of one circular ring will be

$$I_{c.r} = \rho 2\pi r' \delta z (r')^2 = 2\pi \rho (r')^3 \delta z$$

Hence the M.I of the whole disk will be

$$\begin{aligned} \delta I &= \sum_{rings} 2\pi \rho (r')^3 \delta z \\ &= \int_0^r 2\pi \left(\frac{\delta m}{\pi r^2 \delta z} \right) (r')^3 \delta z dr' \\ &= \frac{2\delta m}{r^2} \int_0^r (r')^3 dr' \\ \delta I &= \frac{2\delta m}{4r^2} r^4 = \frac{1}{2} \delta m r^2 \end{aligned}$$

From the similar triangles,

we have

$$\frac{r}{a} = \frac{h-z}{h} \quad \text{or} \quad r = a \frac{h-z}{h}$$

Therefore

$$\delta m = \rho \pi \left(a \frac{h-z}{h} \right)^2 \delta z$$

Since the M.I of the disk is

$$\delta I = \frac{1}{2} \delta m r^2$$

On substituting for δm and r , the M.I of the cone about its axis of symmetry will be

$$\begin{aligned} I &= \frac{1}{2} \int_0^h \rho \pi \left(a \frac{h-z}{h} \right)^2 \left(a \frac{h-z}{h} \right)^2 dz \\ &= \frac{1}{2} \frac{\rho \pi a^4 h}{h^4} \int_0^h (h-z)^4 dz = \frac{1}{2} \frac{\rho \pi a^4 h}{h^4} \int_0^h (z-h)^4 dz \end{aligned}$$

Since the M.I of the disk is

$$\delta I = \frac{1}{2} \delta m r^2$$

On substituting for δm and r , the M.I of the cone about its axis of symmetry will be

$$\begin{aligned} I &= \frac{1}{2} \int_0^h \rho \pi \left(a \frac{h-z}{h} \right)^2 \left(a \frac{h-z}{h} \right)^2 dz \\ &= \frac{1}{2} \frac{\rho \pi a^4 h}{h^4} \int_0^h (h-z)^4 dz = \frac{1}{2} \frac{\rho \pi a^4 h}{h^4} \int_0^h (z-h)^4 dz \\ I &= \frac{\rho \pi a^4 h}{10 h^4} |(z-h)^5|_0^h \\ &= \frac{\rho \pi a^4 h^6}{10 h^4} = \frac{\rho \pi a^4 h^2}{10} \end{aligned}$$

Since we know that

$$\rho = \text{density of the cone}$$

$$= \frac{M}{(1/3)\pi a^2 h}$$

So,

$$I = \frac{3}{10}Ma^2$$

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Module No. 140

Ellipsoid of Inertia

We have obtained the expression of moment of inertia of the given line l in terms of moment and product of inertia w.r.t the coordinate axes OXYZ coordinate system whose origin O lies on the line l .

$$I_l = I = \lambda^2 I_{11} + \mu^2 I_{22} + \nu^2 I_{33} + 2\mu\lambda I_{12} + 2\mu\nu I_{23} + 2\nu\lambda I_{31} \quad (1)$$

For the ellipsoid of inertia, we chose a point P such that $|OP| = 1/\sqrt{I}$. If (x, y, z) are coordinates of point P then

$$\lambda = \frac{x}{|OP|} = x\sqrt{I}, \quad \mu = \frac{y}{|OP|} = y\sqrt{I}, \quad \nu = \frac{z}{|OP|} = z\sqrt{I}$$

On eliminating λ, μ, ν from equation (1), we obtain

$$\begin{aligned} I &= (x\sqrt{I})^2 I_{11} + (y\sqrt{I})^2 I_{22} + (z\sqrt{I})^2 I_{33} + 2(xy\sqrt{I}\sqrt{I})I_{12} + 2(yz\sqrt{I}\sqrt{I})I_{23} + 2(xz\sqrt{I}\sqrt{I})I_{31} \\ I &= I[x^2 I_{11} + y^2 I_{22} + z^2 I_{33} + 2xy I_{12} + 2yz I_{23} + 2zx I_{31}] \\ 1 &= x^2 I_{11} + y^2 I_{22} + z^2 I_{33} + 2xy I_{12} + 2yz I_{23} + 2zx I_{31} \end{aligned}$$

which can also be written as

$$I_{11}x^2 + I_{22}y^2 + I_{33}z^2 + 2I_{12}xy + 2I_{23}yz + 2I_{31}zx = 1$$

Since I_{11}, I_{22}, I_{33} are all positive, equation (2) represents an ellipsoid. This ellipsoid is called ellipsoid of inertia or momental ellipsoid. The momental ellipsoid contains information about moments or product of inertia at given point. If P is any point on the ellipsoid of inertia, then $|OP| = 1/\sqrt{I}$ or $I = 1/OP^2$ i.e. the moment of inertia of a rigid body about any line $\overrightarrow{OP}$ is equals to the reciprocal of the square of the length of $|\overrightarrow{OP}|$.

Module No. 141

Rotational Kinetic Energy

Introduction to Kinetic Energy

Kinetic energy is the energy produced in any body during its motion. It is equal to the half of the product of mass and square of the velocity of the moving body;

$$K.E. = T = \frac{1}{2}mv^2$$

We consider a rigid body in a general state of motion in which it has both translation and rotation w.r.t a fixed coordinate system. We suppose that it has an instantaneous angular velocity ω about a reference point C. We will use a model of a rigid body in which it is considered a collection of large number N of particles which satisfy the constraint of rigidity.

If v is the velocity of C, then the velocity v_i of the i th particle is given by

$$v_i = v + \omega \times r_i$$

If m_i is the mass and v_i the velocity of the i th particle, then the kinetic energy of the i th particle will be

$$T_i = \frac{1}{2}m_i v_i^2$$

therefore the total kinetic energy of the system will be given by

$$\begin{aligned} T &= \sum_{i=1}^N T_i = \sum_{i=1}^N \frac{1}{2}m_i v_i^2 = \sum_{i=1}^N \frac{1}{2}m_i (v + \omega \times r_i)^2 \\ &= \frac{1}{2} \sum_{i=1}^N m_i [v^2 + 2v \cdot \omega \times r_i + (\omega \times r_i)^2] \\ &= \frac{1}{2} \left(\sum_{i=1}^N m_i \right) v^2 + \frac{1}{2} \sum_{i=1}^N m_i 2v \cdot \omega \times r_i + \frac{1}{2} \sum_{i=1}^N m_i (\omega \times r_i)^2 \\ &= \frac{1}{2} M v^2 + v \cdot (\omega \times \sum_{i=1}^N m_i r_i) + \frac{1}{2} \sum_{i=1}^N m_i (\omega \times r_i)^2 \\ &= \frac{1}{2} M v^2 + v \cdot \omega \times \sum_{i=1}^N m_i r_i + \frac{1}{2} \sum_{i=1}^N m_i (\omega \times r_i)^2 \end{aligned} \quad (1)$$

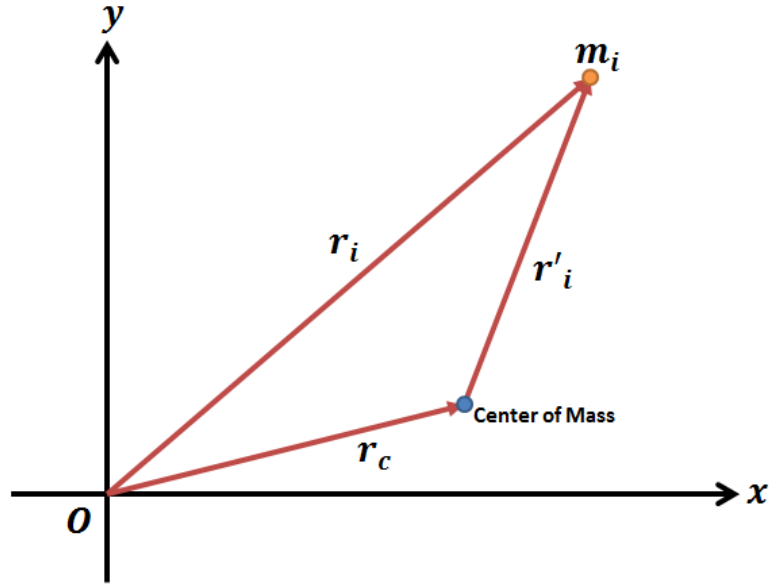
where $\sum_{i=1}^N m_i = M$ is the total mass of the system.

Now by using the definition of position vector center of mass (c.m.), we have

$$r_c = \frac{\sum_i m_i r_i}{\sum_i m_i} \quad (2)$$

Now we will discuss the consequence of referring of the position vectors of particle of the system to the c.m. If the position vector of the i th particle of the system w.r.t the c.s. , then

$$r_i = r'_i + r_c$$



Then by substitution this expression in the definition of c.m.in (2) , we obtain

$$r_c = \frac{\sum_i m_i (r'_i + r_c)}{\sum_i m_i}$$

$$r_c = \frac{\sum_i m_i r'_i}{\sum_i m_i} + \frac{Mr_c}{M}$$

$$r_c = \frac{\sum_i m_i r'_i}{\sum_i m_i} + r_c$$

or we can write

$$r_c = \frac{\sum_i m_i r'_i}{M} + r_c$$

which gives

$$\sum_i m_i r'_i = 0$$

Hence if the reference point C is identified with the center of the mass and origin is taken thereat, the expression $\sum_{i=1}^N m_i r_i = 0$ and therefore

$$T = \frac{1}{2} M v^2 + \frac{1}{2} \sum_{i=1}^N m_i (\omega \times r_i)^2$$

$$T = T_{tr} + T_{rot}$$

where $T_{tr} = \frac{1}{2} M v^2$ is the translational kinetic energy is also equals to the K.E of the center of the mass and $T_{rot} = \frac{1}{2} \sum_{i=1}^N m_i (\omega \times r_i)^2$ is the rotational kinetic energy of the system.

But we have

$$\begin{aligned} (\omega \times r_i)^2 &= (\omega \times r_i) \cdot (\omega \times r_i) \\ &= \omega \cdot r_i \times (\omega \times r_i) \end{aligned}$$

therefore on substitution

$$T_{rot} = \frac{1}{2} \sum_{i=1}^N m_i [\omega \cdot r_i \times (\omega \times r_i)] = \frac{1}{2} \omega \cdot L$$

Thus we have obtained the following formulas for translational and rotational K.E. of a rigid body

$$T_{tr} = \frac{1}{2} M v^2$$

$$T_{rot} = \frac{1}{2} \omega \cdot L \quad (1)$$

As we studied the relation $L = I\omega$

By using this value in relation (1) we obtain the rotational kinetic energy in terms of moment of inertia.

$$T_{rot} = \frac{1}{2} \omega \cdot I\omega = \frac{1}{2} I\omega^2 \quad (2)$$

where I is the moment of inertia of the rigid body about the origin and equation (2) is the required expression for **Rotational Kinetic Energy**.

Module No. 142

Moment of Inertia & Angular Momentum in Tensor Notation

To show that the 3×3 inertia matrix (I_{ij}) is also a Cartesian tensor of rank 3, we proceed as follows. Here we have to distinguish between the particle index and the component index, which denotes particle number, will be denoted by Greek letter α , and will take the value $1, 2, \dots, N$. On the other hand, the component index, which denotes the component number will be denoted by the Latin letters such as i and will take the value $1, 2, 3$.

Using this notation, we can write the angular momentum of the system of N particles as

$$L = \sum_{\alpha} r_{\alpha} \times p_{\alpha} = \sum_{\alpha} r_{\alpha} \times (m_{\alpha} v_{\alpha})$$

where v_{α} is the velocity of α th particle. Continuing we have

$$\sum_{\alpha} m_{\alpha} r_{\alpha} \times v_{\alpha} = \sum_{\alpha} m_{\alpha} r_{\alpha} \times (\omega \times r_{\alpha}), \quad \therefore v_{\alpha} = \omega \times r_{\alpha}$$

Recall that every particle of the rigid body has the same angular velocity at a given t .

Simplification of the above reduces to

$$\begin{aligned} L &= \sum_{\alpha} m_{\alpha} ((r_{\alpha} \cdot r_{\alpha}) \omega - (\omega \cdot r_{\alpha}) r_{\alpha}) \\ &= \sum_{\alpha} m_{\alpha} (r_{\alpha}^2 \omega - (\omega \cdot r_{\alpha}) r_{\alpha}) \end{aligned}$$

But

$$\omega \cdot r_{\alpha} = \sum_{i=1}^3 \omega_i r_{\alpha,i} = \sum_{i=1}^3 \omega_i x_{\alpha,i} = \sum_{j=1}^3 \omega_j x_{\alpha,j}$$

where

$$r_{\alpha} = (x_{\alpha}, y_{\alpha}, z_{\alpha}) = (x_{\alpha,1}, x_{\alpha,2}, x_{\alpha,3}) \equiv x_{\alpha,i}$$

Making substitution and taking the i^{th} component of the expression of angular momentum, we have

$$L_i = \sum_{\alpha} m_{\alpha} \left(r_{\alpha}^2 \omega_i - \left(\sum_j \omega_j x_{\alpha,j} \right) x_{\alpha,i} \right)$$

But $\omega_i = \sum_j \omega_j \delta_{ij}$; where $\delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$, thus

therefore

$$L_i = \sum_{\alpha} m_{\alpha} \left(r_{\alpha}^2 \sum_j \omega_j \delta_{ij} - \sum_j \omega_j x_{\alpha,j} x_{\alpha,i} \right)$$

$$L_i = \sum_{\alpha} m_{\alpha} \sum_j (r_{\alpha}^2 \delta_{ij} - x_{\alpha,j} x_{\alpha,i}) \omega_j$$

$$L_i = \sum_j \omega_j \sum_{\alpha} m_{\alpha} (r_{\alpha}^2 \delta_{ij} - x_{\alpha,j} x_{\alpha,i})$$

which can also be expressed as

$$L_i = \sum_j \omega_j I_{ij} \quad (1)$$

where

$$I_{ij} = \sum_{\alpha} m_{\alpha} (r_{\alpha}^2 \delta_{ij} - x_{\alpha,j} x_{\alpha,i}) \quad (2)$$

Equation (1) shows that the component of angular momentum depends not only on the angular velocity ω but also on the inertia tensor I_{ij} .

Since the angular velocity (ω_i) and the angular momentum (L_i) are known to be cartesian tensor of rank 1, it follows from the quotient theorem I_{ij} that I_{ij} is a tensor of rank 2. It is called the inertia tensor.

Module No. 143

Introduction to Special Moments of Inertia

In this module, we will study some special moment of inertia. In order to introduce the moment of inertia of some specific shapes and special rigid bodies, we assume that the rigid body is of uniform density of particles.

Solid Circular Cylinder

We assume the radius of cylinder is a and mass M about axis of cylinder.

$$I = \frac{1}{2}Ma^2$$

Hollow Circular Cylinder

We assume the radius of cylinder is a and mass M about axis of cylinder.

We consider the thickness of Wall of cylinder is negligible.

$$I = Ma^2$$

Solid Sphere

We assume the radius of sphere is a and mass M about a diameter.

$$I = \frac{2}{5}Ma^2$$

Hollow Sphere

We assume the radius of sphere is a and mass M about a diameter.

We consider the thickness of sphere is negligible.

$$I = Ma^2$$

Rectangular Plate

We consider sides of length a and b , and mass M about an axis perpendicular to the plate through the center of mass.

$$I = \frac{1}{12}M(a^2 + b^2)$$

Thin Rod

We assume the length of rod is a and mass M about an axis perpendicular to the rod through the center of mass.

$$I = \frac{1}{12}Ma^2$$

Triangular Lamina

We assume the height ***h*** and mass ***M*** of lamina.

$$I = \frac{1}{6}Mh^2$$

Right Circular Cone

We assume the radius of the circular cone is ***a*** and mass ***M***.

$$I = \frac{3}{10}Ma^2$$

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Module No. 144

M.I. of the Thin Rod – Derivation

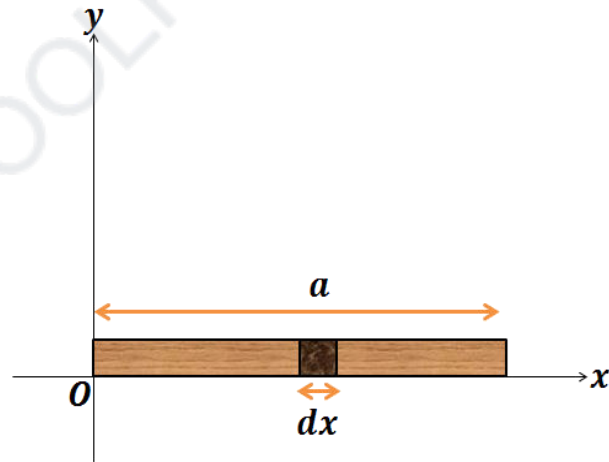
Problem Statement

Calculate the moment of inertia of a uniform rod of length l about an axis perpendicular to the rod and passing through an end point.

Solution

Let the X – axis be chosen along the length of the rod, with origin at one end point as shown in the figure. Let M and a be the mass and length of rod respectively. We suppose the rod to be composed of small elements.

Let dm and dx be the mass and the length of the specific element of the rod at a distance x from the end point O.



Then

$$\frac{dm}{dx} = \frac{M}{a}$$

$$\Rightarrow dm = \frac{M}{a} dx$$

Then the moment of inertia of this element about the given axis is

$$I_{\text{element}} = dm x^2 = \frac{M}{a} x^2 dx$$

Hence the moment of inertia of the whole rod will be

$$\begin{aligned}
 I &= \sum_{\text{all elements}} \frac{M}{a} x^2 dx \\
 &= \frac{M}{a} \int_0^a x^2 dx \\
 &= \frac{M}{a} \cdot \left| \frac{x^3}{3} \right|_0^a = \frac{M a^3}{a \cdot 3} \\
 &= \frac{1}{3} M a^2
 \end{aligned}$$

Module No. 145

M.I. of Hoop or Circular Ring – Derivation

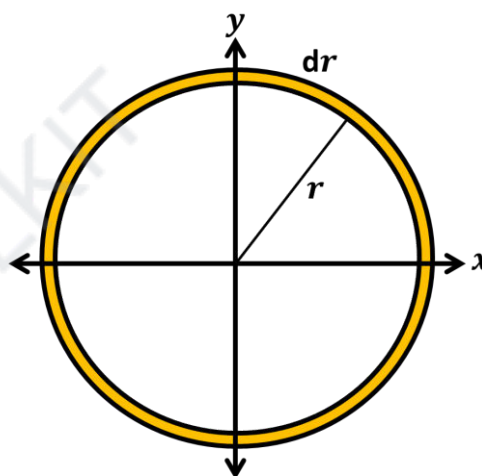
Problem Statement

Calculate the moment of inertia of a hoop of mass M and radius r about an axis passing through its center.

Proof

Let M be the mass and r the radius of the hoop. Then we can define the density of the hoop by

$$\rho = \frac{\text{mass}}{\text{area}} = \frac{M}{2\pi r dr}$$



We consider this hoop to be composed of small masses (δm) each of length δs .

We can write it as

$$\begin{aligned}\rho &= \frac{M}{2\pi r dr} = \frac{\delta m}{dr \delta s} \\ \Rightarrow \delta m &= \frac{M}{2\pi r} \delta s\end{aligned}$$

Moment of inertia of the small portion of the hoop of mass δm about an axis through center and perpendicular to the plane of the ring equals

$$\begin{aligned}I_{\text{particle}} &= \delta m r^2 \\ &= \frac{M}{2\pi r} \delta s r^2 = \frac{M r}{2\pi} \delta s\end{aligned}$$

Therefore the $M.I$ of the whole ring/hoop will be

$$\begin{aligned}
 I &= \frac{Mr}{2\pi} \sum_{\text{particles}} \delta s \\
 I &= \frac{Mr}{2\pi} \int ds \\
 &= \frac{Mr}{2\pi} 2\pi r \\
 &= Mr^2
 \end{aligned}$$

➤ Hence we obtain Mr^2 as the moment of inertia of the hoop.

