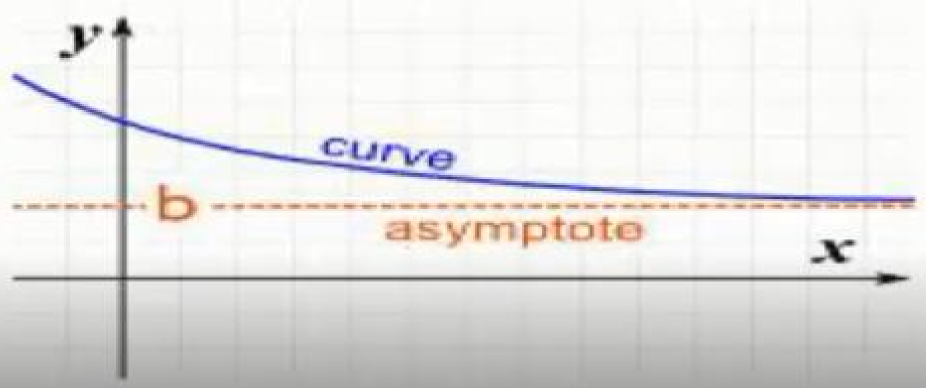
Mth403 short notes

Calculus and Analytical Geometry-II (Virtual University of Pakistan)

Horizontal Asymptote

The graph of $y = f(x)$ has a horizontal asymptote of $y = b$, if and only if, either

$$\lim_{x \to \infty} f(x) = b \quad or \quad \lim_{x \to -\infty} f(x) = b. \tag{1}$$



ASYMPTOTES OF A RATIONAL FUNCTIONS

$$r(x) = \frac{p(x)}{q(x)} = \frac{a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1} x^{m-1} + \cdots + b_0}$$

(1) if $n = m$, then the line $y = \frac{a_n}{b_m}$

(2) if $n < m$, then $y = 0$

(3) if $n > m$, then $r(x)$ becomes unbounded for large values of x.

Asymptotes of a Rational functions

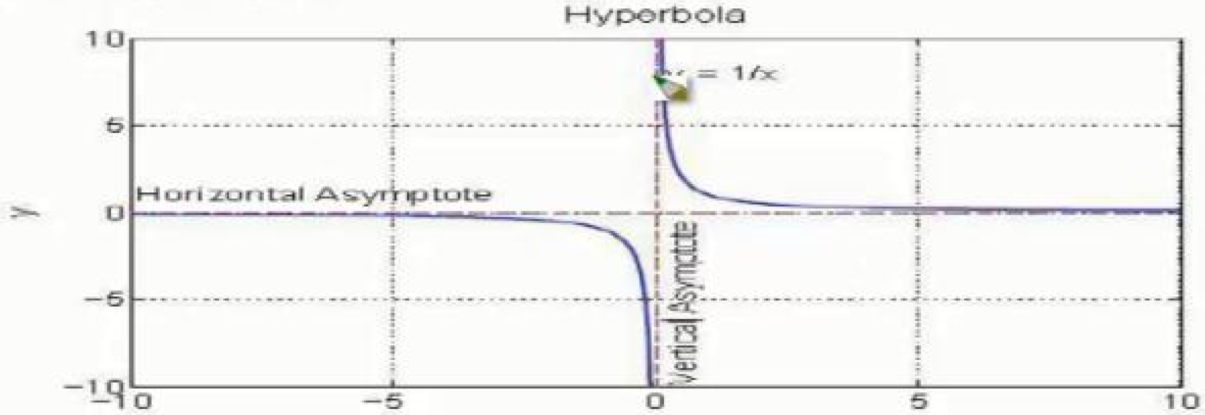
Let r(x) be a rational function with polynomial
$p(x) = a_n x_n + \ldots + a_0$ of degree n in the numeratorand polynomial
$q(x) = b_m x_m + \ldots + b_0$ of degree m in the denominator

- If $n < m$, then $r(x)$ has a horizontal asymptote of $y = 0$.
- If $n > m$, then $r(x)$ becomes unbounded for large values of x (positive or negative).
- If $n = m$, then $r(x)$ has a horizontal asymptote of $y = \frac{a_n}{b_n}$.

EXAMPLE(1):

- Thus there is a vertical asymptote at $x = 0$.

The graph of $y = \frac{1}{x}$ is



EXAMPLE(2):

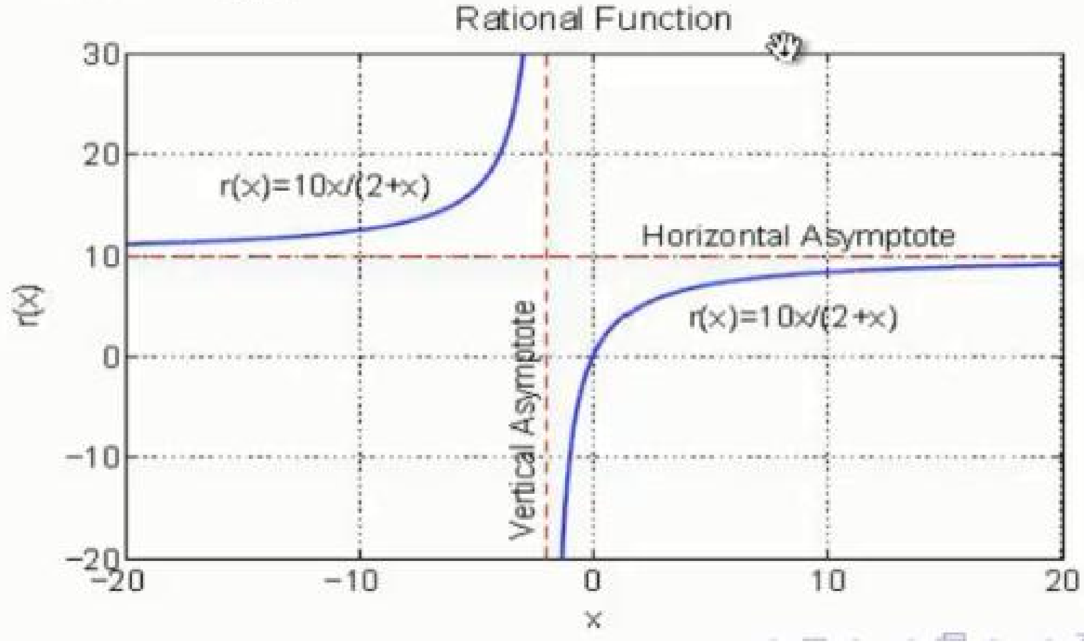
Discuss the Asymptotes of the following rational function

$$r(x) = \frac{10x}{2+x}, \quad (4)$$

- Defined $\forall x \neq -2$
- The numerator and denominator are linear functions (degree of polynomials are the same).
- we can see that as x get large then 2 in the denominator becomes insignificant.
- Thus $r(x) \approx \frac{10x}{x} = 10$
- Thus horizontal asymptote occurs at $x = 10$.

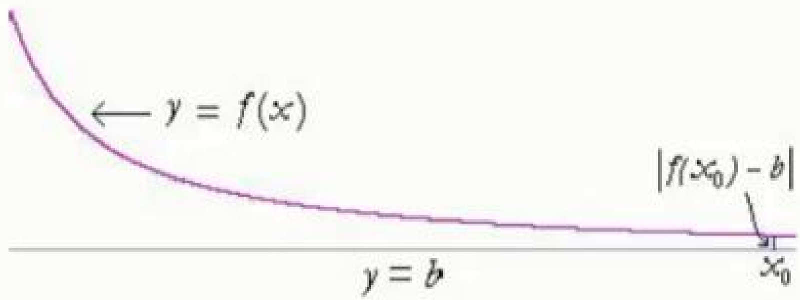
EXAMPLE(2):

The graph of $y = \frac{10x}{2+x}$ is



Asymptote

An asymptote is a line to which the graph gets arbitrarily close. That means that for any distance named, no matter how small, the graph will get within that distance and stay within that distance for some section of the graph with infinite length. More precisely



ASYMPTOTES OF A RATIONAL FUNCTIONS

EXAMPLE(1):

Discuss the Asymptotes of the following rational function

$$r(x) = \frac{1}{x}, \tag{3}$$

- This is defined $\forall x \neq 0$.
- Consider the sequence of numbers $x_n = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \cdots, \frac{1}{k}$ for $k = 1, 2, \cdots, n$.
- These numbers are getting closer and closer to zero.
- Since $r(x_n) = \frac{1}{x_n} = \frac{1}{\frac{1}{n}} = n$.
- So, $r(x_n) = 2, 3, 4, \cdots, k, \cdots$ for $n = 2, 3, 3, \cdots, k, \cdots$, which is getting larger and larger, so approaching the vertical line $x = o$.

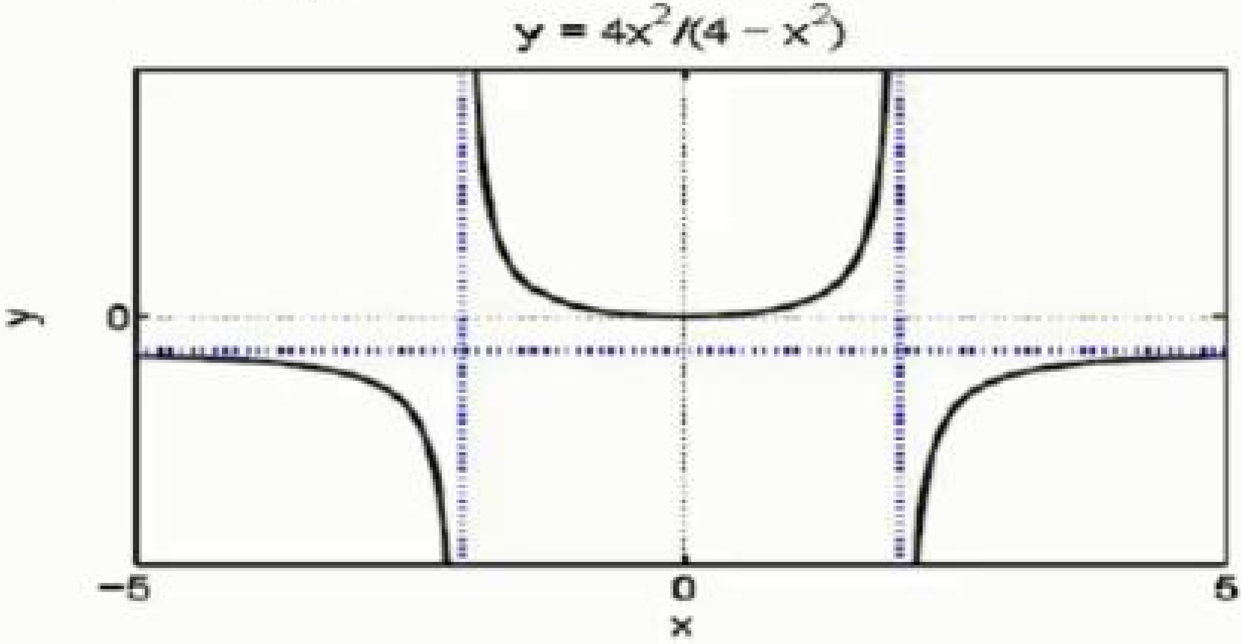
Example(3):

Discuss the Asymptotes of the following rational function

$$r(x) = \frac{4x^2}{4 - x^2}, \tag{5}$$

- Defined $\forall x \neq \pm 2$.
- Clearly the function passes through the origin, so the x and y-intercept is $(x, y) = (0, 0)$.
- Note that this function is an even function, the edge of the domain is $x = 2$, so we see there are vertical asymptotes at $x = 2$.
- Thus, for x large $r(x) \approx \frac{4x^2}{-x^2} = -4$.

