Lecture 39. Periodic Functions

A function $f(x)$ is said to be periodic if its function values repeat at regular intervals of the independent variable. The regular interval b/w repetitions is the period of oscillations.

$$f(x+p) = f(x)$$

$$y = A \sin nx$$

The period of sine is 360° or 2π.
The period of cos is also 360° or 2π.

In $y = 5 \sin 2x$ (amplitude is 5 & period is 180°)

$$y = 5 \sin(2x + 360^\circ)$$

$$= 5 \sin 2\left(\frac{2x}{2} + \frac{360^\circ}{2}\right)$$

$$= 5 \sin 2(x + 180^\circ)$$

Example:-

(i) $y = 3 \sin 5x$ amplitude $= 3$

$$y = 3 \sin(5x + 360)$$

$$= 3 \sin 5\left(\frac{5x}{5} + \frac{360}{5}\right)$$

$$= 3 \sin 5(x + 72^\circ)$$ So period $= 72^\circ$

(ii) $y = 2\cos 3x$

amplitude $= 2$

$y = 2\cos(3x + 2\pi)$

$= 2\cos 3\left(\frac{3x}{3} + \frac{360}{3}\right)$

$= 2\cos 3(x + 120^\circ)$

So period $= 120^\circ$

(iii) $y = \sin\frac{x}{2}$

Amplitude $= 1$

$y = \sin\left(\frac{x}{2} + 360\right)$

$y = \sin\frac{1}{2}\left(\frac{2x}{2} + 360\times 2\right)$

$= \sin\frac{1}{2}(x + 720^\circ)$

So period is 720°

(iv) $y = 4\sin 2x$ amplitude $= 4$

$y = 4\sin(2x + 360)$

$= 4\sin 2\left(\frac{2x}{2} + \frac{360}{2}\right)$

$= 4\sin 2(x + 180^\circ)$ so period $= 180^\circ$

For $y = A \sin nx$

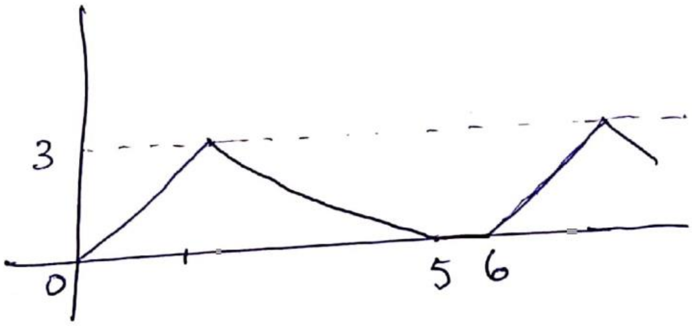
Amplitude $= A$

$$\text{Period} = \frac{360}{n} = \frac{2\pi}{n}$$

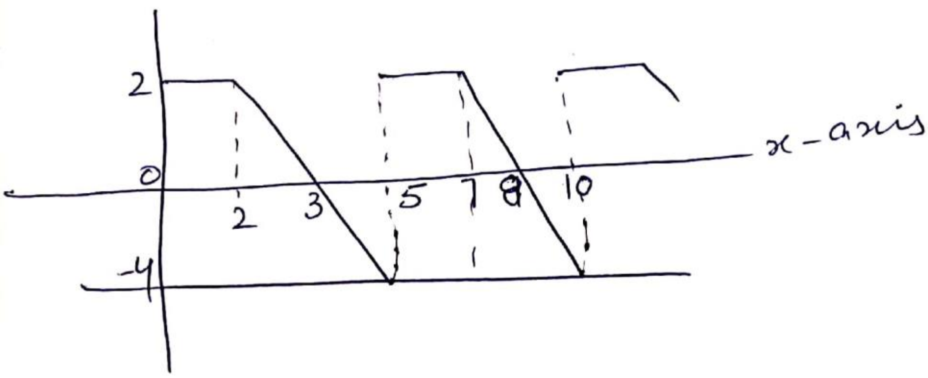
Same goes for $y = A \cos nx$

Examples:-

period = 8



period = 6

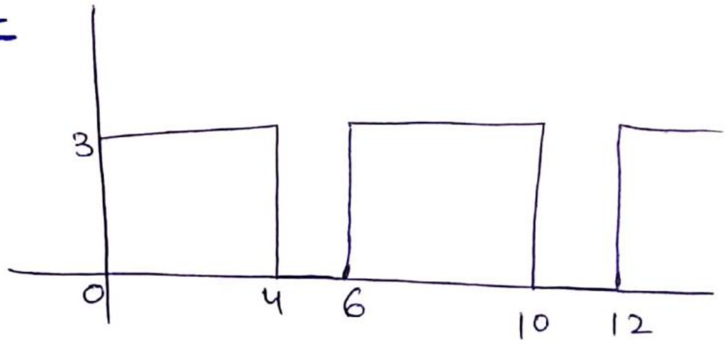


Period = 5

Analytical description of a periodic func

A periodic function can be defined analytically in many cases.

Example:-



If $y=3$, when $0 \le x \le 4$

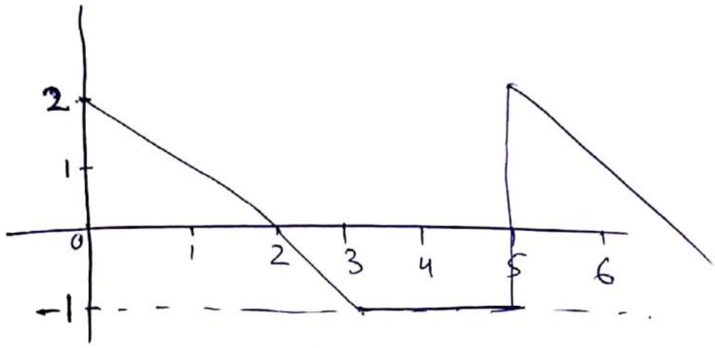
$f(x)=3$

& $y=0$ when $4 \le x \le 6$

$f(x)=0$

$f(x) = f(x+6)$ (function is periodic with period 6)

Example:-



when $0<x<3$

$m = \frac{y_2-y_1}{x_2-x_1} = \frac{-1-0}{3-2} \Rightarrow -1$

$y-y_0 = m(x-x_0)$

$y-2 = -1(x-0)$

$y-2 = -x$

$y = 2-x$

$f(x) = 2-x$ when $0<x<3$

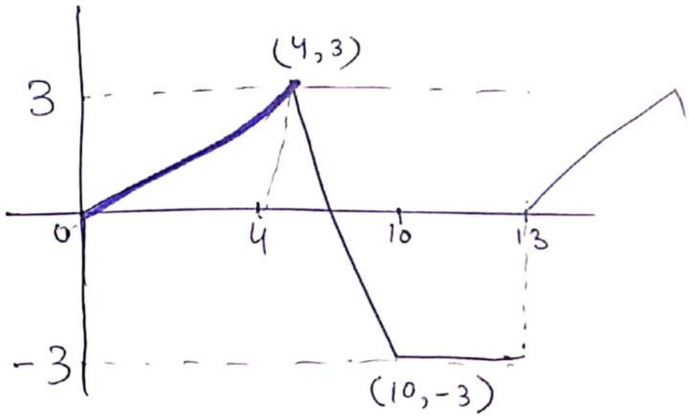
$y = -1$ when $3 < x < 5$

$f(x) = -1$

$f(x) = f(x+5)$

function is periodic with period 5.

Example:-



when $0 < x < 4$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-0}{4-0} \Rightarrow \frac{3}{4}$$

$$y - y_0 = m(x - x_0)$$

$$y - 0 = \frac{3}{4}(x - 0)$$

$y = \frac{3}{4}x$ or $f(x) = \frac{3}{4}x$ when $0 < x < 4$

when $4 < x < 10$

$$m = \frac{-3-3}{10-4} \Rightarrow \frac{-6}{6} \Rightarrow -1$$

$$y - y_0 = m(x - x_0)$$

$$y - 3 = -1(x - 4)$$

$$y - 3 = -x + 4$$

$$y = -x + 4 + 3$$

$$y = 7 - x$$

$f(x) = 7 - x$ where $4 < x < 10$

when $10 < x < 13$

$f(x) = -3$

& $f(x) = f(x+13)$

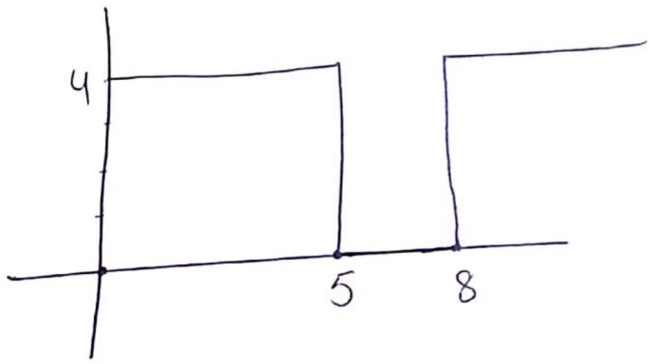
function is periodic with period 13.

Example:- Sketch the graph

1. $f(x) = 4 \qquad 0 < x < 5$

   $f(x) = 0 \qquad 5 < x < 8$

   $f(x) = f(x+8)$



2. $f(x) = 3x - x^2 \qquad 0 < x < 3$

   $f(x) = f(x+3)$

   $f(x) = -x^2 + 3x + 0$ (parabola)

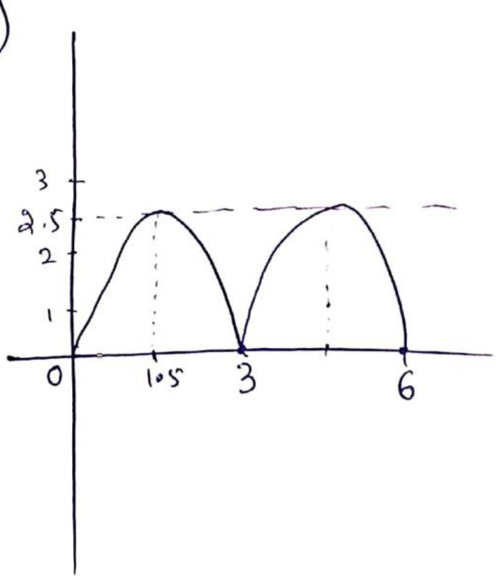
$x = \text{Vertex} = \dfrac{-b}{2a} = \dfrac{-3}{2(-1)} \Rightarrow \dfrac{3}{2} \Rightarrow 1.5$

put $x = 1.5$ in

$f(x) = 3(1.5) - (1.5)^2$

$f(x) = 2.5$

opening downward bcz $a < 0$ (-ve)



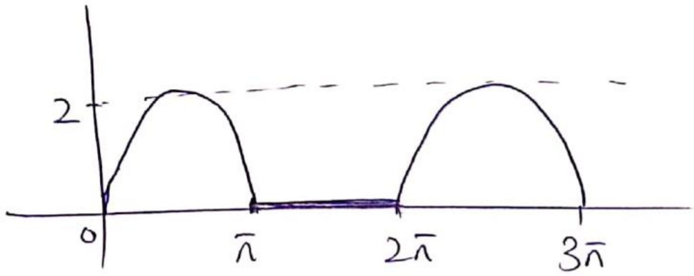
Example:-

$f(x) = 2\sin 2x \qquad 0 < x < \pi$

$f(x) = 0 \qquad \pi < x < 2\pi$

$f(x) = f(x + 2\pi)$

amplitude = 2



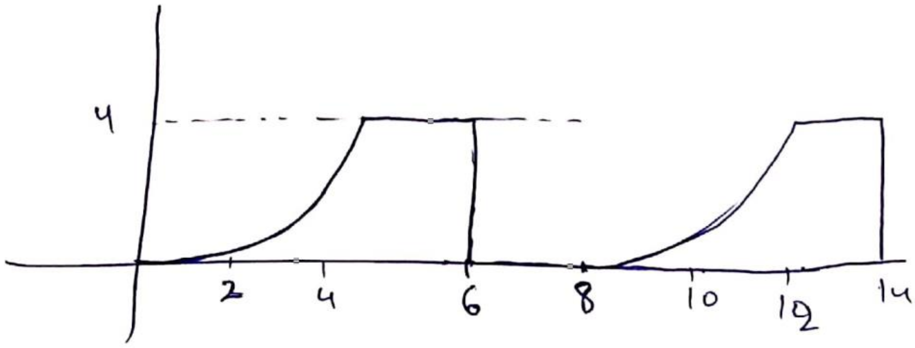
Example:-

$f(x) = \frac{x^2}{4} \qquad 0 < x < 4$ (portion of parabola) a is +ve so opens upward