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Module No. 146

M.I. of Annular Disk - Derivation

Problem

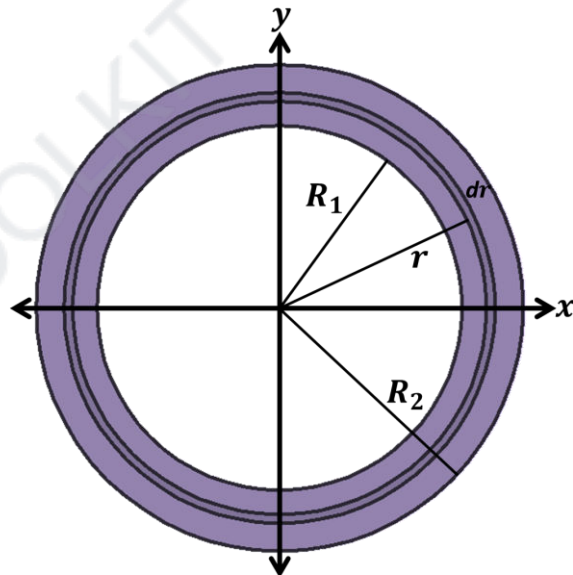
Calculate the moment of inertia of annular disk of mass M . The inner radius of the annulus is R_1 and the outer radius is R_2 about an axis passing through its center.

Solution

Subdivide the annular disk into concentric rings one of which is shown in the fig.

Let the mass of the ring is dm , and the radius be r , then the moment of inertia of the ring will be:

$$I_{ring} = r^2 dm$$



The Surface area of the ring is

$$\text{Area} = (2\pi r)dr = 2\pi r dr$$

Since the surface area of the

annulus is

$$\pi(R_2^2 - R_1^2)$$

Therefore, we can have

$$\frac{dm}{M} = \frac{2\pi r dr}{\pi(R_2^2 - R_1^2)}$$

$$dm = \frac{2r dr}{(R_2^2 - R_1^2)} M$$

Since the moment of Inertia of the ring is:

$$I_{ring} = r^2 dm$$

or

$$I_{ring} = r^2 \frac{2r dr}{R_2^2 - R_1^2} M = \frac{2Mr^3 dr}{R_2^2 - R_1^2}$$

Thus the total M.I of the annular disk will be

$$\begin{aligned} I &= \int_{r=R_1}^{R_2} I_{ring} \\ I &= \int_{r=R_1}^{R_2} \frac{2Mr^3 dr}{R_2^2 - R_1^2} \\ &= \frac{2M}{R_2^2 - R_1^2} \int_{r=R_1}^{R_2} r^3 dr \\ &= \frac{2M}{R_2^2 - R_1^2} \left[\frac{r^4}{4} \right]_{R_1}^{R_2} \\ &= \frac{2M}{R_2^2 - R_1^2} \frac{R_2^4 - R_1^4}{4} \\ I &= \frac{1}{2} M (R_2^2 + R_1^2) \end{aligned}$$

Module No. 147

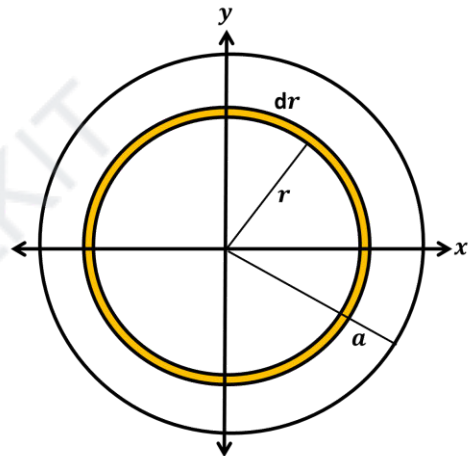
M.I. of a Circular Disk - Derivation

Problem

To find the moment of inertia of a circular disk of radius a , and mass M about the axis of the disk.

Solution

Subdivide the disk into concentric rings one of which is the element (ring) shown in the fig.



Let the mass of the ring is dm , then the moment of inertia of the ring will be:

$$I_{ring} = r^2 dm$$

The Surface area of this element is

$$Area = (2\pi r)dr = 2\pi r dr$$

Since we have

$$\frac{dm}{M} = \frac{2\pi r dr}{\pi a^2}$$

$$dm = \frac{2r dr}{a^2} M$$

Since the moment of Inertia of the ring is:

$$I_{ring} = r^2 dm$$

or

$$I_{ring} = r^2 \frac{2r dr}{a^2} M = \frac{2Mr^3 dr}{a^2}$$

Thus the total moment of inertia of the circular disk will be

$$\begin{aligned} I_{disk} &= \int_{r=0}^a I_{ring} \\ I_{disk} &= \int_{r=0}^a \frac{2Mr^3 dr}{a^2} \\ &= \frac{2M}{a^2} \int_{r=0}^a r^3 dr \\ &= \frac{2M}{a^2} \left[\frac{r^4}{4} \right]_0^a = \frac{2M}{a^2} \frac{a^4}{4} \\ I_{disk} &= \frac{1}{2} Ma^2 \end{aligned}$$

which is M.I of a disk.

Module No. 148

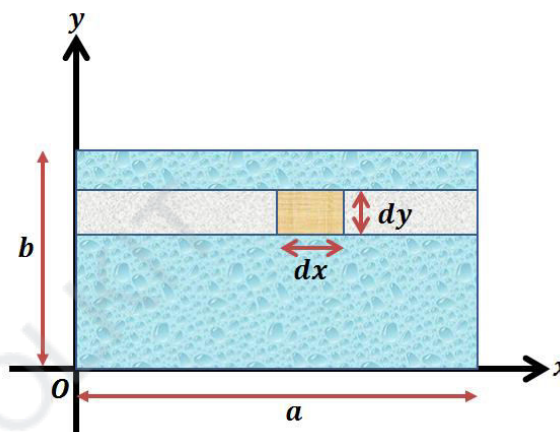
Rectangular Plate – Derivation

Problem Statement

Calculate the inertia matrix of a uniform rectangular plate with sides a and b about its side.

Solution

We consider a rectangular plate (lamina) of sides of length a and b . We consider an element of length dx and dy .



The mass of selected element will be

$$dm = \rho dx dy$$

Its moment of inertia about the y-axis is

$$\rho dx dy x^2 = \rho x^2 dx dy.$$

Thus the total moment of inertia is

$$\begin{aligned} I &= \int_{x=0}^a \int_{y=0}^b \rho x^2 dx dy \\ &= \rho b \int_0^a x^2 dx = \rho b \left[\frac{x^3}{3} \right]_0^a = \frac{1}{3} \rho b a^3 \end{aligned}$$

Since the total mass of rectangular plate is

$$M = \rho ab$$

the moment of inertia will be

$$I = \frac{1}{3} M a^2$$

Module No. 149

M.I. of Square Plate – Derivation

Problem Statement

Calculate the moment of inertia of a uniform square plate with sides a about any axis through its center and lying in the plane of plate.

Solution

Consider a uniform square plate with length of its side to be a , we have to find out M.I. of this plate about any axis, through its center.

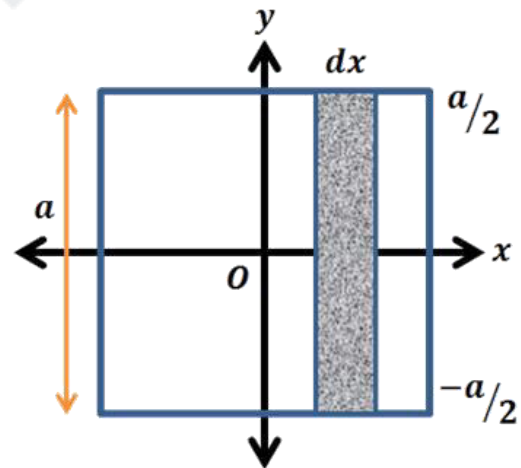
We consider this axis as $y - axis$.

Let ρ be the density of plate and the total area of square plate is a^2 .

So density will be

$$\rho = \frac{M}{a^2}$$

We assume that the plate has been divided into vertical strips. Let us consider a strip from the whole square as shown in figure.



The strips are chosen in this way because each point on a particular strip is approximately the same distance from axis of rotation i.e. y -axis, the mass of the strip is δm and the width of each strip is δx , then the area of the strip will be $a\delta x$, so

$$\delta m = \rho a \delta x$$

Let the distance of the strip from y -axis is x , then the moment of inertia of strip will be

$$\delta I = \delta m x^2 = x^2 \rho a \delta x$$

So the moment of inertia of square is

$$I_{\text{square}} = \sum_{\text{all strips}} x^2 \rho a \delta x$$

$$\begin{aligned}
 I_{square} &= \int_{-a/2}^{a/2} x^2 \rho a dx = \rho a \int_{-a/2}^{a/2} x^2 dx \\
 &= \rho a \left[\frac{x^3}{3} \right]_{-a/2}^{a/2} = \frac{\rho a}{3} \left[\frac{a^3}{8} - \frac{(-a^3)}{8} \right] \\
 &= \frac{\rho a}{3} \left(\frac{a^3}{4} \right) = \frac{\rho a^4}{12}
 \end{aligned}$$

Now substitute $\rho = \frac{M}{a^2}$, we obtain

$$I_{square} = \frac{Ma^2}{12}$$

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Module No. 150

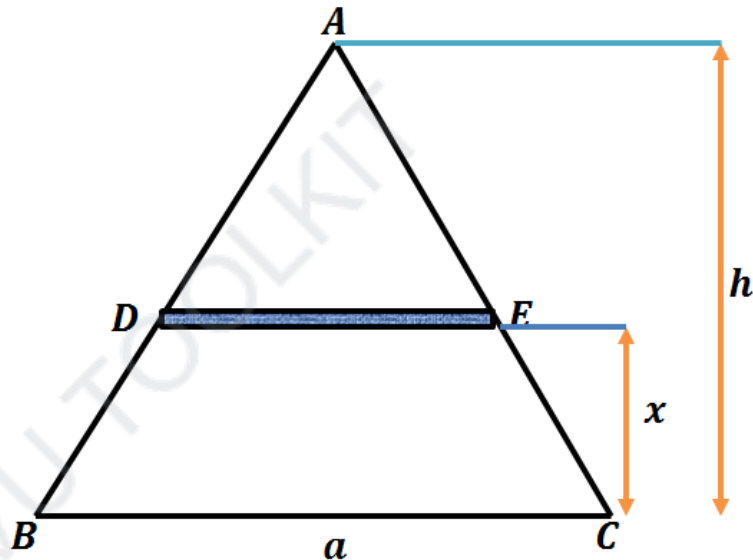
M.I. of Triangular Lamina – Derivation

Problem Statement

Find the moment of inertia of a uniform triangular lamina of mass M about one of its sides.

Solution

Let ABC be the lamina. We will find its M.I. about one of its side BC. We choose the X-axis and the origin as shown in figure.



At a distance x from BC we consider a strip of lamina of width dx . If the mass of the strip is dm then its moment of inertia about BC is $= x^2 dm$.

To find dm we note that the triangles ABC and ADE are similar. Therefore the ratios of sides are the same.

Hence

$$\frac{DE}{BC} = \frac{\text{height of ADE}}{\text{height of ABC}} = \frac{h-x}{h}$$

which gives

$$DE = \frac{h-x}{h} \times BC = \frac{h-x}{h} \times a$$

where h is the height of the triangle ABC at base BC and a is the length of BC. Now

$$\begin{aligned} \delta m &= \rho(\text{area}) = \rho DE \delta x \\ &= a \left(\frac{h-x}{h} \right) \delta x \rho \end{aligned}$$

where ρ is the density.

Therefore the moment of inertia of triangular lamina about the side BC is

$$\begin{aligned}
 I &= \int x^2 dm = \int_0^h a \left(\frac{h-x}{h} \right) x^2 dx \rho \\
 &= \frac{\rho a}{h} \int_0^h x^2 (h-x) dx = \frac{\rho a}{h} \int_0^h (hx^2 - x^3) dx \\
 &= \frac{\rho a}{h} \left[h \frac{x^3}{3} - \frac{x^4}{4} \right]_0^h \\
 &= \frac{\rho a}{h} \left(\frac{h^4}{3} - \frac{h^4}{4} \right) = \frac{\rho a h^3}{12}
 \end{aligned}$$

Now substitute density $\rho = \frac{\text{Mass}}{\text{Area}} = \frac{M}{1/2 ah}$

we obtain

$$I = \frac{1}{6} M h^2$$

Hence the required expression for moment of inertia of the triangular lamina is $\frac{1}{6} M h^2$.

Module No. 151

M.I. of Elliptical Plate along its Major Axis – Derivation

Problem Statement

Find the M.I. of a uniform elliptical plate with semi major axes and semi minor axes a, b respectively about its major axes.

Solution

We consider the elliptical plate as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

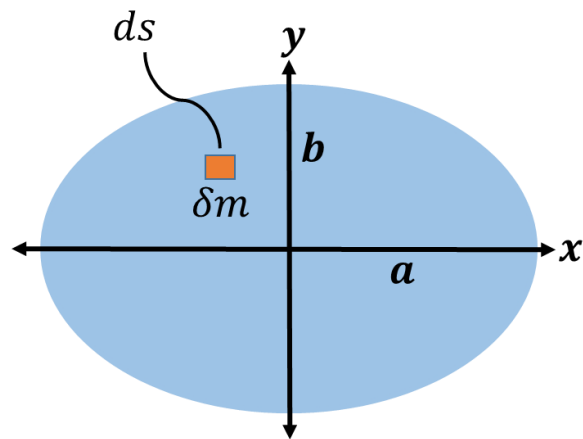
with semi major axes along x-axis as shown in figure.

From (1) we have

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

$$\text{So, let } y_1 = \frac{b}{a} \sqrt{a^2 - x^2},$$

We consider a small element of plate of mass δm of the elliptical plate will be ρds which will be equal to $\rho dx dy$. The moment of inertia if this element along x-axis will be equal to $I = \delta m y^2$.



Then the moment of inertia of whole plate will be

$$I_x = \int (\delta m) y^2 = \int_{plate} \rho ds y^2$$

$$\begin{aligned}
&= \rho \int_{-a}^a \left(\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} y^2 dy \right) dx \\
&= \rho \int_{-a}^a \frac{2y_1^2}{3} dx = \frac{2\rho}{3} \int_{-a}^a \frac{b^3}{a^3} (a^2 - x^2)^{3/2} dx
\end{aligned}$$

due to symmetry, we can write it as

$$= \frac{4\rho b^3}{3a^3} \int_0^a (a^2 - x^2)^{3/2} dx$$

By making use of polar coordinates, substitute $x = a \sin \theta$, then $dx = a \cos \theta$ this integral becomes

$$\begin{aligned}
I_x &= \frac{4\rho b^3}{3a^3} \int_0^{\pi/2} a^3 \cos^3 \theta (a \cos \theta) d\theta \\
&= \frac{4\rho ab^3}{3} \int_0^{\pi/2} \cos^4 \theta d\theta \\
&= \frac{4\rho ab^3}{3} \frac{1 \times 3}{2 \times 4} \times \frac{\pi}{2} \\
&= \frac{1}{4} \rho ab^3 \pi \\
&= \frac{1}{4} ab^3 \pi \times \frac{M}{\pi ab} = \frac{1}{4} Mb^2
\end{aligned}$$

where for elliptical plate, we have

$$\rho = \frac{M}{\pi ab}$$

is the required expression for M.I.

Module No. 152

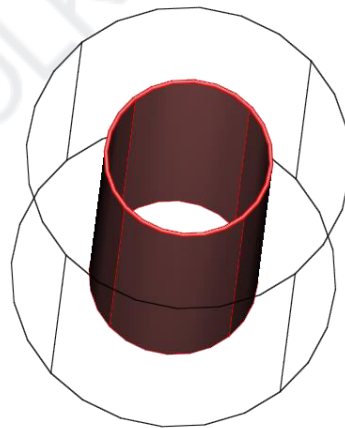
M.I. of a Solid Circular Cylinder - Derivation

Problem Statement

To find the moment of inertia of a solid circular cylinder of radius a , mass M and the height of the cylinder h about the axis of the cylinder.

Solution

Let's subdivide the solid circular cylinder into concentric cylindrical shells/ hollow cylinders, one of which is shown in the fig.



Let the mass of one shell is dm , height is same as h , thickness be dr and the radius be r then the density of the shell will be

$$\rho = \frac{dm}{2\pi r dr h}$$

or

$$dm = 2\pi r \rho dr h$$

Since we have

$$\rho = \frac{M}{V_{cylinder}} = \frac{M}{\pi a^2 h} = \frac{dm}{2\pi r dr h}$$

or

$$dm = \frac{2M}{a^2} r dr$$

Since the moment of inertia of the shell will be:

$$I_{shell} = r^2 dm$$

or

$$I_{shell} = r^2 \frac{2M}{a^2} r dr = \frac{2M}{a^2} r^3 dr$$

Thus the total moment of inertia of the solid circular cylinder will be

$$\begin{aligned} I_{cylinder} &= \int_{r=0}^a I_{shell} \\ I_{cylinder} &= \int_{r=0}^a \frac{2M}{a^2} r^3 dr \\ &= \frac{2M}{a^2} \int_{r=0}^a r^3 dr \\ &= \frac{2M}{a^2} \left[\frac{r^4}{4} \right]_0^a = \frac{2M}{a^2} \frac{a^4}{4} \\ I_{cylinder} &= \frac{1}{2} M a^2 \end{aligned}$$

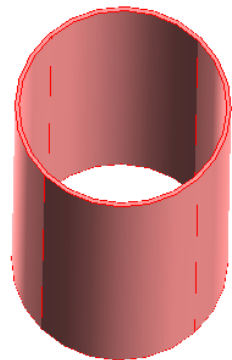
is the M.I of solid cylinder.

Module No. 153

M.I. of Hollow Cylindrical Shell - Derivation

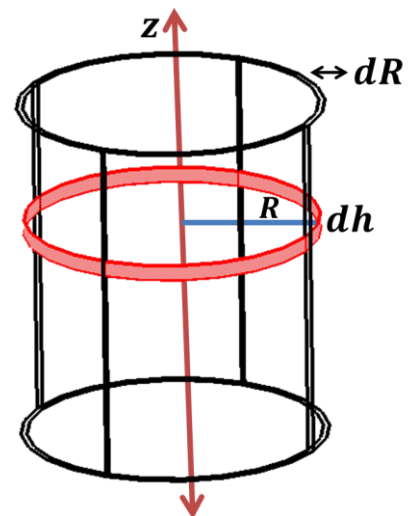
Problem

To find the moment of inertia of a hollow open cylindrical shell of radius R , thickness dR , mass M and height of shell h about the axis of shell.



Solution

Let's subdivide the hollow shell into small hoops/ rings, one of which is shown in the figure.



