MTH644

Measure Theory

By Mahar Azaq Sazdar Muhammadi

Topic #01

Introduction:

Mathematics is a place where you can do things which you can't do in the real world.

(Marcus Du Sautoy)

Why the measur theory?

Do the measurements really need Abstraction and Concrete rigorous grounds?

yes!!.... Due to.... Infinity ∞ and Infinitesimals ε.

Srinivasa Ramanujan: the man who knew infinity(∞)!

There are different ways to look at the size of a set.

$\{0,3,6,9\}$ has order 4, but on real line, these are "Negligible".

For $\{1\}\cup\{1,2\}\cup\{1,2,3\}\cup\ldots, 1\{1\}\cup\{1,2\}\cup\{1,2,3\}\cup\ldots$

$= 1+2+3+\ldots = ?$

But still on real line these are "Negligible".

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So a singal point takes up no real width on the real line.

The mathematical concept of measure is the generalization of lengths (size) in $\mathbb{R}$, are in $\mathbb{R}^2$ volume in $\mathbb{R}^3$ and hence in $\mathbb{R}^4$ and other spaces.

Starting from the lengths of intervals on $\mathbb{R}$.

Intuitively:

I) $|[0,1]| = 1$, and all $(0,1), (0,1], [0,1) \subset [0,1]$

$\Rightarrow |(0,1)|, |(0,1]|, |[0,1)| \leq |[0,1]|$

(Monotonicity preserves with respect to measure in $\mathbb{R}$)

And, $|(2,\infty)| = \infty$

Informally, the measure is a monotonic mapping from a sunsets of $\mathbb{R}$ to non-negative extended real numbers.

II) $[-1/2, 1/2]$ is $[0,1]$ i.e. transleted "1/2" unit to the left. And both have length (measure) "1".

(Translation Invariant)

III) $|(0,2]| = |(0,1] \cup (1,2]|$

$\Rightarrow 2 = 1+1$

Similarly, $(0,1] = (0,1) \cup \{1\} \Rightarrow 1 = 1+0$, means:

"the whole is the sum of its parts" (additivity).

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But for, $|(0,1]| = \left|\bigcup_{x\in(0,1]}\{x\}\right| \Rightarrow 1 = 0+0+0$

Therefore, we take the "whole is the sum of its parts", if a set is decomposed into a countable number of disjoint sets. (Countable Subadditivity)

Topic #02

Ring of Sets:

Let $X \neq \phi$, and "A" be a class or collection of subsets of "X". Then "A" is said to be a ring if:

i) $A \cup B \in A$
ii) $A - B \in A$
$\Big\} \forall\ A, B \in A$

For Examples:

(i) $X = \{a,b,c\}$ and $A = \{\phi, X, \{b\}, \{a,c\}\}$

(all are subset of X.)

(ii) $X \neq \phi$, $P(X) = \{A : A \subseteq X\}$

$\Rightarrow$ $P(X)$ is a ring of "X".

(iii) N, $A = \{A : A \subseteq N$ such that $|A| < \infty\}$ (order of A)

$\Rightarrow A, B \in A \Rightarrow A \cup B \in A$, $A - B \in A$

Topic #3

Sigma Ring:

Let $X \neq \phi$, and "A" be collection of

subsets of "X". Then "A" is said to a σ-ring if:

i) $\forall A, B \in \mathcal{A} \Rightarrow A - B \in \mathcal{A}$

ii) For any sequence $\{A_i\}_{i=1}^{\infty}$ of subsets of "X"

$\Rightarrow \bigcup_{i=1}^{\infty} \{A_i\} \in \mathcal{A}$

Examples:

$X \neq \phi$, $\mathcal{A} = \{\phi, X\}$ is σ-ring.

and $P(X) = \{A : \forall A \subseteq X\}$ is σ-ring.

Topic #04

Ring vs Sigma Ring:

σ-ring is also a ring because σ-ring is satisfied the both properties of ring.

Fo N, $\mathcal{A} = \{A \subset N : |A| < \infty\}$ is ring.
(finite)

$\Rightarrow A - B \in A$

Let $\{n_1\}, \{n_2\}, \{n_3\} \ldots$ be a sequence finite subsets of N.

But $\left| \bigcup_{i=1}^{\infty} \{n_i\} \right| \neq \infty \Rightarrow$ "A" is not σ-ring.

Topic #05

Boolean Algebra (Algebra of Sets):

Let $X \neq \phi$, and "A" is collection of "X". Then "A" is said to a boolean Algebra if

i) $\forall A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$ ii) $\forall A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

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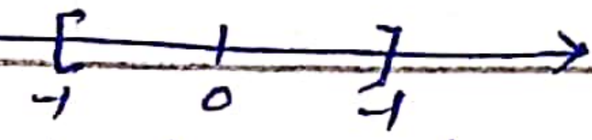
Examples:

(i) $X = \{a, b, c\}$, $\mathcal{A} = \{\phi, X, \{a\}, \{b, c\}\}$

$\mathcal{A}$ is boolean algebra.

(ii) $X \neq \phi$, $P(X) = \{A \subseteq X\}$ is boolean algebra.

(iii) let $X = [-1, 1] \subseteq \mathbb{R}$



and $\mathcal{A} = \{\phi, X, [-1, 0), [0, 1]\}$ is boolean algebra.

Topic #06

Theorem: Let "$\mathcal{A}$" be a ring if subsets of $X \neq \phi$, then (i) $\phi \in \mathcal{A}$, (ii) $A \Delta B \in \mathcal{A}$, (iii) $A \cap B \in \mathcal{A}$

symeteric relation

Proof: (i) $\forall$ $A, B \in \mathcal{A}$ $\Rightarrow$ $A - B \in \mathcal{A}$

In particular if $A = B$.

$A - B = A - A = \phi \in \mathcal{A}$ $\because$ by definition of ring.

(ii) $\forall$ $A, B \in \mathcal{A}$

$\Rightarrow$ $A - B, B - A \in \mathcal{A}$

$\Rightarrow$ $(A - B) \cup (B - A) \in \mathcal{A}$

$\Rightarrow$ $A \Delta B \in \mathcal{A}$

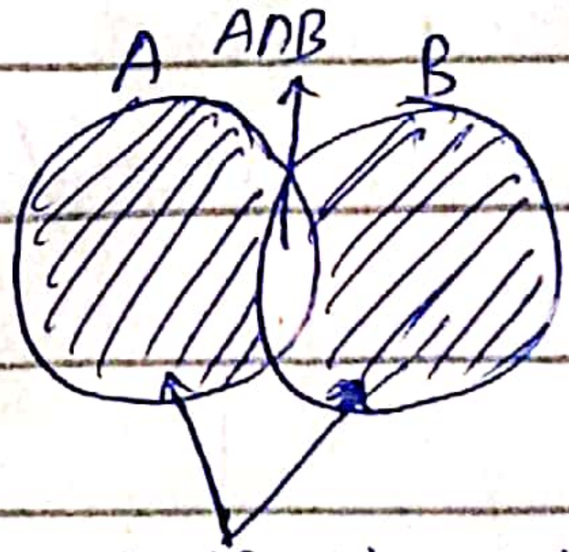
$(A - B) \cup (B - A) = A \Delta B$

A symmetric B

(iii) $\forall$ $A, B \in \mathcal{A}$

$\Rightarrow$ $A \Delta B \in \mathcal{A}$, $A \cup B \in \mathcal{A}$

by definition of ring.



From figure

$A \cup B = [(A - B) \cup (B - A)] \cup (A \cap B) = (A \Delta B) \cup (A \cap B)$

and $A \cap B = [(A \cup B) - (A \Delta B)] \in \mathcal{A}$

$\Rightarrow$ $A \cap B \in \mathcal{A}$

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Topic # 07

Theorem 1: Every Algebra is a Ring.

proof: By definition of Algebra "A" on set $X \neq \phi$;

$\forall A, B \in A \Rightarrow A \cup B \in A$ and $A^c \in A$

$\because A' \in A \; \forall \; A \in A \Rightarrow (A' \cup B) \in A$

$\Rightarrow$ $\because$ By definition of Algebra

$\Rightarrow (A' \cup B)' \in A$

$\Rightarrow A'' \cap B' \in A$ $\because$ By De-Morgan's law

$\Rightarrow A \cap B' \in A$ $\because A'' = A \; \forall$ set "A"

$\Rightarrow A - B \in A$ $\because A \cap B' = A - B \; \forall \; A, B \in X$

$\Rightarrow$ "A" is also a Ring.

Topic # 08

Theorem 2: For a $X \neq \phi$, Then ring "A" if subsets of "X" is also an algebra iff $X \in A$.

proof: p: ring "A" is an algebra.

q: $X \in A$

Given: "p" and To prove: "q";

$\because$ For $X \neq \phi$, "A" is a ring $\Rightarrow \forall A \in A$

$\Rightarrow \phi \in A$ $\because$ $A - A = \phi$ $\Rightarrow (A - A) \in A$

$\Rightarrow \phi' \in A \Rightarrow \phi'^{A} = X \in A$

Now Given: "q" and To prove: "p"

(i) $\because X \in A$ and "A" is a ring

$\Rightarrow \forall A \in A \Rightarrow (X - A) \in A$ $\because$ by definition of ring

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$\Rightarrow A' \in A$

(ii) $\forall A, B \in A \Rightarrow A \cup B \in A$

$\therefore$ (i) and (ii)

$\Rightarrow$ ring "A" of subset of "X" is also an algebra.

Topic # 09

6-Algebra of sets:

let $X \neq \phi$, and "A" be a class of subset of "X", Then "A" is said to be a 6-algebra if (i) $\forall A \in A \Rightarrow A' \in A$

(ii) whenever $\{A_1, A_2, A_3, \ldots\} \subseteq A$

or $\{A_i\}_{i=1}^{\infty} \subseteq A \Rightarrow \bigcup_{i=1}^{\infty} A_i \in A$

Example: (i) for any $X \neq \phi$, $A = \{\phi, X\}$ is a 6-algebra.

(ii) For any $X = \phi$; $P(X) = \{A : A \subseteq X\}$ is also a 6-algebra.

(iii) For $X = [-1, 1] \subseteq R$

$\Rightarrow A = \{\phi, X, [-1, 0), [0, 1]\}$ is also a 6-algebra.

(iv) For $X = N$, $A = \{A \subseteq N ; |A| < \infty \vee A \text{ is countable} \vee A' \text{ is countable}\}$

$A = E = \{2n : n \in N\}$, $E' = O = \{2n-1, n \in N\}$

if $X = R$ (uncountable)

$A = \{A \subset R : |A| < \infty \vee$ "A" countable $\ldots\}$

$Z \subset R$ but Z' is not countable.

(Not Holds)

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Topic #10

Theorem: Let $X \neq \phi$, and "A" is a class of subsets of X; then σ-ring "A" is also a σ-algebra iff $X \in A$.

proof: See Topic#08 for proof.