The graph of $y = \frac{4x^2}{4-x^2}$ is



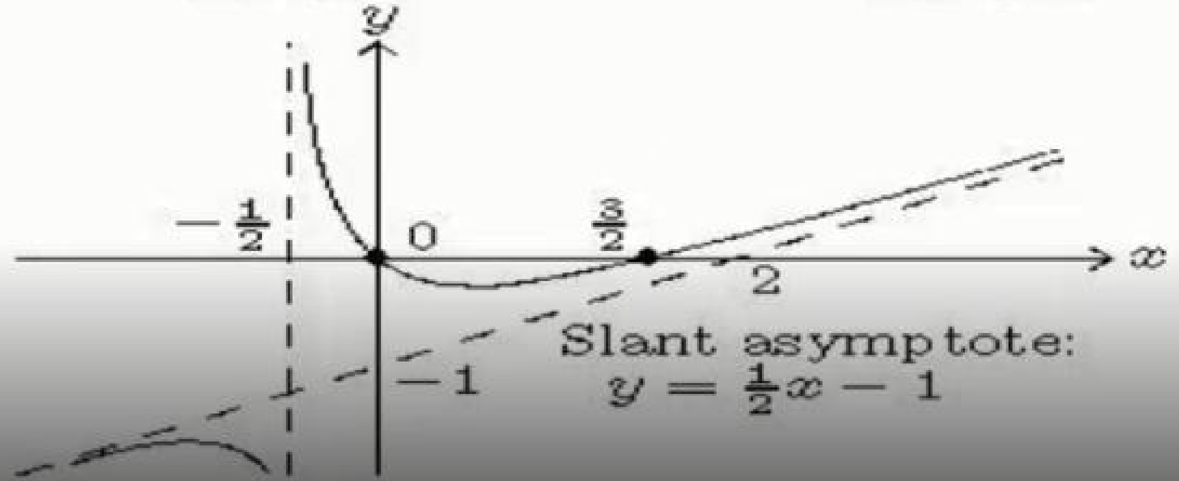
Oblique Asymptote:

- $r(x) = \frac{p(x)}{q(x)} = f(x) + \frac{g(x)}{q(x)}$.
- Obviously $deg(g(x)) < deg(q(x))$.
- Implies $\frac{g(x)}{q(x)} \to 0$ as $x \to \pm\infty$.
- Thus $r(x) \to f(x)$ as $x \to \pm\infty$.
- Particularly if $deg(p(x))$ is one more less than $deg(q(x))$, then $f(x)$ is a linear function whose graph is a non-horizontal line in the plane.
- That line is called oblique asymptote.

ASYMPTOTES OF A RATIONAL FUNCTIONS

OBLIQUE ASYMPTOTE:

$$\frac{x^2 - \frac{3}{2}x}{2x + 1} = \left(\frac{1}{2}x - 1\right) + \frac{1}{2x + 1}$$



Example(2):

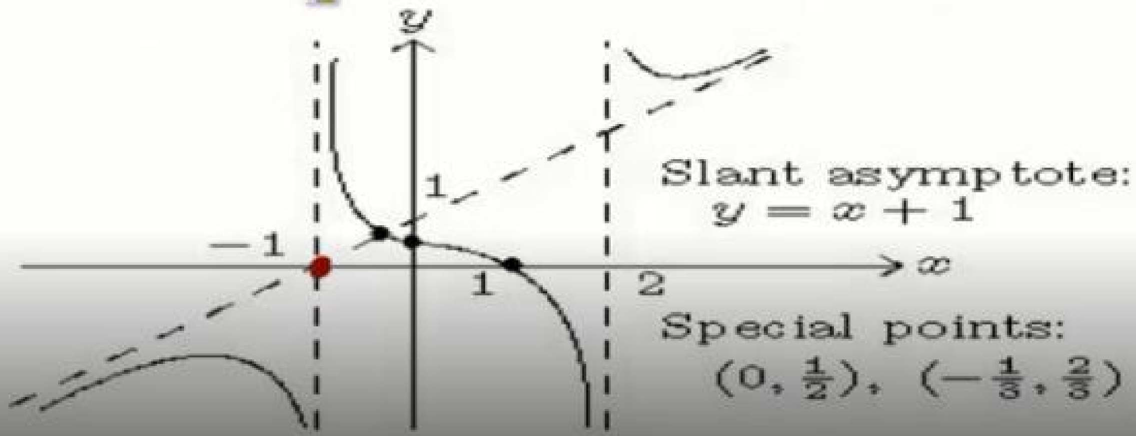
Discuss the Asymptotes of the following rational function

$$
\begin{aligned}
r(x) &= \frac{x^3-1}{x^2-x-2}, \\
&= x+1+\frac{3x+1}{x^2-x-2}. \qquad (7)
\end{aligned}
$$

- Clearly the quotient is a linear function.
- Thus it is our oblique asymptote.
- For more understanding see the graph.

OBLIQUE ASYMPTOTE:

$$\frac{x^3 - 1}{x^2 - x - 2} = (x + 1) + \frac{3x + 1}{(x - 2)(x + 1)}$$



POLAR EQUATION OF A STRAIGHT LINE:

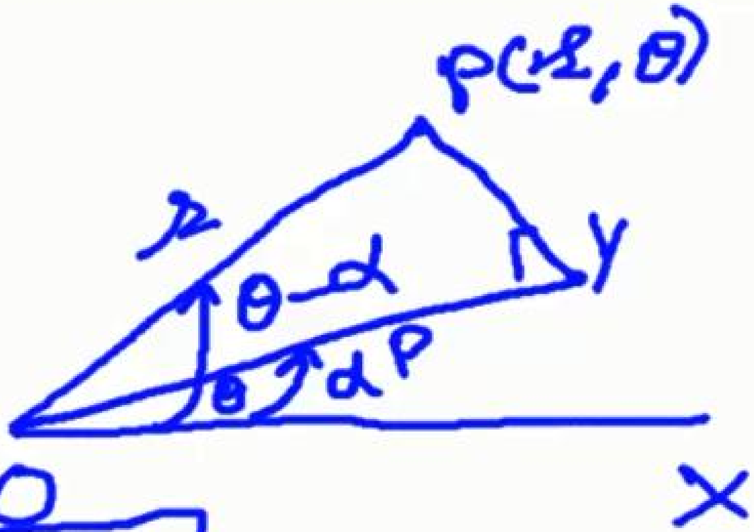
$$p = r\cos(\theta - \alpha)$$

Then

$$\cos(\angle POY) = \frac{p}{r}$$

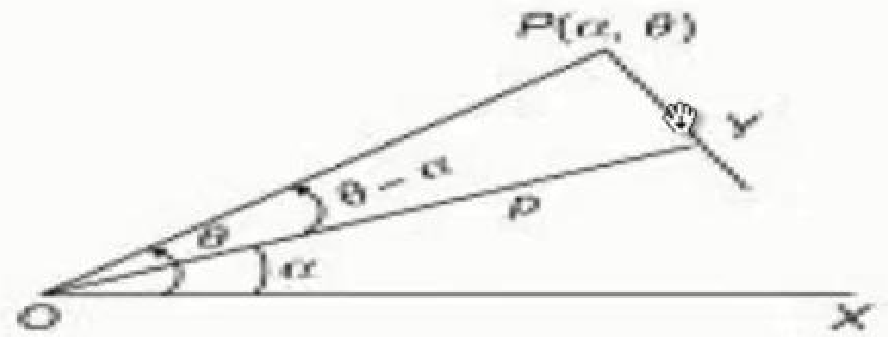
$$\Rightarrow \cos(\theta - \alpha) = \frac{p}{r}$$

$$\Rightarrow \boxed{p = r\cos(\theta - \alpha)}$$



POLAR EQUATION OF A STRAIGHT LINE:

- The polar equation of any line is $p = r\cos(\theta - \alpha)$.
- where p is the length of the perpendicular from pole to the line.
- α is the angle which this perpendicular makes with the initial line.



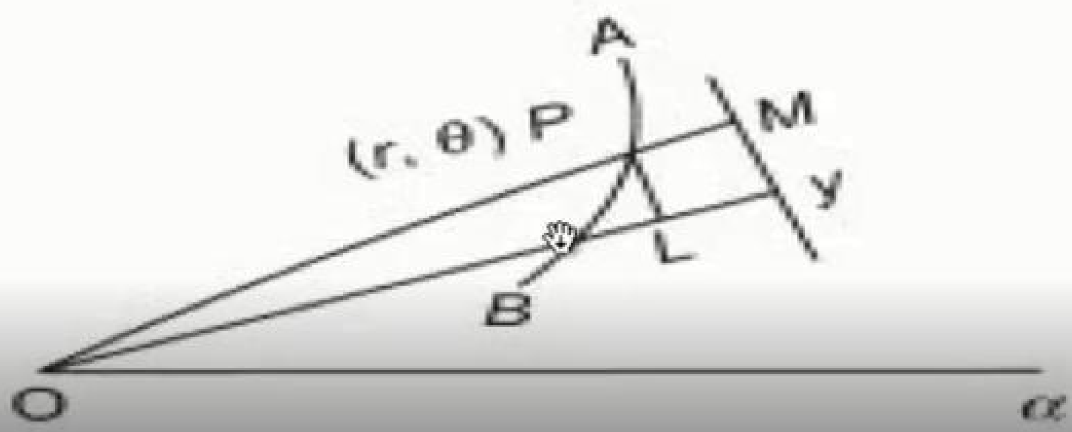
Polar Equation of a Straight Line (Proof):

- Let OX be initial line and P be any point on the line whose polar coordinates are (r, θ).
- Therefore, $OP = r$ and $\angle POX = \theta$.
- Let us draw a perpendicular OY on the line from P produced.
- Now using $\triangle OPY$, we have
-

$$\cos POY = \frac{OY}{OP} = \frac{p}{r}$$

$$p = r\cos POY = r\cos(\theta - \alpha), \qquad (8)$$

- Now we determine the asymptotes of the curve $r = s(\theta)$.
- For this we have to determine the constant p and α.