$f(x) = 4 \qquad 4 < x < 6$

$f(x) = 0 \qquad 6 < x < 8$

$f(x) = f(x+8)$



Useful integrals

Here are some integrals that appear frequently in our work on Fourier Series.

$$\int_{-\pi}^{\pi} \sin nx\,dx = \left[-\frac{\cos nx}{n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{n}\left[-\cos n\pi - (-\cos n(-\pi))\right]$$

$$= \frac{1}{n}\left[-\cos n\pi + \cos n\pi\right] = 0$$

$$\int_{-\pi}^{\pi} \cos nx\,dx = \left[\frac{\sin nx}{n}\right]_{-\pi}^{\pi} \Rightarrow \frac{1}{n}\left[\sin n\pi - (-\sin n\pi)\right]$$

$$= \frac{1}{n}\left[\sin n\pi + \sin n\pi\right] = \frac{1}{n}\left[0+0\right] \Rightarrow 0$$

$$\int_{-\pi}^{\pi} \sin^2 nx\,dx = \int_{-\pi}^{\pi} \frac{1-\cos 2nx}{2}\,dx \Rightarrow \frac{1}{2}\int_{-\pi}^{\pi}\left[1-\cos 2nx\right]dx$$

$$= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\left\{\pi - \frac{\sin 2n\pi}{2n}\right\} - \left\{-\pi + \frac{\sin 2n\pi}{2n}\right\}\right]$$

$$= \frac{1}{2}\left[\pi - 0 + \pi + 0\right] = \frac{1}{2}\left[2\pi\right] = \pi$$

$$\int_{-\pi}^{\pi} \cos^2 nx\,dx = \int_{-\pi}^{\pi} \frac{1+\cos 2nx}{2}\,dx$$

$$\frac{1}{2}\int_{-\pi}^{\pi}(1+\cos 2nx)\,dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi}$$

$$= \pi$$ Ans.

~~Product to sum~~

$$\frac{1}{2}\int_{-\pi}^{\pi} 2\sin nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi}\left[\sin(n+m)x + \sin(n-m)x\right]dx$$

$$= \frac{1}{2}\left\{0+0\right\} = 0$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(n+m)x + \cos(n-m)x\right]dx$$

$$= \frac{1}{2}\left\{0+0\right\} = 0$$

Note:- If $n=m$

$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx$ becomes $\int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = -\frac{1}{2}\int_{-\pi}^{\pi}\left(\cos(n+m)x - \cos(n-m)x\right)dx$$

$$= -\frac{1}{2}\left\{0-0\right\} = 0$$

Note:-
If $n=m$ it becomes $\int_{-\pi}^{\pi} \sin^2 nx = \pi$

Summary of Integrals :-

$$\int_{-\pi}^{\pi} \sin nx \, dx = 0 \quad , \quad \int_{-\pi}^{\pi} \cos nx \, dx = 0$$

$$\int_{-\pi}^{\pi} \cos^2 nx \, dx = \pi \quad , \quad \int_{-\pi}^{\pi} \sin^2 nx \, dx = \pi$$

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \quad , \quad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \; (n \neq m)$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \; (n \neq m) \quad , \quad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \pi \; (n = m)$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \pi \; (n = m)$$

limits can be changed, but the condition is to keep the interval of 2π. like $-\pi$ to π, $\frac{-\pi}{2}$ to $\frac{3\pi}{2}$, 0 to 2π etc.

Lecture 40

Fourier Series

$$f(x) = A_0 + \sum_{n=1}^{\infty} \{a_n \cos nx + b_n \sin nx\}$$

can be written in expanded form

$$f(x) = A_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \dots + (a_n \cos nx + b_n \sin nx)$$

Fourier Coefficients

To find A_0, integrate $f(x)$ from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi} A_0\,dx + \sum_{n=1}^{\infty}\left\{\int_{-\pi}^{\pi} a_n \cos nx\,dx + \int_{-\pi}^{\pi} b_n \sin nx\right\}$$

$$= \left[A_0 x\right]_{-\pi}^{\pi} + \sum_{n=1}^{\infty}\{0 + 0\}$$

$$= A_0(\pi - (-\pi)) \Rightarrow A_0(2\pi) \Rightarrow 2\pi A_0$$

So $\int_{-\pi}^{\pi} f(x)\,dx = 2A_0\pi$

$$\boxed{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = A_0}$$

If $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = a_0$

Then $\boxed{A_0 = \frac{1}{2} a_0}$

To find a_n we multiply $f(x)$ by $\cos mx$ & integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = \int_{-\pi}^{\pi} A_0 \cos mx\,dx + \sum_{n=1}^{\infty}\left\{\int_{-\pi}^{\pi} a_n \cos nx \cos mx\,dx + \int_{-\pi}^{\pi} b_n \sin nx \cos mx\,dx\right\}$$

$$= A_0(0) + \sum_{n=1}^{\infty}\left(a_n(0) + b_n(0)\right)$$

$$= 0 \quad \text{for } n \neq m$$

$$\int_{-\pi}^{\pi} f(x)\cos mx\,dx = 0 + a_n\pi + 0 \Rightarrow a_n\pi \quad \text{for } n = m$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

To find b_n we multiply $f(x)$ by $\sin mx$ & integrate from $-\pi$ to π.

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi} A_0 \sin mx + \sum_{n=1}^{\infty}\left\{\int_{-\pi}^{\pi} a_n \cos nx \sin mx\,dx + \int_{-\pi}^{\pi} b_n \sin nx \sin mx\,dx\right\}$$

$$= 0 \text{ if } n \neq m$$

$$= 0 + 0 + b_n\pi \quad \text{if } n = m$$

$$\int_{-\pi}^{\pi} f(x)\sin nx\,dx = b_n\pi$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = b_n$$

Result for Fourier Series:-

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n\sin nx\}$$

(a) $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = 2\left(\text{mean value of } f(x) \text{ over a period}\right)$

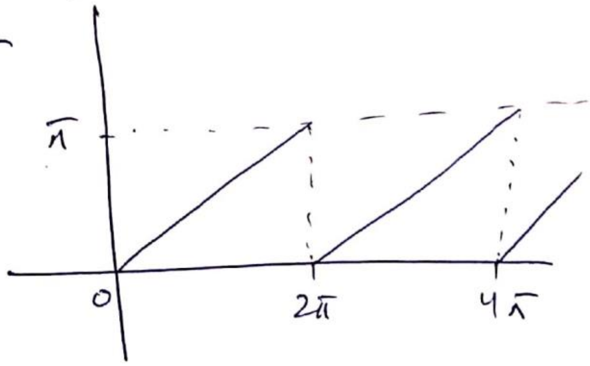
(b) $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 2\left(\text{mean value of } f(x)\cos nx \text{ over a period}\right)$

(c) $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 2\left(\text{mean value of } f(x)\sin nx \text{ over a period}\right)$

In each case $n = 1, 2, 3, 4, \ldots$

Example:- Determine the Fourier Series to represent the periodic function shown.

here limits are $0 < x < 2\pi$



$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\pi - 0}{2\pi - 0}$$

$$m = \frac{\pi}{2\pi} = \frac{1}{2}$$

$$y - y_0 = m(x - x_0)$$

$$y - 0 = \frac{1}{2}(x - 0)$$

$$y = \frac{x}{2}$$

$$f(x) = \frac{x}{2} \quad , \quad 0 < x < 2\pi$$

$$f(x) = f(x + 2\pi) \quad \text{period} = 2\pi$$

Now we'll find Fourier Series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left\{a_n \cos nx + b_n \sin nx\right\}$$

First of all

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\cos nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cos nx \, dx$$

Integrating by parts

$$= \frac{1}{2\pi}\left\{x\int_0^{2\pi}\cos nx \, dx + \int_0^{2\pi}\left[\int_0^{2\pi}\cos nx \, dx \cdot \frac{d}{dx}(x)\right]dx\right\}$$

$$= \frac{1}{2\pi}\left\{x\left[\frac{\sin nx}{n}\right]_0^{2\pi} + \int_0^{2\pi}\frac{\sin nx}{n}\,dx\right\}$$

$$= \frac{1}{2\pi}\left\{x(0) + 0\right\} \Rightarrow \boxed{0 = a_n}$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{4}\right]_0^{2\pi} = \frac{1}{4\pi}\left[x^2\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[(2\pi)^2 - 0^2\right] \Rightarrow \frac{1}{4\pi}(4\pi^2)$$

$$\boxed{a_0 = \pi}$$

Now

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{x}{2}\right)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{2\pi}\left\{x\int_0^{2\pi}\sin nx\,dx - \int_0^{2\pi}\left(\int_0^{2\pi}\sin nx\cdot\frac{dx}{dx}\right)dx\right\}$$

$$= \frac{1}{2\pi}\left\{-x\left[\frac{\cos nx}{n}\right]_0^{2\pi} + \int_0^{2\pi}\left[\frac{\cos nx}{n}\right]_0^{2\pi}dx\right\}$$

$$= \frac{1}{2\pi}\left\{-\frac{2\pi\cos n2\pi}{n} + 0\right\}$$

$$= \frac{1}{2\pi}\left(-\frac{2\pi}{n}\right) = -\frac{1}{n}$$

$a_0 = \pi, \quad a_n = 0, \quad b_n = -\frac{1}{n}$

Now put values in Fourier series

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos nx + b_n \sin nx \right\}$$

$$= \frac{1}{2}\pi + \sum_{n=1}^{\infty} \left\{ -\frac{1}{n} \sin nx \right\}$$

$$= \frac{1}{2}\pi + \left\{ \frac{-1}{1} \sin x - \frac{1}{2} \sin 2x - \frac{1}{3} \sin 3x - \cdots \right\}$$

$$= \frac{1}{2}\pi + \left(-\sin x - \frac{1}{2} \sin 2x - \frac{1}{3} \sin 3x \cdots \right)$$

$$= \frac{\pi}{2} - \left(\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x + \cdots \right)$$

Dirichlet Conditions:-

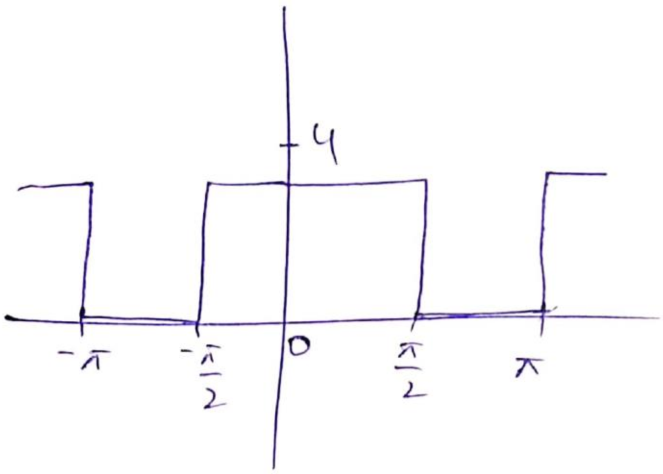
The $f(x)$ must be defined & single-valued

$f(x)$ must be continuous or have a finite number of finite discontinuities within a periodic interval.

$f(x)$ & $f'(x)$ must be piecewise continuous in the periodic interval.

If these dirichlet conditions are satisfied fourier series converges to $f(x_1)$ if $x = x_1$ is a point of continuity.

Example:- Find fourier series for function shown.