Let the mass of one ring is dm , height is dh , thickness be dr and the radius be R , then the mass of one ring will be

$$dm = \rho 2\pi R dR dh$$

Hence the moment of inertia of the ring of radius R will be

$$I_{ring} = dm R^2$$

or

$$I_{ring} = (\rho 2\pi R dR dh) R^2 = 2\rho\pi R^3 dR dh$$

In order to obtain the moment of inertia for the whole hollow cylindrical open shell, we will integrate

$$I = \int I_{ring} = \int_0^h 2\rho\pi R^3 dR dh$$

$$I = 2\rho\pi R^3 dR \int_0^h dh$$

$$= 2\rho\pi R^3 dR h$$

Since the density of the hollow cylindrical shell is

$$\rho = \frac{M}{2\pi R dR h}$$

Therefore

$$I = MR^2$$

is the M.I of hollow cylindrical open shell.

Module No. 154

M.I of Solid Sphere - Derivation

Problem

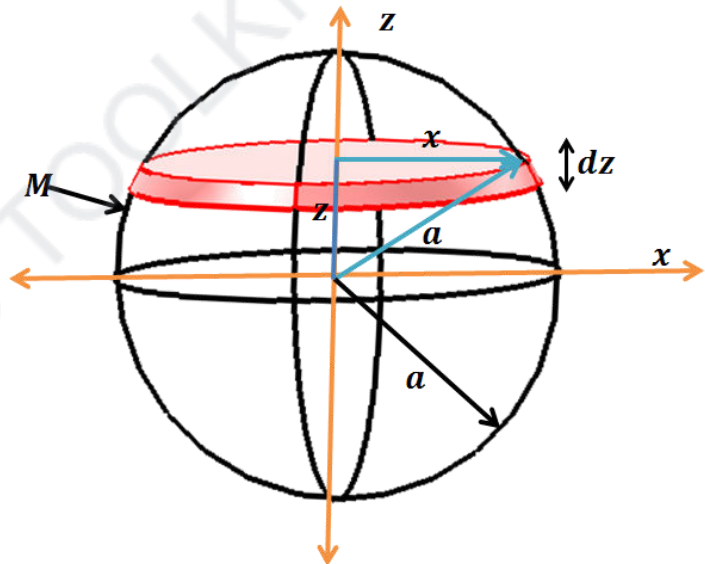
Find the moment of inertia of a uniform solid sphere of radius a and mass M about an axis (the z -axis) passing through the center.

Solution

For a uniform solid sphere, due to symmetry, we have

$$I_{xx} = I_{yy} = I_{zz}$$

In order to calculate the moment of Inertia of the sphere, we split the sphere into thin circular discs, one of which is shown in Figure.



We have already derived the expression for the moment of inertia of a representative disc of radius x , which is

$$I_{disk} = \frac{1}{2} x^2 dm$$

of an elementary disc of mass dm and the radius x .

As we know the mass = (density)(Area of disc)

therefore

$$dm = \rho \pi x^2 dz$$

Hence moment of inertia of the sphere along z-axis will be

$$I_{zz} = \int_{-a}^a \frac{1}{2} \rho \pi x^4 dz$$

Now, to write “ x ” in

terms of z , we make

a triangle as shown in fig,

where

$$a^2 = x^2 + z^2, \Rightarrow x^2 = a^2 - z^2$$

$$I_{zz} = \int_{-a}^a \frac{1}{2} \rho \pi (a^2 - z^2)^2 dz$$

$$= \frac{8}{15} \pi \rho a^5$$

Since the mass of the sphere is

$$M = \frac{4}{3} \pi a^3 \rho$$

Therefore

$$I_{zz} = \frac{2}{5} M a^2$$

Also

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} M a^2$$

is the required moment of inertia of the sphere.

Module No. 155

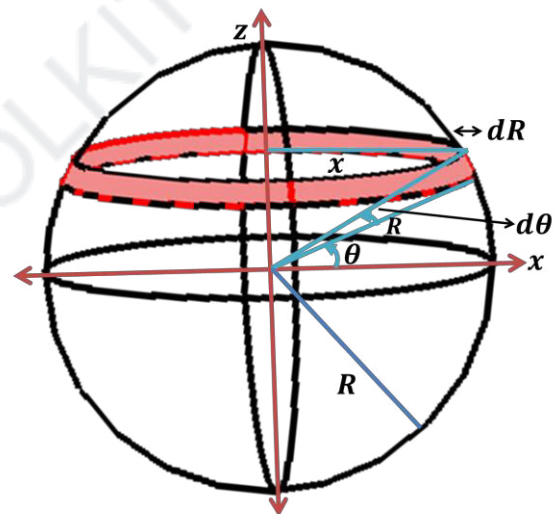
M.I. of the Hollow Sphere – Derivation

Problem

A thin uniform hollow sphere has a radius R and mass M . Calculate its moment of inertia about any axis through its center.

Solution

In order to calculate the moment of inertia of the hollow sphere, we split the hollow sphere into thin hoops (rings), as shown in Figure.



We have already derived the expression for the moment of inertia of a representative hoop of radius x , which is

$$I = dm x^2$$

of an elementary ring of mass dm and the radius x .

The volume of the elementary ring is

$$dV = 2\pi x R d\theta dR$$

As we know the mass = (density)(volume)

$$dm = \rho dV$$

therefore

$$dm = \rho 2\pi x R dR d\theta$$

Hence the moment of inertia of the small ring of radius x will be

$$I_{ring} = dm x^2$$

or

$$I_{ring} = (2\pi x \rho R dR d\theta) x^2 = 2\pi \rho R dR x^3 d\theta$$

which is the moment of inertia of ring of radius x chosen from the hollow sphere.

In order to obtain the moment of inertia for the whole hollow sphere, we will integrate

$$I = \int I_{ring} = \int_{-\pi/2}^{\pi/2} 2\pi \rho R dR x^3 d\theta$$

due to symmetry, we can write

$$I = 4\pi \rho R dR \int_0^{\pi/2} x^3 d\theta$$

To solve the integral, we need to write x in terms of θ . From fig we have

$$x = R \cos \theta$$

The integral becomes,

$$I = 4\pi \rho R dR \int_0^{\pi/2} (R \cos \theta)^3 d\theta$$

$$I = 4\pi \rho R^4 dR \int_0^{\pi/2} \cos^3 \theta d\theta$$

$$I = 4\pi \rho R^4 dR \int_0^{\pi/2} \cos \theta \cos^2 \theta d\theta$$

$$I = 4\pi\rho R^4 dR \int_0^{\pi/2} \cos \theta (1 - \sin^2 \theta) d\theta$$

$$I = 4\pi\rho R^4 dR \int_0^{\pi/2} \cos \theta (1 - \sin^2 \theta) d\theta$$

$$I = 4\pi\rho R^4 dR \left(\int_0^{\pi/2} \cos \theta d\theta - \int_0^{\pi/2} \sin^2 \theta \cos \theta d\theta \right)$$

$$I = 4\pi\rho R^4 dR \left| \sin \theta - \frac{\sin^3 \theta}{3} \right|_0^{\pi/2}$$

$$= 4\pi\rho R^4 dR \left(1 - \frac{1}{3} \right)$$

$$= 4\pi\rho R^4 dR \times \frac{2}{3}$$

$$= \frac{8}{3} \pi \rho R^4 dR$$

Since the density of the hollow sphere is

$$\rho = \frac{M}{V} = \frac{M}{4\pi R^2 dR}$$

Hence M.I of hollow sphere will be

$$I = \frac{2}{3} MR^2$$

Module No. 156

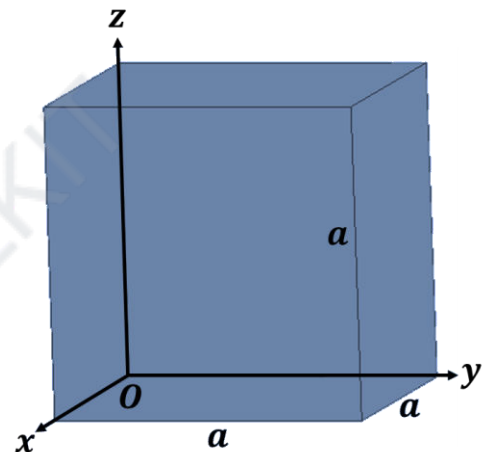
Inertia Matrix / Tensor of solid Cuboid

Problem Statement

Calculate the inertia matrix / inertia tensor of uniform solid cube at one of its corners.

Solution

Let the length of the edges be a of each side and let the axes be chosen along the edges as shown in the figure.



By definition

$$I_{11} \equiv I_{xx} = \int \rho(r)(y^2 + z^2)dV$$

Since the box is made of uniform material, the density ρ must be constant. Therefore

$$\begin{aligned} I_{xx} &= \rho \int_0^a \int_0^a \int_0^a (y^2 + z^2) dx dy dz \\ &= \rho \int_0^a \int_0^a (y^2 + z^2) dy dz \int_0^a dx \\ &= \rho a \int_0^a \int_0^a (y^2 + z^2) dy dz \end{aligned}$$

$$\begin{aligned}
&= \rho a \left[\int_0^a \int_0^a y^2 dy dz + \int_0^a \int_0^a z^2 dy dz \right] \\
&= \rho \left(\frac{a^4}{3} + \frac{a^4}{3} \right) \\
I_{xx} &= \rho a \left(\frac{2a^4}{3} \right) = \rho \left(\frac{2a^5}{3} \right)
\end{aligned}$$

Since we know that for the cube

$$\rho = \frac{M}{a^3}$$

We obtain

$$I_{xx} = \frac{2M}{3} a^2$$

Similarly, due to symmetry, we can write

$$\begin{aligned}
I_{yy} &= \frac{2M}{3} a^2 \\
I_{zz} &= \frac{2M}{3} a^2
\end{aligned}$$

Now for the product of inertia, we have

$$\begin{aligned}
I_{12} &= - \int \rho xy dV \\
&= -\rho \int_0^a \int_0^a \int_0^a xy dx dy dz \\
&= -\rho \int_0^a x dx \int_0^a y dy \int_0^a dz \\
&= -\rho a \frac{a^2}{2} \frac{a^2}{2} = -\rho \frac{a^5}{4}
\end{aligned}$$

Again using

$$\rho = \frac{M}{a^3}$$

We obtain

$$I_{12} = -\frac{Ma^2}{4}$$

Similarly,

$$I_{12} = -\frac{Ma^2}{4}$$

$$I_{12} = -\frac{Ma^2}{4}$$

The required inertia matrix / inertia tensor will be

$$I_{ij} = \begin{bmatrix} \frac{2M}{3}a^2 & -\frac{Ma^2}{4} & -\frac{Ma^2}{4} \\ -\frac{Ma^2}{4} & \frac{2M}{3}a^2 & -\frac{Ma^2}{4} \\ -\frac{Ma^2}{4} & -\frac{Ma^2}{4} & \frac{2M}{3}a^2 \end{bmatrix}$$

Module No. 157

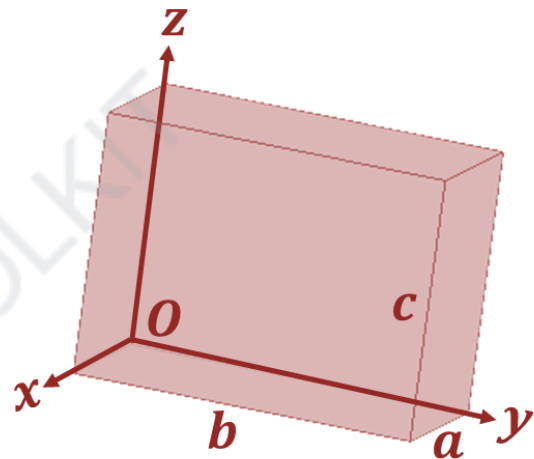
Inertia Matrix / Tensor of solid Cuboid

Problem Statement

Calculate the inertia matrix of uniform solid cuboid (parallelepiped) at one of its corner.

Solution

Let the length of the edges be a, b, c and let the axes be chosen along the edges as shown in the figure.



By definition

$$I_{11} \equiv I_{xx} = \int \rho(r)(y^2 + z^2) dV$$

Since the box is made of uniform material, the density ρ must be constant. Therefore

$$\begin{aligned} I_{xx} &= \rho \int_0^a \int_0^b \int_0^c (y^2 + z^2) dx dy dz \\ &= \rho \int_0^b \int_0^c (y^2 + z^2) dy dz \int_0^a dx \end{aligned}$$

$$\begin{aligned}
&= \rho a \int_0^b \int_0^c (y^2 + z^2) dy dz \\
&= \rho a \left[\int_0^b \int_0^c y^2 dy dz + \int_0^b \int_0^c z^2 dy dz \right] \\
&= \rho a \left(c \frac{b^3}{3} + b \frac{c^3}{3} \right) \\
I_{xx} &= \rho \frac{abc}{3} (b^2 + c^2)
\end{aligned}$$

using relation

$$\begin{aligned}
\rho &= \frac{M}{abc} \\
I_{xx} &= \frac{M}{3} (b^2 + c^2)
\end{aligned}$$

Similarly, due to symmetry, we can write

$$\begin{aligned}
I_{yy} &= \frac{M}{3} (a^2 + c^2) \\
I_{zz} &= \frac{M}{3} (a^2 + b^2)
\end{aligned}$$

Now for the product of inertia, we have

$$\begin{aligned}
I_{12} &= - \int \rho xy dV \\
&= -\rho \int_0^a \int_0^b \int_0^c xy dx dy dz \\
&= -\rho \int_0^a x dx \int_0^b y dy \int_0^c dz \\
&= -\rho c \frac{a^2}{2} \frac{b^2}{2} = -\rho \frac{a^2 b^2 c}{4}
\end{aligned}$$

Again using

$$\rho = \frac{M}{abc}$$

we get

$$I_{12} = -\frac{Mab}{4}$$

Similarly,

$$I_{23} = -\frac{Mbc}{4}$$

$$I_{31} = -\frac{Mac}{4}$$

The required inertia matrix / inertia tensor will be

$$I_{ij} = \begin{bmatrix} \frac{M}{3}(b^2 + c^2) & -\frac{Mab}{4} & -\frac{Mac}{4} \\ -\frac{Mab}{4} & \frac{M}{3}(a^2 + c^2) & -\frac{Mbc}{4} \\ -\frac{Mac}{4} & -\frac{Mbc}{4} & \frac{M}{3}(a^2 + b^2) \end{bmatrix}$$



Module No. 158

M.I of Hemi-Sphere – Derivation

Problem Statement

Find the moment of inertia of a uniform hemisphere of radius a about its axis of symmetry.

Solution

We will use the spherical polar coordinates (r, θ, φ) . Their use makes computational work simpler.

Their range of variation for hemisphere will be

$$0 \leq r < a$$

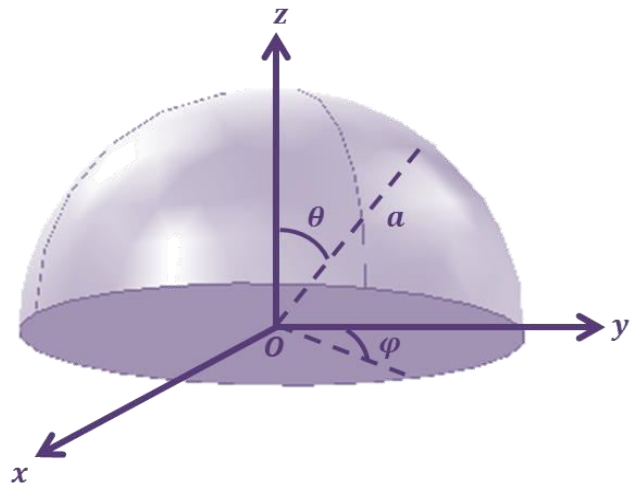
$$0 \leq \theta \leq \pi/2$$

$$0 \leq \varphi \leq 2\pi$$

$$x = r \sin \theta \cos \varphi,$$

$$y = r \sin \theta \sin \varphi, \quad z = r \cos \theta$$

We choose the z-axis as the axis of symmetry.



Hence

$$\begin{aligned}
 I_{zz} &= \int \rho(r)(x^2 + y^2) dV \\
 &= \rho \int (x^2 + y^2) dV
 \end{aligned}$$

Now calculate $x^2 + y^2$ in terms of sp. coordinates

$$\begin{aligned}
 x^2 + y^2 &= r^2(\sin^2 \theta \cos^2 \varphi + \sin^2 \theta \sin^2 \varphi) \\
 &= r^2 \sin^2 \theta (\cos^2 \varphi + \sin^2 \varphi) \\
 &= r^2 \sin^2 \theta
 \end{aligned}$$

and the element of volume in spherical polar coordinates is given by

$$dV = dr(rd\theta)(r \sin \theta d\varphi) = r^2 \sin \theta dr d\theta d\varphi$$

Therefore

$$\begin{aligned}
 I_{zz} &= \rho \int_0^a \int_0^{\pi/2} \int_0^{2\pi} r^4 \sin^3 \theta dr d\theta d\varphi \\
 &= \int_0^a r^4 dr \int_0^{\pi/2} \sin^3 \theta d\theta \int_0^{2\pi} d\varphi \\
 &= 2\pi\rho \frac{a^5}{5} \int_0^{\pi/2} \sin^3 \theta d\theta \\
 &= 2\pi\rho \frac{a^5}{5} \int_0^{\pi/2} \frac{3 \sin \theta - \sin 3\theta}{4} d\theta \\
 &= (2\pi\rho \frac{a^5}{5}) \frac{1}{4} \left[-3 \cos \theta + \frac{\cos 3\theta}{3} \right]_0^{\pi/2} \\
 &= \pi\rho \frac{a^5}{10} [(0) - (-3 + 1/3)] \\
 &= \pi\rho \frac{a^5}{10} \left(\frac{8}{3} \right)
 \end{aligned}$$

$$= 4\pi\rho \frac{a^5}{15}$$

By using the relation for density

$$\rho = \frac{M}{\frac{2}{3}\pi a^3}$$

we obtain

$$I_{zz} = \frac{2}{5}Ma^2$$

which is required expression for moment of inertia of hemisphere.

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Module No. 159

M.I. of Ellipsoid –Derivation

Problem

Find the moment of inertia and product of inertia for the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

w.r.to its axes of symmetry.

Solution

By definition

$$\begin{aligned} I_{11} &\equiv I_{xx} \\ &= \int_{\text{ellipsoid}} \rho(r)(y^2 + z^2) dV \end{aligned}$$

If we put

$$\frac{x}{a} = x', \quad \frac{y}{b} = y', \quad \frac{z}{c} = z'$$

then

$$x = ax', \quad y = by', \quad z = cz'$$

$$\Rightarrow dx = adx', \quad dy = bdy', \quad dz = cdz'$$

and

$$dV = dxdydz = abcdx'dy'dz'$$

Now under the above transformation, the given ellipsoid is transformed into the unit sphere S:

$$x'^2 + y'^2 + z'^2 = 1$$

The integration is now over the region enclosed by the unit sphere.

$$I_{11} = \rho \int_R (b^2 y'^2 + c^2 z'^2) abc dx' dy' dz'$$

or

$$I_{11} = \rho ab^3 c \int_R y'^2 dV' + \rho abc^3 \int_R z'^2 dV'$$

where $dV' = dx' dy' dz'$

Now because of symmetry

$$\int_R y'^2 dV' = \int_R z'^2 dV'$$

Now we solve one integral.

We use the spherical polar coordinates (r, θ, φ) .