CONSTRUCTION:

- Let $P(r,\theta)$ be any point on the curve AB whose equation is $r = p(\theta)$.
- Let us draw a perpendicular OY on the line $p = r\cos(\theta-\alpha)$ which is MY.
- Therefore

$$
\begin{aligned}
PM = LY = OY - OL &= p - OP\cos(\theta-\alpha) \\
&= p - r\cos(\theta-\alpha). \qquad (9)
\end{aligned}
$$

CONSTRUCTION:

- Now $r \to \infty$ as the point recedes to infinity along the curve.Let $\theta \to \theta_1$ where $r \to \infty$.
- Therefore we have $\frac{PM}{r} = \frac{p}{r} - \cos(\theta - \alpha)$ by (9). when $r \to \infty$, $PM \to 0$.
- So that

$$\frac{PM}{r} = PM.\frac{1}{r} \to 0 \; and \; \frac{p}{r} \to 0. \tag{10}$$

CONSTRUCTION:

- Let

$$
\begin{aligned}
\cos(\theta - \alpha) &= 0 \\
\cos(\theta - \alpha) &= \cos(\frac{\pi}{2}) \\
\theta_1 - \alpha &= \frac{\pi}{2} \quad (\because \theta \to \theta_1 \ as \ r \to \infty) \\
\theta_1 &= \alpha + \frac{\pi}{2} \\
\alpha &= \theta_1 - \frac{\pi}{2}.
\end{aligned} \tag{11}
$$

18:48 / 21:20

CONSTRUCTION:

- Again from (9), we have when $r \to \infty$

$$
\begin{aligned}
p &= \lim_{r\to\infty}[r\cos(\theta-\alpha)] \\
&= \lim_{r\to\infty}[r\cos(\theta-\theta_1+\frac{\pi}{2})] \\
&= \lim_{r\to\infty}[r\sin(\theta_1-\theta)] \\
&= \lim_{\theta\to\theta_1}\left[\frac{\sin(\theta_1-\theta)}{\frac{1}{r}}\right] \cdots \frac{0}{0}
\end{aligned}
$$

CONSTRUCTION:

- $$\begin{aligned} p &= \lim_{\theta \to \theta_1} \left[\frac{\cos(\theta_1 - \theta)}{-\frac{1}{r^2}\frac{dr}{d\theta}} \right] \\ &= \frac{\lim_{\theta \to \theta_1}(\cos(\theta_1 - \theta))}{\lim_{\theta \to \theta_1}(-\frac{1}{r^2}\frac{dr}{d\theta})}. \end{aligned} \qquad (13)$$

- Now let $u = \frac{1}{r}$,this implies $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$. Therefore (13) becomes

CONSTRUCTION:

-

$$p = \lim_{\theta \to \theta_1} -\frac{d\theta}{du}. \tag{14}$$

- Hence the asymptotes is

$$\begin{aligned} \lim_{\theta \to \theta_1} -\frac{d\theta}{du} &= r\cos(\theta - \alpha) \\ &= r\cos\left(\theta - \theta_1 + \frac{\pi}{2}\right) \\ &= r\sin(\theta_1 - \theta), \end{aligned}$$

WORKING RULES:

- We have to change r to $\frac{1}{u}$ in the given equation and find out the limit of θ as $u \to 0$.
- Let θ_1 be any one of the angle possible limits of θ.
- We have to determine then $-\frac{d\theta}{du}$ and its limit as $u \to 0$ and $\theta \to \theta_1$.
- Let this limit be p.
- Then $p = r\sin(\theta_1 - \theta)$ is the required asymptote.

EXAMPLE(1):

Find the asymptotes of the curve

$$r\sin(\theta) = 2\cos(2\theta) \tag{16}$$

- **Step1**:(16) implies

$$\begin{aligned} r &= \frac{2\cos 2\theta}{\sin\theta} \\ \frac{1}{r} &= \frac{\sin\theta}{2\cos 2\theta}, \end{aligned} \tag{17}$$

EXAMPLE(1):

- according to working rules , we can write (17) as

$$u = \frac{\sin\theta}{2\cos 2\theta}. \tag{18}$$

- **Step2:** As $u \to 0$ then $\theta \to n\pi,\ \ n \in Z$.
- **Step3**: Now we determine $-\lim_{\theta\to n\pi}(\frac{d\theta}{du})$

EXAMPLE(1):

- according to working rules , we can write (17) as

$$u = \frac{\sin\theta}{2\cos 2\theta}. \tag{18}$$

- **Step2:** As $u \to 0$ then $\theta \to n\pi,\ n \in Z$.

EXAMPLE(1):

$$
\begin{aligned}
-\lim_{\theta\to n\pi}\left(\frac{du}{d\theta}\right) &= \lim_{\theta\to n\pi}\frac{\cos 2\theta\cos\theta-\sin\theta(-2\sin 2\theta)}{\cos^2 2\theta}\\
&= \frac{\cos^2(2n\pi)}{\cos(2\pi)\cos(\pi)-\sin(\pi)(-2\sin(2\pi)}\\
&= \frac{2}{\cos(n\pi)}\\
&= \frac{2}{(-1)^n}. \qquad (19)
\end{aligned}
$$

EXAMPLE(1):

- **Step4:** This limit be p. That is $p = -\frac{2}{(-1)^n}$. Hence, the required asymptotes are

$$
\begin{aligned}
r\sin(n\pi - \theta) &= -\frac{2}{(-1)^n} \\
r[(-1)^{n-1}\sin\theta] &= \frac{2}{(-1)^n} \\
r\sin(\theta) &= 2. \qquad (20)
\end{aligned}
$$

Asymptotes in Polar Coordinates

Exercises):

- Find the equations of the oblique asymptote for the following rational curves
- $\frac{3x^3+2}{x^2-x-1}$
- $\frac{5x^2-3x+1}{x+2}$
- $\frac{x^2+2}{x-2}$
- $\frac{(1-x)^3}{x^2}$
- $\frac{x^3-3x^2}{x^2-1}$
- Find the equati… …rves
- $(x-y)^2(x^2+y$ …
- x^2y+xy^2+xy …

ABSOLUTE MAXIMUM:

- A function f has an absolute maximum (or global maximum) at cif $f(c) \geq f(x)$ for all x in D.
- Where D is the domain of f.
- The number $f(c)$ is called the maximum value of f on D.

ABSOLUTE MINIMA:

- A function f has an absolute minimum (or global minimum) at $d$if $f(d) \leq f(x)$ for all x in D.
- Where D is the domain of f.
- The number $f(d)$ is called the minimum value of f on D.

14:49 / 16:55

Activate Window
Go to Settings to activ

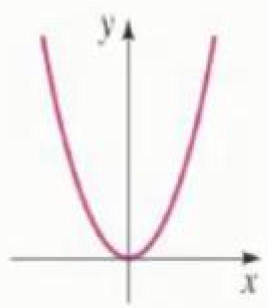
Local Maximum or Minimum:

- A function f has a local maximum (or relative maximum) at c if $f(c) \geq f(x)$ when x is near c.
- This means that $f(c) \geq f(x)$ for all x in some open interval containing c.
- Similarly, f has a local minimum (or relative minimum) at d if $f(d) \leq f(x)$ when x is near $\cancel{c}$. d.

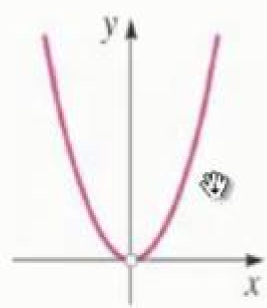
Examples:

Let $f(x) = x^2, \quad x \in \mathbb{R}$

- $f(x)$ has absolute and local minimum at $x = 0$.
- Having no absolute or local maximum.
- The right most diagram showing the function behavior when $x \in (-\infty, 0) \cup (0, \infty)$.



$f(x) = x^2$



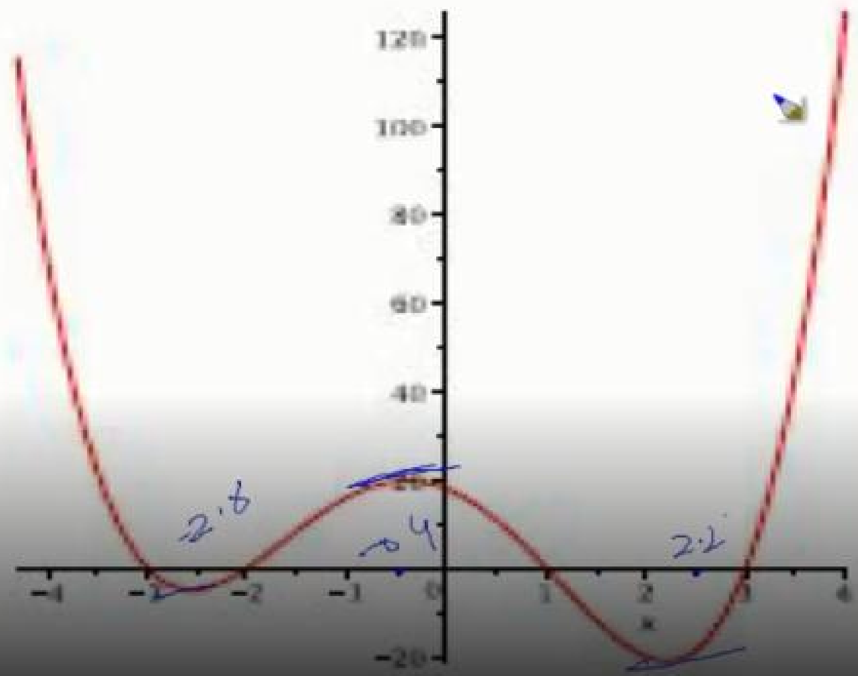
$f(x) = x^2, \; x \in (-\infty, 0) \cup (0, \infty)$

Local Examples:

Let $f(x) = f(x) = x^4 + x^3 - 11x^2 - 9x + 18 =$
$(x-3)(x-1)(x+2)(x+3), \quad x \in \mathbb{R}$

- $f(x)$ has the absolute minimum at $x \approx 2.2$ and has no absolute maximum.
- It has two local minima at $x \approx -2.6$ and $x \approx 2.2$.