$f(x) = 0 \qquad -\pi < x < -\frac{\pi}{2}$

$f(x) = 4 \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$f(x) = 0 \qquad \frac{\pi}{2} < x < \pi$

$f(x) = f(x + 2\pi)$

As

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left\{ a_n \cos nx + b_n \sin nx \right\}$$

Now

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$$

$$= \frac{1}{\pi} \left\{ \int_{-\pi}^{-\pi/2} 0 \, dx + \int_{-\pi/2}^{\pi/2} 4 \, dx + \int_{\pi/2}^{\pi} 0 \, dx \right\}$$

$$= \frac{1}{\pi} 4 [x]_{-\pi/2}^{\pi/2} \Rightarrow \frac{1}{\pi} 4 \left[\frac{\pi}{2} + \frac{\pi}{2} \right]$$

$$= \frac{1}{\pi} 4\left(\frac{2\pi}{2}\right) = 4$$

(b) $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} 0\cos nx\,dx + \int_{-\pi/2}^{\pi/2} 4\cos nx\,dx + \int_{\pi/2}^{\pi} 0\cos nx\,dx\right\}$$

$$= \frac{1}{\pi} 4\left[\left(\frac{\sin nx}{n}\right)\right]_{-\frac{\pi}{2}}^{\pi/2} \Rightarrow \frac{4}{n\pi}\left[\sin n\frac{\pi}{2} + \sin n\frac{\pi}{2}\right]$$

$$= \frac{4}{n\pi} 2\sin n\frac{\pi}{2} \Rightarrow \frac{8}{n\pi}\sin\frac{n\pi}{2}$$

considering different integer values of n

if n is even $\quad a_n = 0$

if $n = 1, 5, 9, \ldots \quad a_n = \frac{8}{n\pi}$

if $n = 3, 7, 11, \ldots \quad a_n = \frac{-8}{n\pi}$

(c) To find b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{-\pi/2} 0 \sin nx\,dx + \int_{-\pi/2}^{\pi/2} 4 \sin nx\,dx + \int_{\pi/2}^{\pi} 0 \sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{ 4\int_{-\pi/2}^{\pi/2} \sin nx\,dx \right\}$$

$$= \frac{4}{\pi}(0) \Rightarrow 0$$

Fourier series is

$$f(x) = 2 + \frac{8}{\pi}\left\{\cos x - \frac{1}{3}\cos 3x + \frac{1}{5}\cos 5x - \cdots\right\}$$

Effect of Harmonics :-

If we include just one cosine term, we can see effect to including subsequent harmonics. As number of terms is increased, the graph gradually approaches to origional wave form. ripples increases in number & decreases in amplitude. A perfect wave is impossible to find, so usually first few terms give accuracy of acceptable level.

Lecture 41 Examples

Example:- Find fourier series for function defined by

$$f(x) = -x \qquad -\pi < x < 0$$

$$f(x) = 0 \qquad 0 < x < \pi$$

$$f(x) = f(x+2\pi)$$

first we'll find a_0, a_n & b_n.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow \frac{1}{\pi}\left\{\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} 0\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{x^2}{2}\right]_{-\pi}^{0}\right\} + 0 \Rightarrow \frac{1}{\pi}\left[-\left(\frac{0^2}{2} - \frac{(-\pi)^2}{2}\right)\right]$$

$$= \frac{1}{\pi}\left[-\left(0 - \frac{\pi^2}{2}\right)\right] \Rightarrow \frac{1}{\pi}\left(\frac{\pi^2}{2}\right) \Rightarrow \frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-x)\cos nx\,dx + \int_{0}^{\pi} 0\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -x\cos nx\,dx + 0\right\}$$

$$-\frac{1}{\pi}\int_{-\pi}^{0} x\cos nx\,dx$$

$$= -\frac{1}{\pi}\left\{\left[x\frac{\sin nx}{n}\right]_{-\pi}^{0} - \frac{1}{n}\int_{-\pi}^{0}(\sin nx)\,dx\right\}$$

$$= -\frac{1}{\pi}\left\{0-0 - \frac{1}{n}\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0}\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{1}{n^2}\left[\cos 0 - \cos n(-\pi)\right]\right\}$$

$$= -\frac{1}{\pi}\left\{\frac{1}{n^2}\left[1-\cos n\pi\right]\right\}$$

$$= \frac{-1}{\pi n^2}\left[1-\cos n\pi\right]$$

now if n is even , $\cos n\pi = 1$

$$a_n = \frac{-1}{\pi n^2}(1-1) = 0$$

if n is odd , $\cos n\pi = -1$

$$a_n = \frac{-1}{\pi n^2}\left[1-(-1)\right] \Rightarrow \frac{-2}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-x)\sin nx\,dx + \int_{0}^{\pi} 0\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -x \sin nx \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[-x \frac{\cos nx}{n}\right]_{-\pi}^{0} + \frac{1}{n}\int_{-\pi}^{0} \cos nx \, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left(\frac{\pi \cos n\pi}{n}\right) + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}\right\}$$

$$= \frac{1}{\pi}\left(\pi \frac{\cos n\pi}{n}\right) \Rightarrow \frac{\cos n\pi}{n}$$

if n is even

$$b_n = \frac{1}{n}$$

if n is odd

$$b_n = -\frac{1}{n}$$

So we have $a_0 = \frac{\pi}{2}$

$a_n = 0$ (n even), $a_n = -\frac{2}{\pi n^2}$ (n odd)

$b_n = \frac{1}{n}$ (n even), $b_n = -\frac{1}{n}$ (n odd)

As $f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty}\left\{a_n \cos nx + b_n \sin nx\right\}$

$$= \frac{1}{2} \cdot \frac{\pi}{2} + \left\{\left(1 \cdot \cos x + 0 + \left(\frac{-2}{\pi 3^2}\right)\cos 3x + 0 + \left(\frac{-2}{\pi(5)^2}\cos 5x\right) + \cdots\right) + \left(\frac{-1}{1}\sin x + \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x \cdots\right)\right\}$$

$$f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right) +$$

$$\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots\right)$$

Odd & Even functions:-

Even functions:-

A function is said to be even if $f(x) = f(-x)$. Also even function is symmetrical about y-axis.

$$y = f(x) = x^2$$

put $x = -x$

$$f(-x) = (-x)^2$$

$$f(-x) = x^2 = f(x)$$

So it is even function.

put $x = 2$ & $x = -2$

$$f(x) = f(2) = x^2 = (2)^2 = 4$$

$$f(-2) = (-2)^2 = 4$$

$$y = \cos x$$

$$f(x) = \cos x$$

$$f(-x) = \cos(-x) = \cos(x)$$

$$f(-x) = f(x)$$

So it is an even function.

Odd function:-

A function $f(x)$ is said to be odd if $f(-x) = -f(x)$. its graph is symmetrical about origin.

$$y = f(x) = x^3$$

put $x = -x$

$$f(-x) = (-x)^3 = -x^3$$

$$f(-x) = -f(x)$$

So it is odd function

put $x = 2$ & -2

$$f(2) = (2)^3 = 8$$

$$f(-2) = (-2)^3 = -8 = -f(2)$$

So $\boxed{f(-2) = -f(2)}$

$$y = f(x) = \sin x$$

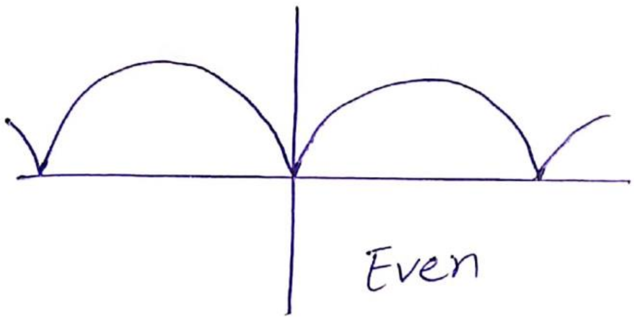
put $x = -x$

$$f(-x) = \sin(-x) = -\sin x$$

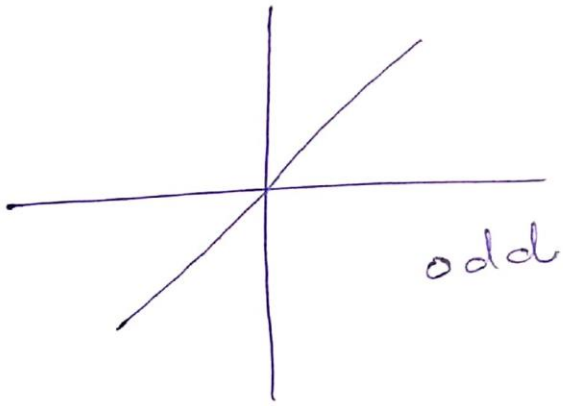
$$f(-x) = -f(x)$$

So it is odd function.

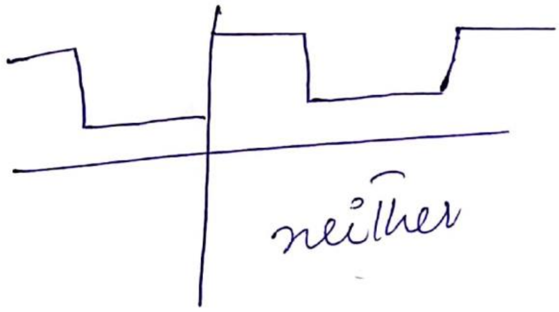
If a function is not symmetric about y-axis & not-symmetric about origin Then it is neither even nor odd.



Even



odd



neither

Products of odd & Even functions:-

even x even = even like $(+)(+)=(+)$

odd x odd = even , $(-)(-)=(+)$

odd x even = odd , $(-)(+)=(-)$

even x odd = odd , $(+)(-)=(-)$

Two even functions:-

$$F(x) = f(x)\,g(x)$$

$$F(-x) = f(-x)\,g(-x)$$

$\because f(-x) = f(x)$
$g(-x) = g(x)$

$$F(-x) = (f(x))(g(x))$$

$$F(-x) = F(x)$$

Two odd functions:

$$F(x) = f(x)\,g(x)$$

$$F(-x) = f(-x)\,g(-x)$$

$\because f(-x) = -f(x)$
$g(-x) = -g(x)$

$$= [-f(x)]\,[-g(x)]$$

$$F(-x) = f(x)\,g(x)$$

$$F(-x) = F(x)$$

One odd & one even func:-

$$F(x) = f(x)\,g(x)$$

when $f(x)$ is odd &
$g(x)$ is even

$$F(-x) = f(-x)\,g(-x)$$

$$= [-f(x)]\,g(x)$$

$$= -\,f(x)\,g(x)$$

$$F(-x) = -\,F(x)$$

Examples:

1. $x^2 \sin 2x$ odd $(E)(O) = O$
2. $x^3 \cos x$ odd $(O)(E) = O$

3. $\cos 2x \cos 3x$ even $(E)(E)=E$

4. $x \sin nx$ even $(O)(O)=E$

5. $3 \sin x \cos 4x$ odd $(O)(E)=O$

6. $(2x+3) \sin 4x$ neither $(N)(O)=N$

7. $\sin^2 x \cos 3x$ Even $(E)(E)=E$

8. $x^3 e^x$ Neither $(O)(N)=N$

9. $(x^4+4) \sin 2x$ Odd $(E)(O)=O$

Two useful facts from odd & even func:-

For an even function

$$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

For an odd function

$$\int_{-a}^{a} f(x)\,dx = 0$$

Theorem:- If $f(x)$ is defined over the interval $-\pi < x < \pi$ & $f(x)$ is even, then fourier series for $f(x)$ contains cosine terms only.

Proof:-

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow \frac{1}{\pi}\,2\int_{0}^{\pi} f(x)\,dx \quad \left(\text{bcz } f(x) \text{ is even}\right)$$

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \quad \left(\text{product of even func. is even}\right)$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx\,dx$$

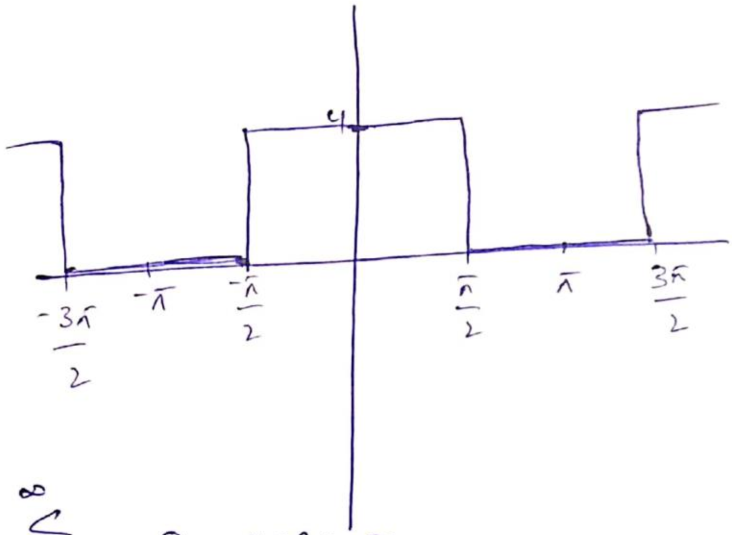
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \quad \left(\text{Product of even \& odd func is odd.}\right)$$

$$= \frac{1}{\pi}(0)$$

$$b_n = 0$$

So proved That There will not be any sine term in fourier series.

Example:- The waveform shown is symmetric about y-axis. The func. is therefore even & there'll be no sine terms in series.