Their range of variation will be

$$0 \leq r < 1, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi$$

$$x = r \sin \theta \cos \varphi, \quad y = r \sin \theta \sin \varphi, \quad z = r \cos \theta$$

and

$$dV' = dr(r d\theta)(r \sin \theta d\varphi) = r^2 \sin \theta dr d\theta d\varphi$$

Thus

$$\begin{aligned} \int_S y'^2 dV' &= \int_0^1 \int_0^\pi \int_0^{2\pi} r^4 \sin^3 \theta \sin^2 \varphi dr d\theta d\varphi \\ &= \int_0^1 r^4 dr \int_0^\pi \sin^3 \theta d\theta \int_0^{2\pi} \sin^2 \varphi d\varphi \\ &= \frac{1}{5} \frac{1}{4} \left| -3 \cos \theta + \frac{\cos 3\theta}{3} \right|_0^\pi \frac{1}{2} \left| \varphi - \frac{\sin 2\varphi}{2} \right|_0^{2\pi} \end{aligned}$$

Thus

$$\int_S y'^2 dV' = \frac{2\pi}{15}$$

Therefore on substitution

$$\begin{aligned} I_{11} &= \rho ab^3 c \frac{2\pi}{15} + \rho abc^3 \frac{2\pi}{15} \\ I_{11} &= \rho abc \frac{2\pi}{15} (b^2 + c^2) \\ &= \frac{M}{(4/3)\pi abc} abc \times \frac{2\pi}{15} (b^2 + c^2) \\ I_{11} &= \frac{M}{10} (b^2 + c^2) \end{aligned}$$

Similarly,

$$\begin{aligned} I_{22} &= \frac{M}{10} (c^2 + a^2) \\ I_{33} &= \frac{M}{10} (a^2 + b^2) \end{aligned}$$

For product of inertia

$$\begin{aligned} I_{12} \equiv I_{xy} &= - \int_{\text{ellipsoid}} xy dV \\ &= -\rho \int_R ax'by'(abcdx'dy'dz') \\ &= -\rho a^2 b^2 c \iiint_R x'y'dx'dy'dz' \end{aligned}$$

Using the polar coordinates (r, θ, φ)

$$\begin{aligned} I_{12} &= -\rho a^2 b^2 c \int_0^{2\pi} \int_0^{\pi} \int_0^1 r \sin \theta \cos \varphi (r \sin \theta \sin \varphi) r^2 \sin \theta dr d\theta d\varphi \\ &= 0 \end{aligned}$$

Similarly, we can obtain

$$I_{23} = 0$$

$$I_{31} = 0$$

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Module No. 160

Example 1 of Moment of Inertia

Problem

Particles of masses $2m$, $3m$ and $4m$ are held in a rigid light framework at points $(0,1,1)$, $(1,1,0)$ and $(-1, 0, 1)$ resp.

Show that the M. I. of the system about the x, y, z – axis are 11, 13, 12 respectively.

Solution

Since the masses are 2, 3 and 4 i.e. $m_1 = 2$, $m_2 = 3$ and $m_3 = 4$, and the points are given by $P_1(0,1,1)$, $P_2(1,1,0)$ and $P_3(-1,0,1)$.

We are required to find out the moment of inertia about x-axis, y-axis and z-axis i.e. I_{xx}, I_{yy}, I_{zz} .

$$\begin{aligned} I_{xx} &= \sum_i m_i (y_i^2 + z_i^2) \\ &= m_1(y_1^2 + z_1^2) + m_2(y_2^2 + z_2^2) + m_3(y_3^2 + z_3^2) \\ &= 2(1 + 1) + 3(1 + 0) + 4(1 + 0) \\ &= 4 + 3 + 4 = 11 \end{aligned}$$

So,

$$\begin{aligned} I_{xx} &= 11 \\ I_{yy} &= \sum_i m_i (x_i^2 + z_i^2) \\ &= m_1(x_1^2 + z_1^2) + m_2(x_2^2 + z_2^2) + m_3(x_3^2 + z_3^2) \\ &= 2(0 + 1) + 3(1 + 0) + 4(1 + 1) \\ &= 2 + 3 + 8 = 13 \end{aligned}$$

So,

$$I_{yy} = 13$$

$$\begin{aligned} I_{zz} &= \sum_i m_i(x_i^2 + y_i^2) \\ &= m_1(y_1^2 + x_1^2) + m_2(x_2^2 + y_2^2) + m_3(x_3^2 + y_3^2) \\ &= 2(0 + 1) + 3(1 + 1) + 4(1 + 0) \\ &= 2 + 6 + 4 = 12 \end{aligned}$$

So,

$$I_{zz} = 12$$

➤ Hence Shown

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Module No. 161

Example 2 of Moment of Inertia

Problem Statement

A boy of mass $M = 30\text{kg}$ is running with a velocity of 3 m/sec on ground just tangentially to a merry-go-round which is at rest. The boy suddenly jumps on the merry-go-round. Calculate the angular velocity acquired by the system. The merry-go-round has a radius of $r = 2\text{m}$ and a mass $m = 120\text{kg}$ and its moment of inertia is 120 kgm^{-2} .

Solution

The merry-go-round rotates about an axis which we regard as passing through its c.m.

Let's give the following notations:

- M.I of boy = I_1
- M.I of merry go round = I_2
- Vel. of boy = v_1
- Vel. of merry go round = v_2
- Angular vel. of boy = ω_1
- Angular vel. of merry go round = ω_2

The moment of inertia I_1 of the boy about the axis of rotation can be found by

$$I_1 = Md^2 = 30 \times 2^2 = 120\text{kgm}^{-2}$$

The moment of inertia I_2 of merry go round is given to be

$$I_2 = 120\text{kgm}^{-2}$$

Since the velocity of the boy about the merry-go-round is $v_1 = 3\text{ms}^{-1}$, therefore his angular velocity ω_1 about the axis of rotation is therefore

$$\omega_1 = \frac{v}{d} = \frac{3}{2} = 1.5 \text{ radian per second}$$

Initially the merry go round is at rest, therefore $v_2 = 0$, and thus its angular velocity $\omega_2 = 0$. We ignore friction and therefore there is no external torque on the system.

Hence by the law of conservation of angular momentum

$$(I_1 + I_2)\omega = I_1\omega_1 + I_2\omega_2$$

$$\omega = \frac{I_1\omega_1 + I_2\omega_2}{(I_1 + I_2)}$$

where ω is the angular velocity of the system when the boy jumps on the merry-go-round.

$$\begin{aligned}\omega &= \frac{120(5) + 0}{120 + 120} \\ &= 0.75 \text{ radian per sec}\end{aligned}$$

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Module No. 162

Example 3 of Moment of Inertia

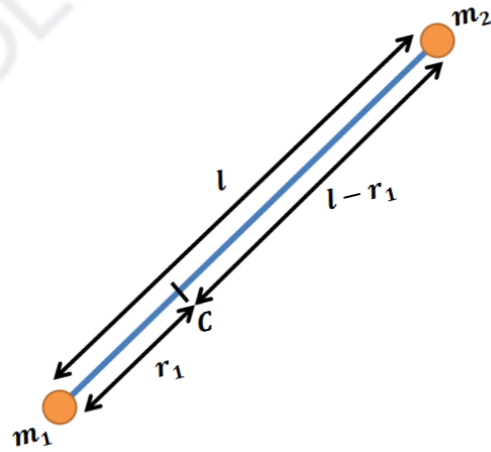
Problem Statement^[1]

Two particles of masses m_1 and m_2 are connected by a rigid massless rod of length l and moves freely in a plane. Show that the M. I. of the system about an axis perpendicular to the plane and passing through the center of mass is Ml^2 where

$$M = \frac{m_1 m_2}{m_1 + m_2}$$

Solution

Let r_1 be the distance of mass m_1 from center of mass C . Then $l - r_1$ is the distance of the mass m_2 from C . Since C is the center of the mass.



So,

$$m_1 r_1 = m_2 (l - r_1)$$

$$\Rightarrow m_1 r_1 = m_2 l - m_2 r_1$$

$$\Rightarrow (m_1 + m_2) r_1 = m_2 l$$

$$\Rightarrow r_1 = \frac{m_2 l}{(m_1 + m_2)}$$

$$\begin{aligned}\Rightarrow l - r_1 &= l - \frac{m_2 l}{(m_1 + m_2)} \\ \Rightarrow l - r_1 &= \frac{m_1 l + m_2 l - m_2 l}{(m_1 + m_2)} \\ \Rightarrow l - r_1 &= \frac{m_1 l}{(m_1 + m_2)}\end{aligned}$$

Thus the M.I. about an axis through C is

$$\begin{aligned}& m_1 r_1^2 + m_2 (l - r_1)^2 \\ &= m_1 \left(\frac{m_2 l}{m_1 + m_2} \right)^2 + m_2 \left(\frac{m_1 l}{m_1 + m_2} \right)^2 \\ &= \frac{m_1 m_2 l^2}{(m_1 + m_2)^2} (m_1 + m_2) = \frac{m_1 m_2}{m_1 + m_2} l^2 \\ &I = M l^2\end{aligned}$$

where

$$M = \frac{m_1 m_2}{m_1 + m_2}$$

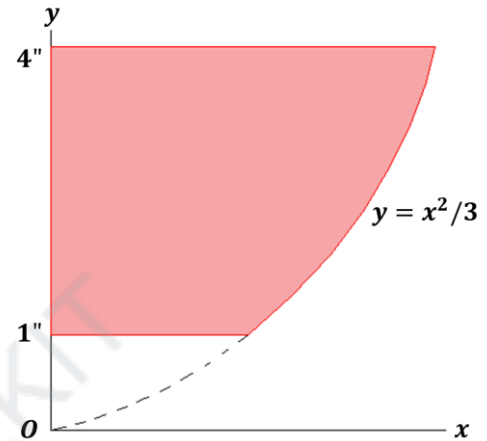
Hence showed.

Module No. 163

Example 4 of Moment of Inertia

Problem (© engineering.unl.edu)

Calculate the moment of inertia of the shaded area given in figure about y-axis.



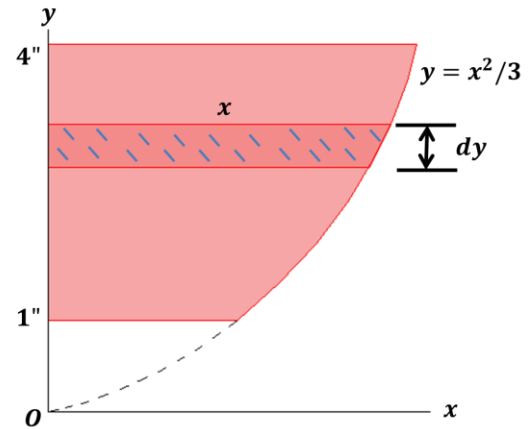
Solution

Here, we have

$$y = x^2/3 \quad (1)$$

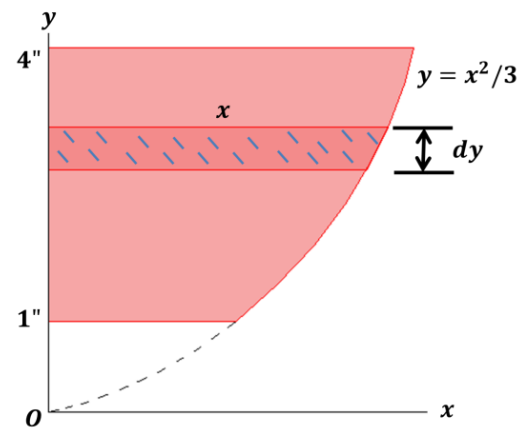
We consider an elementary strip from the shaded area whose M.I is

$$I_{strip} = \frac{1}{3} x^2 (x dy) = \frac{1}{3} x^3 dy$$



Then by integration, we obtain the moment of inertia of the whole shaded area

$$\begin{aligned}
 I_y &= \sqrt{3} \int_1^4 y^{3/2} dy = \sqrt{3} \frac{2}{5} \left| y^{5/2} \right|_1^4 \\
 &= \sqrt{3} \frac{2}{5} \left[(4)^{5/2} - (1)^{5/2} \right] \\
 &= \frac{2\sqrt{3}}{5} \left[(4)^{5/2} - (1)^{5/2} \right] \\
 &= \frac{2\sqrt{3}}{5} (32 - 1) \\
 &= 21.5 \text{ inch}^4
 \end{aligned}$$



By using (1), we get

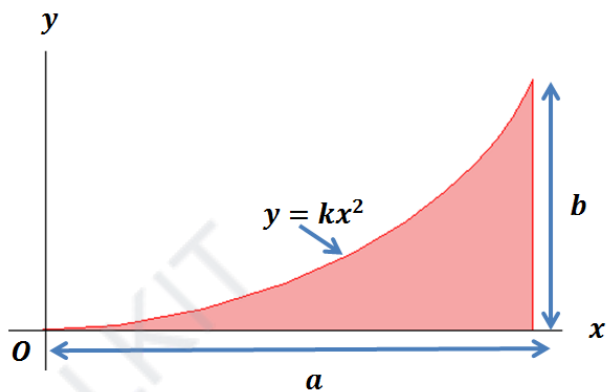
$$I_{strip} = \sqrt{3}y^{3/2}dy$$

Module No. 164

Example 5 of Moment of Inertia

Problem

Determine the moment of inertia of the shaded area shown in figure with respect to each of the coordinate axes.



Solution

Here we have

$$y = kx^2 \quad (1)$$

From fig. we have

$x = a$, $y = b$, then

$$b = ka^2 \Rightarrow k = \frac{b}{a^2}$$

Substituting the value

of k in (1), we obtain

$$y = \frac{b}{a^2}x^2 \text{ or } x = \frac{a}{\sqrt{b}}\sqrt{y}$$

Now the moment of inertia along x-axis will be

$$\begin{aligned}
I_x &= \int_A y^2 dA = \int_0^b y^2 (a - x) dy \\
&= \int_0^b y^2 \left(a - \frac{a}{\sqrt{b}} \sqrt{y} \right) dy \\
&= a \int_0^b y^2 dy - \frac{a}{\sqrt{b}} \int_0^b y^{5/2} dy \\
&= a \left[\frac{y^3}{3} \right]_0^b - \frac{a}{\sqrt{b}} \left[\frac{2}{7} y^{7/2} \right]_0^b \\
&= \frac{ab^3}{3} - \frac{a}{\sqrt{b}} \frac{2}{7} b^{7/2} \\
&= \frac{ab^3}{3} - \frac{2ab^3}{7} \\
I_x &= \frac{ab^3}{21} \\
I_y &= \int_A x^2 dA = \int_0^a x^2 y dx \\
&= \frac{b}{a^2} \left[\frac{x^5}{5} \right]_0^a = \frac{b}{a^2} \frac{a^5}{5} \\
I_y &= \frac{a^3 b}{5}
\end{aligned}$$

Hence

$$I_x = \frac{ab^3}{21}$$

and

$$I_y = \frac{a^3 b}{5}$$

are moment of inertia about x-axis and y-axis respectively.

Module No. 165

Example 5 of Moment of Inertia

Theorem Statement

The moment of inertia of a rigid body about a given axis is equal to the same about a parallel axis through the centroid plus the moment of inertia due to the total mass placed at the centroid, (the last quantity will be referred to as moment of inertia of the centroid).

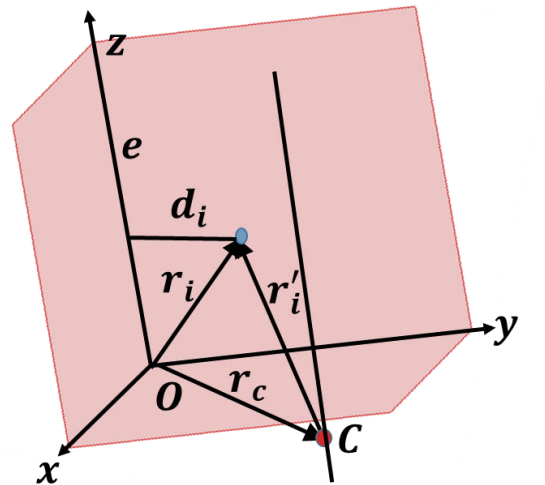
Proof

By definition, if I denotes the M.I about the given axis, and d_i denotes the distance of i^{th} particle from the given axis, then

$$I = \sum_i m_i d_i^2$$

If e is the unit vector in the direction of the axis, then we have the following result

$$d_i^2 = (e \times r_i)^2$$



Let r_c denotes the position vector of the centroid and

r'_i the position vector of the i^{th} particle w.r.t to the centroid, then

$$r_i = r_c + r'_i$$

On making substitutions, we obtain

$$\begin{aligned}
 I &= \sum_i m_i [e \times (r_c + r'_i)]^2 \\
 &= \sum_i m_i [e \times r_c + e \times r'_i]^2 \\
 &= \sum_i m_i [(e \times r_c)^2 + (e \times r'_i)^2 + 2(e \times r_c) \cdot (e \times r'_i)] \\
 &= \sum_i m_i (e \times r_c)^2 + \sum_i m_i (e \times r'_i)^2 \\
 &\quad + 2(e \times r_c) \cdot \sum_i m_i (e \times r'_i) \\
 &= \sum_i m_i d_c^2 + \sum_i m_i d_i'^2 + 2(e \times r_c) \cdot e \times \sum_i m_i r'_i
 \end{aligned}$$

But $\sum_i m_i r'_i = 0$ (We studied in earlier module).

Therefore

$$\begin{aligned}
 I &= (\sum_i m_i) d_c^2 + \sum_i m_i d_i'^2 \\
 &= M d_c^2 + I'
 \end{aligned}$$

where

$$M = \sum_i m_i$$

is the total mass of the system.

or

$$I = I_0 + I'$$

where I_0 denotes the moment of inertia of the centroid, and I' denotes the M.I of the system w.r.to a parallel axis through the centroid.

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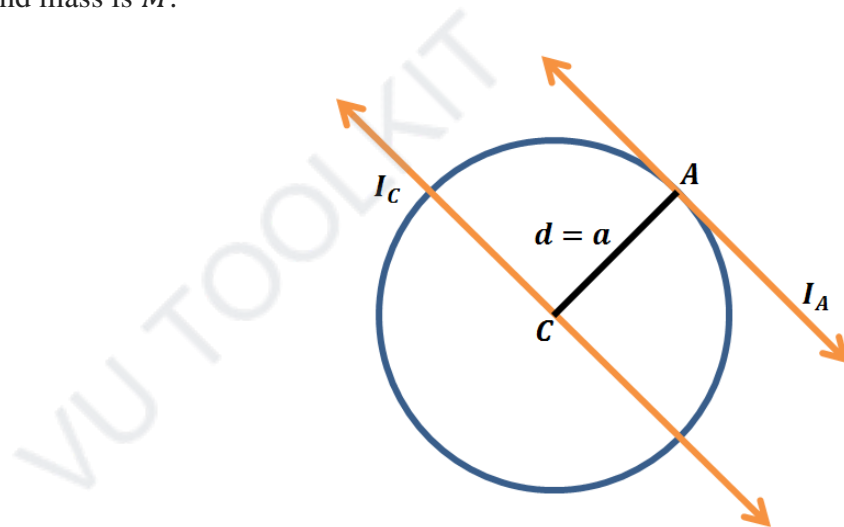
Example 1 of Parallel Axis Theorem

Problem

Use the parallel axis theorem to find the moment of inertia of a solid circular cylinder about a line on the surface of the cylinder and parallel to axis of cylinder.

Solution

Suppose the cross section of cylinder as in figure. Then the axis of the cylinder is passing through the point C , while the line on the surface of cylinder is passing through A . So, we have to find out M.I of circular cylinder about a line passing through the point A whose radius is a (radius of circular cylinder) and mass is M .



By parallel axis theorem

$$I_A = I_C + Ma^2 \quad (1)$$

Since I_C which is the moment of inertia of a solid circular cylinder about an axis passing from the center of mass is defined by

$$I_C = \frac{1}{2} Ma^2 \quad (2)$$

where a is the radius of a solid circular cylinder.