LOCAL EXAMPLES:



CRITICAL NUMBER

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

REMARK(3):

From Fermat's Theorem it follows that if f has a local maximum or minimum at c, then c is a critical number of f.

EXAMPLES:

- If $f(x) = 2x^2 + 5x - 1, \ \ x \in \mathbb{R}$, then $f'(x) = 4x + 5$. Hence the only critical number of f is $x = -\frac{5}{4}$.

$f(x) = (x^2)^{1/3}$

$f'(x) = \frac{2}{3}x^{-1/3} \Rightarrow$ at $x = 0$

5:29 / 7:15

Activate Wind

Go to Settings to ac

EXAMPLES:

- If $f(x) = 2x^2 + 5x - 1, \quad x \in \mathbb{R}$, then $f'(x) = 4x + 5$. Hence the only critical number of f is $x = -\frac{5}{4}$.

$f(x) = \frac{1}{x}$; $x \in \mathbb{R} - \{0\}$

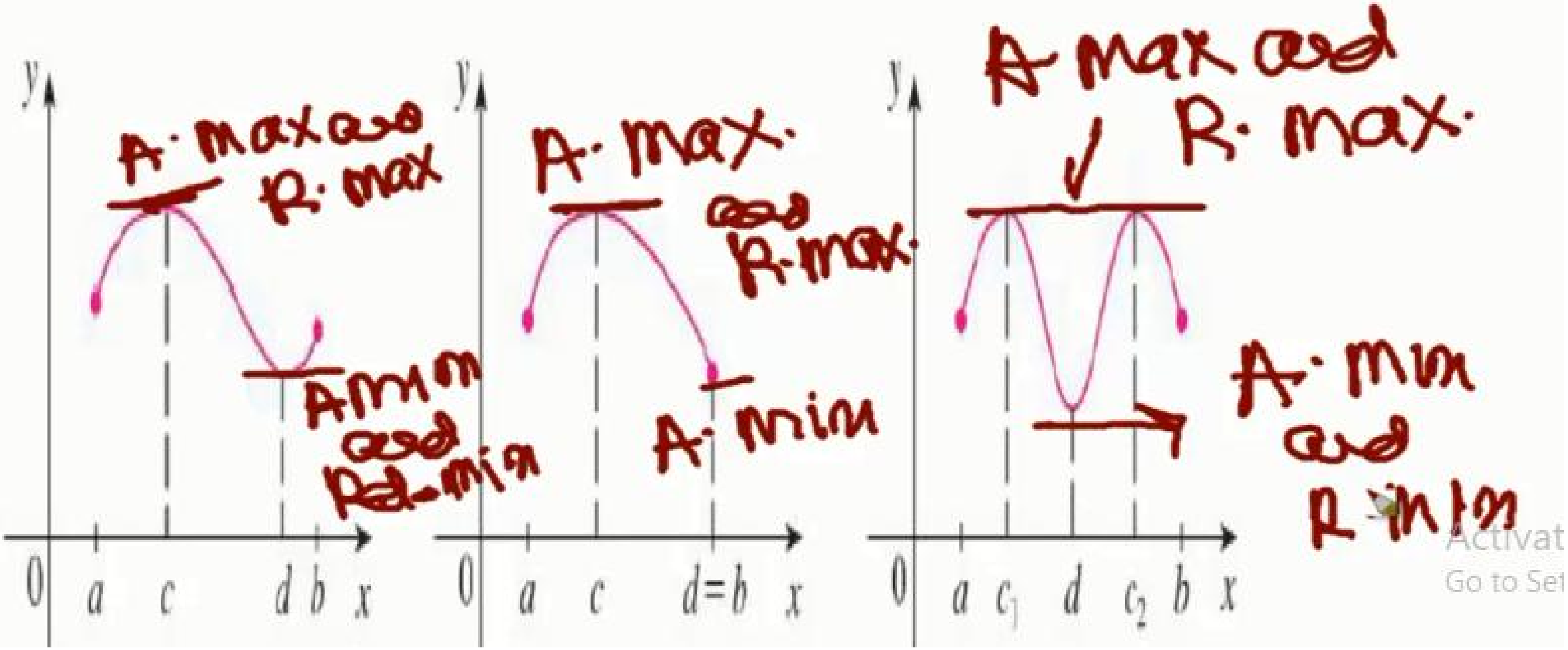
$f'(x) = -\frac{1}{x^2}$ at $\boxed{x = 0}$

EXERCISE

- Find the critical numbers of $f(x) = 2x^3 - 9x^2 + 12x - 5$.
- Find the critical numbers of $f(x) = 2x + 3\sqrt[3]{x^2}$.

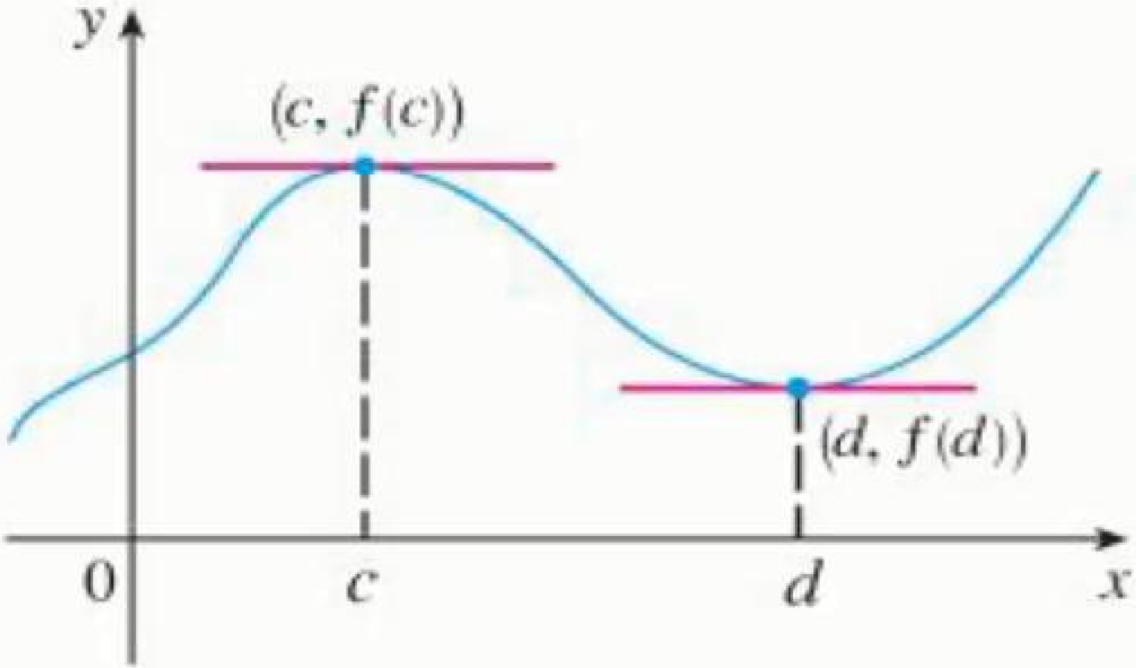
EXTREME VALUE THEOREM

If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ at some number c and an absolute minimum value $f(d)$ at some number d.



FERMATS THEOREM

If f has a local maximum or minimum at some number c, and if $f'(c)$ exists, then $f'(c) = 0$.



FERMATS THEOREM

If f has a local maximum or minimum at some number c ,

$f(x) = x^3 ; x \in \mathbb{R}$

(i) If $f'(x) = 0$

$f'(x) = 3x^2$

$= 0$ at $x = 0$

4:23 / 7:04

Activate
Go to Setti

Fermats Theorem

Remark(1):

- The converse of this theorem is not true.
- In other words, when $f'(c) = 0$, f does not necessarily have a local maximum or minimum.
- For example, if $f(x) = x^3$, then $f'(x) = 3x^2$ equals 0 at $x = 0$.

5:00 / 7:04

FERMATS THEOREM

REMARK(2):

- Sometimes $f'(c)$ does not exist, but some number $f(c)$ is a point of a local maximum or minimum.
- For example, if $f(x) = |x|$, then $f'(0)$ does not exist.
- But $f(x)$ has its local (and absolute) minimum at $x = 0$.

6:58 / 7:04

Activate
Go to Setti

Relative Maxima and Minima

First Derivative Teat

- If $f'(x) > 0$ on an open interval extending left from c and $f'(x) < 0$ on an open interval extending right from c, then f has a relative maximum at c.
- If $f'(x) < 0$ on an open interval extending left from c and $f'(x) > 0$ on an open interval extending right from c, then f has a relative minimum at c.
- If $f'(x)$ has the same sign on both an open interval extending left from c and an open interval extending right from c, then f does not have a relative extremum at c.

Relative Maxima and Minima

- Let $f(x) = 3x^4 - 4x^3 - 12x^2 + 3$. $x \in [-2, 3]$;

$f'(x) = 12x^3 - 12x^2 - 24x$

$0 = 12x^3 - 12x^2 - 24x$

$\Rightarrow x = -1, \; x = 0$ and $x = 2$

$- - - - - \,|\, + + + + \,|\, - - - - \,|\, + - - - \,|\, + + +$ Sign of $f'(x)$

$-1 \qquad 0 \qquad 1 \qquad 2$

Relative Min. Relative Max. Relative Min

Relative Maxima and Minima

Second Derivative Teat

- Let c is a critical point at which $f'(c) = 0$.
- Let $f'(x)$ exists in a neighborhood of c.
- Let $f''(c)$ exists.
- Then f has a relative maximum value at c if $f''(c) < 0$,
- and a relative minimum value at c if $f''(c) > 0$.
- If $f''(c) = 0$, the test is not informative.

1:14 / 6:18

Singular Points

Example

- Discuss the singular points of $f(x) = (x-1)^{\frac{2}{3}} - 3(x-1)$ on $[0, \infty]$.
- Look for values x where $f'(x)$ is not defined, but $f(x)$ is defined.
- $f'(x) = \frac{2}{3(x-1)^{\frac{1}{3}}} - 3$.
- Notice that the denominator is zero when $x = 1$, so that $f'(x)$ is not defined when $x = 1$, even though $f(x)$ is defined when $x = 1$.