Theorem: Arbitrary intersection of σ-algebra on $X \neq \phi$ is also a σ-algebra.

proof: Let $Q = \{A_\alpha : \alpha \in I\}$ be a class of σ-algebra on $X \neq \phi$. clearly $Q \neq \phi$.

To prove $\bigcap_{\alpha=I} A_\alpha$ is also a σ-algebra.

(i) For any $A \subset X$, such that $A \in \bigcap_{\alpha=I} A_\alpha$

$\Rightarrow A \in A_\alpha \quad \forall \alpha \in I$

But each "A_α" is a σ-algebra.

$\Rightarrow A' \in A_\alpha ; \forall \alpha \in I$

$\Rightarrow A' \in \bigcap_{\alpha=I} A_\alpha \longrightarrow (1)$

If $x \in A \cap B$
Then $x \in A$ and
$x \in B$

(ii) Let $\{A_i\}_{i=1}^{\infty}$ be a sequence of subsets of $X \neq \phi$, such that $\{A_i\}_{i=1}^{\infty} \subseteq \bigcap_{\alpha=I} A_\alpha \Rightarrow \{A_i\}_{i=1}^{\infty} \subseteq A_\alpha \ \forall \alpha \in I$

$\because$ each A_α is a σ-algebra

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in A_\alpha \ \forall \alpha \in I$

$\Rightarrow \bigcup_{i=1}^{\infty} A_i \in \bigcap_{\alpha=I} A_\alpha \longrightarrow (2)$

from (1) and (2)

$\Rightarrow \bigcap_{\alpha=I} A_\alpha$ is also a σ-algebra.

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Topic # 11

problem: Give a counter example to show that union of two σ-algebra is not necessarily a σ-algebra.

Solution: let $X = \{a, b, c\}$, and defining two σ-algebra

$$A_1 = \{\phi, X, \{b\}, \{a, c\}\} \text{ and}$$

$$A_2 = \{\phi, X, \{c\}, \{a, b\}\}$$

Now taking

$$A_1 \cup A_2 = \{\phi, X, \{b\}, \{c\}, \{a, c\}, \{a, b\}\}$$

Taking finite (countable) sequence of subsets of $A_1 \cup A_2$ i.e. $\{b\}, \{c\}$, Then

$$\{b\} \cup \{c\} = \{b, c\} \notin A_1 \cup A_2$$

$\Rightarrow$ $A_1 \cup A_2$ is not a σ-algebra.

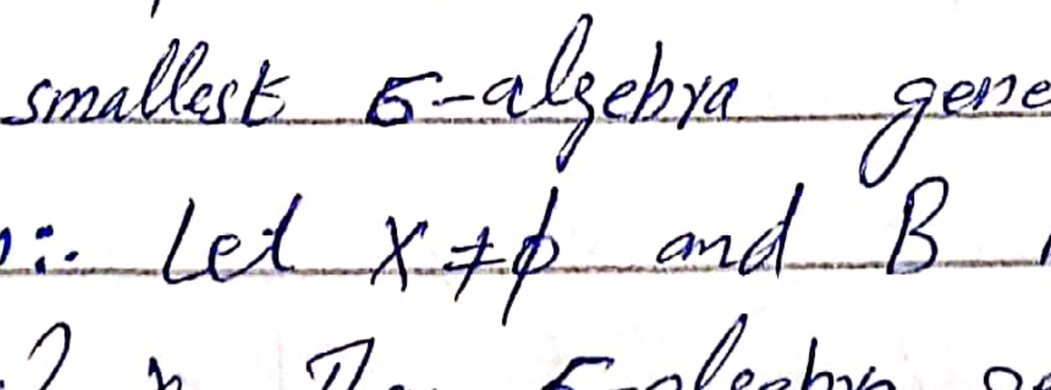
Topic # 12

Sigma (σ) - Algebra Generated by a Class:

For $X \neq \phi$, $A \subseteq X$ and $B = \{A\}$, Then $\{\phi, X, A, A'\}$ is a smallest σ-algebra generated by "B".

Definition: Let $X \neq \phi$ and B be a family of subsets of X, Then σ-algebra generated by "B" defined as the interseetion of all σ-algebra containing "B"

$$\bigcap_{i \in I} \{A_i : A_i \text{ is a } \sigma\text{-algebra}, B \subseteq A_i, \forall\, i \in I\}$$



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Topic #13

Theorem: Let "B" be a family of subsets of $X \neq \phi$, Then there exist a smallest sigma algebra containing "B".

proof: let $\mathcal{Q} = \{A_i : B \subseteq A_i \ \forall \ i \in I\}$

$\because P(X)$ is a σ-algebra containing "B"

$\Rightarrow \mathcal{Q} \neq \phi$.

We will prove:

(i) Existance (ii) Minimality

(i) $G = \bigcap_{i \in I} \{A_i : B \subseteq A_i, \ \forall \ i \in I\} \longrightarrow (1)$

$\because$ Arbitrary intersection of σ-algebra is a σ-algebra and $B \subseteq A_i, \ \forall \ i \in I \Rightarrow B \subseteq \cap A_i = G$.

$\Rightarrow$ "G" is a σ-algebra containing "B".

(ii) Let "J" is a σ-algebra containing "B".

$\Rightarrow J \in \mathcal{Q} \Rightarrow \bigcap_{i \in I} A_i \subseteq J \quad \because A \cap B \subseteq A$

$\Rightarrow G \subseteq J \Rightarrow$ "G" is a smallest σ-algebra.

Topic #14

Theorem: Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of elements of σ-algebra "A" of subsets of $X \neq \phi$, Then there exists also a sequence $\{F_n\}_{n=1}^{\infty}$ of ~~elements~~ subsets of "A" "X" such that

(i) $F_n \in A \ \forall \ n \in \mathbb{N}$, (ii) $F_n \subseteq E_n \ \forall \ n \in \mathbb{N}$,

(iii) $F_n \cap F_m = \phi, \ \forall \ m \neq n$, (iv) $\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} F_i$

proof: (i) Let we constract:

$F_1 = E_1$, $F_2 = E_2 - E_1$, $F_3 = E_3 - (E_1 \cup E_2)$

by induction: $F_n = E_n - (E_1 \cup E_2 \cup E_3 \cdots \cup E_{n-1})$

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$\Rightarrow F_n = E_n \cap (E_1 \cup E_2 \cup \dots \cup E_{n-1})'$ $\because A-B = A \cap B'$

$= E_n \cap (E_1' \cap E_2' \cap \dots \cap E_{n-1}')$ By De Morgan's Law

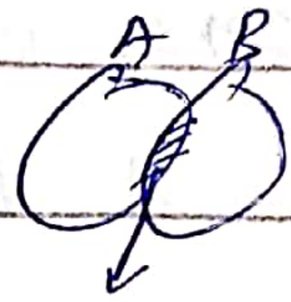
$\because E_n \in A$ and $\because E_1, E_2, \dots, E_{n-1} \in A$

$\Rightarrow E_1', E_2', \dots, E_{n-1}' \in A$

$\Rightarrow F_n = E_n \cap (E_1' \cap E_2' \cap \dots \cap E_{n-1}') \in A \longrightarrow (1)$

(ii) From eq. (1)

$\Rightarrow F_n \subseteq E_n \quad \forall \quad n \in \mathbb{N} \longrightarrow (2)$



$A \cap B \subseteq A, B$

(iii) $\forall \ m \neq n \Rightarrow m \leq n$.

Say without loss of generality $m < n$.

$\Rightarrow$ From (2) $F_m \subseteq E_m \Rightarrow F_m \cap F_n \subseteq E_m \cap F_n \longrightarrow (3)$

$\Rightarrow E_m \cap F_n = E_m \cap [E_n \cap (E_1' \cap E_2' \cap \dots \cap E_m' \cap \dots E_{n-1}')]$

$\Downarrow$ From eq. (1)

$E_m \cap F_n = \underbrace{(E_m \cap E_m')}_{\phi} \cap [E_n \cap (E_1' \cap E_2' \cap \dots \cap E_{n-1}')]$

$E_m \cap F_n = \phi$

$\Rightarrow F_m \cap F_n \subseteq \phi \Rightarrow F_m \cap F_n = \phi \quad \forall \ m \neq n$

(iv) $\because F_n \subseteq E_n \quad \forall \ n \in \mathbb{N}$

$\Rightarrow \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} E_n \longrightarrow (3)$

Let $y \in \bigcup_{n=1}^{\infty} E_n \Rightarrow y \in E_n$ for some $n \in \mathbb{N}$

~~And Say~~ Suppose "m" be a least positive integer

$y \in E_m$ but $y \notin E_1, E_2, \dots, E_{m-1}$

$\Rightarrow y \notin E_1 \cup E_2 \cup \dots \cup E_{m-1}$

$\Rightarrow y \in E_m - (E_1 \cup E_2 \cup \dots \cup E_{m-1})$

$\Rightarrow y \in E_m \Rightarrow y \in \bigcup_{m=1}^{\infty} E_m = \bigcup_{n=1}^{\infty} E_n$

$y \in A$ & $y \notin B$

$\Rightarrow y \in A - B$

$\sim q$ = negation q

$\therefore \Rightarrow \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} F_n \subseteq \bigcup_{n=1}^{\infty} E_n$

$\Rightarrow \boxed{\bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} F_n}$

Topic #15

An Important Result of Analysis:

Theorem: Suppose that for any $\varepsilon > 0$, if $a < b + \varepsilon$, then $a \leq b$.

proof: p: $a < b + \varepsilon$ and q: $a \leq b$

proof scheme: $\begin{cases} p \to \text{true} \text{ and } \sim q \to \text{true} \\ \text{and seek for contradiction.} \end{cases}$

$p \Rightarrow a < b + \varepsilon$ and $\sim q \Rightarrow a > b$

$\Rightarrow (a - b) > 0$

But we have given from p: $a < b + \varepsilon$

$\therefore$ In particular taking $\varepsilon = (a - b) > 0$

$\therefore$ $p \Rightarrow a < b + (a - b) \Rightarrow a < a$

i.e. a contradiction.

$\therefore$ $a < b + \varepsilon \Rightarrow a \leq b$; $\varepsilon \to 0$ arbitrarily $(\varepsilon > 0)$

$\begin{cases} p \to q \text{ if } p \to \text{true} \\ q \to \text{false}; \; p \to \sim q \to \text{false} \end{cases}$

Topic # 16

Set Fnction:

Let $X \neq \phi$ and $\mathcal{A} \subseteq p(X)$, be a class of subsets of X, then a function defined on "$\mathcal{A}$" as: $\mu : \mathcal{A} \to [0, \infty) \cup \{\infty\}$, i.e. every element (set) is mapped to a non-negative

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real extended real number.