$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left\{\int_{0}^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_{0}^{\pi/2} 4\,dx + 0\right\}$$

$$= \frac{2}{\pi}\left\{4\left[x\right]_0^{\pi/2}\right\} \Rightarrow \frac{2}{\pi}\left(4\frac{\pi}{2}\right) \Rightarrow \boxed{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi} f(x)\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} 4\cos nx\,dx + \int_{\pi/2}^{\pi} 0\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{4\left[\frac{\sin nx}{n}\right]_0^{\pi/2} + 0\right\}$$

$$= \frac{2}{\pi}\cdot\frac{4}{n}\cdot\left(\sin n\cdot\frac{\pi}{2} - \sin n(0)\right]$$

$$\boxed{a_n = \frac{8}{n\pi}\sin\frac{n\pi}{2}}$$

But $\sin\frac{n\pi}{2} = 0$ (for even values of n)

$= 1$ (for $n = 1, 5, 9, \ldots$

$= -1$ (for $n = 3, 7, 11, \ldots$

$\boxed{a_n = \frac{8}{n\pi}}$ for $n = 1, 5, 9$

$\boxed{a_n = -\frac{8}{n\pi}}$ for $n = 3, 7, 11$

$$f(x) = \frac{1}{2}(4) + \left[\frac{8}{\pi}\cos x + 0 - \frac{8}{3\pi}\cos 3x + 0 + \frac{8}{5\pi}\cos 5x + \cdots\right]$$

$$= 2 + \frac{8}{\pi}\left[\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - \frac{\cos 7x}{7} + \frac{\cos 9x}{9} + \cdots\right]$$

Theorem 2:-

If $f(x)$ is an odd function over $[-\pi,\pi]$ Then Fourier Series for $f(x)$ contains sine terms only.

Since $f(x)$ is odd, so

$$\int_{-\pi}^{\pi} f(x)\,dx = 0$$

$$\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx = 0$$

$$\int_{-\pi}^{0} f(x)\,dx = -\int_{0}^{\pi} f(x)\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \Rightarrow 0 \qquad \left(\begin{matrix} f(x) \text{ is} \\ \text{odd} \end{matrix}\right)$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \qquad \left([O][E]=[O]\right.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad \left((O)(O)=(E)\right)$$

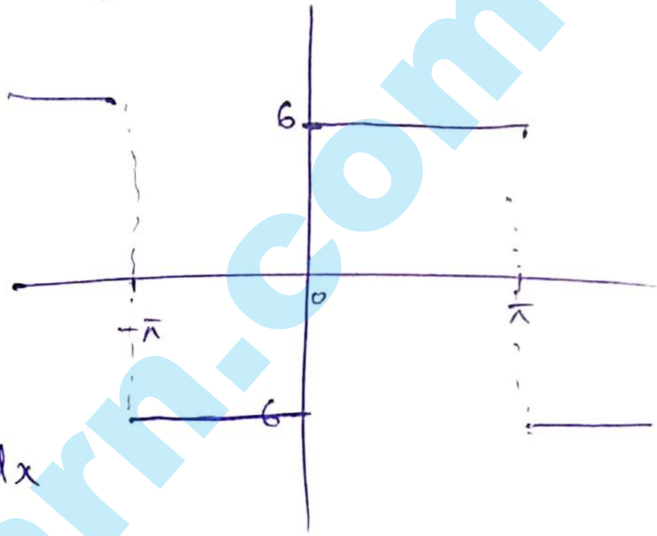
$$= \frac{1}{\pi}\cdot 2\int_{0}^{\pi} f(x)\sin nx\,dx$$

So fourier series contain sine terms only.

Example:- $f(x) = -6 \qquad -\pi < x < 0$

$f(x) = 6 \qquad 0 < x < \pi$

$f(x) = f(x+2\pi)$



function is odd, so

$a_0 = 0, \quad a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} 6 \sin nx \, dx \Rightarrow \frac{2}{\pi} \cdot 6 \left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{12}{n\pi}\left[-\cos n\pi - (-\cos 0)\right]$$

$$= \frac{12}{n\pi}\left[-\cos n\pi + 1\right] \Rightarrow \frac{12}{n\pi}(1 - \cos n\pi)$$

if n is even

$$b_n = \frac{12}{n\pi}(1 - 1) \Rightarrow 0$$

if n is odd

$$b_n = \frac{12}{n\pi}(1 - (-1)) \Rightarrow \frac{24}{n\pi}$$

Fourier Series will be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

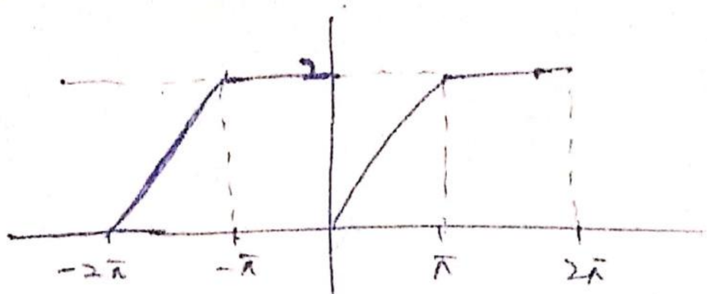
$$= \frac{24}{\pi} \sin x + 0 + \frac{24}{3\pi} \sin 3x + 0 + \frac{24}{5\pi} \sin 5x + \cdots$$

$$= \frac{24}{\pi} \left[\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right].$$

If $f(x)$ is neither even nor odd, we'll must find expressions for a_0, a_n, b_n in full.

Examples: Lecture 42.

Example:- Determine the Fourier Series for the function shown.



This is neither even nor odd.

Therefore we'll find a_0, a_n, b_n.

As Fourier Series are

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

$$m = \frac{2-0}{\pi-0} = \frac{2}{\pi}$$

$$y - y_0 = m(x - x_0)$$

$$y - 0 = \frac{2}{\pi}(x - 0)$$

$$y = \frac{2}{\pi}x$$

$$f(x) = \frac{2}{\pi}x \quad \text{when} \quad 0 < x < \pi$$

$$f(x) = 2 \quad \text{when} \quad \pi < x < 2\pi$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\,dx + \int_{\pi}^{2\pi} f(x)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} \frac{2}{\pi}x\,dx + \int_{\pi}^{2\pi} 2\,dx\right.$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi} \frac{1}{\pi}x\,dx + \int_{\pi}^{2\pi} dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\cdot\left[\frac{x^2}{2}\right]_0^{\pi} + \left[x\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\left(\frac{\pi^2}{2}\right) + \left(2\pi - \pi\right)\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2} + \pi\right\} \Rightarrow \frac{2}{\pi}\left\{\frac{\pi + 2\pi}{2}\right\} \Rightarrow \frac{2}{\pi}\left\{\frac{3\pi}{2}\right\}$$

$$\boxed{a_0 = 3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \Rightarrow \frac{1}{\pi}\left[\int_0^{\pi} f(x)\cos nx\,dx + \int_{\pi}^{2\pi} f(x)\cos nx\,dx\right.$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} \frac{2}{\pi}x\cos nx\,dx + \int_{\pi}^{2\pi} 2\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\left(\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \cancel{\frac{1}{\pi}}\int_0^{2\pi} \frac{\sin nx}{n}\,dx\right) + \left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n\pi}\left((0) - \left(-\frac{\cos nx}{n}\right)\right)_0^{\pi} + 0\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n\pi}\left[\left(\frac{\cos nx}{n}\right)\right]_0^{\pi}\right.$$

$$= \frac{2\left[\cos nx\right]_0^{\pi}}{n^2\pi^2} \Rightarrow \frac{2}{n^2\pi^2}\left(\cos n\pi - \cos 0^\circ\right)$$

$$= \frac{2}{n^2\pi^2}\left(\cos n\pi - 1\right)$$

$= 0$ if n is even

$= \frac{-4}{\pi^2 n^2}$ if n is odd.

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi}\left(\frac{2}{\pi}x\sin nx\right)dx + \int_{\pi}^{2\pi} 2\sin nx\,dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\left(\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \int_{\pi}\left(-\frac{\cos nx}{n}\right)dx\right) + \cancel{2}\left[\frac{-\cos nx}{n}\right]_{\pi}^{2\pi}\right.$$

$$= \frac{2}{\pi}\left\{\frac{1}{\cancel{n}\pi}\left[\left(\frac{-\pi\cos n\pi}{n}\right) + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_{\pi}^{2\pi}\right] \cancel{q}\left(\frac{\cos 2\pi n - \cos n\pi}{n}\right)\right.$$

$$= \frac{2}{\pi}\left\{-\frac{\cancel{\cos n\pi}}{n} - \frac{\cos 2\pi n}{n} + \frac{\cancel{\cos n\pi}}{n}\right\}$$

$$= \frac{2}{\pi}\left(-\frac{\cos 2\pi n}{n}\right)$$

$$= \frac{-2}{n\pi} \cos 2\pi n$$

$$= \frac{-2}{n\pi}(1)$$

$$\boxed{b_n = \frac{-2}{n\pi}}$$

So fourier series is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\{a_n \cos nx + b_n \sin nx\}$$

$$= \frac{1}{2}(3) + \left\{\frac{-4}{\pi^2}\cos x + 0 - \frac{4}{9\pi^2}\cos 3x + 0 \mp \frac{4}{25\pi^2}\cos 5x + \cdots\right\}$$

$$+ \left\{-\frac{2}{\pi}\sin x - \frac{2}{2\pi}\sin 2x - \frac{2}{3\pi}\sin 3x - \frac{2}{4\pi}\sin 4x \cdots\right\}$$

$$= \frac{3}{2} - \frac{4}{\pi^2}\left\{\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right\} +$$

$$- \frac{2}{\pi}\left\{\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \frac{1}{4}\sin 4x + \cdots\right\}$$

Sum of a Fourier series at a point of discontinuity:-

At $x = x_1$ series converges to $f(x_1)$

A particular point of interest occurs at a point of finite discontinuity or jump of $f(x)$.

At $x = x_1$ the function appears to have two distinct values y_1 & y_2.

Fourier series that results is now converging to the average of y_1 & y_2

(i-e) $= \frac{1}{2}(y_1 + y_2)$.

4

Example:-

Consider the function

$$f(x) = 0 \qquad -\pi < x < 0$$

$$f(x) = a \qquad 0 < x < \pi$$

$$f(x) = f(x + 2\pi)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} a\,dx\right\}$$

$$= \frac{1}{\pi}\left\{ a\,[x]_0^{\pi} \right\} \Rightarrow \frac{1}{\pi}\,a(\pi - 0) = \frac{1}{\pi}\,a(\pi)$$

$$\boxed{a_0 = a}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\cos nx\,dx + \int_{0}^{\pi} a\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{ 0 + a\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$$

$$\Rightarrow \frac{1}{\pi}\{a \cdot 0\} \Rightarrow \boxed{0 = a_n}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0 \sin nx\,dx + \int_{0}^{\pi} a \sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{0 + a\left[\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{a}{n}\left[-\left(\cos n\pi - \cos 0^\circ\right)\right]\right\}$$

$$= \frac{a}{n\pi}\left[-\left(\cos n\pi - 1\right)\right] \Rightarrow \frac{a}{n\pi}\left[\left(1 - \cos n\pi\right)\right]$$

$\boxed{b_n = 0}$ if n is even.

$\boxed{b_n = \frac{2a}{n\pi}}$ if n is odd.

So $f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$

$$= \frac{1}{2}(a) + 0 + \frac{2a}{\pi}\sin x + 0 + \frac{2a}{3\pi}\sin 3x + 0 + \frac{2a}{5\pi}\sin 5x + \cdots$$

$$= \frac{a}{2} + \frac{2a}{\pi}\left\{\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \cdots\right\}$$

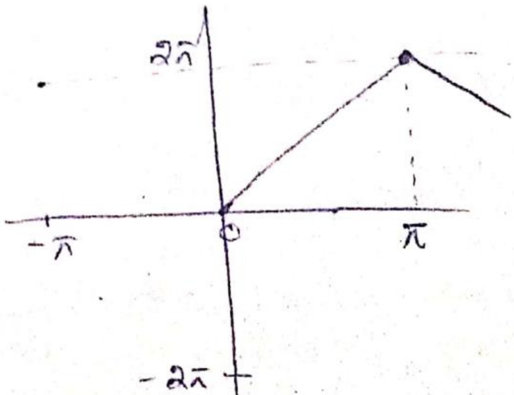
A finite discontinuity or jump occurs at $x=0$. If we substitute $x=0$, all sine terms vanishes. & we get $f(x)=\frac{a}{2}$ which is average of two func. values at $x=0$.