SINGULAR POINTS

DEFINITIONS

- A Point on the curve through three branches of the curve pass is called Triple Point.
- A Point on the curve through which r branch of the curve pass is called Multiple Point of rth order.
- A Double Point P on a curve is called a Node if two real branches of a curve pass through P and two tangents at which are real and different. Thus the point P shown is a Node.

SINGULAR POINTS

DEFINITIONS

- A Point on the curve through three branches of the curve pass is called Triple Point.
- A Point on the curve through which r branch of the curve pass is called Multiple Point of rth order.
- A Double Point P on a curve is called a Node if two real branches of a curve pass through P and two tangents at which are real and different. Thus the point P shown is a Node.
- A Double Point Q on a curve is called a Cusp if two real branches of a curve pass through Q and two tangents at which are real and coincident. Thus the point Q shown in the adjoining two figures is a Cusp.

WORKING RULE FOR INVESTIGATING THE NATURE OF THE DOUBLE POINT AT THE ORIGIN.

- Find the tangents at the origin by equating to zero the lowest degree terms present in the equation of the curve. If origin is a double point, then we shall get tangents real and imaginary.
- If the two tangents at the origin are imaginary, then the origin is a conjugate point.
- If the two tangents at the origin are real and different, then the tangent is a node or a conjugate point.
- If the two tangents at the origin are real and coincident, then the origin is a cusp or a conjugate point.

Singular Points

Remark.

To study the nature of the curve near origin

- If the tangents at the origin are $y^2 = 0$, solve the equation of the curve for y, neglecting all terms of y having powers above second. If for small non zero values of x, the values of y are real, then the branches of the curve through the origin are also real, otherwise they are imaginary.
- If the tangents at the origin are $x^2 = 0$, solve the given equation of the curve for x instead of y and proceed as mentioned in the above point.
- In other cases, solve for y or x, whichever is convenient.

Singular Points

Example

Determine the nature of the singular point $(0,0)$ of
$f(x,y) = x^3 + y^3 - 3xy$

- According to the last discussions, the tangents at the origin are $x = 0$ and $y = 0$
- Hence the origin is either a node or isolated point.
- When x is so small, the equation of the curve is $y^2 = 3x$ which represent two real branches through the origin.
- Hence the origin is a node.

SINGULAR POINTS

EXAMPLE

Determine the nature of the singular point $(0, 0)$ of $f(x, y) = (x^2 + y^2)(2a - x) - b^2x$.

- According to the last discussions, the tangents at the origin are $x = 0$.
- Hence the origin is either a cusp or isolated point.
- To see whether the origin is a cusp or an isolated point. Solve the equation of a curve for x.
- By neglecting y^2, the equation of the curve can also be written as

3:27 / 4:13

EXAMPLE

$$-x^3 - b^2x + 2ax^2 = 0$$
$$x^2 - 2ax + b^2 =$$
$$x = \frac{2a \pm \sqrt{4a^2 - 4(1)(b^2)}}{2}$$
$$x = a \pm \sqrt{a^2 - b^2}.$$

Hence origin is a cusp if $a^2 - b^2 > 0$.

WORKING RULES TO CALCULATE SINGULAR POINTS AND ITS NATURE USING PARTIAL DERIVATIVES

- If $f(x, y)$ be a curve then its singular points are the simultaneous solutions of $f(x, y) = 0$, $f_x(x, y) = 0$ and $f_y(x, y) = 0$.
- The values of tangents at singular points are the roots of $f_{yy}(y'(x))^2 + 2f_{xy}y'(x) + f_{xx} = 0$.
- If f_{xx}, f_{xy} and f_{yy} are not all zero then the point (x, y) will be a double point.
- If $(f_{xy})^2 - f_{xx}f_{yy} > 0$ the point (x, y) would be a node.
- If $(f_{xy})^2 - f_{xx}f_{yy} = 0$ the point (x, y) would be a cusp.
- If $(f_{xy})^2 - f_{xx}f_{yy} < 0$ the point (x, y) would be an isolated point

Discuss and sketch the graph of

$$x(x^2+y^2)=a(x^2-y^2)$$

$$f(x,y)=x(x^2+y^2)-a(x^2-y^2).$$

① $\Rightarrow \boxed{f(x,-y)=f(x,y)}$

② $x^2-y^2=0 \Rightarrow x^2=y^2$

$\Rightarrow \boxed{y=\pm x}$

origin is node.

③ Asymptotes

$$y^2=\frac{ax^2-x^3}{x+a}$$

$x \to -a$

$y \to \pm\infty$

$x=-a$ vertical $\parallel$ to y-axis

$x=a$

Domain sol.

$$y=\pm\frac{x\sqrt{a-x}}{\sqrt{x+a}}$$

...s and Sketch the graph of

$$y^2 = x^2(4-x^2)$$

$$f(x,y) = y^2 - x^2(4-x^2)$$

Symmetrically

$$f(x,-y) = f(x,y)$$

$$f(-x,y) = f(x,y)$$

$$y^2 - 4x^2 = 0$$

$$y = \pm 2x$$

$$\Rightarrow$$

(3) It has not asymptote

(4) $0 = x^2(4-x^2)$

$x = 0$, $x = \pm 2$

$y = 0$

(5)

$$y = \pm x\sqrt{(4-x^2)}$$

$$4 - x^2 \geq 0$$

$$\Rightarrow -x^2 \geq -4$$

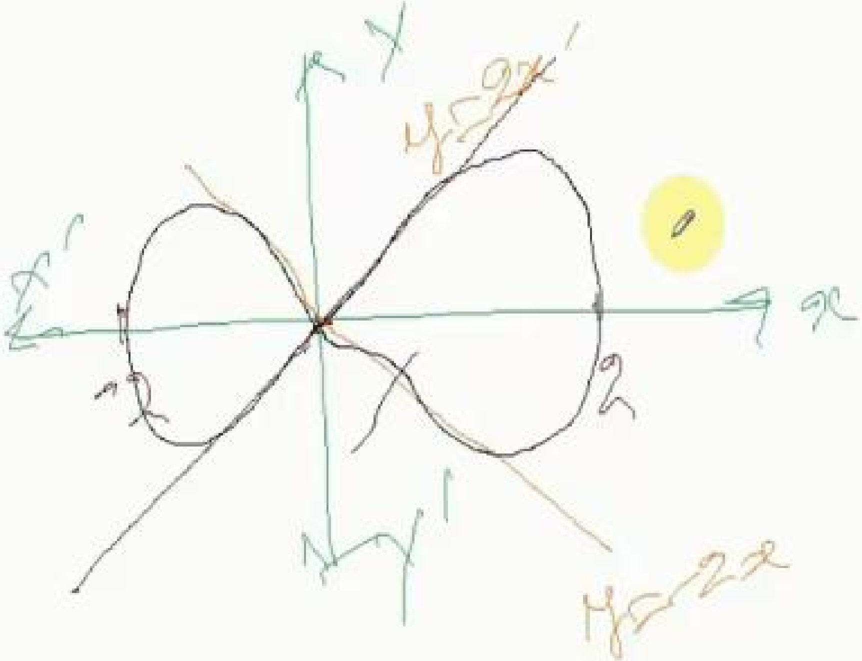
$$\Rightarrow \boxed{x^2 \leq 4}$$

$$[-2, 2]$$

Discuss and Sketch the graph of

$$y^2 = x^2(4-x^2)$$

$$\Rightarrow f(x,y) = y^2 - x^2(4-x^2)$$



(3) It has not asymptote.

(4) $0 = x^2(4-x^2)$

$x=0$ & $x=\pm 2$

$y=0$

(5)

$$y = \pm x\sqrt{4-x^2}$$

$$4-x^2 \geq 0$$

$$\Rightarrow -x^2 \geq -4$$

$$\Rightarrow x^2 \leq 4$$

$[-2, 2]$

Discuss and Sketch the graph of

$3ay^2 = x^2(x-a)$

$f(x,y) = 3ay^2 - x^2(x-a)$

① $\Rightarrow f(x,-y) = f(x,y)$

② $x^2 + 3y^2 = 0$

$y^2 = -\frac{x^2}{3}$

$\Rightarrow$ Isolated point.

③ Asymptotes: No

$0 = x^2(x-a)$

$\Rightarrow x=0$ or $x=a$

Domain:

$y^2 = \frac{x^2(x-a)}{3a}$

$\Rightarrow y = \pm \frac{x\sqrt{(x-a)}}{3a}$

$x-a \geq 0$

$\Rightarrow x \geq a$

Activate
Go to Setti

Discuss and Sketch the graph of

$3ay^2 = x^2(x-a)$

$f(x,y) = 3ay^2 - x^2(x-a)$

$0 = x^2(x-a)$

$\Rightarrow x = 0$ or $x = a$

Domain:

$y^2 = \frac{x^2(x-a)}{3a}$

$\Rightarrow y = \pm \frac{x\sqrt{(x-a)}}{3a}$

$x - a \geq 0$

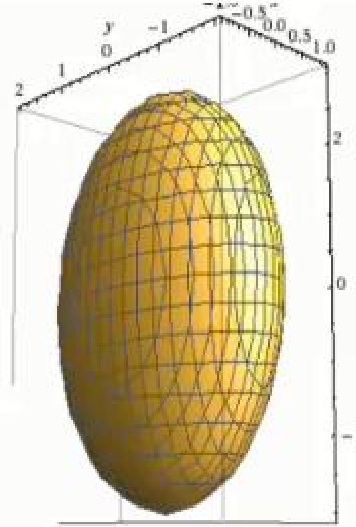
$\Rightarrow x \geq a$

Surface:

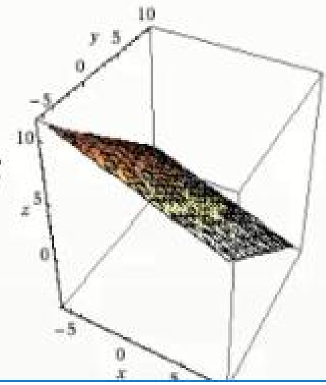
(x, y, z)

$f(x, y, z) = 0$

$x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ is ellipsoid



$x + 2y + 3z = 6$ is plane



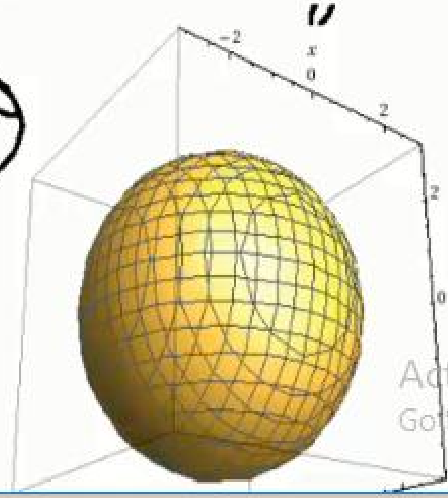
Activate Windo
Go to Settings to ac

Quadratic Surface: A quadratic surface is defined as the set of all points which satisfy second degree eq.