By substituting equation (2) in equation (1) we have

$$I_A = \frac{1}{2} Ma^2 + Ma^2$$

$\Rightarrow$

$$I_A = \left(\frac{1}{2} + 1\right) M a^2$$

$$I_A = \frac{3}{2} M a^2$$

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Example 2 of Parallel Axis Theorem

Problem

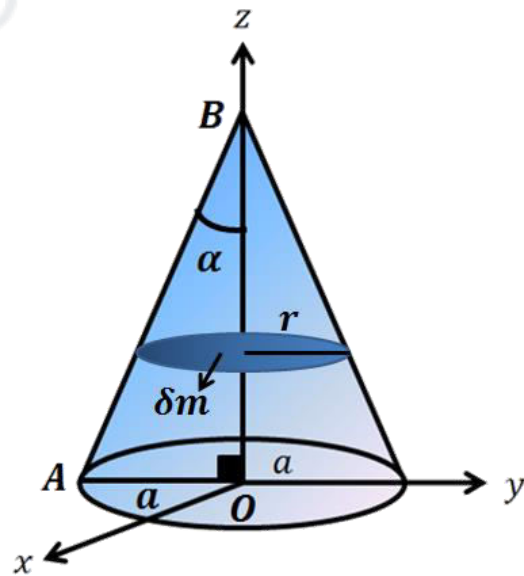
Prove that the moment of inertia of a uniform right circular cone using parallel axis theorem of mass m , height h and semi vertical angle α about a diameter of its base is

$$Mh^2(3 \tan^2 \alpha + 2)/20$$

Solution

In the case of M.I about its diameter, we consider the elementary disc of mass δm whose moment of inertia about a diameter will be

$$\delta I_0 = \frac{1}{4} r^2 \delta m$$



We note that the diameter passes through the center (which is also the centroid) of the elementary disc.

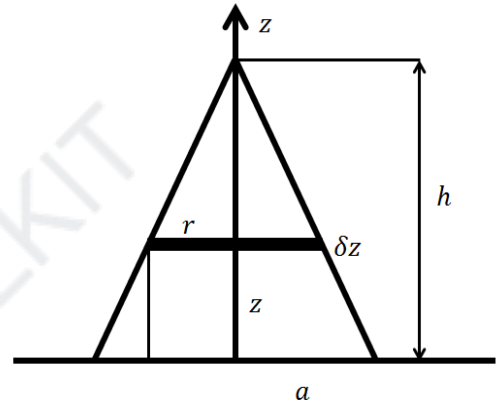
Hence by parallel axis theorem, the M.I. δI of the elementary disc about a parallel axis (parallel diameter) at the base is given by

$$\begin{aligned}
 \delta I &= \delta I_0 + (\delta m)z^2 \\
 &= \frac{1}{4}r^2\delta m + \delta m z^2 = \delta m\left(\frac{1}{4}r^2 + z^2\right) \\
 &= \rho\pi r^2\delta z\left(\frac{1}{4}r^2 + z^2\right) = \rho\pi\left(\frac{1}{4}r^4 + r^2z^2\right)\delta z
 \end{aligned}$$

From the similar triangles,

we have

$$\frac{r}{a} = \frac{h-z}{h} \quad \text{or} \quad r = a\frac{h-z}{h}$$



$$\begin{aligned}
 \text{Therefore } &= \rho\pi\left[\frac{a^4}{4h^4}(h-z)^4 + \frac{a^2}{h^2}(h-z)^2z^2\right]\delta z \\
 &= \rho\pi\left[\frac{a^4}{4h^4}(h-z)^4 + \frac{a^2}{h^2}(h^2z^2 - 2hz^3 + z^4)\right]\delta z
 \end{aligned}$$

Therefore M.I of complete right circular cone about a diameter is given by

$$\begin{aligned}
 I &= \rho\pi \int_0^h \left\{ \frac{a^4}{4h^4}(h-z)^4 + \frac{a^2}{h^2}(h^2z^2 - 2hz^3 + z^4) \right\} \delta z \\
 I &= \rho\pi \left(\frac{a^4}{4h^4} \frac{h^5}{5} + \frac{a^2}{h^2} \frac{h^5}{30} \right)
 \end{aligned}$$

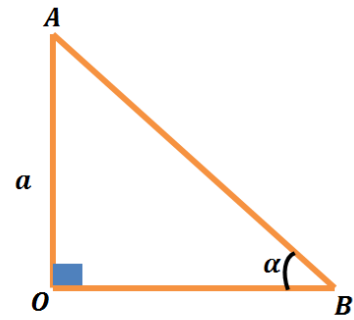
Since we know that $\rho = \frac{M}{(1/3)\pi a^2 h}$

$$I = \frac{M}{20}(3a^2 + 2h^2)$$

Since the semi vertical angle of
the right circular cone is α ,

So by right triangle AOB, we have

$$\tan \alpha = \frac{AO}{OB} = \frac{a}{h}$$



$$a = h \tan \alpha$$

Therefore

$$I = \frac{M}{20} [3(h \tan \alpha)^2 + 2h^2]$$

or

$$I = \frac{Mh^2}{20} [3 \tan^2 \alpha + 2]$$

Hence proved.

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Example 3 of Parallel Axis Theorem

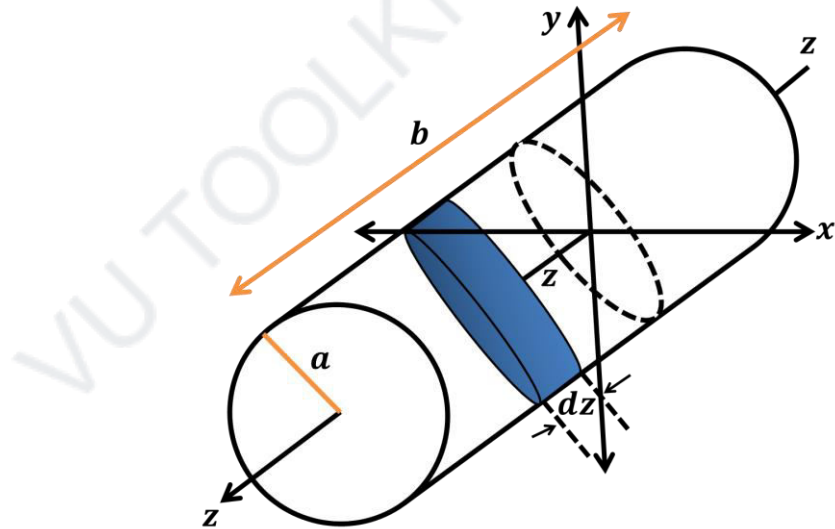
Problem^[1]

Find the moment of inertia of a uniform circular cylinder of length b and radius a about an axis through the center and perpendicular to the central axis, namely I_x or I_y .

Solution

Consider the elementary disc of mass dm and thickness dz located at a distance z from the xy plane. Then the moment of inertia about a diameter will be

$$\delta I_0 = \frac{1}{4} a^2 dm$$



Then, using the parallel-axis theorem, moment of inertia of the thin disc about the x – axis will be

$$dI_x = \frac{1}{4} a^2 dm + z^2 dm$$

where $dm = \rho \pi a^2 dz$.

Thus

$$\begin{aligned}
 I_x &= \rho \pi a^2 \int_{-b/2}^{b/2} \left(\frac{1}{4} a^2 + z^2 \right) dz \\
 &= \rho \pi a^2 \left(\frac{1}{4} a^2 b + \frac{1}{12} b^3 \right)
 \end{aligned}$$

but the mass of the cylinder m is $\rho \pi a^2 b$,

therefore

$$I_x = m \left(\frac{1}{4} a^2 + \frac{1}{12} b^2 \right)$$

Due to symmetry we have

$$I_x = I_y = m \left(\frac{1}{4} a^2 + \frac{1}{12} b^2 \right)$$

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Example 4 of Parallel Axis Theorem

Problem

Calculate the moment of inertia I_{cm} for a uniform rod of length l and mass M rotating about an axis through the center, perpendicular to the rod.

Solution

In order to calculate the moment of inertia through the center of mass c.m., we use parallel axes theorem.

In a transparent notation

$$I_l = I_{cm} + Md^2 \quad (1)$$

where d is the distance between the origin and the center of mass and $d = l/2$.

Also I_l is the moment of inertia of rod about one of its end (which we calculated earlier).

Here

$$I_l = \frac{1}{3}Ml^2 \quad (2)$$

From (1), we have

$$I_{cm} = I_l - Md^2$$

Substituting value from (2) in (1)

$$I_{cm} = \frac{1}{3}Ml^2 - M\left(\frac{l}{2}\right)^2 = \frac{1}{3}Ml^2 - \frac{1}{4}Ml^2$$

$$I_{cm} = \frac{1}{12}Ml^2$$

which is required moment of inertia about its center of mass.

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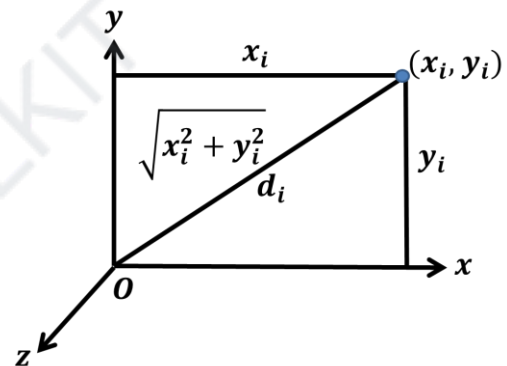
Perpendicular Axis Theorem

Theorem Statement

The moment of inertia of a **plane rigid body** about an axis perpendicular to the body is equal to the sum of the moment of inertia about two mutually perpendicular axis lying in the plane of the body and meeting at the common point with the given axis.

Proof

We choose the coordinates such that XY axes lie in the plane of the rigid body and the Z -axis is perpendicular to it.



Then the theorem can be stated as

$$I_{33} = I_{11} + I_{22}$$

or

$$I_{zz} = I_{xx} + I_{yy}$$

where $I_{11} = I_{xx}$ etc. are M.I about the x -axis etc.

Since the z -axis has been

chosen to be perpendicular

to the lamina body, therefore

$$I_{33} = I_{zz} = \sum_i m_i d_i^2$$

where d_i is the distance of the i^{th} particle (lying in the xy -plane) from the z -axis.

If we denote the coordinate of this particle by (x_i, y_i) , then

$$d_i^2 = x_i^2 + y_i^2$$

and therefore

$$\begin{aligned} I_{33} = I_{zz} &= \sum_i m_i (x_i^2 + y_i^2) \\ &= \sum_i m_i x_i^2 + \sum_i m_i y_i^2 \\ &= I_{xx} + I_{yy} \end{aligned}$$

➤ Hence the proof.

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Example 1 of Perpendicular Axis Theorem

Problem Statement

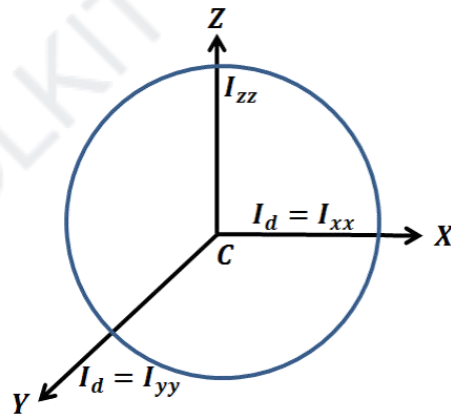
Find the moment of inertia of a uniform circular plate (or disc) about any diameter.

Proof

Since we have deduced that moment of inertia of a uniform circular disc is

$$I_{disc} = \frac{1}{2} Ma^2$$

which is about a line passing through center and perpendicular to the plane.



Considering the same axis with same terms in 3-dim body (i.e. z – axis passing through center of mass and perpendicular to xy – plane), we have using perpendicular axis theorem

$$I_{zz} = I_{xx} + I_{yy} \quad (1)$$

We have to find out M.I. of uniform circular plate (disc) about any of its diameters which are along x – axis and y – axis (i.e. we have to find out I_{xx} or I_{yy} both of them are equal to I_d)

Since

$$I_{xx} = I_d = I_{yy}$$

So equation (1) becomes

$$I_{zz} = I_d + I_d$$

or

$$I_{zz} = 2I_d$$

$$I_d = \frac{1}{2} I_{zz}$$

$$I_d = \frac{1}{2} \left(\frac{1}{2} M a^2 \right)$$

$$(\therefore I_{zz} = I_{disc} = \frac{1}{2} M a^2)$$

$$I_d = \frac{1}{4} M a^2$$

where M is the mass of disc and a is its radius.

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Example 2 of Perpendicular Axis Theorem

Problem Statement

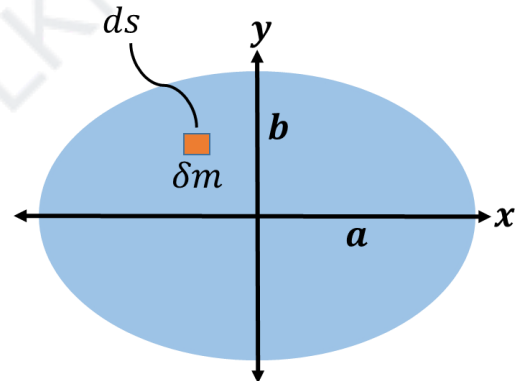
Find the M.I. of a uniform elliptical lamina with semi major axes and semi minor axes a, b respectively about respective axes (x, y, z-axis) using perpendicular axis theorem.

Proof

We consider the elliptical plate as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

with semi major axes along x-axis as shown in figure.



From (1) we have

$$y = \pm \frac{b}{a} \sqrt{a^2 - x^2}$$

$$\text{let } y_1 = \frac{b}{a} \sqrt{a^2 - x^2}$$

We consider a small element of mass δm of the elliptical plate, then we will have

$$\delta m = \rho ds = \rho dx dy$$

The moment of inertia of this element along x-axis will be equal to $I = \delta m y^2$.

Then the moment of inertia of whole plate will be

$$\begin{aligned} I_{xx} &= \int (\delta m) y^2 = \int_{plate} \rho ds y^2 \\ &= \rho \int_{-a}^a \left(\int_{-y_1}^{y_1} y^2 dy \right) dx \\ &= \rho \int_{-a}^a \frac{2y_1^3}{3} dx = \frac{2\rho}{3} \int_{-a}^a \frac{b^3}{a^3} (a^2 - x^2)^{3/2} dx \end{aligned}$$

due to symmetry, we can write it as

$$= \frac{4\rho b^3}{3a^3} \int_0^a (a^2 - x^2)^{3/2} dx$$

By making use of polar coordinates,

substitute $x = a \sin \theta$, then $dx = a \cos \theta$ this integral becomes

$$\begin{aligned} I_{xx} &= \frac{4\rho b^3}{3a^3} \int_0^{\pi/2} a^3 \cos^3 \theta (a \cos \theta) d\theta \\ &= \frac{4\rho ab^3}{3} \int_0^{\pi/2} \cos^4 \theta d\theta \\ &= \frac{1}{4} \rho ab^3 \pi \end{aligned}$$

for elliptical plate, we have

$$\rho = \frac{M}{\pi ab}$$

Thus

$$I_{xx} = \frac{1}{4} ab^3 \pi \times \frac{M}{\pi ab}$$

$$I_{xx} = \frac{1}{4} M b^2$$

Similarly, about y-axis

$$\begin{aligned} I_{yy} &= \int (\delta m) x^2 = \int_{plate} \rho ds x^2 \\ &= \rho \int_{-b}^b \left(\int_{-x_1}^{x_1} x^2 dx \right) dy \\ &= \rho \int_{-b}^b \frac{2x_1^2}{3} dy = \frac{2\rho}{3} \int_{-b}^b \frac{b^3}{a^3} (a^2 - y^2)^{3/2} dy \end{aligned}$$

due to symmetry, we can write it as

$$= \frac{4\rho b^3}{3a^3} \int_0^b (a^2 - y^2)^{3/2} dy$$

Using polar coordinates, substitute $y = b \sin \theta$, then $dy = b \cos \theta d\theta$ this integral becomes

$$\begin{aligned} I_{yy} &= \frac{4\rho a^3}{3b^3} \int_0^{\pi/2} b^3 \cos^3 \theta (b \cos \theta) d\theta \\ &= \frac{4\rho a^3 b}{3} \int_0^{\pi/2} \cos^4 \theta d\theta \\ &= \frac{4\rho a^3 b}{3} \frac{3}{8} \times \frac{\pi}{2} \\ &= \frac{1}{4} \rho a^3 b \pi \\ &= \frac{1}{4} a^3 b \pi \times \frac{M}{\pi a b} \\ I_{yy} &= \frac{1}{4} M a^2 \end{aligned}$$

now using perpendicular axis theorem, we obtain

$$\begin{aligned}I_{zz} &= I_{xx} + I_{yy} \\&= \frac{1}{4}Mb^2 + \frac{1}{4}Ma^2 \\I_{zz} &= \frac{1}{4}M(a^2 + b^2)\end{aligned}$$

which is M.I. of elliptical lamina along z-axis

Module No. 173

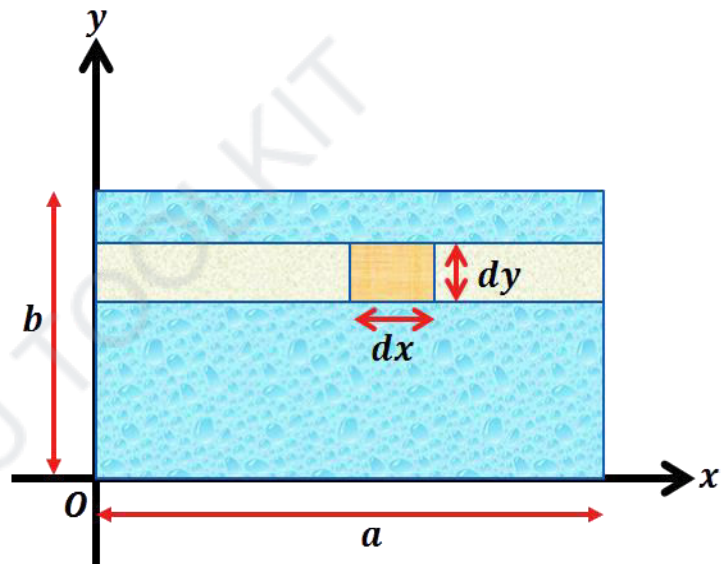
Example 3 of Perpendicular Axis Theorem

Problem

Find the moment of inertia of a rectangular plate with sides a and b about an axis perpendicular to the plate and passing through a vertex using perpendicular axis theorem.

Solution

We consider a rectangular plate (lamina) of sides of length a and b . We consider an element of length dx and dy as shown in figure.



We find the M.I about $y - axis$.

The mass of selected element will be

$$dm = \rho dx dy$$

It's moment of inertia about the $y - axis$ is

$$I_{element} = \rho dx dy x^2 = \rho x^2 dx dy.$$

where x is the perpendicular distance from the element to the $y - axis$.

Thus the total moment of inertia about y-axis is

$$I_y = \int_{x=0}^a \int_{y=0}^b \rho x^2 dx dy$$

$$= \rho b \int_0^a x^2 dx = \rho b \left[\frac{x^3}{3} \right]_0^a = \frac{1}{3} \rho b a^3$$

Since the density of the rectangular plate is

$$\rho = \frac{M}{ab}$$

the moment of inertia will be

$$I_y = \frac{1}{3} M a^2$$

In the similar manner, we will calculate the moment of inertia about x-axis

The total moment of inertia about x-axis is

$$I_x = \int_{x=0}^a \int_{y=0}^b \rho y^2 dx dy$$

$$= \rho a \int_0^b y^2 dy = \rho a \left[\frac{y^3}{3} \right]_0^b = \frac{1}{3} \rho a b^3$$

Using the relation of the total mass of rectangular plate

$$M = \rho ab$$

Then the moment of inertia will be

$$I_x = \frac{1}{3} M b^2$$

Thus by using perpendicular axes theorem, we obtain the moment of inertia along z-axis

$$I_z = I_x + I_y$$

$$I_z = \frac{1}{3} M b^2 + \frac{1}{3} M a^2$$

$$I_z = \frac{1}{3}M(a^2 + b^2)$$

which are the required moments of inertia of rectangle along x, y, z -axis.