For examples: (i) $[0,\infty) \subseteq \mathbb{R} \Rightarrow \mu[0,\infty) = \infty$

$(0,1) \subseteq \mathbb{R} \Rightarrow \mu(0,1) = 1-0 = 1$

(ii) $\mu : P(\mathbb{N}) \to [0,\infty) \cup \{\infty\} \quad \forall \quad E \subseteq P(\mathbb{N})$

if $E = \{1,2,3\} \Rightarrow \mu(E) = O(E) = 3$

$E = \{1,2,3,\dots\} \Rightarrow \mu(E) = O(E) = \infty$

Topic #17

Measure on a σ-Algebra:

Let $X \neq \phi$ and "A" be a σ-algebra on X, then a set function $\mu : A \to [0,\infty) \cup \{\infty\}$ is said to be (i) Finitely subadditive if

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) \leq \sum_{i=1}^{n} \mu(E_i)$$

(ii) Countably subadditive if

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i)$$

(iii) Finitely additive if

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i), \quad \forall \quad E_i \cap E_j = \phi \quad \forall\, i \neq j$$

(iv) Countably additive if

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i), \quad \forall \quad E_i \cap E_j = \phi \quad \forall\, i \neq j$$

(v) Monotone if

$\mu(A) \leq \mu(B)$ whenever $A \subseteq B$

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$\mu : A \to [0, \infty) \cup \{\infty\}$

measure on "A" if

(i) $\mu(\phi) = 0$

(ii) $\mu(A) \geq 0 \quad \forall \; A \in A$

(iii) μ is ~~finite~~ countably additive

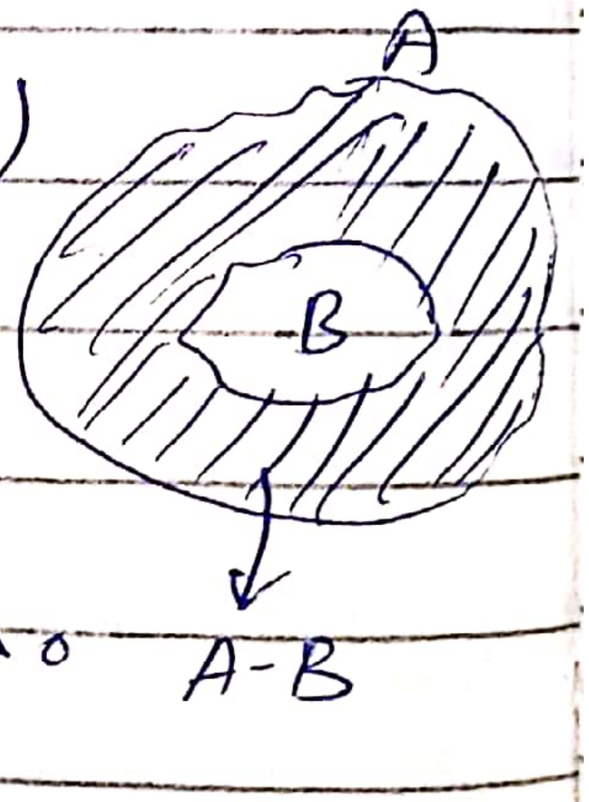
$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i) \quad \forall \; E_i \cap E_j = \phi \quad \forall \; i \neq j$$

Topic # 18

Monotonicity of Measure:

(i) If $\phi \neq A, B \in A$ such that $B \subseteq A$, then:

$$\mu(B) \leq \mu(A)$$



proof: $\because B \subseteq A \Rightarrow A = B \cup (A - B)$

$\Rightarrow \mu(A) = \mu[B \cup (A - B)]$

$\mu(A) = \mu(B) + \mu(A - B) \longrightarrow (1)$

$\Rightarrow \mu(A) \geq \mu(B)$

$\Rightarrow \mu(B) \leq \mu(A)$

$\begin{cases} a \neq b + c \\ a \geq b, c \geq 0 \\ \Rightarrow a \geq b \end{cases}$

Topic # 19

(ii) If $\mu(B) < \infty$, then $\mu(A - B) = \mu(A) - \mu(B)$

proof: From eq. (1) $\Rightarrow \mu(A) - \mu(B) = \mu(A - B)$

$[0, \infty]$ ← subtraction — Difference Operation on sets

Topic #20

Measurable Space:

For $X \neq \phi$, Let "A" be the σ-algebra on "X" Then the triple (X, A, μ) is called measurable space and members of "A" are called measurable sets.

"A" → class of subsets of "X".

Topic #21

Theorem:- Let (X, A, μ) be a measure space Then if $\exists$ a $\phi \neq A \in A$, such that $\mu(A) < \infty$ then $\mu(\phi) = 0$

proof: $\because A = A \cup \phi$

$\Rightarrow \mu(A) = \mu(A \cup \phi) = \mu(A) + \mu(\phi)$

By countable additivity of "μ"

$\Rightarrow \mu(\phi) = 0$

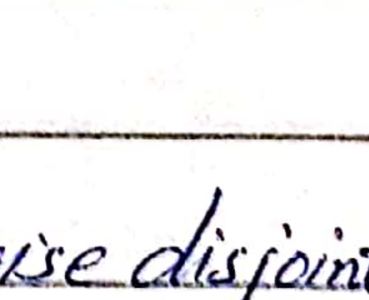
Topic #22

Theorem: Let $\{A_n\}$ is a sequence of pair wise disjoint subsets of "X" in σ-algebra "A" then for any $T \in A$,

$\mu(T) = \sum_{n=1}^{\infty} \mu(A_n \cap T) + \mu\left(T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)^c\right)$.

proof: $\because$ For any $T \in A \Rightarrow T = T \cap X$

$\Rightarrow T = T \cap \left[\left(\bigcup_{n=1}^{\infty} A_n\right) \cup \left(\bigcup_{n=1}^{\infty} A_n\right)^c\right]$



$\Rightarrow T = \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)\right] \cup \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)^c\right]$

$A \cup A^c = X$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

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$$\begin{cases} T \cap \left(\bigcup_{n=1}^{\infty} A_n\right) = T \cap (A_1 \cup A_2 \cup A_3 \cup \cdots) \\ = (T \cap A_1) \cup (T \cap A_2) \cup \cdots \\ = \bigcup_{n=1}^{\infty} (T \cap A_n) \end{cases}$$

$$\Rightarrow T = \left[\bigcup_{n=1}^{\infty} (T \cap A_n)\right] \cup \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)^c\right]$$

$$\Rightarrow \mu(T) = \mu\left\{\left[\bigcup_{n=1}^{\infty} (T \cap A_n)\right] \cup \left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)^c\right]\right\}$$

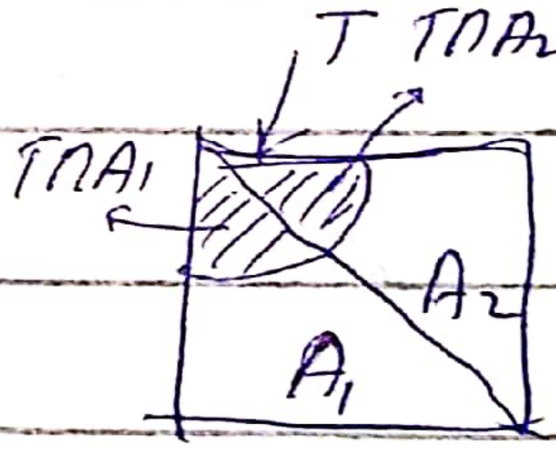
By additive property of "μ"

$$\Rightarrow \mu(T) = \mu\left[\bigcup_{n=1}^{\infty} (T \cap A_n)\right] + \mu\left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)^c\right]$$

$\because$ All $T \cap A_n$'s are disjoint

$\therefore$ by additive property of "μ"

$$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu(B_1 \cup B_2 \cup \cdots)$$



$$= \mu(B_1) + \mu(B_2) + \cdots$$

$$= \sum_{n=1}^{\infty} \mu(B_n)$$

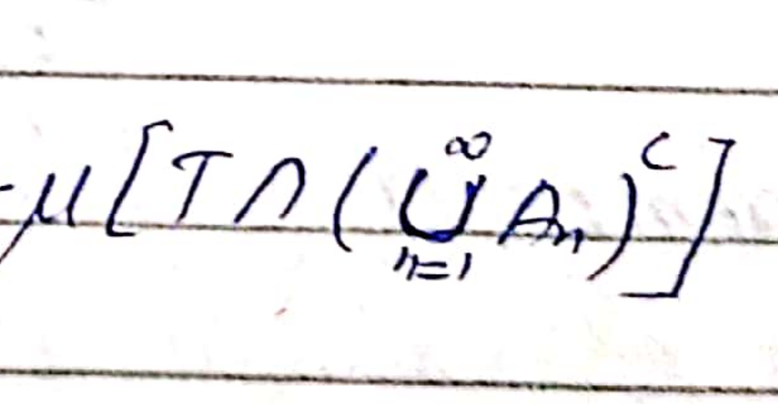
$$\Rightarrow \mu(T) = \sum_{n=1}^{\infty} \mu(T \cap A_n) + \mu\left[T \cap \left(\bigcup_{n=1}^{\infty} A_n\right)^c\right]$$

Topic #23

Theorem: In a measure space (X, A, μ), if $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F) = \mu(E \cup F) = \mu(E \cap F)$

proof: $\because E \Delta F = (E - F) \cup (F - E)$

$\Rightarrow \mu(E \Delta F) = \mu[(E - F) \cup (F - E)]$



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E symmetric F

$\Rightarrow \mu(E \Delta F) = \mu(E-F) + \mu(F-E)$

Given that $\mu(E \Delta F) = 0$

$\Rightarrow 0 = \mu(E-F) + \mu(F-E)$

$\Rightarrow \mu(E-F) = 0$ and $\mu(F-E) = 0$

$a+b=0$ $a, b \geq 0$ $\Rightarrow a=0$ and $b=0$

Note that $F = (F-E) \cup (F \cap E)$

$\because F-E$ and $F \cap E$ are disjoint

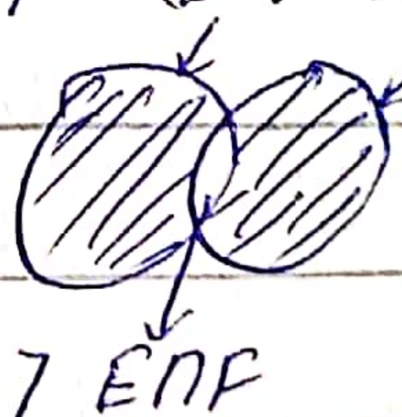
$\Rightarrow \mu(F) = \mu[(F-E) \cup (F \cap E)]$

$\Rightarrow \mu(F) = \mu(F-E) + \mu(F \cap E)$

$\Rightarrow \mu(F) = 0 + \mu(F \cap E)$

$\Rightarrow \mu(F) = \mu(F \cap E) \rightarrow (1)$

$E \Delta F = (E-F) \cup (F-E)$



Also $E \cup F = (E \Delta F) \cup (E \cap F)$

$\Rightarrow \mu(E \cup F) = \mu[(E \Delta F) \cup (E \cap F)]$

$\Rightarrow \mu(E \cup F) = \mu(E \Delta F) + \mu(E \cap F)$

$\Rightarrow \mu(E \cup F) = 0 + \mu(E \cap F)$

$\Rightarrow \mu(E \cup F) = \mu(E \cap F) \rightarrow (2)$

Now $E = (E-F) \cup (E \cap F)$

$E-F$ and $E \cap F$ are disjoint

$\Rightarrow \mu(E) = \mu[(E-F) \cup (E \cap F)]$

$\Rightarrow \mu(E) = \mu(E-F) + \mu(E \cap F)$

$\Rightarrow \mu(E) = 0 + \mu(E \cap F)$

$\Rightarrow \mu(E) = \mu(E \cap F) \rightarrow (3)$

From (1), (2) and (3) $\Rightarrow \mu(E) = \mu(F) = \mu(E \cup F) = \mu(E \cap F)$

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Topic #24

Finite Measure:

For $\phi \neq X$ the triplet (X, A, μ) is said to be finite measure space if $\mu(X) < \infty$ and "μ" is called finite measure.