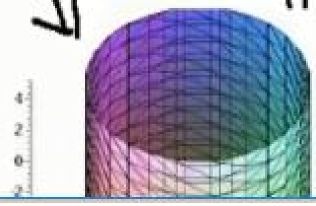
$$Ax^2+By^2+Cz^2+Dxy+Eyz+Fxz+Gx+Hy+Iz+J=0$$

where A, B, C, D, E and F are not all zero.

Sphere $\quad x^2+y^2+z^2=9$

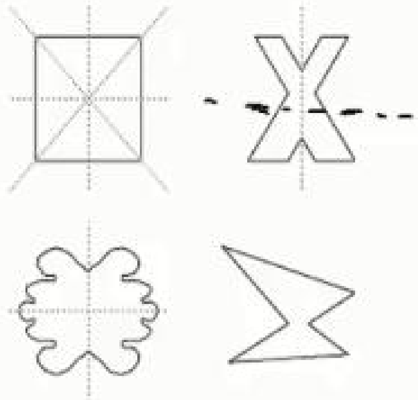


Cylinder $\quad x^2=4y$



Properties of Surfaces

(1) Symmetry



If $f(x,y,z)=0 \Rightarrow f(x,y,-z)=0$

xy-plane

$x+y+z^2=3$ $\qquad x+y+(-z)^2=3$

$\Rightarrow x+y+z^2=3$

If $f(x,y,z)=0 \Rightarrow f(-x,y,z)=0$

yz-plane

Similarly

If $f(x,y,z)=0 \Rightarrow f(x,-y,z)=0$

$\Rightarrow$ xz-plane

If $f(x,y,z)=0 \Rightarrow f(x,-y,-z)=0 \Rightarrow$ x-axis

$f(x,y,z)=0 \Rightarrow f(-x,y,-z)=0 \Rightarrow$ y-axis

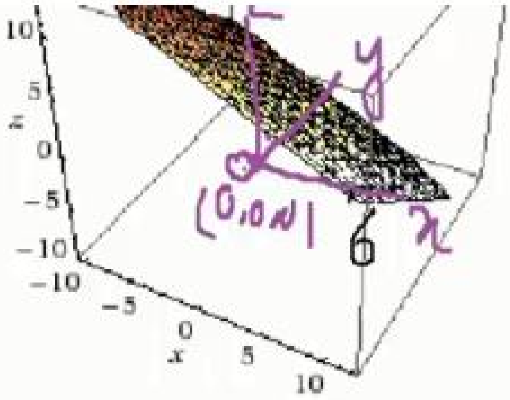
$f(x,y,z)=0 \Rightarrow f(-x,-y,z)=0 \Rightarrow$ z-axi

If $f(x,y,z)=0 \Rightarrow f(-x,-y,-z)=0.$

$\Rightarrow$ Symmetric about origin O

Intercepts:

- x-intercept.
- y-intercept
- z-intercept.



$$2x+3y+4z=12$$

$y=0,\ z=0$

$$2x+3(0)+4(0)=12 \Rightarrow 2x=12 \Rightarrow x=6$$

$x=0,\ z=0$

$$2(0)+3y+4(0)=12 \Rightarrow 3y=12 \Rightarrow y=4$$

$x=0,\ y=0$

$$2(0)+3(0)+4z=12 \Rightarrow z=4$$

Traces:

$$x^2 + y + z^2 = 25$$

$$z = 3$$

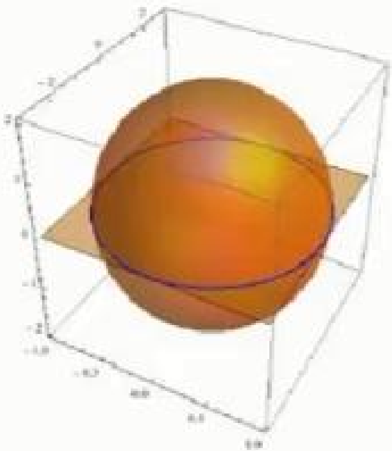
$$x^2 + y^2 + 3^2 = 25$$

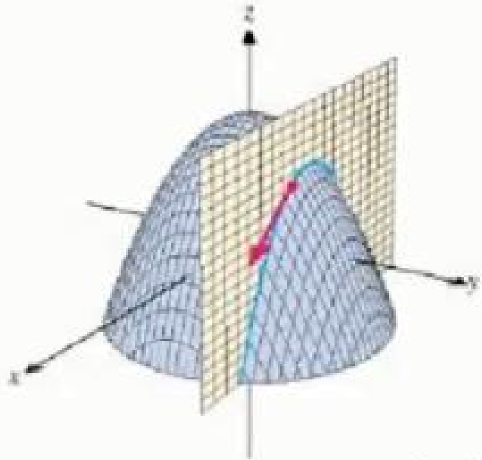
$$x^2 + y^2 + 9 = 25$$

$$x^2 + y^2 = 25 - 9$$

$$x^2 + y^2 = 16$$

circle

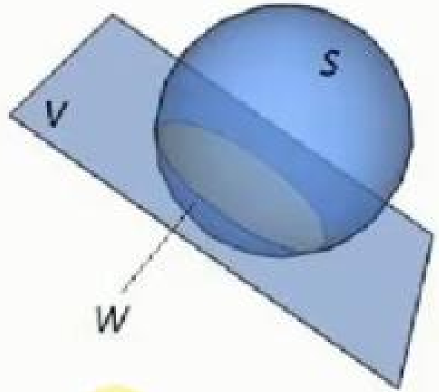




Sections:

eq. of Surface
eq. of plane

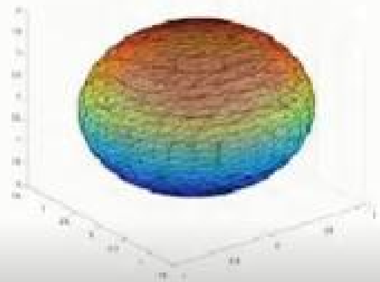
(1)



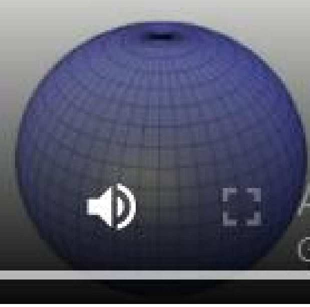
Boundedness:

finite → bounded
Infinite → unbounded

(2)



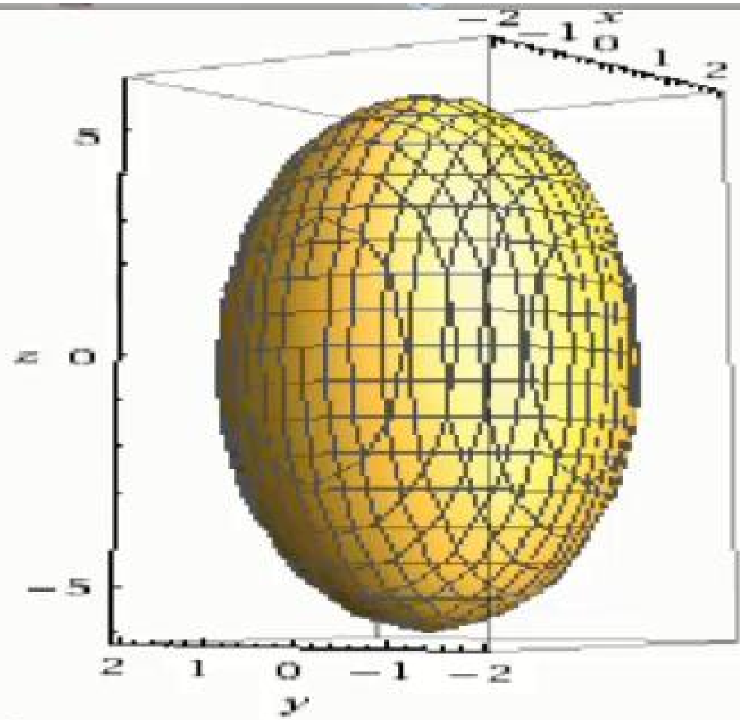
(3)



0:51 / 1:27

Example. Discuss the surface

$S: f(x,y,z) = x^2 + y^2 + \frac{z^2}{9} - 4 = 0$



Sol.

Symmetry.

(i) **about xy-plane**

If $f(x,y,-z) = f(x,y,z) = 0$

$f(x,y,-z) = (x)^2 + (y)^2 + \frac{(-z)^2}{9} - 4 = 0$

$= x^2 + y^2 + \frac{z^2}{9} - 4 = 0 = f(x,y,z)$

$\Rightarrow$ Symmetrical about xy-plane

(ii) about yz-plane

$$f(-x, y, z) = (-x)^2 + (y)^2 + \frac{z^2}{9} - 4 = x^2 + y^2 + \frac{z^2}{9} - 4$$

$$= f(x, y, z) = 0$$

$\Rightarrow$ Symmetrical about yz-plane.