$y=0$ & $y=a$

$$= \frac{1}{2}(0+a) \Rightarrow \frac{a}{2}.$$

Half Range Series:-

Sometimes a ~~series~~ function of period 2π is defined over range 0 to π. instead of $-\pi$ to π or 0 to 2π.



if it is even function it should be symmetrical about y-axis having only cosine values.

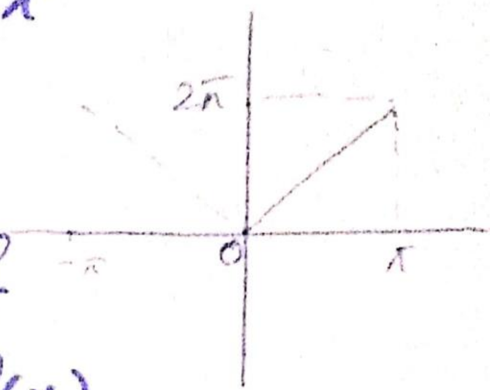
If it is odd func. it should be symmetric about origin. having only sine terms.

Example:-

A function $f(x)$ is defined by

$f(x) = 2x \quad 0 < x < \pi$

$f(x) = f(x+2\pi)$



Obtain a cosine series of half Range to represent $f(x)$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} 2x\,dx \Rightarrow \frac{2}{\pi}\, 2\left[\left(\frac{x^2}{2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 - 0^2\right] = \frac{2}{\pi}(\pi^2) \Rightarrow \boxed{2\pi = a_0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \Rightarrow \frac{2}{\pi}\int_0^{\pi} 2x\cos nx\,dx$$

$$= \frac{4}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right\}$$

$$= \frac{4}{\pi}\left\{\frac{\pi\sin n\pi}{n} - 0 - \frac{1}{n}\left[-\frac{\cos nx}{n}\right]_0^{\pi}\right\}$$

$$= \frac{4}{\pi}\left(0 - 0 - \frac{1}{n^2}\left[-(\cos n\pi - \cos 0^\circ)\right]\right)$$

$$= \frac{4}{\pi}\left(-\frac{1}{n^2}(1 - \cos n\pi)\right)$$

$$= \frac{-4}{n^2\pi}(1 - \cos n\pi)$$

$$= \frac{4}{n^2\pi}(\cos n\pi - 1)$$

$\boxed{a_n = 0}$ if n is even

$\boxed{a_n = -\frac{8}{n^2\pi}}$ if n is odd.

$\boxed{b_n = 0}$ bcz $f(x)$ is even & contain only cosine terms.

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{1}{2}(2\pi) + -\frac{8}{\pi}\cos x + 0 - \frac{8}{9\pi}\cos 3x + 0 - \frac{8}{25\pi}\cos 5x + \cdots$$

$$= \pi - \frac{8}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \cdots\right)$$

Example:- Determine a half range sine series to represent the function $f(x)$ defined by $f(x) = 1+x \quad 0 < x < \pi$

$$f(x) = f(x+2\pi)$$

it is odd function, so $a_0 = 0$ & $a_n = 0$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} (1+x)\sin nx \, dx$$

$$= \frac{2}{\pi}\left\{\left[(1+x)\left[\frac{-\cos nx}{n}\right]\right]_0^{\pi} - \int_0^{\pi} \frac{-\cos nx}{n} \, dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{(1+\pi)\cos n\pi}{n} + \frac{+1}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{(1+\pi)\cos n\pi}{n} + \frac{1}{n} + 0\right\}$$

$$= \frac{2}{n\pi}\left(1 - (1+\pi)\cos n\pi\right)$$

if n is even

$$= \frac{2}{n\pi}\left(1-(1+\pi)(1)\right) \Rightarrow \frac{2}{n\pi}\left(1-1+\pi\right)$$

$$= \frac{2}{n\pi}(\pi) \Rightarrow \boxed{\frac{2}{n}}$$

if n is odd

$$= \frac{2}{n\pi}\left(1-(1+\pi)(-1)\right) \Rightarrow \frac{2}{n\pi}\left(1+1+\pi\right)$$

$$= \boxed{\frac{4+2\pi}{n\pi}}$$

putting values in

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$= 0 + 0 + \frac{4+2\pi}{\pi}\sin x + \frac{2}{2}\sin 2x + \frac{4+2\pi}{3\pi}\sin 3x$$

$$+ \frac{2}{4}\sin 4x + \cdots$$

$$f(x) = \frac{4+2\pi}{\pi}\sin x + \sin 2x + \frac{4+2\pi}{3\pi}\sin 3x +$$

$$\frac{1}{2}\sin 4x + \cdots$$

Lecture 43

Function with Periods Over Than 2π

If $y = f(x)$ is defined in the range $-\frac{T}{2}$ to $\frac{T}{2}$ i.e has a period T, we can convert this to an interval of 2π by changing units of independent variable. usually independent variable is time(t) in physical oscillations. Periodic interval is usually T.

$$f(t) = f(t+T)$$

Each cycle is completed in T seconds & frequency is $f = \frac{1}{T}$. If angular velocity, ω radians per second, defined by $\omega = 2\pi f$, then $\omega = \frac{2\pi}{T}$ & $T = \frac{2\pi}{\omega}$

also $x = \omega t$ so fourier series will be

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

can also be written as

$$f(t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} B_n(\sin(n\omega t + \varphi_n))$$

$n = 1, 2, 3, \ldots\ldots$

Fourier Co-efficients:-

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

where

$$a_0 = \frac{2}{T}\int_0^T f(t)\,dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos n\omega t\,dt = \frac{\omega}{\pi}\int_0^{\frac{2\pi}{\omega}} f(t)\cos n\omega t\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin n\omega t\,dt = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\,dt$$

limits can be 0 to T, $-\frac{T}{2}$ to $\frac{T}{2}$, $-\frac{\pi}{\omega}$ to $\frac{\pi}{\omega}$, 0 to $\frac{2\pi}{\omega}$ etc.

Example:- Determine the fourier series for a periodic function defined by

$$f(t) = 2(1+t) \qquad -1 < t < 0$$

$$f(t) = 0 \qquad 0 < t < 1$$

$$f(t) = f(t+2)$$

As
$$f(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \{ a_n \cos n\omega t + b_n \sin n\omega t \}$$

$$a_0 = \frac{2}{T} \int_{-1}^{1} f(t)\, dt \Rightarrow \frac{2}{2} \int_{-1}^{1} f(t) dt$$

$$= \int_{-1}^{0} 2(1+t) dt + \int_{0}^{1} 0 \; dt$$

$$= 2\left[t + \frac{t^2}{2} \right]_{-1}^{0} + 0 \Rightarrow 2\left[\left(0 + \frac{0^2}{2}\right) - \left(-1 + \frac{(-1)^2}{2}\right) \right]$$

$$= 2\left(- \left(-1 + \frac{1}{2} \right) \right) \Rightarrow \cancel{2[-(}$$

$$= 2\left(1 - \frac{1}{2}\right) = 2\left(\frac{1}{2}\right) \Rightarrow \boxed{1}$$

$$a_n = \frac{2}{T} \int_{-1}^{0} f(t) \cos n\omega t \; dt$$

$$= \frac{2}{2}\int_{-1}^{0} 2(1+t)\cos n\omega t\,dt + \int_{0}^{1} 0 \cos n\omega t\ dt$$

$$= 2\left[\left[(1+t)\ \frac{\sin n\omega t}{n\omega}\right]_{-1}^{0} - \int_{-1}^{0} \frac{\sin n\omega t}{n\omega}\ dt\right] + 0$$

$$= 2\left[0 - 0 - \frac{1}{n\omega}\left[\frac{-\cos n\omega t}{n\omega}\right]_{-1}^{0}\right]$$

$$= 2\left[\frac{1}{n^2\omega^2}\left[1 - \cos n\omega\right]\right]$$

$$a_n = \frac{2}{n^2\omega^2}\left(1 - \cos n\omega\right)$$

as $\omega = \frac{2\pi}{T} \Rightarrow \frac{2\pi}{2} \Rightarrow \pi$

So $a_n = \frac{2}{n^2\omega^2}\left(1 - \cos n\pi\right)$

$a_n = 0$ (n even)

$a_n = \frac{4}{n^2\omega^2}$ (n odd)

$$b_n = \frac{2}{T}\int_{-1}^{1} f(x) \sin n\omega t \, dt$$

$$= \frac{2}{2}\left[\int_{-1}^{0} 2(1+t) \sin n\omega t \, dt + \int_{0}^{1} 0 \sin n\omega t \, dt\right]$$

$$= 2\left\{\left[(1+t)\left(-\frac{\cos n\omega t}{n\omega}\right)\right]_{-1}^{0} - \int_{-1}^{0} \frac{-\cos n\omega t}{n\omega} dt\right\}$$

$$= 2\left\{\frac{-1}{n\omega} + \frac{1}{n\omega}\left[\left(\frac{\sin n\omega t}{n\omega}\right)\right]_{-1}^{0}\right\}$$

$$= 2\left\{\frac{-1}{n\omega} + \frac{1}{n^2\omega^2}\left(0 - (-\sin n\omega)\right)\right\}$$

$$= 2\left(-\frac{1}{n\omega} + \frac{1}{n^2\omega^2} \sin n\omega\right)$$

As we derived for a_n, $\omega = \pi$

$$= 2\left(-\frac{1}{n\omega}\right) \Rightarrow \boxed{\frac{-2}{n\omega} = b_n}$$

So fourier series'll be

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{1}{2}(1) + \left\{\frac{4}{\omega^2}\cos\omega t + 0 + \frac{4}{9\omega^2}\cos 3\omega t + 0 + \frac{4}{25\omega^2}\cos 5\omega t + \cdots\right\} + \left\{\frac{-2}{\omega}\sin\omega t - \frac{2}{2\omega}\sin 2\omega t - \frac{2}{3\omega}\sin 3\omega t - \frac{2}{4\omega}\sin 4\omega t + \cdots\right\}$$

$$= \frac{1}{2} + \frac{4}{\omega^2}\left\{\cos\omega t + \frac{1}{9}\cos 3\omega t + \frac{1}{25}\cos 5\omega t + \cdots\right\} + -\frac{2}{\omega}\left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \frac{1}{4}\sin 4\omega t + \cdots\right\}$$

Half Range Series:-

Even function:- Half Range cosine series

$y = f(t)$, $0 < t < \frac{T}{2}$

$y = f(t+T)$. Symmetrical about y-axis

In even func. $b_n = 0$ So

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega t$$

where

$$a_0 = \frac{4}{T}\int_0^{T/2} f(t)\,dt$$

,

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt$$

Odd functions:-

Half Range sine series

$y = f(t)$, $0 < t < \frac{T}{2}$

$f(t) = f(t+T)$, symmetrical about origin

$$a_0 = 0,\quad a_n = 0,\quad b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\,dt$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t.$$

Example:- A function $f(t)$ is defined by $f(t) = 4 - t$ $\quad 0 < t < 4$

Form a half range cosine series to represent the function.

$$a_0 = \frac{4}{T}\int_0^{4} f(t)\,dt \Rightarrow \frac{4}{8}\int_0^{4}(4-t)\,dt$$

$$a_0 = \frac{1}{2}\left[4t - \frac{t^2}{2}\right]_0^4 \Rightarrow \frac{1}{2}\left[\left(4(4) - \frac{(4)^2}{2}\right) - (0)\right]$$

$$= \frac{1}{2}\left(16 - \frac{16}{2}\right) \Rightarrow \frac{1}{2}(16-8) \Rightarrow \frac{8}{2} \Rightarrow \boxed{4 = a_0}$$

$$a_n = \frac{4}{T}\int_0^4 f(t)\cos n\omega t\,dt$$

$$= \frac{4}{8}\int_0^4 (4-t)\cos n\omega t\,dt$$

$$= \frac{1}{2}\left[(4-t)\frac{\sin n\omega t}{n\omega}\right]_0^4 - \int_0^4 \frac{\sin n\omega t}{n\omega}(-1)\,dt$$

$$= \frac{1}{2}\left[0 - 0 + \frac{1}{n\omega}\left[\frac{-\cos n\omega t}{n\omega}\right]_0^4\right.$$

$$= \frac{1}{2}\left(-\frac{1}{n^2\omega^2}(\cos 4n\omega - 1)\right)$$

$$= -\frac{1}{2n^2\omega^2}(\cos 4n\omega - 1) \qquad \boxed{\text{As } \omega = \frac{2\pi}{T} \Rightarrow \frac{2\pi}{8} \Rightarrow \frac{\pi}{4}}$$

$$= -\frac{1}{2n^2\omega^2}\left(\cos 4n\left(\frac{\pi}{4}\right) - 1\right) \Rightarrow \frac{-1}{2n^2\omega^2}(\cos n\pi - 1)$$

$$\underline{a_n = 0 \quad (n \text{ even})}$$

$$\underline{a_n = \frac{1}{n^2\omega^2} \quad (n \text{ odd})}$$

$$f(t) = \frac{1}{2}(4) + \frac{1}{w^2}\cos wt + 0 + \frac{1}{9w^2}\cos 3wt + 0 + \frac{1}{25w^2}\cos 5wt + \dots$$

$$= 2 + \frac{1}{w^2}\left(\cos wt + \frac{1}{9}\cos 3wt + \frac{1}{25}\cos 5wt + \dots\right)$$