Sigma(σ)-Finite Measure:

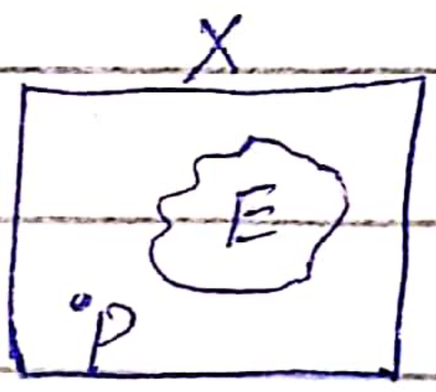
For $\phi \neq X$, the triple (X, A, μ) is σ-finite measure if and only if $\mu(X) < \infty$ and $\exists$ a sequence $\{A_n\}$ of subsets of X, such that

$$\bigcup_{n=1}^{\infty} A_n = X \text{ and } \mu(A_n) < \infty \quad \forall A_n \in A$$

For example:

(i) $X \neq \phi$, $\mu : p(X) \to [0, \infty]$

$$\mu(E) = \begin{cases} 0 & \text{if } p \notin E \\ 1 & \text{if } p \in E \end{cases}$$



for any fixed point "p" in "X".

(ii) $\mathbb{Z} = \{0, \pm 1, \pm 2, \cdots\}$

$\mu : p(\mathbb{Z}) \to [0, \infty]$

$$\mu(A) = \begin{cases} |A| = c(A) = \#(A) & \text{if "A" is finite} \\ \infty & \text{if "A" is infinite.} \end{cases}$$

Counting measure.

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Topic #25

For $\phi \neq X$, defining; $\mu : P(X) \to [0, \infty]$, as

$$\mu(A) = \begin{cases} 0 & ; p \notin A \\ 1 & ; p \in A \end{cases}$$, for fixed point $p \in X$, Then show that "μ" is a finite measure on "X."

Proof: (i) $\because p \in \phi \Rightarrow \mu(\phi) = 0$

by given definition of "μ"

(ii) for any $A \subset X$, $\forall A \in \mathcal{A}$; by definition

$\mu(A) \geq 0$

(iii) Let $\{A_n\}$ be the sequence of pairwise disjoint subset of "X" in "$\mathcal{A}$".

μ on σ-algebra on "X"
i) $\mu(\phi) = 0$
ii) $\mu(A) \geq 0$
iii) $\{A_n\} \to$ disjoint $\in \mathcal{A}$
$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{i=1}^{\infty} \mu(A_n)$
- Finite μ $\mu(X) < \infty$

Case I: let $p \in \left(\bigcup_{n=1}^{\infty} A_n\right) \Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 1$

$\Rightarrow p \in A_k$ for some fixed $k = n$.

$\Rightarrow \mu(A_k) = 1$ and $\mu(A_n) = 0 \quad n \neq k$

$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) = 1 = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$

Case II: $p \notin \left(\bigcup_{n=1}^{\infty} A_n\right) \Rightarrow p \notin A_n \ \forall n$

$\Rightarrow \mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0$ and $\mu(A_n) = 0 \ \forall n \in \mathbb{N}$

$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) = 0$

So $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$

Now, Trivially $p \in X \Rightarrow \mu(X) = 1$

$\Rightarrow \mu(X) < \infty$

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$\eta \rightarrow$ Eta

Topic #26

Closed Operations of Measure:-

(i) Sum of Measures:

Let "μ" and "η" are measure on σ-algebra of $X \neq \phi$, Then

(i) $\mu(\phi) = 0$, $\eta(\phi) = 0$

(ii) $\mu(A) \geq 0$, $\eta(A) \geq 0$, $\forall A \in \mathcal{A}$

(iii) For any sequence $\{A_n\}$ in $\mathcal{A}$ of disjoint pairs

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$$

and

$$\eta\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \eta(A_n)$$

(1) $(\mu+\eta)(\phi) = \mu(\phi) + \eta(\phi) = 0+0 = 0$

(2) $(\mu+\eta)(A) = \mu(A) + \eta(A) \geq 0 \quad \because \mu(A), \eta(A) \geq 0$

(3) $(\mu+\eta)\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right) + \eta\left(\bigcup_{n=1}^{\infty} A_n\right)$

$$= \sum_{n=1}^{\infty} \mu(A_n) + \sum_{n=1}^{\infty} \eta(A_n)$$

$$= \sum_{n=1}^{\infty} \left(\mu(A_n) + \eta(A_n)\right)$$

$$= \sum_{n=1}^{\infty} (\mu+\eta)(A_n)$$

For any say $\alpha = \mathbb{R}$

$\alpha\mu(A) = \alpha(\mu A)$ scaler composition.

Counter case: For $A \in \mathcal{A}$

$\mu(A) \geq 0$, $\eta(A) \geq 0$

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But $\mu(A) - \eta(A) < 0$

if $\eta(A) > \mu(A)$

$\Rightarrow$ "$\mu - \eta$" is not defined

Topic #27

Outer Measure:

For $X \neq \phi$, a function $\mu^* : P(X) \to [0, \infty]$ is said to be outer measure if:

(i) $\mu^*(\phi) = 0$

(ii) whenever $A \subseteq B$: $\mu^*(A) \leq \mu^*(B)$

(iii) If $\{A_n\}$ be a sequence of subsets of "X" i.e $\{A_n\}_{n=1}^{\infty} \subseteq P(X)$. Then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)$$

For Example $X \neq \phi$: define $\mu^* : P(X) \to [0, \infty]$

$$\mu^*(A) = \begin{cases} 0 & \text{if } A = \phi \\ 1 & \text{if } A \neq \phi \end{cases}$$

Sol: (i) If $\phi = A \Rightarrow \mu^*(A) = 0$ trivially true.

(ii) If $A \subseteq B \Rightarrow \mu^*(A) \leq \mu^*(B)$

(a) $A = B = \phi \Rightarrow \mu^*(A) = 0 = \mu^*(B)$

(b) $A = \phi, B \neq \phi \Rightarrow \mu^*(A) = 0 < \mu^*(B) = 1$

(c) $A \neq B \neq \phi \Rightarrow \mu^*(A) = 1 = \mu^*(B)$

In all three cases: $\mu^*(A) \leq \mu^*(B)$

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(iii) Let $\{A_n\}$ be the sequence of subsets in "X".

(a) If all $A_n = \phi \quad \forall\, n \in N \Rightarrow \left(\bigcup_{n=1}^{\infty} A_n\right) = \phi$

$\Rightarrow \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) = 0 = \mu^*(A_n) = \sum_{n}^{\infty} \mu^*(A_n)$

(b) If $A_n \neq \phi$ for some $n \in N$

$\Rightarrow \mu^*(A_n) = 1$ for some $n \in N$

$\Rightarrow \sum_{n=1}^{\infty} \mu(A_n) \geq 1 \longrightarrow (1)$

and If $A_n \neq \phi \Rightarrow \bigcup_{n=1}^{\infty} A_n \neq \phi$

$\Rightarrow \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 \longrightarrow (2)$

From (1) and (2)

$\Rightarrow \mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$

Topic # 28

Lebesgue Outer Measure:

For $\phi \neq E \subseteq R$, the Lebesgue measure of "E" defined on $P(R)$ as $m^*: P(R) \to [0, \infty) \cup \{\infty\}$ and given by:

$m^*(E) = Inf\left\{\sum_{n=1}^{\infty} l(I_n) : A \subseteq \bigcup_{n=1}^{\infty} I_n \text{ where } \{I_n\} \text{ is a sequence of open intervals covering "E"}\right\}$

Here $l(I_n) \to$ length of nth interval $\{I_n\}$.

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Topic #29

Non-negativity and Monotonicity of Lebesgue Outer Measure:

Since $l(I_n) \geq 0, \forall\ n \in \mathbb{N}$

$\Rightarrow \sum_{n=1}^{\infty} l(I_n) \geq 0 \quad \because E \subseteq \bigcup_{n=1}^{\infty} I_n$

$\{I_n\}$ on $\mathbb{R}$

$\Rightarrow \inf\left\{\sum_{n=1}^{\infty} l(I_n) : E \subseteq \bigcup_{n=1}^{\infty} I_n, \{I_n\} \text{ be the sequence of interval covering } E\right\} \geq 0$

$\Rightarrow m^*(E) \geq 0$

Let $\phi \neq A, B \in P(\mathbb{R})$ such that $A \subseteq B$ and $\{I_n\}$ be a sequence of intervals covering "B" in $\mathbb{R}$.