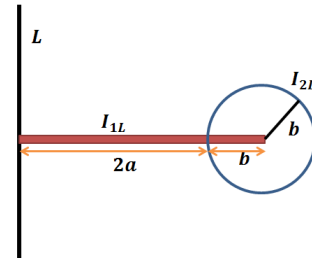
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Module No. 174

Problem of Moment of Inertia

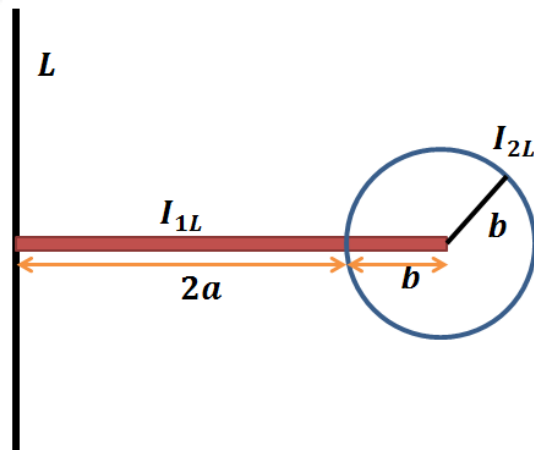
Problem

Find the moment of inertia about the line of an apparatus (as shown in figure) consisting of a sphere of mass M and radius b attached to a rod of length $2a$ & mass m .



Solution

Since from the figure it is clear that b is the radius of the sphere whose mass is M and is attached to the rod of length $2a$ whose mass is m .



Therefore

$$I_L = I_{1L} + I_{2L}$$

where I_{1L} is the M.I. of a rod of length $2a$ about L and I_{2L} is the M.I. of a sphere of radius b about the line L , then

$$\begin{aligned}
 I_{1L} &= \int_0^{2a} \rho x^2 dx \\
 &= \rho \left[\frac{x^3}{3} \right]_0^{2a} = \rho \frac{8}{3} a^3
 \end{aligned}$$

using relation for ρ , we get

$$I_{1L} = \frac{4}{3} ma^2 \quad (2)$$

Now

$$I_{2L} = I' + Md^2$$

where I' is the MI of sphere about its diameter

$$I_{2L} = \frac{2}{5} Mb^2 + M(2a + b)^2 \quad (3)$$

Substituting equation (2) & (3) in (1), we get

$$I_L = \frac{4}{3} ma^2 + \frac{2}{5} Mb^2 + M(2a + b)^2$$

which is the required M.I. of the apparatus.

Module No. 175

Existence of Principle Axes

Introduction

The axes relative to which product of inertia are zero are called the principal axes and the moment of inertia along these axes are called principal moment of inertia.

Theorem Statement

For a rigid body, there exist a set of three mutually orthogonal axes called principal axes relative to which the product of inertia are zero and $\vec{\omega}$ and L are considered along the same direction.

Proof

We assume that there exists an axis through a point O of the rigid body such that angular velocity $\vec{\omega}$ and angular momentum L of the body are parallel to this axis. Then we can write

$$L = I\vec{\omega} \quad (1)$$

where I is a constant of proportionality.

Form equation (1), we have

$$L_1 = I\omega_1, \quad L_2 = I\omega_2, \quad L_3 = I\omega_3$$

Also from the general theory of angular momentum

$$L_i = \sum_j I_{ij} \omega_j \quad (2)$$

From equation (1) and (2), we have

$$L_1 = I_{11}\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = I\omega_1$$

$$L_2 = I_{21}\omega_1 + I_{22}\omega_2 + I_{23}\omega_3 = I\omega_2$$

$$L_3 = I_{31}\omega_1 + I_{32}\omega_2 + I_{33}\omega_3 = I\omega_3$$

which can also be written as

$$(I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0$$

$$I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 = 0$$

$$I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 = 0 \quad (3)$$

Equation (3) is a system of three homogeneous linear algebraic equation in the unknown $\omega_1, \omega_2, \omega_3$.

This system will have a non-trivial solution, i.e. $(\vec{\omega} \neq 0)$, if the determinant of the matrix of coefficients is zero. i.e.

$$\begin{vmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0 \quad (4)$$

This is a cubic equation in I and will in general have three roots. Equation (4) is called characteristic equation of the matrix (I_{ij}) .

The roots of equation (4) are called eigen values of the inertia matrix (I_{ij}) .

The problem of finding principal moment of inertia and directions of inertia has been reduced to that of finding the eigenvalues and eigenvectors of a symmetric 3×3 matrix.

The following results from the eigenvalue theory of matrices will deduce further results about the principal moments and directions (axes) of inertia.

Related Theorems

➤ Theorem 1

A 3×3 symmetrical matrix has three real eigenvalues, which may be distinct or repeated.

➤ Theorem 2

The eigenvectors of a symmetrical matrix corresponding to distinct eigenvalues are orthogonal.

➤ Theorem 3

It is always possible to find three mutually orthogonal eigenvectors for a 3×3 symmetric matrix, whether the eigenvalues are distinct or repeated.

Results

In the light of above theorems, we deduce the following results about the inertia matrix I_{mat} at point O of a rigid body.

- i. The principal of moment of inertia are always real numbers. This is obviously physical because the moment of inertia is defined as the quantity $\sum_i m_i d_i^2$ where m_i and d_i are both real.

- ii. When all three principal moments I_1, I_2, I_3 are distinct, then by the theorem 2, three mutually orthogonal principal axes can be found.
- iii. When $I_1 = I_2$ but $I_1 \neq I_3$, then by the theorem 3, we can still determine three mutually orthogonal principal axes, because the inertia matrix is symmetric.

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Module No. 176

Determination of Principal Axes of Other Two When One is known

In many instances a body possesses sufficient symmetry so that at least one principal axis can be found by inspection, i.e.; the axis can be chosen so as to make two of the three products of inertia vanish.

We consider a plane rigid body i.e. a 2D body; for example a plate of uniform thickness. Such a system can be regarded as a coplanar distribution of mass.

Since there are three mutually orthogonal principal axes, one of them must be perpendicular to the plane of the body.

The other two axes will lie in the plane of lamina.

We choose X and Y axes in the plane of lamina, and the Z axis perpendicular to its plane, as shown in figure.

$$I_{xz} = I_{yz} = 0 \quad \text{whereas } I_{xy} \neq 0 \quad (1)$$

Since we have obtained the following equations for principal axes.

$$(I_{11} - I)\omega_1 + I_{12}\omega_2 + I_{13}\omega_3 = 0$$

$$I_{21}\omega_1 + (I_{22} - I)\omega_2 + I_{23}\omega_3 = 0$$

$$I_{31}\omega_1 + I_{32}\omega_2 + (I_{33} - I)\omega_3 = 0 \quad (2)$$

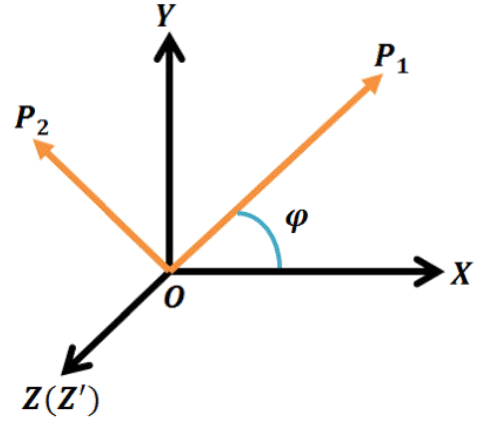
By making use of (1), and using the first two equations of (2), we obtain

$$(I_{11} - I)\omega_1 + I_{12}\omega_2 = 0$$

$$I_{21}\omega_1 + (I_{22} - I)\omega_2 = 0$$

Since the principal axis perpendicular to the plane of lamina is supposed to be known, and we are interested in determining the principal axes in XY -plane, therefore we didn't consider third eq. of (2).

Let OP_1 , OP_2 denote the principal axes in the XY -plane and let φ denote the angle between the OP_1 and the X -axis.



We define

$$\tan \varphi = \omega_2 / \omega_1$$

which can also be written as

$$\frac{\omega_1}{\cos \varphi} = \frac{\omega_2}{\sin \varphi} = k, \quad (3)$$

where k is any arbitrary constant.

Substituting for ω_1 , ω_2 from equation (3) in (2) we have, after simplification

$$(I_{11} - I) \cos \varphi + I_{12} \sin \varphi = 0$$

$$I_{21} \cos \varphi + (I_{22} - I) \sin \varphi = 0$$

These equations can be put in the form

$$I_{11} - I = -I_{12} \frac{\sin \varphi}{\cos \varphi}, \quad I_{22} - I = -I_{12} \frac{\cos \varphi}{\sin \varphi}$$

Module No. 177

Determination of Principal Axes by Diagonalizing the Inertia Matrix

Introduction

Suppose a rigid body has no axis of symmetry. Even so, the tensor that represents the moment of inertia of such a body, is characterized by a real, symmetric 3×3 matrix that can be diagonalized. The resulting diagonal elements are the values of the principal moments of inertia of the rigid body.

The axes of the coordinate system, in which this matrix is diagonal, are the principal axes of the body, because all products of inertia have vanished.

Thus, finding the principal axes and corresponding moments of inertia of any rigid body, symmetric or not, is virtually the same as to diagonalizing its moment of inertia matrix.

Explanation

There are a number of ways to diagonalize a real, symmetric matrix. We present here a way that is quite standard.

First, suppose that we have found the coordinate system (principal axes) in which all products of inertia vanish and the resulting moment of inertia tensor is now represented by a diagonal matrix whose diagonal elements are the principal moments of inertia.

Let e_i be the unit vectors that represent this coordinate system, that is, they point along the direction along the three principal axes of the rigid body.

If the moment of inertia tensor is "dotted" with one of these unit vectors, the result is equivalent to a simple multiplication of the unit vector by a scalar quantity, i.e.

$$Ie_i = \lambda e_i \quad (1)$$

The quantities λ_i are just the principal M.I about their respective principal axes. The problem of finding the principal axes is one of finding those vectors e_i that satisfy the condition

$$(I - \lambda I)e_i = 0 \quad (2)$$

In general this condition is not satisfied for any arbitrary set of orthonormal unit vectors e_i . It is satisfied only by a set of unit vectors aligned with the principal axes of the rigid body.

Any arbitrary xyz coordinate system can always be rotated such that the coordinate axes line up with the principal axes. The unit vectors specifying these coordinate axes then satisfy the condition in eq (2). This condition is equivalent to vanishing of the following determinant

$$|I - \lambda I| = 0 \quad (3)$$

Explicitly, this equation reads

$$\begin{vmatrix} I_{11} - I & I_{12} & I_{13} \\ I_{21} & I_{22} - I & I_{23} \\ I_{31} & I_{32} & I_{33} - I \end{vmatrix} = 0$$

It is a cubic in λ , namely,

$$-\lambda^3 + A\lambda^2 + B\lambda + C = 0 \quad (4)$$

in which A , B , and C are functions of the I 's. The three roots λ_1 , λ_2 and λ_3 are the three principal moments of inertia.

We now have the principal moments of inertia, but the task of specifying the components of the unit vectors representing the principal axes in terms of our initial coordinate system remains to be solved.

Here we can make use of the fact that when the rigid body rotates about one of its principal axes; the angular momentum vector is in the same direction as the angular velocity vector.

Let the angles of one of the principal axes relative to the initial xyz coordinate system be α , β and γ and let the body rotate about this axis. Therefore, a unit vector pointing in the direction of this principal axis has components $(\cos \alpha, \cos \beta, \cos \gamma)$.

Using eq (1),

$$Ie_1 = \lambda_1 e_1$$

where λ_1 , the first principal moment of the three $(\lambda_1, \lambda_2, \lambda_3)$, is obtained by solving eq (4).

In matrix form

$$\begin{bmatrix} I_{11} - \lambda_1 & I_{12} & I_{13} \\ I_{21} & I_{22} - \lambda_1 & I_{23} \\ I_{31} & I_{32} & I_{33} - \lambda_1 \end{bmatrix} \begin{bmatrix} \cos \alpha \\ \cos \beta \\ \cos \gamma \end{bmatrix} = 0$$

- The direction cosines may be found by solving the above equations.
- The solutions are not independent. They are subject to the constraint

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

- In other words the resultant vector e_1 specified by these components is a unit vector.

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Module No. 178

Relation of Fixed and Rotating Frames of Reference

In order to derive the relationship between fixed and rotating frames of reference, we will study the following theorem^[1,2].

Rotating Axes Theorem

➤ Theorem Statement

If a time dependent vector function $\vec{A}$ is represented by $\vec{A}_f$ and $\vec{A}_r$ in fixed and rotating coordinate system, then

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \omega \times \vec{A}_r$$

where it is understood that the origins of the two systems coincide at $t = 0$

➤ Proof

We denote the fixed and rotating coordinate systems by $OX_0Y_0Z_0$ and $OXYZ$ and denote the associated unit vectors by $\{i_0, j_0, k_0\}$ and $\{i, j, k\}$.

Consider a vector A which is changing with time. To an observer fixed relative to $OXYZ$ system, the time rate of change of $A = A_1\hat{i} + A_2\hat{j} + A_3\hat{k}$ will be

$$\frac{d}{dt}A_r = \frac{d}{dt}A_1\hat{i} + \frac{d}{dt}A_2\hat{j} + \frac{d}{dt}A_3\hat{k}$$

where $\frac{dA_r}{dt}$ denotes the time derivative of A relative to the rotating frame of reference.

However, the time rate of change of A relative to the fixed system $OX_0Y_0Z_0$ symbolized by the $\frac{dA_f}{dt}$ needs to be found.

To the fixed observer the unit vectors i, j, k actually change with time.

Thus

$$\begin{aligned}
\frac{d}{dt}A_f &= \frac{d}{dt}(A_1\hat{i} + A_2\hat{j} + A_3\hat{k}) \\
&= \frac{dA_1}{dt}\hat{i} + \frac{dA_2}{dt}\hat{j} + \frac{dA_3}{dt}\hat{k} + A_1\frac{d\hat{i}}{dt} + A_2\frac{d\hat{j}}{dt} + A_3\frac{d\hat{k}}{dt} \\
\frac{dA_f}{dt} &= \frac{dA_r}{dt} + A_1\frac{d\hat{i}}{dt} + A_2\frac{d\hat{j}}{dt} + A_3\frac{d\hat{k}}{dt} \quad (1)
\end{aligned}$$

Since $\hat{i}$ is a unit vector, $d\hat{i}/dt$ is perpendicular to $\hat{i}$. Then $\frac{d\hat{i}}{dt}$ must lie in the plane of $\hat{j}$ and $\hat{k}$. Therefore

$$\frac{d\hat{i}}{dt} = \alpha_1\hat{j} + \alpha_2\hat{k} \quad (2)$$

Similarly,

$$\frac{d\hat{j}}{dt} = \alpha_3\hat{k} + \alpha_4\hat{i} \quad (3)$$

$$\frac{d\hat{k}}{dt} = \alpha_5\hat{i} + \alpha_6\hat{j} \quad (4)$$

Form $\hat{i} \cdot \hat{j} = 0$, differentiation yields

$$\hat{i} \cdot \frac{d\hat{j}}{dt} + \hat{j} \cdot \frac{d\hat{i}}{dt} = 0 \Rightarrow \hat{i} \cdot \frac{d\hat{j}}{dt} = -\hat{j} \cdot \frac{d\hat{i}}{dt}$$

But from (2), we have

$$\begin{aligned}
\hat{j} \cdot \frac{d\hat{i}}{dt} &= \alpha_1 \text{ and } \hat{i} \cdot \frac{d\hat{j}}{dt} = \alpha_4 \\
&\Rightarrow \alpha_4 = -\alpha_1
\end{aligned}$$

Similarly form $\hat{i} \cdot \hat{k} = 0$ we obtain

$$\hat{i} \cdot \frac{d\hat{k}}{dt} + \hat{k} \cdot \frac{d\hat{i}}{dt} = 0 \Rightarrow \hat{i} \cdot \frac{d\hat{k}}{dt} = -\hat{k} \cdot \frac{d\hat{i}}{dt}$$

From (3), we have

$$\begin{aligned}
\hat{k} \cdot \frac{d\hat{i}}{dt} &= \alpha_2 \text{ and } \hat{k} \cdot \frac{d\hat{j}}{dt} = \alpha_5 \\
&\Rightarrow \alpha_5 = -\alpha_2
\end{aligned}$$

and from $\hat{j} \cdot \hat{k} = 0$ we obtain

$$\hat{j} \cdot \frac{d\hat{k}}{dt} + \hat{k} \cdot \frac{d\hat{j}}{dt} = 0 \Rightarrow \hat{j} \cdot \frac{d\hat{k}}{dt} = -\hat{k} \cdot \frac{d\hat{j}}{dt}$$

From (4), we have

$$\hat{k} \cdot \frac{d\hat{j}}{dt} = \alpha_2 \text{ and } \hat{j} \cdot \frac{d\hat{k}}{dt} = \alpha_5$$

$$\Rightarrow \alpha_6 = -\alpha_3$$

Then

$$\frac{d\hat{i}}{dt} = \alpha_1 \hat{j} + \alpha_2 \hat{k}$$

$$\frac{d\hat{j}}{dt} = \alpha_3 \hat{k} - \alpha_1 \hat{i}$$

$$\frac{d\hat{k}}{dt} = -\alpha_2 \hat{i} - \alpha_3 \hat{j}$$

follows that

$$A_1 \frac{d\hat{i}}{dt} + A_2 \frac{d\hat{j}}{dt} + A_3 \frac{d\hat{k}}{dt} = (-\alpha_1 A_2 - \alpha_2 A_3) \hat{i} + (-\alpha_1 A_1 - \alpha_3 A_3) \hat{j} + (-\alpha_2 A_1 - \alpha_3 A_2) \hat{k}$$

where

$$\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

The vector quantity $\vec{\omega}$ is the angular velocity of the moving system relative to the fixed system.

Thus from (1) and (5), we obtain

$$\left(\frac{d\vec{A}}{dt} \right)_f = \left(\frac{d\vec{A}}{dt} \right)_r + \omega \times \vec{A}$$

Module No. 179

Equation of Motion in Rotating Frame of Reference

There are two cases to be discussed in this article. First is when the origins of fixed and rotating coordinate system coincide and other is when the origins of two system are distant.

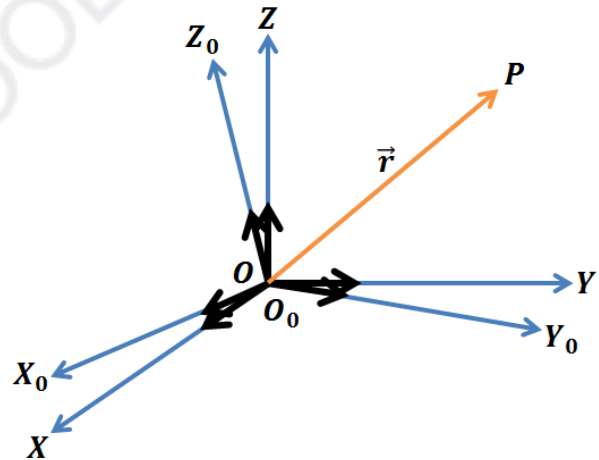
Case I

In this case we consider the origins of the fixed and rotating coordinate system coincide. This case was earlier discussed in detail, where it was found that:

To an observer fixed relative to $OXYZ$ system, the time rate of change of $r = r_1\hat{i} + r_2\hat{j} + r_3\hat{k}$ will be

$$\frac{d}{dt}r_r = \frac{d}{dt}r_1\hat{i} + \frac{d}{dt}r_2\hat{j} + \frac{d}{dt}r_3\hat{k} \quad (1)$$

where $\frac{dr_r}{dt}$ denotes the time derivative of r relative to the rotating frame of reference.