Example:- A function $f(t)$ is defined by

$f(t) = 3+t \qquad 0<t<2$

$f(t) = f(t+4)$

obtain half range sine series.

function is odd so $a_n = 0$, $a_0 = 0$

$$b_n = \frac{4}{T}\int_0^2 f(t)\sin nwt\,dt$$

$$= \frac{4}{4}\int_0^2 (3+t)\sin nwt\,dt$$

$$= \left[(3+t)\frac{-\cos nwt}{nw}\right]_0^2 - \int_0^2 \frac{-\cos nwt}{nw}\,dt$$

$$= -\frac{5\cos 2nw}{nw} + \frac{3}{nw} + \frac{1}{nw}\left[\frac{\sin nwt}{nw}\right]_0^2$$

$$= -\frac{5\cos 2nw}{nw} + \frac{3}{nw} + \frac{1}{n^2w^2}\sin 2nw$$

$$= \frac{1}{n\omega}\left[3 - 5\cos 2n\omega\right] + \frac{1}{n^2\omega^2}\sin 2n\omega$$

But $\omega = \frac{2\pi}{T} = \frac{2\pi}{4} \Rightarrow \frac{\pi}{2}$

$$= \frac{1}{n\omega}\left[3 - 5\cos n\pi\right] + \frac{1}{n^2\omega^2}\sin n\pi$$

$$b_n = \frac{-2}{n\omega} \quad (n \text{ even})$$

$$b_n = \frac{8}{n\omega} \quad (n \text{ odd})$$

$$f(t) = \sum_{n=1}^{\infty} b_n \sin n\omega t$$

$$= \frac{8}{\omega}\sin\omega t + \left(\frac{-2}{2\omega}\right)\sin 2\omega t + \frac{8}{3\omega}\sin 3\omega t +$$

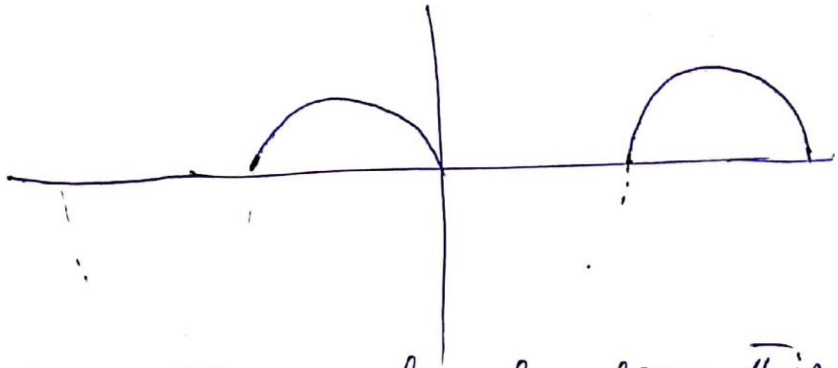
$$\left(\frac{-2}{4\omega}\right)\sin 4\omega t + \frac{8}{5\omega}\sin 5\omega t + \cdots$$

$$= \frac{2}{\omega}\left(4\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{4}{3}\sin 3\omega t +\right.$$

$$\left.\frac{1}{4}\sin 4\omega t + \frac{4}{5}\sin 5\omega t + \cdots\right.$$

Half Wave Rectifier:-

Half wave rectifier clips the -ve portion of wave.



Fourier series will be found like this.

$u(t) = 0 \qquad -\frac{T}{2} < t < 0$

$u(t) = E\sin\omega t \qquad 0 < t < \frac{T}{2}$

here $T = \frac{2\pi}{\omega}$

$$a_0 = \frac{2}{T}\int_{-T/2}^{T/2} u(t)\,dt$$

$$= \frac{2}{T}\left\{\int_{-T/2}^{0} 0\,dt + \int_{0}^{T/2} E\sin\omega t\,dt\right\}$$

$$= \frac{2}{2\pi/\omega}\left\{0 + \int_{0}^{\frac{2\pi}{\omega 2}} E\sin\omega t\,dt\right\}$$

$$= \frac{\omega}{\pi}\int_{0}^{\pi/\omega} E\sin\omega t\,dt \Rightarrow \frac{\omega}{\pi}E\left|\frac{-\cos\omega t}{\omega}\right|_{0}^{\pi/\omega}$$

$$= -\frac{\omega}{\pi}\frac{E}{\omega}\left(\cos\omega\left(\frac{\pi}{\omega}\right) - \cos 0\right)$$

$$= \frac{-E}{\pi}\left(-1-1\right) \Rightarrow -\frac{E}{\pi}(-2)$$

$$\boxed{a_0 = \frac{2E}{\pi}}$$

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} u(t)\cos n\omega t\,dt$$

$$= \frac{2}{2\pi/\omega}\int_{-T/2}^{0} 0\cos n\omega t\,dt + \int_{0}^{T/2} E\sin\omega t\cos n\omega t\,dt$$

$$= \frac{\omega}{2\pi}E\int_{0}^{\pi/\omega} 2\sin\omega t\cos n\omega t\,dt$$

$$= \frac{\omega}{2\pi}E\int_{0}^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt$$ for $n=1$ $a_n = 0$

if $n \neq 1$

$$= \frac{\omega E}{2\pi}\left[\frac{-\cos(1+n)\omega t}{(1+n)\omega} + \frac{-\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$= +\frac{\omega E}{2\pi}\left[\frac{-\cos(1+n)\pi + 1}{(1+n)\omega} + \frac{-\cos(1-n)\pi + 1}{(1-n)\omega}\right] -$$

If n is odd

$$\boxed{a_n = 0}$$

If n is even

$$a_n = \frac{E}{2\pi}\left(\frac{2}{1+n} + \frac{2}{1-n}\right)$$

$$= \frac{E}{2\pi}\left(\frac{2-2n+2+2n}{(1+n)(1-n)}\right) \Rightarrow \frac{E}{2\pi}\left(\frac{4}{1^2-n^2}\right)$$

$$a_n = \frac{4E}{2\pi(1-n^2)} \Rightarrow \boxed{\frac{2E}{\pi(1-n^2)} = a_n}$$

For $b_n = \frac{2}{T}\int_{-T/2}^{T/2} u(t)\sin n\omega t\, dt$

$$= \frac{2}{2\pi/\omega}\int_{-T/2}^{0} 0 \sin n\omega t + \int_{0}^{T/2} E \sin\omega t \sin n\omega t\, dt$$

$$= \frac{\omega E}{-2\pi}\int_0^{\pi/\omega} -2\sin\omega t \sin n\omega t\, dt$$

for $n=1$

$$b_n = \frac{-\omega E}{2\pi}\int_0^{\pi/\omega}(\cos 2\omega t - 1)$$

$$= \frac{-\omega E}{2\pi}\left[\frac{\sin 2\omega t}{2\omega} - t\right]_0^{\frac{\pi}{\omega}}$$

$$b_n = \frac{E}{2}$$

$$= -\frac{\omega E}{2\pi}\int_0^{\pi/\omega}\left(\cos(1+n)\omega t - \cos(1-n)\omega t\right)$$

if $n \neq 1$

$$= \frac{-\omega E}{2\pi}\left[\frac{\sin(1+n)\omega t}{(1+n)\omega} - \frac{\sin(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$= -\frac{\omega E}{2\pi}\left[\frac{\sin(1+n)\pi}{(1+n)\omega} - \frac{\sin(1-n)\pi}{(1-n)\omega}\right]$$

$$b_n = 0$$

$$u(t) = \frac{1}{2}a_0 + \sum_{n=2}^{\infty}\left(a_n \cos n\omega t + b_n \cos n\omega t\right)$$

$$= \frac{1}{2}\left(\frac{2E}{\pi}\right) + \left\{0 + \frac{2E}{\pi(-3)}\cos 2\omega t + 0 + \frac{2E}{\pi(-15)}\cos 4\omega t + \cdots\right\} + \frac{E}{2}\sin \omega t$$

$$= \frac{E}{\pi} - \frac{2E}{\pi}\left\{\frac{1}{3}\cos 2\omega t + \frac{1}{15}\cos 4\omega t + \cdots\right\} + \frac{E}{2}\sin \omega t$$

Lecture 44

Laplace Transforms

Laplace Transform of a function F(t) is denoted by $L\{F(t)\}$ & is defined as the integral of $F(t)e^{-st}$ b/w the limits $t=0$ & $t=\infty$

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

s is always assumed to be +ve & $F(t)e^{-st}$ converges to zero at $t \to \infty$. whatever the function F(t) is.

for example for $F(t) = a$

$$L(a) = \int_0^\infty a e^{-st}\,dt \Rightarrow a\int_0^\infty e^{-st}\,dt$$

$$= a\left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{a}{s}\left[e^{-st}\right]_0^\infty \Rightarrow \frac{-a}{s}\left[\frac{1}{e^{st}}\right]_0^\infty$$

$$= \frac{-a}{s}(0-1) \Rightarrow \frac{a}{s} \quad \text{Ans.}$$

Example:- Find Laplace Transform of The form e^{at} That $F(t) = e^{at}$

$$L(e^{at}) = \int_0^\infty e^{at} e^{-st}\, dt$$

$$= \int_0^\infty e^{at-st}\, dt$$

$$= \int_0^\infty e^{-t(s-a)}\, dt \Rightarrow \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= -\frac{1}{s-a}\left[\frac{1}{e^{t(s-a)}}\right]_0^\infty \Rightarrow \frac{-1}{s-a}\left[0-1\right]$$

$$L(e^{at}) = \frac{1}{s-a}$$

So we have two standard Transforms.

$$L\{a\} = \frac{a}{s} \quad , \quad L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{4\} = \frac{4}{s} \quad , \quad L\{e^{4t}\} = \frac{1}{s-4}$$

$$L\{-5\} = \frac{-5}{s} \quad , \quad L\{e^{-5t}\} = \frac{1}{s+5}$$

Laplace Transform is always a function of s.

Show That $L\{t^3\} = \dfrac{3!}{s^4}$

$$L\{t^3\} = \int_0^\infty t^3 e^{-st}\, dt =$$

$$= \left[t^3 \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot 3t^2\, dt$$

$$= 0 - 0 + \frac{1}{s} \int_0^\infty 3t^2 e^{-st} dt$$

$$= \frac{1}{s}\left[\left[3t^2 \frac{e^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot 6t\right]$$

$$= \frac{1}{s}\left[0 - 0 + \frac{6}{s}\int_0^\infty t e^{-st}\, dt\right]$$

$$= \frac{1}{s}\left[\frac{6}{s}\int_0^\infty t\, e^{-st}\, dt\right] \Rightarrow \frac{6}{s^2}\int_0^\infty t e^{-st} dt$$

$$= \frac{6}{s^2}\left\{\left[\frac{te^{-st}}{-s}\right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} \cdot 1\, dt\right\}$$

$$= \frac{6}{s^2}\left\{0 - 0 + \frac{1}{s}\int_0^\infty e^{-st}\, dt\right\}$$

$$= \frac{6}{s^3}\int_0^\infty e^{-st} dt \Rightarrow \frac{6}{s^3}\left\{\left[\frac{e^{-st}}{-s}\right]_0^\infty\right\}$$

$$\Rightarrow \frac{-6}{s^4}\left[e^{-st}\right]_0^\infty \Rightarrow \frac{-6}{s^4}\left[0 - 1\right] \Rightarrow \frac{3.2.1}{s^4}$$

$$= \frac{3!}{s^4}$$

Complex Numbers power of i

$i=\sqrt{-1}$, $i^2=-1$, $i^4=1$

Solve

$i^9 = i^8 \cdot i = (i^4)^2 \cdot i = i$

$i^{20} = (i^4)^5 = (1)^5 = 1$

$i^{30} = (i^2)^{15} = (-1)^{15} = -1$

$i^{15} = i^{14} \cdot i = (i^2)^7 \cdot i = (-1)^7 \cdot i = -i$

Complex numbers:-

$z = 3+5i$ is called a complex number where 3 is real part & 5 is imaginary part.