$\Rightarrow B \subseteq \bigcup_{n=1}^{\infty} I_n \Rightarrow A \subseteq \bigcup_{n=1}^{\infty} I_n$

$\Rightarrow m^*(A) \leq \sum_{n=1}^{\infty} l(I_n)$, $B \subseteq \bigcup_{n=1}^{\infty} I_n$

$\Rightarrow m^*(A)$ is lower bound of $\left\{\sum_{n=1}^{\infty} l(I_n) : B \subseteq \bigcup_{n=1}^{\infty} I_n\right\}$

$\Rightarrow m^*(A) \leq \inf\left\{\sum_{n=1}^{\infty} l(I_n) : B \subseteq \bigcup_{n=1}^{\infty} I_n\right\}$

$\Rightarrow m^*(A) \leq m^*(B)$

$\{x\} \subset \mathbb{R}, \forall \varepsilon > 0$

$\{x\} \subseteq (x-\varepsilon, x+\varepsilon)$ | $A \subseteq B$

$\Rightarrow m^*\{x\} \leq \frac{2\varepsilon}{2} = \varepsilon$ | $m^*(A) \leq m^*(B)$

$\Rightarrow m^*\{x\} = 0$ where $\varepsilon \to 0$ arbitrary

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Topic #30

Countable Subadditivity of Lebesgue Outer Measure:

Theorem: For any arbitrary seq $\{A_n\}_{n=1}^{\infty} \subseteq P(\mathbb{R})$ $m^*(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} m^*(A_n)$ i.e. m^* is countably subadditive.

proof: (i) For fixed $n \in \mathbb{N}$ if $m^*(A_n) = \infty$

$\Rightarrow m^*(\bigcup_{n=1}^{\infty} A_n) = \infty = m^*(A_n) = \sum_{n=1}^{\infty} m^*(A_n)$

(ii) For $m^*(A_n) < \infty$, $\forall\ n \in \mathbb{N}$

But $m^*(A_n) = Inf\left\{\sum_{m=1}^{\infty} l(I_{nm}) : A_n \subseteq \bigcup_{m=1}^{\infty} I_{nm}\right.$,

where $\{I_{nm}\}_{m=1}^{\infty}$ is open covering of $\{A_n\}$

For $\varepsilon > 0$, $\Rightarrow \frac{\varepsilon}{2n} > 0$

$\Rightarrow$ there exists a sequence $\{I_{nm}\}_{m=1}^{\infty}$

$$\sum_{m=1}^{\infty} l(I_{nm}) < m^*(A) + \frac{\varepsilon}{2n} \longrightarrow (1)$$

$\because A_n \subseteq \bigcup_{m=1}^{\infty} I_{nm} \Rightarrow \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} (\bigcup_{m=1}^{\infty} I_{nm})$

Here, countable union $\bigcup_{n=1}^{\infty} (\bigcup_{m=1}^{\infty} I_{nm})$ is countable sets : $\bigcup_{m=1}^{\infty} I_{nm}$; is countable.

$\Rightarrow m^*(\bigcup_{n=1}^{\infty} A_n) = Inf\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} l(I_{nm}) :\right.$

$\bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} (\bigcup_{m=1}^{\infty} I_{nm})$, $\longrightarrow$

, where: $\{I_{nm}\}$ is sequence of open intervals covering $\bigcup_{n=1}^{\infty} A_n$ }

$$m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} l(I_{nm})$$

From (1) $\Rightarrow$ $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty}\left(m^*(A_n) + \frac{\varepsilon}{2^n}\right)$

$$= \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n}$$ → Geometric series.

$$= \sum_{n=1}^{\infty} m^*(A_n) + \varepsilon \cdot 1$$

$\Rightarrow$ $m^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m^*(A_n)$

$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

$S = \frac{a}{1-r}$

$S = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$

Topic # 31

Lebesgue Outer Measure of Countable set:

Since every countable set can expressed in the form of sequence: i.e. 1, 2, 3, ---- n, ---

$\Rightarrow$ $A = \{x_1, x_2, x_3, \cdots x_n, \cdots\} = \bigcup_{n=1}^{\infty} \{x_n\}_{n=\phi}^{\infty}$

$$0 \leq m^*(A) = m^*\left(\bigcup_{n=1}^{\infty} \{x_n\}_{n=1}^{\infty}\right) \leq \sum_{n=1}^{\infty} m^*\{x_n\} = 0$$

$\because$ Lebesgue outer Measure of $\{x\} = 0$

$\Rightarrow$ $m^*(A) = 0$

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Topic #32

Theorem: For any $\phi \neq A \subseteq \mathbb{R}$ and any $\varepsilon > 0$, there is an open set "U" such that "$A \subseteq U$" and $m^*(U) \leq m^*(A) + \varepsilon$

proof: By definition

$m^*(A) = \inf\left\{ \sum_{n=1}^{\infty} l(I_n) \cdot A \subseteq \bigcup I_n \right\}$, where $\{I_n\}_{n=1}^{\infty}$ is a sequence of open intervals covering "A" in $\mathbb{R}$ and $l(I_n)$ is the length of nth interval

For any $\varepsilon > 0$, $\exists$ $\{I_n\}_{n=1}^{\infty}$ in $\mathbb{R}$

such that $m^*(A) + \varepsilon > \sum_{n=1}^{\infty} l(I_n)$

$\Rightarrow \sum_{n=1}^{\infty} l(I_n) < m^*(A) + \varepsilon \longrightarrow (1)$

Union of open interval in "$\mathbb{R}$" is an open set $\Rightarrow$ thereexists U an open set.

$\bigcup_{n=1}^{\infty} I_n = U$ (open set)

$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty} I_n\right) = m^*(U)$

$\Rightarrow m^*(U) = m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \longrightarrow (2)$

$\because m^*\left(\bigcup_{n=1}^{\infty} I_n\right) \leq \sum_{n=1}^{\infty} m^*(I_n)$

From (2) $\Rightarrow m^*(U) \leq \sum_{n=1}^{\infty} m^*(I_n) = \sum_{n=1}^{\infty} l(I_n)$

From (1) $\Rightarrow m^*(U) < m^*(A) + \varepsilon$

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Topic #33

The Lebesgue Outer Measure of an Interval:

Theorem: Lebesgue outer measure of an interval is its lengths. i.e. $m^*(a,b) = m^*[a,b] = m^*(a,b] = m^*[a,b) = b-a$.

proof: Case I: Let $I=[a,b]$ in $\mathbb{R}$. By definition:

$m^*(I) = Inf\left\{\sum_{i=1}^{\infty} \ell(I_i) : I \subseteq \bigcup_{i=1}^{\infty} I_i, \text{ where } \{I_i\}_{i=1}^{\infty} \text{ are open intervals covering } I \text{ in } \mathbb{R} \text{ and } \ell(I_i) \text{ is length of } i\text{th interval}\right\}$ ⟶ ①

In particular for $\varepsilon>0$, $(a-\varepsilon, b+\varepsilon)$ will also serve as covering $I=[a,b]$.

i.e. $[a,b] \subset \left(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2}\right)$

By definition ① $\Rightarrow m^*(I) < \ell\left(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2}\right) = b-a+\varepsilon$

$\Rightarrow m^*(I) \leq b-a$, ; $\varepsilon>0$ and $\varepsilon \to 0$ arbitrary. ⟶ ②

For proving; $m^*(I) \geq b-a$, let $\{I_k\}_{k=1}^{\infty}$ be a countable family of open intervals covering $I=[a,b]$

$\because [a,b]$ is bounded and closed | i.e. $[a,b] \subseteq \bigcup_{k=1}^{\infty} I_k$

$\Rightarrow$) By Heine Borel ; there exist a finite sub-cover (collection) $\{I_j\}_{j=1}^{n} \Rightarrow [a,b] \subseteq \bigcup_{j=1}^{n} I_j$.

$\Rightarrow \ell[a,b] \leq \sum_{j=1}^{n} \ell(I_j)$

$\Rightarrow \sum_{j=1}^{n} \ell(I_j) \geq b-a$ ⟶ ③

i.e true for all families of sequence of open intervals covering $I=[a,b]$.

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$$\Rightarrow \sum_{j=1}^{n} \ell(I_j) \leq \sum_{k=1}^{\infty} \ell(I_k)$$

Taking Infimum in (3)

$$\text{Inf}\left\{\sum_{j=1}^{n} \ell(I_j) : I \subseteq \bigcup_{j=1}^{n} I_j ; \text{whenever } \{I_j\}_{j=1}^{n} \text{ is a covering of } I\right\} \geq b-a$$

$$m^*(I) \geq b-a \quad (4)$$

From (2) and (4) $\Rightarrow \boxed{m^*([a,b]) = b-a} \longrightarrow (5)$

Case II: If $I = (a,b) \Rightarrow (a,b) \subset [a,b]$

$\Rightarrow m^*(a,b) \leq m^*[a,b] \longrightarrow (6)$ by monotonicity of m^*

for $\varepsilon > 0$; $\left[a+\frac{\varepsilon}{2}, b-\frac{\varepsilon}{2}\right] \subset (a,b)$

$$\Rightarrow m^*\left[a+\frac{\varepsilon}{2}, b-\frac{\varepsilon}{2}\right] \leq m^*(a,b)$$

$$\Rightarrow (b-a)-\varepsilon \leq m^*(a,b)$$

$$\Rightarrow b-a \leq m^*(a,b) \longrightarrow (7)$$

From (5), (6) and (7) $\Rightarrow \boxed{m^*(a,b) = b-a} \longrightarrow (8)$

Case III: If $I = (a,b]$ or $[a,b)$

$$\Rightarrow (a,b) \subseteq I \subset [a,b]$$

$$\Rightarrow m^*(a,b) \leq m^*(I) \leq m^*[a,b]$$

$$\Rightarrow b-a \leq m^*(I) \leq b-a \quad \text{From (5) and (8)}$$

$$\Rightarrow m^*(I) = b-a$$

$$\Rightarrow \boxed{m^*(a,b] = m^*[a,b) = b-a}$$

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Topic #34

Determining Uncountability by Lebesgue Outer Measure:

Theorem: Intervals in "$\mathbb{R}$" are uncountable.

proof: Let $[a,b]$ be a interval and it is countable.

$\therefore \quad m^*[a,b]=0 \quad \because$ it is countable

But $m^*[a,b]=\ell[a,b]=b-a\neq 0$, if $a\neq b$

i.e contradiction.

$\Rightarrow$ $[a,b]$ in $\mathbb{R}$ is uncountable.

Topic #35

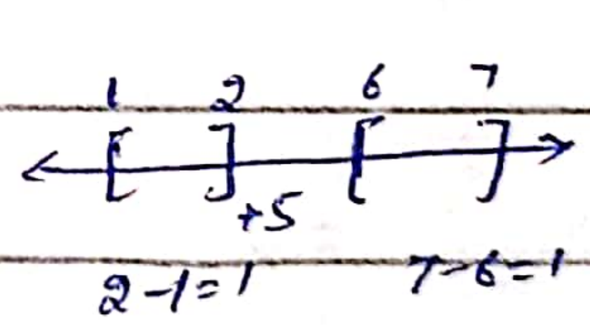
Translation Invariance of Lebesgue Outer Measure:

Theorem: Show that m^* is translation invariant:

proof: Let $\phi \neq E \subset \mathbb{R}$ and

$x \in E \Rightarrow P+E=\{P+x, x\in E\}$,

i.e in translation of "E" by "p" units.