However, the time rate of change of r relative to fixed system $OX_0Y_0Z_0$ symbolized by $\frac{dr_f}{dt}$ will be

$$\frac{dr_f}{dt} = \left(\frac{dr}{dt}\right)_f = \left(\frac{dr}{dt}\right)_r + \omega \times r \quad (2)$$

or in operator form, we can write

$$\left(\frac{d}{dt}\right)_f = \left(\frac{d}{dt}\right)_r + \omega \times$$

Differentiating both sides of (2) w.r.t the fixed coordinate system, we have

$$\left(\frac{d}{dt}\right)_f v_f = \left(\frac{d}{dt}\right)_f (v_r + \omega \times r)$$

by applying operator, we have

$$\begin{aligned} \left(\frac{d}{dt}\right)_f v_f &= \left(\frac{d}{dt} + \omega \times\right)_r (v_r + \omega \times r) \\ &= a_r + 2\omega \times \frac{dr}{dt} + \dot{\omega} \times r + \omega \times (\omega \times r) \end{aligned}$$

or

$$a_f = a_r + 2\omega \times v_r + \dot{\omega} \times r + \omega \times (\omega \times r) \quad (3)$$

where

$$a_f = \frac{dv_f}{dt}, \quad a_r = \frac{dv_r}{dt}$$

are the acceleration in the fixed and rotating coordinate systems. The relation (3) is referred as Coriolis theorem. The term $2\omega \times v_r$ is called Coriolis acceleration, whereas the term $\omega \times (\omega \times r)$ is called centripetal acceleration.

The equations of motion in fixed and moving/ rotating coordinate system are

$$F = ma_f, \quad F' = ma_r$$

where F and F' are the total forces in the fixed and the rotating coordinate systems.

On making substitution for a_f from (3) in equation $F = ma_f$, we obtain

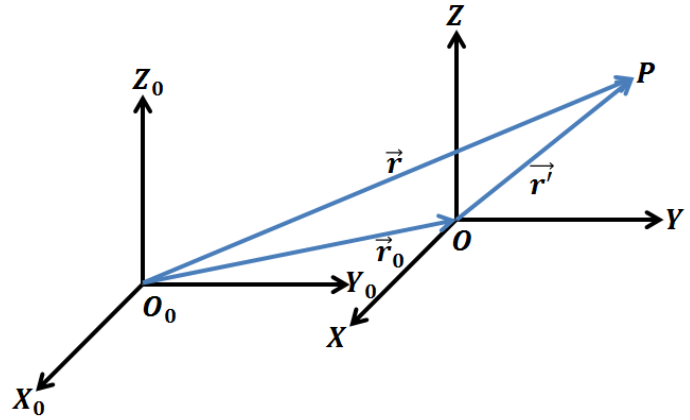
These forces are called fictitious or apparent or inertial forces. They do not have physical forces and do not arise from interactions of the particles.

Case II

In this case we consider the origins of fixed and rotating coordinate systems are not coincident. We have

$$\begin{aligned} (r)_f &= R = (r)_r + r_0 \\ &= x\hat{i} + y\hat{j} + z\hat{k} + r_0 \end{aligned}$$

where r_0 is the position vector of the origin of the rotating system w.r.t the origin of the fixed system.



Therefore

$$\left(\frac{dr}{dt}\right)_f = \frac{dr_0}{dt} + \frac{d}{dt}(x\hat{i} + y\hat{j} + z\hat{k})$$

$$v_f = v_0 + (\dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k}) + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt}$$

$$v_f = v_0 + v_r + \omega \times r$$

where $r \equiv (r)_r$.

Similarly the acceleration in the two systems will be related by

$$a_f = a_0 + a_r + 2\omega \times v_r + \dot{\omega} \times r + \omega \times (\omega \times r)$$

Module No. 180

Example 1 of Equation of Motion in Rotating Frame of Reference

Problem Statement

The angular velocity of a rotating coordinate system $OXYZ$ relative to a fixed coordinate system $OX_0Y_0Z_0$ is

$$\vec{\omega} = 2t\hat{i} - t^2\hat{j} + (2t + 4)\hat{k}$$

where t is the time and $\{\hat{i}, \hat{j}, \hat{k}\}$ unit vectors associated with $OXYZ$. The position vector of a particle at time t in the body system ($OXYZ$ system) is given by

$$r = (t^2 + 1)\hat{i} - 6t\hat{j} + 4t^3\hat{k}$$

To Find

- The apparent and true velocities at time $t = 1$.
- The apparent and true acceleration at time $t = 1$.

Solution

The apparent velocity is given by

$$\begin{aligned} v_r &= \left(\frac{dr}{dt} \right)_r = \frac{d}{dt} [(t^2 + 1)\hat{i} - 6t\hat{j} + 4t^3\hat{k}] \\ &= \hat{i} \frac{d}{dt} (t^2 + 1) - \hat{j} \frac{d}{dt} 6t + \hat{k} \frac{d}{dt} 4t^3 \\ &= 2t\hat{i} - 6\hat{j} + 12t^2\hat{k} \end{aligned}$$

Therefore the apparent velocity at time $t = 1$ is given by

$$v_r(t = 1) = 2\hat{i} - 6\hat{j} + 12\hat{k}$$

The true velocity at any time t is given by

$$v_f = v_r + \vec{\omega} \times r$$



$$= 2\hat{i} - 6\hat{j} + 12\hat{k} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2t & -t^2 & 2t+4 \\ t^2+1 & -6t & 4t^3 \end{vmatrix}$$

Therefore

$$\begin{aligned} v_f(t=1) &= 2\hat{i} - 6\hat{j} + 12\hat{k} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 6 \\ 2 & -6 & 4 \end{vmatrix} \\ &= 34\hat{i} - 2\hat{j} + 2\hat{k} \end{aligned}$$

ii. For apparent acceleration

$$\begin{aligned} a_r &= \frac{dv_r}{dt} = \frac{d}{dt}(2t\hat{i} - 6\hat{j} + 12t^2\hat{k}) \\ &= 2\hat{i} + 24t\hat{k} \end{aligned}$$

Therefore

$$a_r(t=1) = 2\hat{i} + 24\hat{k}$$

For true acceleration we have

$$a_f = a_r + 2\omega \times v_r + \dot{\omega} \times r + \omega \times (\omega \times r) \quad (1)$$

now

$$\omega(t=1) = 2\hat{i} - \hat{j} + 6\hat{k}$$

and

$$\dot{\omega} = 2\hat{i} - 2t\hat{j} + 2\hat{k}$$

$$\dot{\omega}(t=1) = 2\hat{i} - 2\hat{j} + 2\hat{k}$$

Since $r = (t^2 + 1)\hat{i} - 6t\hat{j} + 4t^3\hat{k}$ and

$v_r = 2t\hat{i} - 6\hat{j} + 12t^2\hat{k}$, therefore

$$\dot{\omega} \times r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 2 \\ 2 & -6 & 4 \end{vmatrix} = 4\hat{i} - 4\hat{j} - 8\hat{k}$$

and

$$\begin{aligned}
 2\omega \times v_r &= 2 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 6 \\ 2 & -6 & 12 \end{vmatrix} \\
 &= 2(24\hat{i} - 12\hat{j} - 10\hat{k})
 \end{aligned}$$

Also since

$$\omega \times r = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 6 \\ 2 & -6 & 4 \end{vmatrix} = 32\hat{i} + 4\hat{j} - 10\hat{k}$$

Therefore

$$\begin{aligned}
 \omega \times (\omega \times r) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 6 \\ 32 & 4 & -10 \end{vmatrix} \\
 &= -14\hat{i} + 212\hat{j} + 40\hat{k}
 \end{aligned}$$

Hence on making substitution, we have

$$\begin{aligned}
 a_f &= (2\hat{i} + 24t\hat{k}) + (4\hat{i} - 4\hat{j} - 8\hat{k}) + 2(24\hat{i} - 12\hat{j} - 10\hat{k}) + (-14\hat{i} + 212\hat{j} + 40\hat{k}) \\
 &= 40\hat{i} + 184\hat{j} + 36\hat{k}
 \end{aligned}$$

which is required true acceleration.

Module No. 181

Example 2 of Equation of Motion in Rotating Frame of Reference

Problem Statement

A coordinate system OXYZ rotates with angular velocity $\vec{\omega} = 2\hat{i} - 3\hat{j} + 5\hat{k}$ relative to the fixed coordinate system $OX_0Y_0Z_0$ both systems having the same origin. If $\vec{A} = \sin t \hat{i} - \cos t \hat{j} + e^{-t} \hat{k}$ where $\hat{i}, \hat{j}, \hat{k}$ refer to the rotating coordinate system OXYZ then find $\frac{d\vec{A}}{dt}$ and $\frac{d^2\vec{A}}{dt^2}$ w.r.to

- Fixed system
- The rotating system

Solution

- To find $\frac{d\vec{A}}{dt}$ and $\frac{d^2\vec{A}}{dt^2}$ in the fixed system, we have to apply the following formulas

$$\left(\frac{d\vec{A}}{dt}\right)_f = \left(\frac{d\vec{A}}{dt}\right)_r + \vec{\omega} \times \vec{A} \quad (1)$$

$$\left(\frac{d^2\vec{A}}{dt^2}\right)_f = \left(\frac{d^2\vec{A}}{dt^2}\right)_r + 2\vec{\omega} \times \frac{d\vec{A}}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{A} + \vec{\omega} \times (\vec{\omega} \times \vec{A}) \quad (2)$$

$$\left(\frac{d\vec{A}}{dt}\right)_r = \left[\frac{d\vec{A}}{dt}\right]_r = \frac{d\vec{A}_r}{dt}$$

$$= \frac{d}{dt} (\sin t \hat{i} - \cos t \hat{j} + e^{-t} \hat{k})$$

$$= \cos t \hat{i} + \sin t \hat{j} - e^{-t} \hat{k}$$

$$\vec{\omega} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 5 \\ \sin t & -\cos t & e^{-t} \end{vmatrix}$$

$$\vec{\omega} \times \vec{A} = (-3e^{-t} + 5 \cos t) \hat{i} - (2e^{-t} - 5 \sin t) \hat{j} + (-2 \cos t + 3 \sin t) \hat{k}$$

$$\begin{aligned} \left(\frac{d\vec{A}}{dt}\right)_f &= \cos t \hat{i} + \sin t \hat{j} - e^{-t} \hat{k} + (-3e^{-t} + 5 \cos t) \hat{i} - (2e^{-t} - 5 \sin t) \hat{j} \\ &\quad + (-2 \cos t + 3 \sin t) \hat{k} \end{aligned}$$

$$\begin{aligned}
&= (\cos t + 5 \cos t - 3e^{-t})\hat{i} + (5 \sin t + \sin t - 2e^{-t})\hat{j} + (-e^{-t} - 2 \cos t + 3 \sin t)\hat{k} \\
&= (6 \cos t - 3e^{-t})\hat{i} + (6 \sin t - 2e^{-t})\hat{j} + (-e^{-t} - 2 \cos t + 3 \sin t)\hat{k}
\end{aligned}$$

Now for solving equation (2), we proceed as follows

$$\begin{aligned}
\left(\frac{d^2 \vec{A}}{dt^2}\right)_r &= \frac{d}{dt} \left(\frac{d\vec{A}}{dt}\right)_r = \frac{d}{dt} (\cos t \hat{i} + \sin t \hat{j} - e^{-t} \hat{k}) \\
&= (-\sin t \hat{i} + \cos t \hat{j} + e^{-t} \hat{k}) \\
2\vec{\omega} \times \left(\frac{d\vec{A}}{dt}\right)_r &= 2(2\hat{i} - 3\hat{j} + 5\hat{k}) \times (\cos t \hat{i} + \sin t \hat{j} - e^{-t} \hat{k}) \\
&= (4\hat{i} - 6\hat{j} + 10\hat{k}) \times (\cos t \hat{i} + \sin t \hat{j} - e^{-t} \hat{k}) \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -6 & 10 \\ \cos t & \sin t & -e^{-t} \end{vmatrix} \\
&= 2\vec{\omega} \times \left(\frac{d\vec{A}}{dt}\right)_r \\
&= (6e^{-t} - 10 \sin t)\hat{i} + (4e^{-t} + 10 \cos t)\hat{j} + (4 \sin t + 6 \cos t)\hat{k} \\
\vec{\omega} &= 2\hat{i} - 3\hat{j} + 5\hat{k} \Rightarrow \dot{\vec{\omega}} = \frac{d\vec{\omega}}{dt} = 0
\end{aligned}$$

So,

$$\begin{aligned}
\frac{d\vec{\omega}}{dt} \times \vec{A} &= 0 \times (\sin t \hat{i} - \cos t \hat{j} + e^{-t} \hat{k}) = 0 \\
\vec{\omega} \times (\vec{\omega} \times \vec{A}) &= (2\hat{i} - 3\hat{j} + 5\hat{k}) \times [(2\hat{i} - 3\hat{j} + 5\hat{k}) \times (\sin t \hat{i} - \cos t \hat{j} + e^{-t} \hat{k})] \\
&= (2\hat{i} - 3\hat{j} + 5\hat{k}) \times [(-3e^{-t} + 5 \cos t)\hat{i} - (2e^{-t} - 5 \sin t)\hat{j} + (-2 \cos t + 3 \sin t)\hat{k}] \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -3 & 5 \\ -3e^{-t} + 5 \cos t & -2e^{-t} + 5 \sin t & -2 \cos t + 3 \sin t \end{vmatrix} \\
&= (6 \cos t - 9 \sin t - 25 \sin t + 10e^{-t})\hat{i} + (4 \cos t - 6 \sin t - 15e^{-t} + 25 \cos t)\hat{j} + (4e^{-t} - 10 \sin t + 9e^{-t} - 15 \cos t)\hat{k}
\end{aligned}$$

$$\begin{aligned}
&= (6 \cos t - 34 \sin t + 10e^{-t})\hat{i} + (29 \cos t - 6 \sin t - 15e^{-t})\hat{j} \\
&\quad + (-4e^{-t} + 10 \sin t + 9e^{-t} - 15 \cos t)\hat{k} \\
&= (-\sin t \hat{i} + \cos t \hat{j} + e^{-t} \hat{k}) + (6e^{-t} - 10 \sin t)\hat{i} + (4e^{-t} + 10 \cos t)\hat{j} + (4 \sin t + 6 \cos t)\hat{k} \\
&\quad + 0 + (6 \cos t - 34 \sin t + 10e^{-t})\hat{i} + (29 \cos t - 6 \sin t - 15e^{-t})\hat{j} + (10 \sin t \\
&\quad - 13e^{-t} + 15 \cos t)\hat{k} \\
\left(\frac{d^2 \vec{A}}{dt^2}\right)_f &= (6 \cos t - 45 \sin t + 16e^{-t})\hat{i} + (40 \cos t - 6 \sin t - 11e^{-t})\hat{j} \\
&\quad + (14 \sin t - 12e^{-t} + 21 \cos t)\hat{k}
\end{aligned}$$

ii. The rotating system

$$\begin{aligned}
\left(\frac{d\vec{A}}{dt}\right)_r &= \left[\frac{d\vec{A}}{dt}\right]_r = \frac{d\vec{A}_r}{dt} \\
&= \frac{d}{dt}(\sin t \hat{i} - \cos t \hat{j} + e^{-t} \hat{k}) \\
&= \cos t \hat{i} + \sin t \hat{j} - e^{-t} \hat{k} \\
\left(\frac{d^2 \vec{A}}{dt^2}\right)_r &= \frac{d}{dt} \left(\frac{d\vec{A}}{dt}\right)_r \\
&= \frac{d}{dt}(\cos t \hat{i} + \sin t \hat{j} - e^{-t} \hat{k}) \\
&= (-\sin t \hat{i} + \cos t \hat{j} + e^{-t} \hat{k})
\end{aligned}$$

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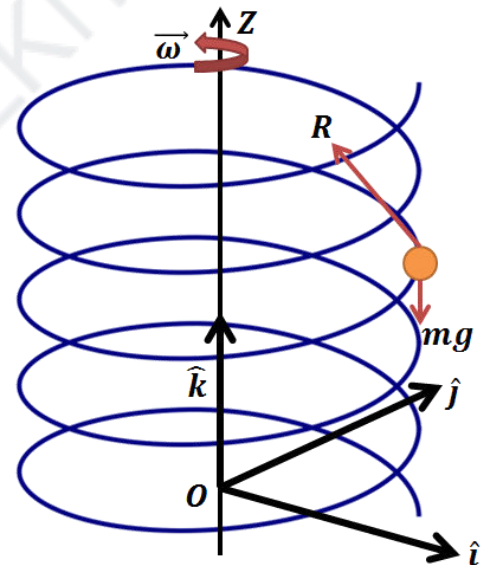
Example 3 of Equation of Motion in Rotating Frame of Reference

Problem Statement

A bead slides on a smooth helix whose central axis is vertical. The helix is forced to rotate about its central axis with constant angular speed ω . Find the equation of motion of the bead relative to the helix.

Solution

We choose a coordinate $OXYZ$ fixed in the helix such that Z -axis is coincident with the axis of the helix.



Then the parametric eqs. of the helix are given by

$$x = a \cos \theta, y = a \sin \theta,$$

$$z = b\theta$$

In vector form these can be written as

$$\vec{r} = a \cos \theta \hat{i} + a \sin \theta \hat{j} + b\theta \hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ point along the coordinate axis. Here $\hat{k}$ is constant but $\hat{i}, \hat{j}$ are variable.

The equation of motion in the rotating coordinate system is given by

$$ma_r = F - 2m\omega \times v_r - m\dot{\omega} \times r - m\omega \times (\omega \times r) \quad (1)$$

In this problem, since ω is constant,

$$\omega = \omega \hat{k}, \quad \dot{\omega} = 0 \quad (2)$$

Also

$$v_r = \dot{r} = (-a \sin \theta \hat{i} + a \cos \theta \hat{j} + b \hat{k}) \dot{\theta} \quad (3)$$

and

$$\begin{aligned} a_r = \ddot{r} &= \frac{d}{dt} [(-a \sin \theta \hat{i} + a \cos \theta \hat{j} + b \hat{k}) \dot{\theta}]_r \\ &= (-a \sin \theta \hat{i} + a \cos \theta \hat{j} + b \hat{k}) \ddot{\theta} + (-a \cos \theta \hat{i} - a \sin \theta \hat{j}) \dot{\theta}^2 \end{aligned} \quad (4)$$

If F denotes the external force in the fixed (space) coordinate system, then

$$F = -mg \hat{k} + R \quad (5)$$

where R is the reaction of the helix on the bead.