(iii) about xz-plane

$$f(x, -y, z) = x^2 + (-y)^2 + \frac{z^2}{9} - 4 = x^2 + y^2 + \frac{z^2}{9} - 4 = 0$$

$$= f(x, y, z)$$

Activate
Go to Settin

symmetrical about xz-plane

(iv) <u>about x-axis</u>

$$f(x,-y,-z) = f(x,y,z) = 0$$

$\Rightarrow$ symmetrical about x-axis

(v) <u>about y-axis</u>

$$f(-x, y, -z) = f(x,y,z) = 0$$

(v) about y-axis

$$f(-x, y, -z) = f(x, y, z) = 0$$

$$f(-x, y, -z) = (-x)^2 + (y)^2 + \frac{(-z)^2}{9} - 4$$

$$= x^2 + y^2 + \frac{z^2}{9} - 4 = 0$$

$$= f(x, y, z)$$

Symmetrical about y-axis

(vi) Symmetrical about z axis $f(-x, -y, z) = f(x, y, z) = 0$

(v) Symmetry about origin

if $f(-x, -y, -z) = 0 \Rightarrow f(x, y, z) = 0$

Traces:

(i) yz-plane

put $x=0$ in

$$x^2+y^2+\frac{z^2}{9}-4=0$$

$$0^2+y^2+\frac{z^2}{9}-4=0 \Rightarrow y^2+\frac{z^2}{9}-4=0$$

$$\Rightarrow y^2+\frac{z^2}{9}=4 \Rightarrow \frac{y^2}{4}+\frac{z^2}{36}=1 \Rightarrow \text{ellipse}$$

(ii) xy-plane

put $z=0$

$$x^2+y^2-4=0 \Rightarrow x^2+y^2=4=2^2 \Rightarrow \text{circle}$$

(ii) xy-plane

put $z=0$

$$x^2+y^2-4=0 \Rightarrow x^2+y^2=4=2^2 \Rightarrow \text{circle}$$

(iii) xz-plane

put $y=0$

$$x^2+\frac{z^2}{9}-4=0 \Rightarrow x^2+\frac{z^2}{9}=4$$

$$\Rightarrow \frac{x^2}{4}+\frac{z^2}{36}=1 \Rightarrow \text{ellipse}$$

Intercepts

x-intercept put $y=0, z=0$

$$x^2-4=0 \Rightarrow x^2=4 \Rightarrow x=\pm 2$$

$(2,0,0)$, $(-2,0,0)$

y-intercept $x=0, z=0$

$$y^2-4=0 \Rightarrow y^2=4 \Rightarrow y=\pm 2$$

$(0,2,0)$ and $(0,-2,0)$

y-intercept $x=0, z=0$

$y^2-4=0 \Rightarrow y^2=4 \Rightarrow y=\pm 2$

$(0,2,0)$ and $(0,-2,0)$

z-intercept

$y=0, x=0$

$\frac{z^2}{9}-4=0 \Rightarrow \frac{z^2}{9}=4 \Rightarrow z^2=36 \Rightarrow z=\pm 6$

$(0,0,6)$ and $(0,0,-6)$

Example Find the intercepts of the surface

$$x^2+4y^2+5xz-2x+y-3=0$$

Sol.

x-intercept: $y=0,\ z=0$

$$x^2+4(0)^2+5x(0)-2x+0-3=0$$

$$x^2-2x-3=0$$

$$x^2-3x+x-3=0$$

$$x(x-3)+1(x-3)=0$$

$$(x-3)(x+1)=0$$

$$\Rightarrow x-3=0 \quad \text{or} \quad x+1=0$$

$$\Rightarrow x=3 \quad \text{or} \quad x=-1$$

$(3,0,0)$ and $(-1,0,0)$

y-intercept put $x=0,\ z=0$

$$(0)^2+4y^2+5(0)(z)-2(0)+y-3=0$$

$$4y^2+y-3=0$$

$$4y^2+4y-3y-3=0$$

$$4y(y+1)-3(y+1)=0$$

$$(y+1)(4y-3)=0$$

$$y+1=0\ ,\quad 4y-3=0$$

$$y=-1 \qquad y=3/4$$

$(0,-1,0)$ and $(0,3/4,0)$

Act
Go t

z-intercept

$x=0,\ y=0$

$0^2+4(0)^2+5(0)(z)-2(0)+0-3=0$

$-3=0$

which is a false eq.

Example Find the trace of the Surface $x^2+y^2+2z^2-3x+5y-8z=0$ in (i) xy-plane (ii) yz-plane (iii) xz-plane

Sol. (i) xy-plane

put $z=0$

$$x^2+y^2+2(0)^2-3x+5y-8(0)=0$$

$$x^2+y^2-3x+5y=0 \Rightarrow \text{eq. of circle}$$

(ii) **yz-plane**

put $x=0$

$$(0)^2+y^2+2z^2-3(0)+5y-8z=0$$

$$y^2+2z^2+5y-8z=0 \Rightarrow \text{ellipse}$$

Act
Go t

(iii) xz-plane

put $y=0$

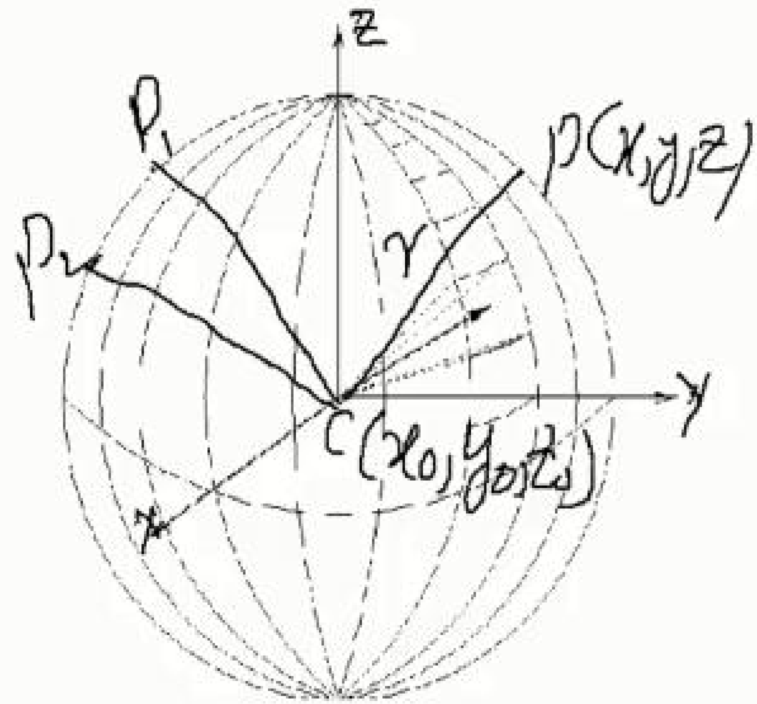
$$x^2+(0)^2+2z^2-3x+5(0)-8z=0$$

$$x^2+2z^2-3x-8z=0$$

$\Rightarrow$ ellipse

Sphere:

Sphere is the set of all points in the space that are equidistant from a fixed point. The fixed point is called the Centre of the sphere and the constant distance from the points of the sphere to the Centre of the circle is called as the radius of the sphere.



$$\overline{CP} = r$$

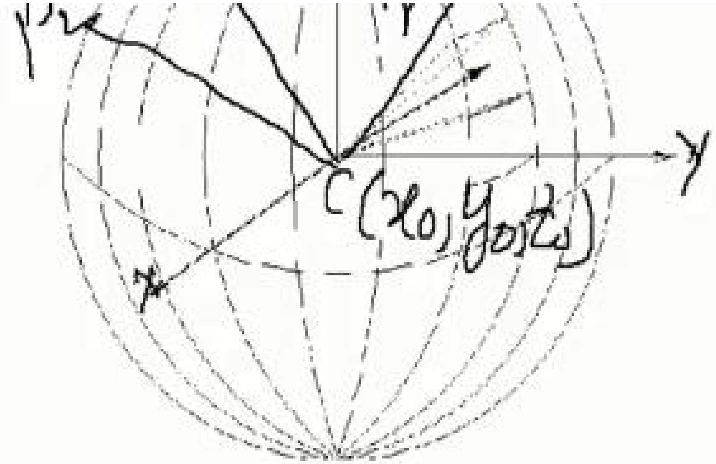
$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2$$

eq. of sphere

Act
Go t

circle is called as the radius of the sphere.



$$\overline{CP} = r$$

$$\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2} = r$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2 = r^2$$

eq. of sphere

$C(x_0, y_0, z_0) = C(0,0,0)$, radius $= r$

$$(x-0)^2+(y-0)^2+(z-0)^2 = r^2$$

$$x^2+y^2+z^2 = r^2$$

eq. of sphere $C(0,0,0)$, radius $= r$

Act
Go t

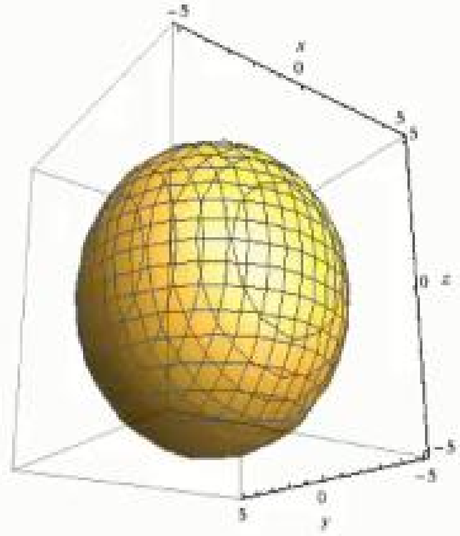
Properties of Sphere

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$

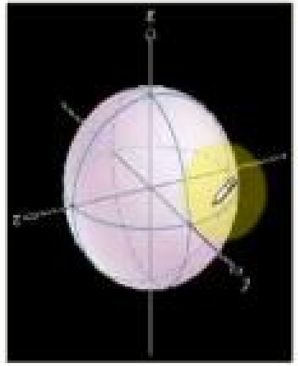
centre (x_0, y_0, z_0) , radius $= r$

centre $(0,0,0)$,

$$x^2+y^2+z^2=r^2$$



$x^2 + y^2 + z^2 = 25$



x-intercept

$y = 0, z = 0$

$x^2 + 0^2 + 0^2 = 25$

$x^2 = 25$

$x = \pm 5$

$(\pm 5, 0, 0)$

y-intercept $\quad x = 0, \ z = 0$

z-intercept $\quad x = 0, \ y = 0$

3:04 / 5:14

Sections

Traces

$$x^2 + y^2 + z^2 = 25$$

$x = 0$, or $y = 0$, $z = 0$

$x = 0 \quad y^2 + z^2 = 25 \rightarrow$ circle,

$y = 0 \quad x^2 + z^2 = 25 \rightarrow$ circle

$z = 0 \quad x^2 + y^2 = 25 \rightarrow$ Circ

Boundness

Sphere is a bounded Surface

Symmetry $x^2+y^2+z^2=25$

$(x,y,z) \rightarrow (x,y,-z)$

$x^2+y^2+(-z)^2=25 \Rightarrow x^2+y^2+z^2=25 \rightarrow$ Symmetrical about xy-plane

xz, yz origin

Example:- Find the eq. of sphere with centre (2,1,3) and radius 4.