By definition:

$$m^*(E)=Inf\left\{\sum_{n=1}^{\infty}\ell(I_n) : E\subset\bigcup_{n=1}^{\infty}I_n, \text{ where } \{I_n\}_{n=1}^{\infty} \text{ is sequence of intervals covering } E\right\}$$

let $I_n=(a_n,b_n) \Rightarrow \ell(I_n)=b_n-a_n \longrightarrow (1)$

If $E\subseteq\bigcup_{n=1}^{\infty}I_n \Rightarrow P+E\subseteq\bigcup_{n=1}^{\infty}J_n$, such that

$J_n=(p+a_n, p+b_n) \Rightarrow \ell(J_n)=b_n-a_n=\ell(I_n)$

$\Rightarrow \sum_{n=1}^{\infty}\ell(I_n)=\sum_{n=1}^{\infty}\ell(J_n) \longrightarrow (2)$

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From ① and ②

$\Rightarrow m^*(E) = \inf\left\{\sum_{n=1}^{\infty} l(J_n) : p+E \subseteq \bigcup_{n=1}^{\infty} J_n, \text{ where } \{J_n\}_{n=1}^{\infty} \text{ sequence: open intervals covering } p+E\right\} = m^*(E+p).$

Topic #36

Least Length of Open Covering of All Rationals in an Interval:

Theorem: If $E = \{p \in Q : 0 < p < 1\}$ and $\{I_n\}_{n=1}^{k}$ is the finite sequence of open intervals such that $A \subseteq \bigcup_{n=1}^{k} I_n$, then $\sum_{n=1}^{k} l(I_n) \geq 1$

proof:

$\because A \subseteq \bigcup_{n=1}^{k} I_n \Rightarrow (0,1) \subseteq \bigcup_{n=1}^{k} I_n$

$\Rightarrow m^*(0,1) \leq m^*\left(\bigcup_{n=1}^{k} I_n\right)$ by monoticity of m^*

$\leq \sum_{n=1}^{k} m^*(I_n)$ $\because m^*$ is sub-additive

$m^*(0,1) \leq \sum_{n=1}^{k} l(I_n) \longrightarrow ①$

But $m^*(0,1) = l(0,1) = 1 - 0 = 1$

From ① $\Rightarrow 1 \leq \sum_{n=1}^{k} l(I_n)$

$\Rightarrow \sum_{n=1}^{k} l(I_n) \geq 1$

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Topic # 37

Lebesgue Measurable Sets:

$A \neq \phi \neq E \subseteq R$ is said to be lebesgue measurable or simply measurable if $\forall A \subseteq R$; we have, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \longrightarrow (1)$

Note: $A = A \cap R = A \cap (E \cup E^c)$

$A = (A \cap E) \cup (A \cap E^c)$

$\Rightarrow m^*(A) = m^*[(A \cap E) \cup (A \cap E^c)]$

$\Rightarrow m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c) \longrightarrow (2)$

In order to prove (1); we will go for:

$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c) \longrightarrow (3)$

Topic # 38

Lebesgue Measurability of Complement of a set:

Theorem: A set "E" is measurable if and only if its complement "E^c" is measurable.

proof: $\because E \subseteq R$ is measurable if $\forall A \subseteq R$

$\Leftrightarrow m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

$\Leftrightarrow m^*(A) = m^*(A \cap E^c) + m^*(A \cap E^{cc})$

$\Leftrightarrow E^c$ is measurable.

Topic # 39

Theorem: "ϕ" and R are measurable.

proof: $m^*(A) = m^*(A) + m^*(\phi)$

$\because m^*(\phi) = 0$

$\phi^c = R$
$R^c = \phi$

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$\Rightarrow$ $m^*(A) = m^*(A \cap R) + m^*(A \cap \phi)$

$\Rightarrow$ $m^*(A) = m^*(A \cap R) + m^*(A \cap R^c)$

$A = A \cap R$

$\phi = A \cap \phi$

$\Rightarrow$ R is measurable.

$\Rightarrow$ $R^c = \phi$ is also measurable. $\because$ {From previous theorem

Topic #40

Theorem: If $m^*(E) = 0$, then $E \subseteq R$ is measurable.

proof: For any $\phi \neq A \subseteq R$; $E \cap A \subseteq E$

$\Rightarrow$ $m^*(A \cap E) \leq m^*(E)$ $\because$ m^* is monotonic.

$\Rightarrow$ $m^*(A \cap E) \leq 0 \longrightarrow (1)$ $\because$ given

But $m^*(A \cap E) \geq 0 \longrightarrow (2)$ By definition.

$\therefore$ (1) and (2) $\Rightarrow$ $m^*(A \cap E) = 0$ or $0 = m^*(A \cap E)$ $\longrightarrow (3)$

Similarly $\because$ $A \cap E^c \subseteq A$

$\Rightarrow$ $m^*(A \cap E^c) \leq m^*(A)$

$\Rightarrow$ $m^*(A) \geq m^*(A \cap E^c)$ $\longrightarrow (4)$

$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

$m^*(A) \geq$ " "

$\therefore$ From (3) and (4) $\Rightarrow$

$m^*(A) + 0 \geq m^*(A \cap E) + m^*(A \cap E^c)$

$\Rightarrow$ "E" is measurable.

Topic #41

Theorem: Union and Intersection of two measurable sets is measurable.

proof: $\forall$ $A \subseteq R$, $E \rightarrow$ measurable

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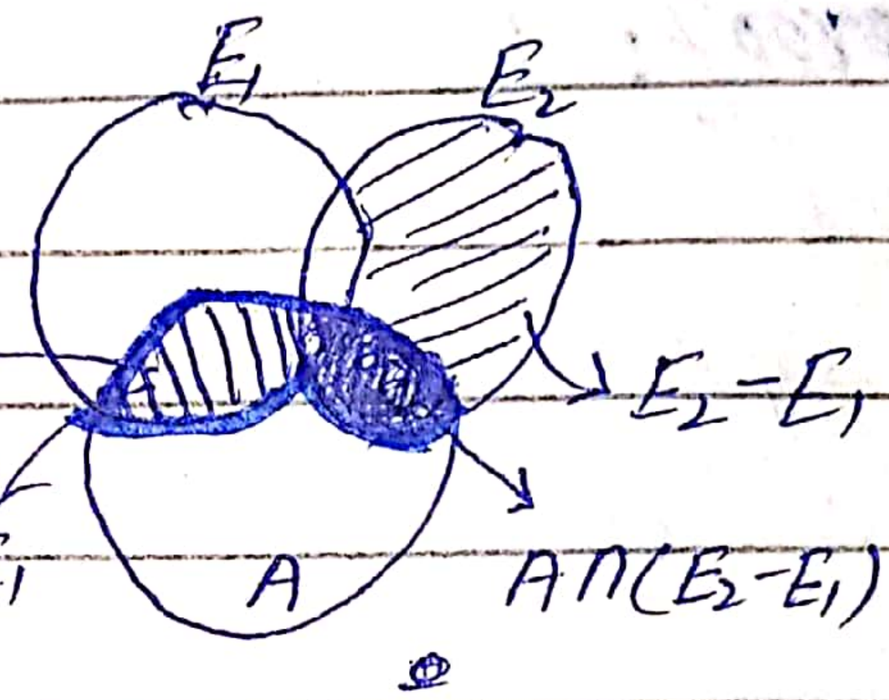
$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

let E_1, E_2 are two measurable sets.

To prove $E_1 \cup E_2$ is measurable:

$$m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c] \rightarrow (1)$$

$$R.H.S = m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c]$$



$$= (A \cap E_1) \cup (A \cap (E_2 - E_1))$$

$$= E_2 \cap E_1' = E_1' \cap E_2$$

$$\Rightarrow R.H.S = m^*[(A \cap E_1) \cup (A \cap (E_1' \cap E_2))] + m^*[A \cap (E_1' \cap E_2')]$$

By sub-additivity of m^*.

$$\Rightarrow R.H.S \leq m^*(A \cap E_1) + m^*(A \cap E_1' \cap E_2) + m^*(A \cap E_1' \cap E_2')$$

$\because$ "E_2" is measurable and using test set $(A \cap E_1')$

$$\Rightarrow m^*(A \cap E_1') = m^*(A \cap E_1' \cap E_2) + m^*(A \cap E_1' \cap E_2')$$

$$\therefore R.H.S \leq m^*(A \cap E_1) + m^*(A \cap E_1')$$

Since again "E_1" is measurable.

$$R.H.S \leq m^*(A)$$

$$\Rightarrow m^*(A) \geq m^*[A \cap (E_1 \cup E_2)] + m^*[A \cap (E_1 \cup E_2)^c] \rightarrow (A)$$

$\because$ E_1 and E_2 are measurable,

$\Rightarrow$ E_1' and E_2' are measurable.

$\Rightarrow$ $(E_1' \cup E_2')$ is also measurable.

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$\Rightarrow (E_1' \cup E_2')' = E_1 \cap E_2$ is measurable.

Topic # 42

Theorem: For any $\phi \neq A \subset R$ and $\{E_i\}_{i=1}^{n}$ be a family of pairwise disjoint measurable sets, then

$$m^*[A \cap (\bigcup_{i=1}^{n} E_i)] = \sum_{i=1}^{n} m^*(A \cap E_i) \rightarrow (A) \quad n \in N$$

proof: We will prove this by mathematical induction. For $n=1$, result is self-evident.

Suppose (A) is true for $n = k-1$;

$$\therefore (A) \Rightarrow m^*[A \cap (\bigcup_{i=1}^{k-1} E_i)] = \sum_{i=1}^{k-1} m^*(A \cap E_i) \rightarrow (B)$$

Consider: $A \cap (\bigcup_{i=1}^{k} E_i) \cap E_k = A \cap [(\bigcup_{i=1}^{k} E_i) \cap E_k]$

$$= A \cap [(E_1 \cup E_2 \cup \cdots \cup E_k) \cap E_k]$$

$$A \cap (\bigcup_{i=1}^{k} E_i) \cap E_k = A \cap [\underbrace{(E_1 \cap E_k)}_{=\phi} \cup \underbrace{(E_2 \cap E_k)}_{=\phi} \cup \cdots \cup (E_k \cap E_k)]$$ ✓

$\because E_i \cap E_j = \phi \quad \forall i \neq j$

$$\Rightarrow A \cap (\bigcup_{i=1}^{k} E_i) \cap E_k = A \cap E_k \rightarrow (1)$$

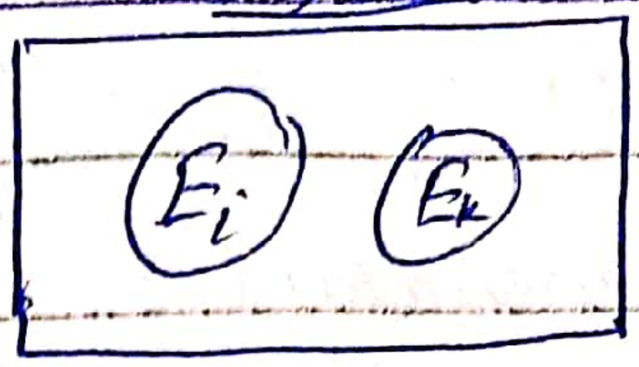
Now

$$A \cap (\bigcup_{i=1}^{k} E_i) \cap E_k' = A \cap [(\bigcup_{i=1}^{k} E_i) \cap E_k']$$

$$= A \cap [(E_1 \cup E_2 \cup \cdots \cup E_k) \cap E_k']$$

$$= A \cap [(E_1 \cap E_k') \cup (E_2 \cap E_k') \cup \cdots \cup \underbrace{(E_k \cap E_k')}_{\phi}]$$

Disjoint



$E_i \subseteq E_k'$

$\Rightarrow E_i \cap E_k' = E_i$

$\Rightarrow A \cap (\bigcup_{i=1}^{k} E_i) \cap E_k' = A \cap (E_1 \cup E_2 \cup \cdots \cup E_{k-1})$

$\Rightarrow A \cap (\bigcup_{i=1}^{k} E_i) \cap E_k' = A \cap (\bigcup_{i=1}^{k-1} E_i) \longrightarrow (2)$

Taking "$A \cap (\bigcup_{i=1}^{k} E_i)$" as test set and using Lebesgue measurability of "E_k":

$$m^*\left[A \cap (\bigcup_{i=1}^{k} E_i)\right] = m^*\left[A \cap (\bigcup_{i=1}^{k} E_i) \cap E_k\right] + m^*\left[A \cap (\bigcup_{i=1}^{k} E_i) \cap E_k'\right] \quad \text{By defi}$$

$$m^*\left[A \cap (\bigcup_{i=1}^{k} E_i)\right] = m^*(A \cap E_k) + m^*\left[A \cap (\bigcup_{i=1}^{k-1} E_i)\right]$$

$$= m^*(A \cap E_k) + \sum_{i=1}^{k-1} m^*(A \cap E_i)$$

$$m^*\left[A \cap (\bigcup_{i=1}^{k} E_i)\right] = \sum_{i=1}^{k} m^*(A \cap E_i)$$

$\therefore$ Truth of $n = k-1$ also implies its truth for $n = k$. This proves the given theorem.