Before making substitution from equation (2-5) into equation (1), we obtain simplified expression for the second, and the fourth terms of (1), (the third term being zero).

$$\begin{aligned} \omega \times v_r &= \omega \hat{k} \times (-a \sin \theta \hat{i} + a \cos \theta \hat{j} + b \hat{k}) \dot{\theta} \\ &= \omega a (-\sin \theta \hat{j} - \cos \theta \hat{i}) \dot{\theta} \end{aligned} \quad (6)$$

and

$$\begin{aligned} \omega \times r &= \omega \hat{k} \times (a \cos \theta \hat{i} + a \sin \theta \hat{j} + b \hat{k}) \\ &= \omega a (\cos \theta \hat{j} - \sin \theta \hat{i}) \\ \omega \times (\omega \times r) &= \omega^2 a \hat{k} \times (\cos \theta \hat{j} - \sin \theta \hat{i}) \\ &= \omega^2 a (-\cos \theta \hat{i} - \sin \theta \hat{j}) \end{aligned} \quad (7)$$

Therefore on making substitutions from (4), (5), (6) and (7) into (1), we obtain

$$m(-a \sin \theta \hat{i} + a \cos \theta \hat{j} + b \hat{k}) \ddot{\theta} - ma(\cos \theta \hat{i} + \sin \theta \hat{j}) = -mg \hat{k} + R + 2m\omega a(\sin \theta \hat{j} + \cos \theta \hat{i}) + ma\omega^2(\cos \theta \hat{i} + \sin \theta \hat{j})$$

$$m(-a \sin \theta \hat{i} + a \cos \theta \hat{j} + b \hat{k})\ddot{\theta} - ma(\cos \theta \hat{i} + \sin \theta \hat{j}) = -mg\hat{k} + R + (2m\omega a + m\omega^2)(\sin \theta \hat{j} + \cos \theta \hat{i}) \quad (8)$$

Now if we write

$$v_r = (-a \sin \theta \hat{i} + a \cos \theta \hat{j} + b \hat{k})\dot{\theta} = u\dot{\theta}$$

Then it is clear that the vector u is tangent to the helix. Since R is normal to the helix,

$$R \cdot v_r = R \cdot u\dot{\theta} = 0$$

Rewriting (8) in terms of u (after eliminating the common factor m), we have

$$u\ddot{\theta} - a(\cos \theta \hat{i} + \sin \theta \hat{j})\dot{\theta}^2 = -g\hat{k} + R + (2\omega a + a\omega^2)(\cos \theta \hat{i} + \sin \theta \hat{j}) \quad (9)$$

Taking dot product of both sides of (9) with u , we have

$$u \cdot u\ddot{\theta} - au \cdot (\cos \theta \hat{i} + \sin \theta \hat{j})\dot{\theta}^2 = -gk \cdot u + R \cdot u + (2\omega a + a\omega^2)(\cos \theta \hat{i} + \sin \theta \hat{j}) \cdot u \quad (10)$$

Now

$$u \cdot u = a^2 \sin^2 \theta + a^2 \cos^2 \theta + b^2 = a^2 + b^2$$

$$\hat{k} \cdot u = \hat{k} \cdot (-a \sin \theta \hat{i} + a \cos \theta \hat{j} + b \hat{k}) = b$$

$$R \cdot u = 0$$

$$(\sin \theta \hat{j} + \cos \theta \hat{i}) \cdot u = (\sin \theta \hat{j} + \cos \theta \hat{i}) \cdot (-a \sin \theta \hat{i} + a \cos \theta \hat{j} + b \hat{k})$$

$$= a \cos \theta \sin \theta - a \cos \theta \sin \theta$$

$$= 0$$

Therefore (10) reduces to

$$(a^2 + b^2)\ddot{\theta} = -gb$$

or

$$\ddot{\theta} = -\frac{gb}{(a^2 + b^2)}$$

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Example 4 of Equation of Motion in Rotating Frame of Reference

Problem Statement

Prove that the centrifugal force acting on a particle of mass m on the earth's surface is the vector directed away from the earth's center and perpendicular to the angular velocity vector $\vec{\omega}$.

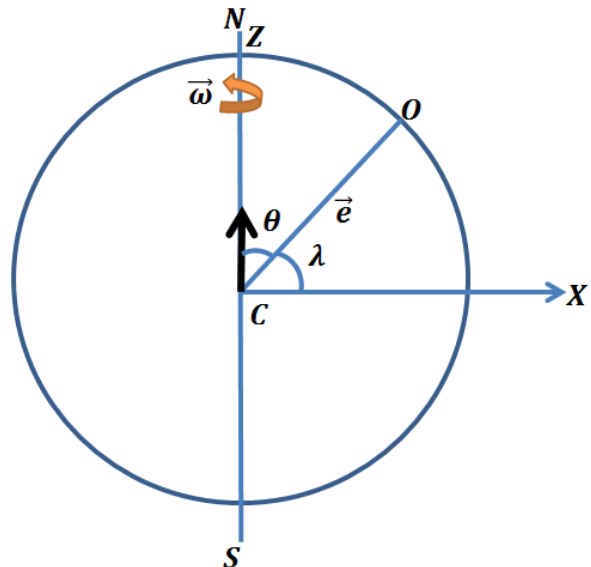
- Show that its magnitude is $m\omega^2 r_e \cos \lambda$.
- Determine the places on the surface of the earth where centrifugal force will be maximum or minimum.

Solution

- Since centrifugal force is $-m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)$.

We have to show that it's vector is directed away from center of the earth and perpendicular to the angular velocity vector $\vec{\omega}$ and its magnitude is $m\omega^2 r_e \cos \lambda$

where λ is the latitude as shown in figure and r_e is the radius of the earth.



It is quite clear that the particle is moving away from the center of earth and centrifugal force is produced in the opposite direction of the path of the particle.

It can be observed from the given figure that the movement of the particle (i.e. $m\vec{\omega} \times (\vec{\omega} \times r_e)$) is perpendicular to the angular velocity $\vec{\omega}$ and vector $\vec{\omega} \times r_e$, becomes centrifugal force in the opposite direction of the particle, so its centrifugal force is also perpendicular to $\vec{\omega}$ and $\vec{\omega} \times r_e$.

Now we move onto the proof of magnitude of centrifugal force, since

$$\vec{F}_{cf} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)$$

We need to show that

$$|\vec{F}_{cf}| = |-m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)| = m\omega^2 r_e \cos \lambda$$

Now we assume that the motion takes place in the XZ-plane.

Then,

$$\begin{aligned}\hat{e} &= \hat{k} \cos \theta + \hat{i} \sin \theta \\ &= \hat{k} \cos(\pi/2 - \lambda) + \hat{i} \sin(\pi/2 - \lambda) \\ \hat{e} &= \hat{k} \sin \lambda + \hat{i} \cos \lambda\end{aligned}$$

Consider the centrifugal force

$$= -m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)$$

If $\hat{e}$ denotes a unit vector along the NS-axis, then we can write $\vec{\omega} = \omega\hat{e}$ and the expression becomes $\vec{\omega} \times (\vec{\omega} \times \vec{r}_e) = \omega^2 \hat{e} \times (\hat{e} \times \vec{r}_e)$

$$\begin{aligned}-\omega^2 \hat{e} \times (\hat{e} \times \vec{r}_e) &= -\omega^2 [(\hat{e} \cdot \vec{r}_e)\hat{e} - (\hat{e} \cdot \hat{e})\vec{r}_e] \\ &= -\omega^2 [(\hat{e} \cdot \vec{r}_e)\hat{e} - \vec{r}_e]\end{aligned}$$

Now

$$\begin{aligned}\hat{e} \cdot \vec{r}_e &= (\hat{k} \sin \lambda + \hat{i} \cos \lambda) \cdot r_e \hat{k} \\ &= r_e \cos \theta = r_e \cos(\pi/2 - \lambda) = r_e \sin \lambda\end{aligned}$$

So expression (1) becomes

$$\begin{aligned}-\vec{\omega} \times (\vec{\omega} \times \vec{r}_e) &= -\omega^2 r_e \sin \lambda (\hat{k} \sin \lambda + \hat{i} \cos \lambda) + \omega^2 r_e \hat{k} \\ &= (-\omega^2 r_e \sin^2 \lambda + \omega^2 r_e) \hat{k} - \omega^2 r_e \sin \lambda \cos \lambda \hat{i} \\ &= (-\omega^2 r_e (1 - \cos^2 \lambda) + \omega^2 r_e) \hat{k} - \omega^2 r_e \sin \lambda \cos \lambda \hat{i}\end{aligned}$$

$$\begin{aligned}
|\vec{F}_{cf}| &= |-m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)| = |-m((- \omega^2 r_e (1 - \cos^2 \lambda) + \omega^2 r_e) \hat{k} - \omega^2 r_e \sin \lambda \cos \lambda \hat{i})| \\
&= |-m(\omega^2 r_e \cos^2 \lambda) \hat{k} + m\omega^2 r_e \sin \lambda \cos \lambda \hat{i}| \\
&= m\sqrt{(\omega^2 r_e \cos^2 \lambda)^2 + (\omega^2 r_e \sin \lambda \cos \lambda)^2} \\
&= m\sqrt{\omega^4 r_e^2 \cos^4 \lambda + \omega^4 r_e^2 \cos^2 \lambda \sin^2 \lambda} \\
&= m\sqrt{\omega^4 r_e^2 \cos^4 \lambda + \omega^4 r_e^2 \cos^2 \lambda (1 - \cos^2 \lambda)} \\
&= m\sqrt{\omega^4 r_e^2 \cos^4 \lambda + \omega^4 r_e^2 \cos^2 \lambda - \omega^4 r_e^2 \cos^4 \lambda} \\
&= m\sqrt{\omega^4 r_e^2 \cos^2 \lambda} \\
|\vec{F}_{cf}| &= m\omega^2 r_e \cos \lambda
\end{aligned}$$

Hence the required result.

The centrifugal force will be maximum at $\lambda = 0$, where $\cos \lambda = 1$ (maximum) i.e. at equator and minimum will be at NS-poles i.e. at $\lambda = \pi/2$ where $\cos \lambda = 0$ (minimum).

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Module No. 184

Example 5 of Equation of Motion in Rotating Frame of Reference

Problem Statement

A coordinate system $OXYZ$ is rotating with angular velocity $\vec{\omega} = 5\hat{i} - 4\hat{j} - 10\hat{k}$ relative to fixed coordinate system $OX_0Y_0Z_0$ both systems having the same origin. Find the velocity of the particle at rest in the $OXYZ$ system at the point $(3,1,-2)$ as seen by an observer in the fixed system.

Solution

Since the given angular velocity is

$$\vec{\omega} = 5\hat{i} - 4\hat{j} - 10\hat{k}$$

and the point $P(3,1,-2)$ at rest in $OXYZ$ system.

Then

$$\vec{OP} = \vec{r} = 3\hat{i} + \hat{j} - 2\hat{k}$$

Also, we know the equation of motion in case when the origins of the coordinate systems coincide each other

$$\vec{v}_f = \vec{v}_r + \vec{\omega} \times \vec{r}$$

But we have to find out the velocity of the particle at rest in $OXYZ$ system at $P(3,1,-2)$.

i.e.

$$\begin{aligned}\vec{v}_f &= \vec{v}_0 + \vec{\omega} \times \vec{r} \\ \vec{v}_f &= (0,0,0) + (5\hat{i} - 4\hat{j} - 10\hat{k}) \times (3\hat{i} + \hat{j} - 2\hat{k}) \\ &= (0\hat{i} + 0\hat{j} + 0\hat{k}) + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -4 & -10 \\ 3 & 1 & -2 \end{vmatrix} \\ &= (0\hat{i} + 0\hat{j} + 0\hat{k}) + ((8 + 10)\hat{i} - (-10 + 30)\hat{j} + (5 + 12)\hat{k})\end{aligned}$$

$$= (0\hat{i} + 0\hat{j} + 0\hat{k}) + (18\hat{i} - 20\hat{j} + 17\hat{k})$$

Hence

$$\vec{v}_f = 18\hat{i} - 20\hat{j} + 17\hat{k}$$

is the required velocity of the system.

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Module No. 185

General Motion of a Rigid Body

In general motion of the body, (i.e. no point of the body is fixed in space), let $\mathbf{F}^{ext}$ be the total external force on the rigid body and $\mathbf{G}_c^{ext}$ the total external torque about its center of mass (i.e. centroid). Then the equations of motion are

$$M\mathbf{a}_c = \mathbf{F}^{ext} \quad (1)$$

and

$$\dot{\mathbf{L}}_c = \mathbf{G}_c^{ext} \quad (2)$$

where $\mathbf{a}_c$ is the acceleration of the c.m. and $\mathbf{L}_c$ is the total angular momentum about it.

Now we resolve the vectors $\mathbf{a}_c, \mathbf{F}^{ext}, \mathbf{G}_c^{ext}$ and $\mathbf{L}$ along the unit vector i, j, k taken along the principal axes at the mass center.

The triad of vectors i, j, k may be inferred to as a principal triad. It will be assumed to be permanently a principal triad.

Let $\vec{\Omega}$ be its angular velocity. If the triad is fixed in the body then $\vec{\Omega} = \vec{\omega}$, the angular velocity of the body.

Now using the operator: $\left(\frac{d}{dt}\right)_f = \left(\frac{d}{dt}\right)_r + \omega \times$

$$\left(\frac{d\vec{F}}{dt}\right)_f = \left(\frac{d\vec{F}}{dt}\right)_r + \vec{\omega} \times \vec{F}$$

which relates the rate of change of a vector in a fixed (i.e. inertial) frame and a rotating frame, we have (on dropping the suffix r)

$$a_f \equiv \left(\frac{d\vec{v}}{dt}\right)_f = \frac{d\vec{v}}{dt} + \vec{\Omega} \times \vec{v} \quad (\because \mathbf{v}_r = \mathbf{v})$$

$$\text{or } a_f = \frac{dv}{dt} + \vec{\Omega} \times \vec{v} \quad (3)$$

where $\mathbf{v} = v_1 i + v_2 j + v_3 k$ is the velocity of the mass center (in the rotating coordinate system). Substituting for $a_f = a_c$ from equation (3) into (1), we obtain

$$M\left(\frac{d\mathbf{v}}{dt} + \vec{\Omega} \times \mathbf{v}\right) = \mathbf{F}^{ext}$$

which is equivalent to

$$\left. \begin{aligned} M(\dot{v}_1 + \Omega_2 v_3 + \Omega_3 v_2) &= F_1 \\ M(\dot{v}_2 + \Omega_3 v_1 + \Omega_1 v_3) &= F_2 \\ M(\dot{v}_3 + \Omega_1 v_2 + \Omega_2 v_1) &= F_3 \end{aligned} \right\} \quad (4)$$

From (2), on using

$$\left(\frac{dL}{dt}\right)_f = \frac{dL}{dt} + \Omega \times L$$

and the relation $L = I_1 \omega_1 \hat{i} + I_2 \omega_2 \hat{j} + I_3 \omega_3 \hat{k}$, we obtain the equations

$$I_1 \dot{\omega}_1 \hat{i} + I_2 \dot{\omega}_2 \hat{j} + I_3 \dot{\omega}_3 \hat{k} + (\Omega_2 L_3 - \Omega_3 L_2) \hat{i} + (\Omega_2 L_1 - \Omega_1 L_3) \hat{j} + (\Omega_1 L_2 - \Omega_2 L_1) \hat{k} = \vec{G}$$

From this vector equation we obtain the following three scalar equations

$$I_1 \dot{\omega}_1 + \Omega_2 L_3 - \Omega_3 L_2 = G_1$$

$$I_2 \dot{\omega}_2 + \Omega_3 L_1 - \Omega_1 L_3 = G_2$$

and

$$I_3 \dot{\omega}_3 + \Omega_1 L_2 - \Omega_2 L_1 = G_3$$

where on using the results

$$L_1 = I_1 \omega_1, L_2 = I_2 \omega_2, L_3 = I_3 \omega_3$$

we have

$$\left. \begin{aligned} I_1 \dot{\omega}_1 + \omega_3 \Omega_2 I_3 - \omega_2 \Omega_3 I_2 &= G_1 \\ I_2 \dot{\omega}_2 + \omega_1 \Omega_3 I_1 - \omega_3 \Omega_1 I_3 &= G_2 \\ I_3 \dot{\omega}_3 + \omega_2 \Omega_1 I_2 - \omega_1 \Omega_2 I_1 &= G_3 \end{aligned} \right\} \quad (5)$$

where I_1, I_2, I_3 denote principal M.I at the centroid of the body.

The set of eqs (4) and (5) constitute six equations for the components of velocity of centroid and the components of angular velocity of the body.

Module No. 186

Equation of Motion Relative to Coordinate System Fixed on Earth

We derived the equation of motion when the origins of fixed and rotating coordinate systems are distant.

$$\vec{a}_f = \vec{a}_0 + \vec{a}_r + 2\vec{\omega} \times \vec{v}_r + \dot{\vec{\omega}} \times \vec{r} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) \quad (1)$$

From (1) the most general form of the equation of motion in a moving frame can be written as

$$m \left(\frac{d^2 \vec{r}}{dt^2} \right)_r = \vec{F} - m\dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times \vec{v}_r - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - m\vec{a}_0 \quad (2)$$

where the subscript r refers to the rotating frame, $\vec{F}$ is the total external force in the fixed coordinate system and $\vec{a}_0$ is the acceleration of the origin O in moving (rotating) coordinate system, $OXYZ$. In the case of the earth, which is a rotating coordinate system, we choose the origin as a point fixed on the earth.

We choose the fixed coordinate system at the center C of the earth and denote it by $CX_0Y_0Z_0$. For a particle near the surface of the earth

$$\vec{F} = m\vec{g} \quad (3)$$

where $\vec{g}$ is the acceleration due to gravity.

$\vec{a}_0$, the acceleration of the origin O is the centripetal acceleration of O due to the rotation of the earth. It may be represented as

$$\vec{a}_0 = \vec{\omega} \times (\vec{\omega} \times \vec{r}_e) \quad (4)$$

Therefore on substitution from (3) and (4) into (2), we obtain

$$m \left(\frac{d^2 \vec{r}}{dt^2} \right)_r = m\vec{g} - m\dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times \vec{v}_r - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e) \quad (5)$$

which gives the full equation of motion for a particle of mass m w.r.t. a coordinate system fixed on the earth.

Next we obtain a simple form of (5) taking into account the fact that the angular velocity $\vec{\omega}$ of the earth is nearly constant both in magnitude and direction, (which is along NS i.e. North-South polar line), and of small magnitude. In fact

$$\begin{aligned}
\omega &= |\vec{\omega}| = \frac{2\pi \text{ radian}}{24 \text{ hours}} \\
&= \frac{2\pi}{86400} \text{ radian per second} \\
&= 7.27 \times 10^{-5} \text{ rad/sec}
\end{aligned}$$

Since $\vec{\omega}$ may be taken as constant, $\dot{\vec{\omega}} = 0$. Since the fourth term on R.H.S. of (5) has the magnitude $m\omega^2 r$ (where r is the distance of the particle from the NS-axis), is negligibly small because of ω^2 . However the fifth term $m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)$, the last term of equation (5) is not negligible because $|\vec{r}_e|$ (the magnitude of the radius of the earth) is very large.

Hence equation (5) can be written as

$$m \left(\frac{d^2 \vec{r}}{dt^2} \right)_r = m\vec{g} - 2m\vec{\omega} \times v_r - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e)$$

which is a second order differential equation in $\vec{r}$ and may also written as

$$m \frac{d^2 \vec{r}}{dt^2} + 2m\vec{\omega} \times \frac{d\vec{r}}{dt} = m\vec{g} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}_e) \quad (6)$$

where we have dropped the subscript r , it being understood that the terms on L.H.S of (6) refer to the rotating coordinate system.

If $\hat{e}$ denotes a unit vector along the NS-axis, then we can write $\vec{\omega} = \omega\hat{e}$, and equation of motion takes the form

$$\ddot{\vec{r}} + 2\omega\hat{e} \times \dot{\vec{r}} = \vec{g} - \omega^2\hat{e} \times (\hat{e} \times \vec{r}_e)$$