In general, $a+bi$ is a complex number where a is real & b is imaginary part.

So

Complex no. = real part + (imaginary part)i

Conjugate Complex numbers:-

For a complex no. $a+bi$, the complex no. $a-bi$ is conjugate of $a+bi$. Conjugate complex no. are identical except the sign in the middle of brackets.

$(4+5i)$ & $(4-5i)$ are conjugate Complex no.

$(6+2i)$ & $(2+6i)$ are not conjugate.

$(5-3i)$ & $(-5+3i)$ are not conjugate.

Remember:-

Product of a complex no. & its conjugate is a real no. entirely.

$$(3+4i)(3-4i) \Rightarrow (3)^2-(4i)^2 \Rightarrow 9-16i^2 \Rightarrow 9-16(-1)$$

$$= 9+16 = \underline{25}$$

$$(a+bi)(a-bi) \Rightarrow (a)^2-(bi)^2 \Rightarrow a^2-b^2i^2$$

$$= a^2-b^2(-1) \Rightarrow \underline{a^2+b^2}$$

Euler Formula:-

As we know that series expansion of e^x, $\cos x$ & $\sin x$ are

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\cdots$$

$$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\cdots\cdots$$

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\cdots\cdots$$

$$e^{it} = 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots$$

$$= 1 + it - \frac{t^2}{2!} - \frac{it^3}{3!} + \frac{t^4}{4!} + \frac{it^5}{5!} - \frac{t^6}{6!} - \frac{it^7}{7!} + \cdots$$

$$= \left\{1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right\} + i\left\{t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right\}$$

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots$$

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

So

$$e^{it} = \cos t + i \sin t$$

$$R(e^{it}) = \cos t \quad \& \quad I(e^{it}) = \sin t$$

The Laplace Transformation of $F(t) = \sin at$

$$L(\sin at) = L\{I(e^{iat})\} = \int_0^\infty I\, e^{iat} e^{-st} dt \Rightarrow I \int_0^\infty e^{iat-st} dt$$

$$= I \int_0^\infty e^{-t(s-ia)} dt = I\left[\frac{e^{-t(s-ia)}}{-(s-ia)}\right]_0^\infty$$

$$= I\left\{\frac{-1}{(s-ia)}[0-1]\right\} = I\left\{\frac{1}{(s-ia)}\right\}$$

Rationalizing it

$$= I\left\{ \frac{1}{s-ia} \times \frac{s+ia}{s+ia} \right\} \Rightarrow I\left\{ \frac{s+ia}{s^2 - i^2a^2} \right\}$$

$$= I\left\{ \frac{s+ia}{s^2+a^2} \right\} \Rightarrow I\left[\frac{s}{s^2+a^2} + \frac{ia}{s^2+a^2} \right]$$

$$\boxed{L\{\sin at\} = \frac{a}{s^2+a^2}}$$

Similarly for cos

$$L\{\cos at\} = R\left[\frac{s}{s^2+a^2} + \frac{ia}{s^2+a^2} \right]$$

$$\boxed{L\{\cos at\} = \frac{s}{s^2+a^2}}$$

The Transform of $F(t) = t^n$ where n is +ve

As $L\{t^n\} = \int_0^\infty t^n e^{-st} dt$

$$= \left[\frac{t^n e^{-st}}{-s} \right]_0^\infty - \int_0^\infty \frac{e^{-st}}{-s} . n t^{n-1} dt$$

$$= 0 - 0 + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st} dt$$

$$L\{t^n\} = \frac{n}{s}\int_0^\infty t^{n-1} e^{-st}\,dt$$

If $I_n = \int_0^\infty t^n e^{-st}\,dt$ then $I_{n-1} = \int_0^\infty t^{n-1} e^{-st}\,dt$

So $I_n = \frac{n}{s} I_{n-1}$ (replaced n by $n-1$)

again replacing n by $n-1$

$$I_{n-1} = \frac{n-1}{s} I_{n-2}$$

again replacing n by $n-1$

$$I_{n-2} = \frac{n-2}{s} I_{n-3}$$

So

$$I_n = \int_0^\infty t^n e^{-st}\,dt = \frac{n}{s} I_{n-1} = \frac{n}{s}\cdot\frac{n-1}{s} I_{n-2} = \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdot I_{n-3}$$

So finally

$$I_n = \frac{n}{s}\cdot\frac{n-1}{s}\cdot\frac{n-2}{s}\cdot\frac{n-3}{s}\cdot\frac{n-4}{s}\cdot \ldots\ldots \frac{2}{s}\cdot\frac{1}{s}\cdot I_0$$

But $I_0 = L\{t^0\} = L\{1\} = \int_0^\infty e^{-st}\,dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty \Rightarrow \frac{-1}{s}\left[e^{-st}\right]_0^\infty \Rightarrow \frac{-1}{s}(0-1) \Rightarrow \frac{1}{s}$$

$$I_n = \frac{n}{s} \cdot \frac{n-1}{s} \cdot \frac{n-2}{s} \cdot \frac{n-3}{s} \cdot \frac{n-4}{s} \cdots\cdots \frac{2}{s} \cdot \frac{1}{s} \cdot \frac{1}{s}$$

$$I_n = \frac{n!}{s^n \cdot s} \Rightarrow \frac{n!}{s^{n+1}}$$

$$\boxed{L\{t^n\} = \frac{n!}{s^{n+1}}}$$

(e.g)

$$L\{t\} = \frac{n!}{s^{n+1}} = \frac{1!}{s^{1+1}} \Rightarrow \frac{1}{s^2}$$

$$L\{t^2\} = \frac{2!}{s^{2+1}} = \frac{2!}{s^3} \Rightarrow \frac{2}{s^3}$$

$$L\{t^3\} = \frac{3!}{s^{3+1}} = \frac{3!}{s^4} \Rightarrow \frac{6}{s^4}$$

Laplace Transform of $F(t) = \sinh at$ & $F(t) = \cosh at$

$$\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right) \quad \& \quad \cosh at = \frac{1}{2}\left(e^{at} + e^{-at}\right)$$

$F(t) = \sinh at$

$$L\{\sinh at\} = \int_0^\infty \sinh at \, e^{-st} dt \Rightarrow \int_0^\infty \frac{1}{2}\left(e^{at} - e^{-at}\right) e^{-st} dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{at} \cdot e^{-st} - e^{-at} e^{-st}\right) dt \Rightarrow \frac{1}{2}\int_0^\infty \left(e^{at-st} - e^{-at-st}\right) dt$$

$$= \frac{1}{2}\int_0^\infty \left(e^{-t(s-a)} - e^{-t(s+a)}\right) dt$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^{\infty} - \left[\frac{e^{-t(s+a)}}{-(s+a)}\right]_0^{\infty}\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{-(s-a)}[0-1] - \frac{1}{-(s+a)}[0-1]\right\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] \Rightarrow \frac{1}{2}\left[\frac{s+a-s+a}{(s-a)(s+a)}\right]$$

$$= \frac{1}{2}\left[\frac{2a}{s^2-a^2}\right] \Rightarrow \boxed{\frac{a}{s^2-a^2} = L\{\sin h\, at\}}$$

$F(t) = \cosh at$

$$L\{\cos h\, at\} = \int_0^{\infty} \cos h\, at \; e^{-st}\, dt$$

$$= \int_0^{\infty} \frac{1}{2}\left(e^{at} + e^{-at}\right) e^{-st}\, dt$$

$$= \frac{1}{2}\int_0^{\infty} \left(e^{at}\cdot e^{-st} + e^{-at}\cdot s^{-st}\right) dt$$

$$= \frac{1}{2}\int_0^{\infty} \left(e^{at-st} + e^{-at-st}\right) dt$$

$$= \frac{1}{2}\int_0^{\infty} \left(e^{-t(s-a)} + e^{-t(s+a)}\right) dt$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^{\infty} + \left[\frac{e^{-t(s+a)}}{-(s+a)}\right]_0^{\infty}\right\} dt$$

$$= \frac{1}{2}\left\{ \frac{1}{-(s-a)}\left[0-1\right] + \frac{1}{-(s+a)}(0-1) \right\}$$

$$= \frac{1}{2}\left\{ \frac{1}{s-a} + \frac{1}{s+a} \right\} \Rightarrow \frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)} \right]$$

$$= \frac{1}{2}\left\{ \frac{2s}{s^2-a^2} \right\} \Rightarrow \boxed{\frac{s}{s^2-a^2} = L\{\cosh at\}}$$

Several Standard Results:-

- $L\{a\} = \frac{a}{s}$
- $L\{e^{at}\} = \frac{1}{s-a}$
- $L\{t^n\} = \frac{n!}{s^{n+1}}$
- $L\{\sin at\} = \frac{a}{s^2+a^2}$
- $L\{\cos at\} = \frac{s}{s^2+a^2}$
- $L\{\sinh at\} = \frac{a}{s^2-a^2}$
- $L\{\cosh at\} = \frac{s}{s^2-a^2}$

Example:-

a) $L\{2\sin 3t + \cos 3t\}$

$$= 2L\{\sin 3t\} + L\{\cos 3t\}$$

$$= 2\left[\frac{3}{s^2+3^2}\right] + \left[\frac{s}{s^2+3^2}\right] \Rightarrow \frac{6}{s^2+9} + \frac{s}{s^2+9}$$

$$= \frac{6+s}{s^2+9^2}$$ Ans.

$$L\{4e^{2t}+3\cosh 4t\}$$

$$= 4L\{e^{2t}\}+3L\{\cosh 4t\}$$

$$= 4\cdot\frac{1}{s-2}+3\cdot\frac{s}{s^2-4^2}$$

$$= \frac{4}{s-2}+\frac{3s}{s^2-16}$$

$$= \frac{4(s^2-16)+3s(s-2)}{(s-2)(s^2-16)}$$

$$= \frac{4s^2-64+3s^2-6s}{(s-2)(s^2-16)}$$

$$= \frac{7s^2-6s-64}{(s-2)(s^2-16)}$$

Lecture 45

Theorems

Theorem 1:

The first Shift Theorem:-

The first Shift theorem states That if $L\{F(t)\} = f(s)$ Then $L\{e^{-at} F(t)\} = f(s+a)$.

The Transform $L\{e^{-at} F(t)\}$ is Thus The same as $L\{F(t)\}$ with s everywhere in The result replaced by $(s+a)$.

Example:- $L\{\sin 2t\} = \frac{2}{s^2+4}$, Then

$$L\{e^{-3t}\sin 2t\} = \frac{2}{(s+3)^2+4} \Rightarrow \frac{2}{s^2+9+6s+4}$$

$$= \frac{2}{s^2+6s+13}$$

Example:-

$L\{t^2\} = \frac{2}{s^3}$; $L\{t^2 e^{4t}\}$ is same, s replaced by $(s-4)$, So

$$L\{t^2 e^{4t}\} = \frac{2}{(s-4)^3}$$

Theorem 2:-

Multiplying by t

If $L\{F(t)\} = f(s)$ Then $L\{t\,F(t)\} = -\frac{d}{ds}\{f(s)\}$

Example:- $L\{\sin 2t\} = \frac{2}{s^2+4}$

$$L\{t \sin 2t\} = -\frac{d}{ds}\left[\frac{2}{s^2+4}\right] \Rightarrow -2\frac{d}{ds}(s^2+4)^{-1}$$

$$= -2\left[-1\,(s^2+4)^{-1-1}(2s)\right] \Rightarrow 2\,(s^2+4)^{-2}(2s)$$

$$= \frac{2(2s)}{(s^2+4)^2} \Rightarrow \frac{4s}{(s^2+4)^2}$$

Example:- $L\{\cos 3t\} = \frac{s}{s^2-9}$

$$L\{t \cos 3t\} = -\frac{d}{dx}\left[\frac{s}{s^2-9}\right]$$

$$= -\left[\frac{(s^2-9)(1) - s(2s)}{(s^2-9)^2}\right] \Rightarrow \frac{-(s^2-9-2s^2)}{(s^2-9)^2}$$

$$= \frac{-(-s^2-9)}{(s^2-9)^2} \Rightarrow \frac{s^2+9}{(s^2-9)^2}$$

if

$$L\{t^2 \cos 3t\} = \frac{-d}{ds}\left[\frac{s^2+9}{(s^2-9)^2}\right]$$

$$= -\left[\frac{(s^2-9)^2(2s) - (s^2+9)\,2(s^2-9)(2s)}{(s^2-9)^4}\right]$$

$$= -\left[\frac{(s^2-9)\left[(s^2-9)(2s) - 4s(s^2+9)\right]}{(s^2-9)^4}\right]$$

$$= -\left[\frac{2s^3 - 18s - 4s^3 - 36s}{(s^2-9)^3}\right] \Rightarrow -\left[\frac{-2s^3 - 54s}{(s^2-9)^3}\right]$$

$$= \frac{2s^3 + 54s}{(s^2-9)^3} \Rightarrow \frac{2s(s^2+54)}{(s^2-9)^3}$$

So if

$$L\{F(t)\} = f(s) \quad \text{Then}$$

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n}\{f(s)\}$$

Theorem:-

Dividing by t

If $L\{F(t)\} = f(s)$, Then

$$L\left\{\frac{F(t)}{t}\right\} = \int_s^\infty f(s)\,ds$$

Example:- Determine $L\left\{\frac{\sin at}{t}\right\}$

As $L\{\sin at\} = \frac{a}{s^2+a^2}$

$$L\left\{\frac{\sin at}{t}\right\} = \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$\because \frac{d}{dx}\,Tan^{-1}\left(\frac{x}{a}\right) = \frac{a}{x^2+a^2}$

$$= \left[Tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty \Rightarrow \frac{\pi}{2} - Tan^{-1}\left(\frac{s}{a}\right)$$