Topic #43

Theorem: The class "M" of all measurable sets is a σ-algebra.

$\phi = X$; $\mathcal{A} \subseteq P(X)$ (subsets of "X")

(i) $\forall A \in \mathcal{A} \Rightarrow A' \in \mathcal{A}$

(ii) $A, B \in \mathcal{A} \Rightarrow A \cup B \in \mathcal{A}$

(iii) $\{E_i\}_{i=1}^{\infty} \rightarrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$

proof: (i) If "E" is measurable

$\rightarrow$ "E'" is also measurable

$\rightarrow E \in$ "M"

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(ii) If E_1, E_2 are measurable $\Rightarrow$ $E_1 \cup E_2$ is also measurable.

$\Rightarrow E_1 \cup E_2 \in M$

(iii) Let $\{E_i\}_{i=1}^{\infty}$ be the sequence of measurable sets in "M", Then we will show

$$\bigcup_{i=1}^{\infty} E_i = E \in M$$

Constracting: $F_1 = E_1$, $F_2 = E_2 - E_1 \Rightarrow F_2 \cap F_1 = \phi$

$F_3 = E_3 - (E_1 \cup E_2) \Rightarrow F_3 \cap F_2 = \phi$

$\vdots$ $\vdots$

$F_n = E_n - (E_1 \cup E_2 \cup \cdots \cup E_{n-1}) = E_n - \left(\bigcup_{i=1}^{n-1} E_i\right)$

$\Rightarrow \{F_i\}_{i=1}^{\infty}$ is a sequence of pairwise disjoint measurable sets.

$$\Rightarrow \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i \longrightarrow (1)$$

Let $H_n = \bigcup_{i=1}^{n} F_i$ - i.e. a measurable as well being finite union of measurable sets.

$$\Rightarrow H_n = \bigcup_{i=1}^{n} F_i \subseteq \bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} E_i = E$$

$\Rightarrow H_n' \supseteq E' \Rightarrow \forall A \subseteq R, A \cap H_n' \supseteq A \cap E'$

$\Rightarrow m^*(A \cap H_n') \geq m^*(A \cap E') \longrightarrow (2)$

| $A \subseteq B$
 $A' \supseteq B'$

$\because$ "H_n" is measurable and taking $\phi \neq A \subseteq R$ as test set:

$\Rightarrow m^*(A) = m^*(A \cap H_n) + m^*(A \cap H_n')$ * By defi:

$$\Rightarrow m^*(A) \geq m^*(A \cap H_n) + m^*(A \cap E')$$

$$m^*(A) \geq m^*\left[A \cap \left(\bigcup_{i=1}^{n} F_i\right)\right] + m^*(A \cap E')$$

$$m^*(A) \geq \sum_{i=1}^{n} m^*(A \cap F_i) + m^*(A \cap E')$$

$\because$ L.H.S is independent of "n", so for $n \to \infty$

$$\therefore m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap F_i) + m^*(A \cap E')$$

$$\Rightarrow m^*(A) \geq m^*\left[A \cap \left(\bigcup_{i=1}^{\infty} F_i\right)\right] + m^*(A \cap E')$$

$$\Rightarrow m^*(A) \geq m^*\left[A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right] + m^*(A \cap E')$$

$$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap E')$$

$\Rightarrow$ "E" is also measurable.

$\Rightarrow E \in M$

$\Rightarrow$ "M" is a σ-algebra.

Topic #44

Theorem: If "F" is measurable set and $m^*(F \Delta G) = 0$, then "G" is measurable set.

proof: $G - F \subseteq (F \Delta G)$

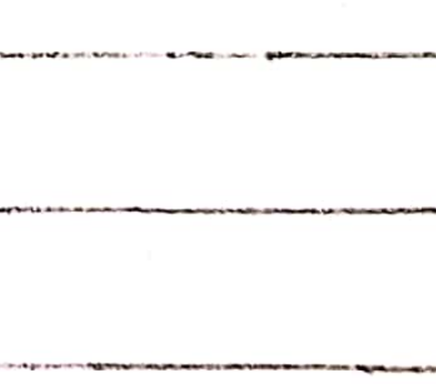
$\Rightarrow m^*(G-F) \leq m^*(F \Delta G) = 0$

$\Rightarrow m^*(G-F) = 0$

$\because m^*$ is non-negative.

$\Rightarrow (G-F)$ is measurable.

Similarly, $(F-G)$ is also measurable.



$F \Delta G = (G-F) \cup (F-G)$

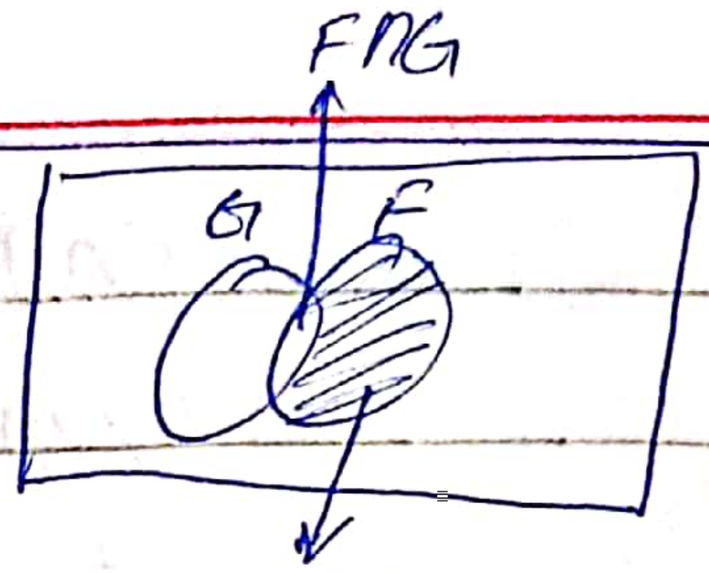
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$\because$ $F \cap G = F \cap (F - G)'$

is measurable being

intersection of two measurable sets.

$\therefore$ $G = (F \cap G) \cup (G - F)$ is also measurable.



Topic # 45

G-Delta and F-Sigma Sets:

G_δ-Set

A $\phi \neq A \subseteq \mathbb{R}$ is said to be G_δ-set if it can be written as countable intersection of open sets.

For Example:

$$\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) = (a, b)$$

(a, b) is trivially G_δ-set.

Note: G_δ-set is measurable being the countable intersection of measurable sets.

F_σ-set

A $\phi \neq A \subseteq \mathbb{R}$ is said to be F_σ-set if it can be written as countable union of closed sets.

For Example:

$$(a, b) = \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right]$$

(a, b) is also F_σ-set.

Note: F_σ-set is measurable being the countable ~~intersection~~ union of measurable sets.

Note: $G_\delta' = F_\sigma$ and $F_\sigma' = G_\delta$

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Topic #46

Theorem: Given any $\phi \neq A \subseteq \mathbb{R}$ and $\varepsilon > 0$, there is a G_δ set "G" with $A \subseteq G$ and $m^*(A) = m^*(G)$.

Proof: $\because \forall\ A \subseteq \mathbb{R}$ and for any $\varepsilon > 0$, there exist an open set say "U" such that $A \subseteq U$ and $m^*(U) < m^*(A) + \varepsilon$ $\longrightarrow (1)$

In particular for $\varepsilon = \frac{1}{n}$, let "$U_n$" an open set, such that $A \subseteq U_n$

$\Rightarrow m^*(A) \leq m^*(U_n) \longrightarrow (2)$

and $m^*(U_n) < m^*(A) + \frac{1}{n} \longrightarrow (3)$

Since $A \subseteq U_n \Rightarrow A \subseteq \bigcap_{n=1}^{\infty} U_n = G_\delta$ (say) $\longrightarrow (4)$

i.e. G_δ-set being intersection of measurable sets. (open sets)

$\Rightarrow m^*(G_\delta) = m^*\left(\bigcap_{n=1}^{\infty} U_n\right) \leq m^*(U_n)$

$\Rightarrow m^*(G_\delta) < m^*(A) + \frac{1}{n}$ (From (3))

As for $n \to \infty \Rightarrow \varepsilon = \frac{1}{n} \to 0$

$\therefore m^*(G_\delta) \leq m^*(A) \longrightarrow (5)$

From (4) $\Rightarrow A \subseteq G_\delta \Rightarrow m^*(A) \leq m^*(G_\delta) \longrightarrow (6)$

& From (5) and (6) $\Rightarrow m^*(A) = m^*(G_\delta)$

$A \subseteq B$

$\Rightarrow m^*(A) \leq m^*(B)$

$A \subseteq U$

$\Rightarrow m^*(A) \leq m^*(U)$

$m^*(A) + \varepsilon > m^*(U)$

$\Rightarrow m^*(U) < m^*(A) + \varepsilon$

where $\varepsilon > 0$

$A \cap B \subseteq A$

$m^*(A \cap B) \leq m^*(A)$

Written by

Mahar Afaq Safdar Muhammadi

MSc Mathematics & MA Islamic Studies

(Scholar)

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MTH644
Measure Theory

Written By
Mahar Afaq Safdar Muhammaeli
(MSc Mathematics and MA Islamic Studies)

Topic # 47

Theorem: Interval (a,∞) is measurable $\forall\ a\in R$.

Proof:

Taking test set say $A\subseteq R$ and we only need to prove:

$\because\ m^*(A)=m^*(A\cap U)+m^*(A\cap U')$

$\because\ m^*(a,b)=l(a,b)=b-a$

$$m^*(A)\geq m^*[A\cap(a,\infty)]+m^*[A\cap(-\infty,a]] \longrightarrow (1)$$

Let $A_\alpha = A\cap(a,\infty)$ and $A_\beta = A\cap(-\infty,a]$

$\therefore (1)\Longrightarrow m^*(A)\geq m^*(A_\alpha)+m^*(A_\beta) \longrightarrow (2)$

Case I: $m^*(A)=+\infty$, then (2) is obviously.