Sol:-

$$C(x_0, y_0, z_0) = C(2,1,3), \quad \text{radius} = r = 4$$

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$$

$$(x-2)^2 + (y-1)^2 + (z-3)^2 = 4^2$$

$$x^2 - 2(x)(2) + 2^2 + y^2 - 2(y)(1) + 1^2 + z^2 - 2(z)(3) + 3^2 = 4^2$$

$$x^2 - 4x + 4 + y^2 - 2y + 1 + z^2 - 6z + 9 = 16$$

$$x^2 + y^2 + z^2 - 4x - 2y - 6z + 4 + 1 + 9 - 16 = 0$$

$$x^2 + y^2 + z^2 - 4x - 2y - 6z - 2 = 0$$

Example:- Find the centre and radius of the sphere

$$x^2+y^2+z^2+6x+4y-10z-11=0$$

Sol

$$x^2+6x+y^2+4y+z^2-10z=11$$

$$x^2+2(x)(3)+3^2+\underline{y^2+2(y)(2)+2^2}+\underline{z^2-2(z)(5)+5^2}=11+3^2+2^2+5^2$$

$$(x+3)^2+(y+2)^2+(z-5)^2=11+9+4+25$$

$$(x-(-3))^2+(y-(-2))^2+(z-5)^2=49=7^2$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$

$$(x-(-3))^2+(y-(-2))^2+(z-5)^2=49=7^2$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2$$

$$c(x_0,y_0,z_0)=c(-3,-2,5)$$

radius $= r = 7$

Example:- Find the eq. of sphere with centre $(2,3,-1)$ and tangent to the plane $3x-2y+6z=1$

Sol:

Distance of any pt. $P(x_0, y_0, z_0)$ from a plane $ax+by+cz=d$ is given by.

$$\text{distance} = \frac{|ax_0+by_0+cz_0-d|}{\sqrt{a^2+b^2+c^2}}$$

$$\text{distance} = \frac{|3(2)-2(3)+6(-1)-1|}{\sqrt{3^2+(-2)^2+6^2}}$$

Acti
Go to

$$\text{radias} = \frac{|6-6-6-1|}{\sqrt{9+4+36}}$$

$$\text{radias} = \frac{|-7|}{\sqrt{49}} = \frac{7}{7} = 1$$

$r = 1 \quad , \quad C(2, 3, -1)$

$$(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2 = r^2$$

$$(x-2)^2 + (y-3)^2 + (z-(-1))^2 = 1^2$$

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)=r^2$$

$$(x-2)^2+(y-3)^2+(z-(-1))^2=1^2$$

$$x^2-4x+4+y^2-6y+9+z^2+2z+1=1$$

$$x^2+y^2+z^2-4x-6y+2z+14-1=0$$

$$x^2+y^2+z^2-4x-6y+2z+13=0$$

How to recognize Sphere:

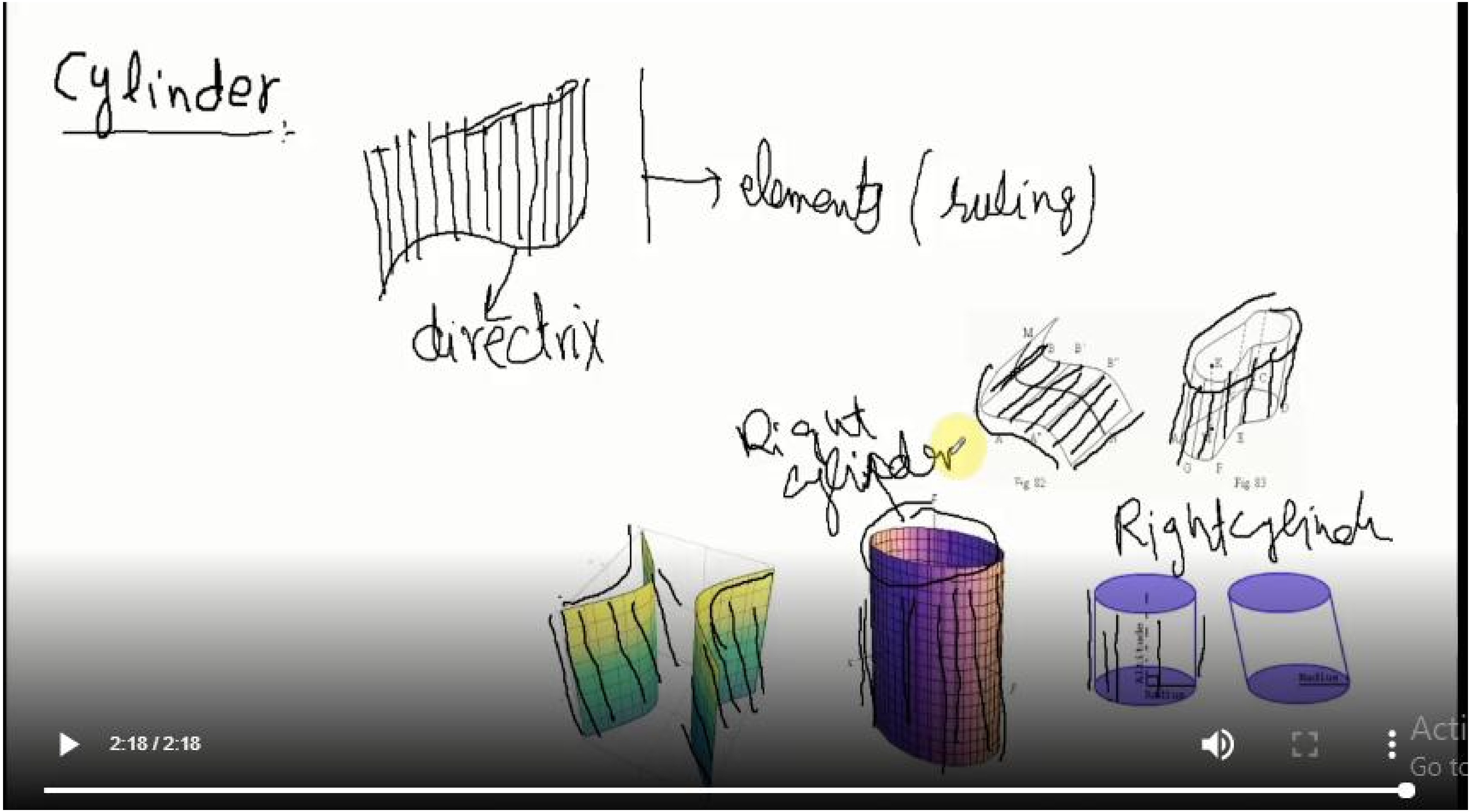
Coefficients x^2, y^2, z^2

xy, xz, yz

$x^2+y^2+z^2=16 \longrightarrow$ Sphere

$x^2+y^2+z^2-5x-7y=35 \longrightarrow$ Sphere

$2x^2+y^2-z^2=42 \longrightarrow$ Not Sphere

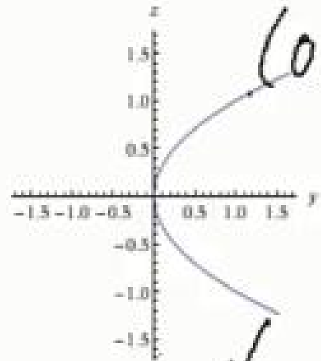

Cylinder
element (ruling)
directrix
Right cylinder
Right cylinder
Fig 82
Fig 83
Altitude
Radius
Radius
2:18 / 2:18

Example:

Find an equation of the cylinder with directrix $C: y - z^2 = 0$ and having elements parallel to $\bar{n} = [1 \ \ 2 \ \ 3]$.

Sol.

$$\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$



$(0, y_1, z_1)$

$y - z^2 = 0$

$y = z^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y_1 \\ z_1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$x = t, \quad y = y_1 + 2t, \quad z = z_1 + 3t$

$y = y_1 + 2x \Rightarrow y_1 = y - 2x$

$x = t, \quad y = y_1 + 2t, \; z = z_1 + 3t$

$y = y_1 + 2x \Rightarrow y_1 = y - 2x$

$z = z_1 + 3x$

$z_1 = z - 3x$

$y - z^2 = 0$

$y_1 - z_1^2 = 0$

$y - 2x - (z - 3x)^2 = 0$

or $\quad y - 2x = (z - 3x)^2$

Acti
Go to

$y - 2x - (z - 3x) = 0$

or $y - 2x = (z - 3x)^2$

trace of the cylinder at $x = d$.

$y - 2d = (z - 3d)^2$

Example:

Find an equation of the right cylinder whose directrix is the circle with centre(5,3,0) and the radius 4.

Sol

Eq. of directrix.

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

$$c(a,b,c)=c(5,3,0),\quad r=4$$

$$(x-5)^2+(y-3)^2+(z-0)^2=4^2$$

$$x^2-10x+25+y^2-6y+9+z^2=4^2$$

$$x^2+y^2+z^2-10x-6y+34=16$$

$$x^2+y^2+z^2-10x-6y+18=0$$

Example:

Find an equation of the right cylinder whose directrix is the circle with centre(5,3,0) and the radius 4.

Sol.

Eq. of directrix.

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

$$c(a,b,c)=c(5,3,0),\quad r=4$$

$$(x-5)^2+(y-3)^2+(z-0)^2=4^2$$

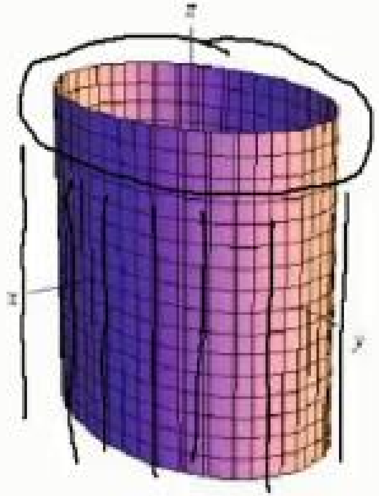
$$x^2-10x+25+y^2-6y+9+z^2=4^2$$

$$x^2+y^2+z^2-10x-6y+34=16$$

$$x^2+y^2+z^2-10x-6y+18=0$$ which is the eq of Cylin

Elliptic cylinder:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$\frac{x^2}{a^2} + \frac{z^2}{b^2} = 1$$

$$\frac{y^2}{a^2} + \frac{z^2}{b^2} = 1$$

Example Discuss the surface $8x^2+15y^2-5=0$

Sol

$$8x^2+15y^2-5=0$$

$$8x^2+15y^2=5$$

$$\frac{8x^2}{5}+3y^2=1$$

$$\frac{x^2}{5/8}+\frac{y^2}{1/3}=1 \Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$