=

Example:- Determine $L\left\{\frac{1-\cos 2t}{t}\right\}$

As $L\{1-\cos 2t\} = \frac{1}{s} - \frac{s}{s^2+4}$

$$L\left\{\frac{1-\cos 2t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2+4}\right)ds = \int_s^\infty \left(\frac{1}{s} - \frac{2s}{2(s^2+4)}\right)ds$$

$$= \left[\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \left[\frac{1}{2}\cdot 2\ln s - \frac{1}{2}\ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\ln s^2 - \ln(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left\{\ln \frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \ln \frac{s^2}{s^2+4}\right]$$

$$= -\frac{1}{2}\ln\frac{s^2}{s^2+4} \Rightarrow \ln\left[\frac{s^2}{s^2+4}\right]^{-\frac{1}{2}}$$

$$= \ln\left[\frac{s^2+4}{s^2}\right]^{\frac{1}{2}} \Rightarrow \ln\sqrt{\frac{s^2+4}{s^2}}$$

Standard forms:-

$F(t)$	$L\{f(t)\}=f(s)$
a	$\frac{a}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$

$F(t)$	$L\{f(t)\}=f(s)$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$t^n =$	$\frac{n!}{s^{n+1}}$

Inverse Transforms:-

Here is reverse process, i-e given a laplace Transform & we have to find the function of t to which it belongs. We know that

$$\frac{a}{s^2+a^2} = L\{\sin at\}$$

$$L^{-1}\left[\frac{a}{s^2+a^2}\right] = \sin at$$, here L^{-1} is indicating inverse Transform, not a reciprocal.

$$\Rightarrow L^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

$$\Rightarrow L^{-1}\left\{\frac{s}{s^2+25}\right\} = \cos 5t$$

$$\Rightarrow L^{-1}\left\{\frac{4}{s}\right\} = 4$$

$$\Rightarrow L^{-1}\left\{\frac{12}{s^2-9}\right\} = \underline{4 \sinh 3t}$$

$$\Rightarrow L^{-1}\left\{\frac{3s}{s^2-s-6}\right\} = ?$$

We'll Solve it using partial fraction rules.

Rules of Partial Fractions:.

⇒ Numerator's degree must be lower than denominator, otherwise divide it first.

⇒ Factorize denominator into prime factors.

⇒ A linear factor $(s+a)$ gives partial fraction

⇒ $\frac{A}{s+a}$ is a constant to be determined.

⇒ A repeated factor $(s+a)^2$ gives $\frac{A}{s+a} + \frac{B}{(s+a)^2}$

⇒ Similarly $(s+a)^3$ gives $\frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{C}{(s+a)^3}$

⇒ A quadratic factor (s^2+px+q) gives $\frac{Ps+Q}{s^2+ps+q}$

⇒ Similarly $(s^2+px+q)^2$ gives $\frac{Ps+Q}{s^2+ps+q} + \frac{Rs+T}{s^2+ps+q}$

⇒ So $\frac{s-19}{(s-2)(s-5)}$ has partial fractions $\frac{A}{s-2} + \frac{B}{s-5}$

⇒ ε | $\frac{3s^2-4s+11}{(s+3)(s-2)^2} = \frac{A}{s+3} + \frac{B}{s-2} + \frac{C}{(s-2)^2}$

Example:- $L^{-1}\left\{\dfrac{5s+1}{s^2-s-12}\right\}$

$$\frac{5s+1}{s^2-(4-3)s-12} = \frac{5s+1}{s^2-4s+3s-12} = \frac{5s+1}{s(s-4)+3(s-4)}$$

let

$$\frac{5s+1}{(s-4)(s+3)} = \frac{A}{s-4} + \frac{B}{s+3}$$

multiply b/s by $(s-4)(s+3)$

$$5s+1 = A(s+3) + B(s-4)$$

put $s=4$

$$5(4)+1 = A(4+3)+0$$

$$21 = 7A$$

$$\boxed{3 = A}$$

put $s=-3$

$$5(-3)+1 = A(0) + B(-3-4)$$

$$-15+1 = -7B$$

$$-14 = -7B$$

$$\frac{-14}{-7} = B$$

$$\boxed{2 = B}$$

So $$\frac{5s+1}{s^2-s-12} = \frac{3}{s-4} + \frac{2}{s+3}$$

$$L^{-1}\left\{\frac{5s+1}{s^2-s-12}\right\} = 3e^{4t} + 2e^{-3t}$$

Determine $L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\}$

$$\frac{9s-8}{s^2-2s} \Rightarrow \frac{9s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

multiply b/s by $s(s-2)$

$$9s-8 = A(s-2) + Bs(\cancel{s-2})$$

put $s-2=0$, $s=2$

$$9(2)-8 = 0 + B(2)$$

$$10 = 2B$$

$$\boxed{5 = B}$$

put $s=0$

$$9(0)-8 = A(0-2) + 0$$

$$-8 = -2A$$

$$8 = 2A$$

$$\boxed{4 = A}$$

So $\frac{9s-8}{s^2-2s} = \frac{4}{s} + \frac{5}{s-2}$

$$L^{-1}\left\{\frac{9s-8}{s^2-2s}\right\} = 4 + 5e^{2t}$$

Table of inverse Transforms:-

$f(s)$	$F(t)$
$\frac{a}{s}$	a
$\frac{1}{s+a}$	e^{-at}
$\frac{n!}{s^{n+1}}$	t^n
$\frac{1}{s^n}$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{a}{s^2+a^2}$	$\sin at$
$\frac{s}{s^2+a^2}$	$\cos at$
$\frac{a}{s^2-a^2}$	$\sinh at$
$\frac{s}{s^2-a^2}$	$\cosh at$

Transforms of derivatives:-

$F'(t)$ denotes first derivative of $F(t)$ w.r.t t

$F''(t)$ denotes second derivative of $F(t)$ w.r.t t

Then

$$L\{F'(t)\} = \int_0^\infty e^{-st} F'(t)\,dt$$

Integrating by parts

$$= \left[e^{-st} F(t) \right]_0^\infty - \int_0^\infty F(t)\left(e^{-st} \cdot (-s)\right) dt$$

$$= 0 - (1)(F(0)) + s\int_0^\infty e^{-st} F(t)\,dt$$

$$L\{F'(t)\} = -F(0) + s\,L\{F(t)\}$$

Similarly

$$L\{F''(t)\} = -F'(0) + s\,L\{F'(t)\}$$

$$= -F'(0) + s\left[-F(0) + sL\{F(t)\}\right]$$

$$= -F'(0) - sF(0) + s^2 L\{F(t)\}$$

$$= s^2 L\{F(t)\} - sF(0) - F'(0)$$

$$L\{F'''(t)\} = s^3 L\{F(t)\} - s^2F(0) - sF'(0) - F''(0)$$

$$L\{F^{iv}(t)\} = s^4 L\{F(t)\} - s^3F(0) - s^2F'(0) - sF''(0) - F'''(0$$

Differential Equ. & Its solution:-

Solve $\frac{dx}{dt} - 2x = 4$ given that at $t=0$ & $x=1$

$$L\left\{\frac{dx(t)}{dt} - 2x(t)\right\} = L\{4\}$$

$$L\left[\frac{d}{dt}x(t)\right] - 2L\{x(t)\} = L\{4\}$$

$$-x(0) + sL[x(t)] - 2L[x(t)] = \frac{4}{s}$$

$$-x(0) + L[x(t)](s-2) = \frac{4}{s}$$

$$-1 + L[x(t)](s-2) = \frac{4}{s}$$

$\because$ at $t=0$, $x=1$
So $x(0)=1$

$$L[x(t)](s-2) = \frac{4}{s} + 1$$

$$L[x(t)](s-2) = \frac{4+s}{s}$$

$$L[x(t)] = \frac{4+s}{s(s-2)}$$

$$x(t) = L^{-1}\left[\frac{4+s}{s(s-2)}\right]$$

Now by partial fractions.

$$\frac{4+S}{S(S-2)} = \frac{A}{S} + \frac{B}{S-2}$$

multiply b/s by $S(S-2)$

$$4+S = A(S-2) + B(S)$$

put $S=0$

$$4 = A(-2) + 0$$

$$\frac{4}{-2} = A$$

$$\boxed{-2 = A}$$

put $S-2=0$, $S=2$

$$4+2 = A(0) + B(2)$$

$$6 = 2B$$

$$\boxed{3 = B}$$

So $$\frac{4+S}{S(S-2)} = \frac{-2}{S} + \frac{3}{S-2}$$

$$L^{-1}\left[\frac{4+S}{S(S-2)}\right] = -2 + 3e^{2t}$$

Solution of differential equ by laplace

(a) Rewrite equ in term of Laplace Transform.

(b) Insert given initial conditions.

(c) Re-arrange equ algebrically to give Transform of solution.

(d) Determine inverse Transform to obtain the particular Solution.

Solve the equ.

$$\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 2e^{3t}$$ given that at

$t=0$, $x=5$ & $\frac{dx}{dt} = 7$

means $x(0)=5$ $x'(0)=7$

$$x''(t) - 3x'(t) + 2x(t) = 2e^{3t}$$

$$L[x''(t)] - 3L[x'(t)] + 2L[x(t)] = 2L[e^{3t}]$$

$$s^2 L\{x(t)\} - sx(0) - x'(0) - 3[sL(x(t)) - x(0)] + 2L\,x(t) = \frac{2}{s-3}$$

$$s^2 L\{x(t)\} - s(5) - 7 - 3[sL(x(t))] + 3(5) + 2L(x(t)) = \frac{2}{s-3}$$

$$s^2 L\{x(t)\} - 3sL(x(t)) + 2L(x(t)) = \frac{2}{s-3} + 7 + 5s - 15$$

$$L\{x(t)\}[s^2 - 3s + 2] = \frac{2}{s-3} + 5s - 8$$

$$L\{x(t)\}(s-1)(s-2) = \frac{2 + 5s^2 - 15s - 8s + 24}{(s-3)}$$

$$L\{x(t)\} = \frac{5s^2 - 23s + 26}{(s-1)(s-2)(s-3)}$$

Now partial fractions

$$L\{x(t)\} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3}$$

multiply b/s by $(s-1)(s-2)(s-3)$

$$5s^2 - 23s + 26 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

put $s-1=0$, $s=1$

$$5 - 23 + 26 = A(-1)(-2) + 0 + 0$$

$$31 - 23 = 2A$$

$$8 = 2A$$

$$\boxed{4 = A}$$

put $s=2$

$$20 - 46 + 26 = 0 + B(2-1)(2-3) + 0$$

$$46 - 46 = -B$$

$$\boxed{0 = B}$$

put $s = 3$

$$45 - 69 + 26 = 0 + 0 + C(3-1)(3-2)$$

$$71 - 69 = 2C$$

$$2 = 2C$$

$$\boxed{1 = C}$$

$$L\{x(t)\} = \frac{4}{s-1} + \frac{0}{s-2} + \frac{1}{s-3}$$

$$x(t) = L^{-1}\left[\frac{4}{s-1} + \frac{1}{s-3}\right] \Rightarrow 4e^{t} + e^{3t}$$ Ans.

Thank you!

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