Case II: if $m^*(A)<\infty$ and for $A\subseteq R\ \exists$ a seq. $\{I_n'\}_{n=1}^{\infty}$ such that $A\subseteq \bigcup_{n=1}^{\infty} I_n' \longrightarrow (A)$

$\forall\ \varepsilon>0$, Then there exists another such sequence $\{I_n\}$ such that

$(A)\Rightarrow m^*(A)\leq m^*\left(\bigcup_{n=1}^{\infty} I_n\right)$ "

$A\subseteq \bigcup_{n=1}^{\infty} I_n$ and $m^*(A)+\varepsilon \geq m^*\left(\bigcup_{n=1}^{\infty} I_n'\right) \longrightarrow (a)$

$\because$ on R; $m^*(I_n)=l(I_n)$

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$\therefore m^*\left(\bigcup_{n=1}^{\infty} I_n\right) = \sum_{n=1}^{\infty} l(I_n)$ → not disjoint

$\longrightarrow$ (b)

(a) $\Rightarrow \sum_{n=1}^{\infty} l(I_n) < m^*(A) + \epsilon$ $\longrightarrow$ (3)

Let $M_n = I_n \cap (a, \infty)$ and $N_n = I_n \cap (-\infty, a]$

$M_n \cap N_n = \{I_n \cap (a,\infty)\} \cap \{I_n \cap (-\infty, a]\}$

$= I_n \cap \{(a,\infty) \cap (-\infty, a]\}$

$M_n \cap N_n = I_n \cap \phi = \phi$

Now $M_n \cup N_n = \{I_n \cap (a,\infty)\} \cup \{I_n \cap (-\infty, a]\}$

$= I_n \cap \{(a,\infty) \cup (-\infty, a]\}$

$= I_n \cap R$

$M_n \cup N_n = I_n$

$\Rightarrow l(I_n) = l(M_n) + l(N_n)$ $\longrightarrow$ (4)

$\because A_\alpha = A \cap (a,\infty)$ and $A \subseteq \bigcup_{n=1}^{\infty} I_n$

$\Rightarrow A_\alpha \subseteq \left(\bigcup_{n=1}^{\infty} I_n\right) \cap (a,\infty)$

$\Rightarrow A_\alpha \subseteq (I_1 \cup I_2 \cup \dots) \cap (a,\infty)$

$\Rightarrow A_\alpha \subseteq \{I_1 \cap (a,\infty)\} \cup \{I_2 \cap (a,\infty)\} \cup \dots$

$\Rightarrow A_\alpha \subseteq \bigcup_{n=1}^{\infty} \{I_n \cap (a,\infty)\}$

$\Rightarrow A_\alpha \subseteq \bigcup_{n=1}^{\infty} M_n$

$\Rightarrow m(A_\alpha) \leq m^*\left(\bigcup_{n=1}^{\infty} M_n\right)$ $\longrightarrow$ (c)

Similarly $A_\beta \subseteq \bigcup_{n=1}^{\infty} N_n$

$\Rightarrow m(A_\beta) \leq m^*\left(\bigcup_{n=1}^{\infty} N_n\right)$ $\longrightarrow$ (d)

Adding (c) and (d)

$m(A_\alpha) + m(A_\beta) \leq m^*\left(\bigcup_{n=1}^{\infty} M_n\right) + m^*\left(\bigcup_{n=1}^{\infty} N_n\right)$

$\leq \sum_{n=1}^{\infty} m^*(M_n) + \sum_{n=1}^{\infty} m^*(N_n)$

$\because M_n \cap N_n = \phi \quad \forall n \in \mathbb{N}$

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$$m^*(A_\alpha) + m^*(A_\beta) \leq \sum_{n=1}^{\infty} l(M_n) + \sum_{n=1}^{\infty} l(N_n)$$

$$\leq \sum_{n=1}^{\infty} (l(M_n) + l(N_n))$$

$$m^*(A_\alpha) + m^*(A_\beta) \leq \sum_{n=1}^{\infty} l(I_n) \quad \text{From (4)}$$

$$\Rightarrow m^*(A_\alpha) + m^*(A_\beta) < m^*(A) + \varepsilon \quad \text{From (3)}$$

$\because \varepsilon \to 0$ arbitrarily;

$$\therefore m^*(A_\alpha) + m^*(A_\beta) \leq m^*(A)$$

or $m^*(A) \geq m^*(A_\alpha) + m^*(A_\beta)$

Hence proved; (a, ∞) is measurable.

Topic #48

Theorem: Open interval $(a,b) \subset \mathbb{R}$ is measurable.

proof: $\because$ (a, ∞) is measurable

$\Rightarrow (a,\infty)^c = (-\infty, a]$ is also measurable.

And $(-\infty, b - \frac{1}{n}]$ is also measurable $\forall\ n \in \mathbb{N}$ as $n \to \infty$

and $(-\infty, b) = \bigcup_{n=1}^{\infty} (-\infty, b - \frac{1}{n}]$ is also measurable

because being countable union of measurable set

$\Rightarrow (a,\infty) \cap (-\infty, b) = (a,b)$ i.e also measurable.

being intersection of two measurable set

Open set in $\mathbb{R}$:

Any open set in $\mathbb{R}$ say U can be written as countable union of disjoint open intervals

i.e $U = \bigcup_{n=1}^{\infty} \{I_n\}$ and each open intervals is

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measurable and countable union of measurable set is measurable.

Topic #49

Theorem: A subset $\phi \neq E \subset \mathbb{R}$ is measurable $\Leftrightarrow$ for given $\varepsilon > 0$, $\exists$ an open set $U \supseteq E$ with $m^*(U-E) < \varepsilon$.

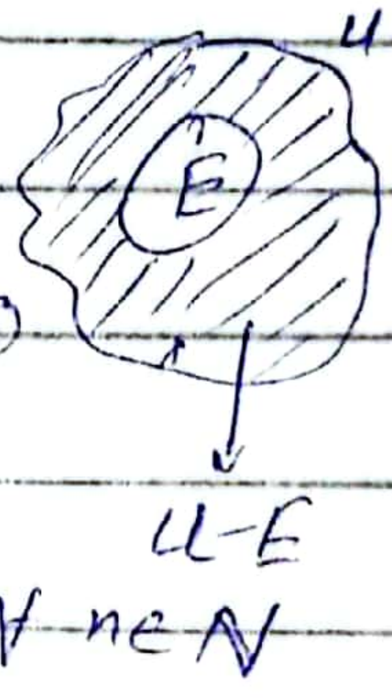
proof: Suppose E is measurable

Case I: If $m^*(E) < \infty$, $\forall\ \phi \neq E \subset \mathbb{R}$, there exists an open set U such that $E \subset U$ and for any $\varepsilon > 0$, $m^*(U) < m^*(E) + \varepsilon \longrightarrow (a)$

$E \subset U \Rightarrow U = E \cup (U-E)$

$\Rightarrow m^*(U) = m^*(E) + m^*(U-E)$

$\Rightarrow m^*(U-E) = m^*(U) - m^*(E) < \varepsilon$. from (a)



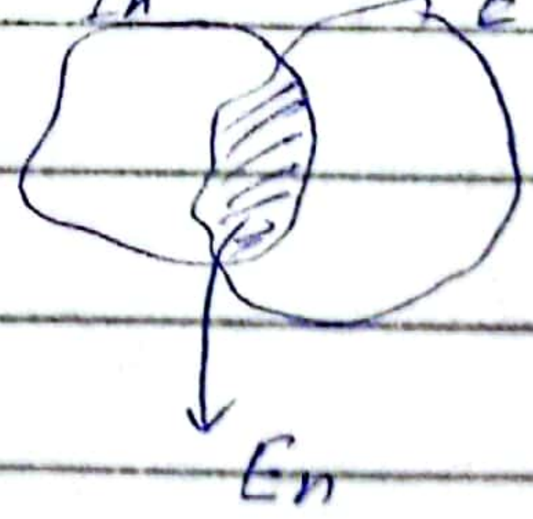
Case II: If $m^*(E) = \infty$

$\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$, where $\ell(I_n) < \infty\ \forall\ n \in \mathbb{N}$

and $I_j \cap I_k = \phi$, $j \neq k$

put $E_n = E \cap I_n \Rightarrow E_n \subseteq I_n\ \forall\ n \in \mathbb{N}$

$\Rightarrow m^*(E_n) \leq m^*(I_n) = \ell(I_n) < \infty$



By case I $\forall\ E_n$, $\exists$ an open set say "V_n" such that $E_n \subseteq V_n$, for $\varepsilon > 0$

and $m^*(V_n - E_n) < \frac{\varepsilon}{2^n} \longrightarrow (b)$

Also $V = \bigcup_{n=1}^{\infty} V_n$, an open set.

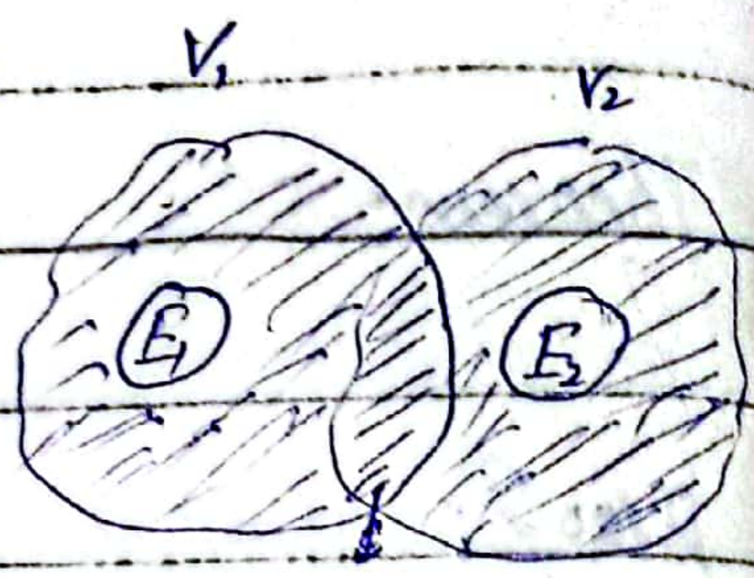
$\Rightarrow \bigcup_{n=1}^{\infty} E_n = \bigcup_{n=1}^{\infty} (E \cap I_n) = E \cap \left(\bigcup_{n=1}^{\infty} I_n\right) = E \cap \mathbb{R} = E$

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$E = \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} V_n = V$ $\quad \because E_n \subseteq V_n \ \forall\, n \in \mathbb{N}$

$\Rightarrow V - E = \bigcup_{n=1}^{\infty} V_n - \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} (V_n - E_n)$

$\Rightarrow m^*(V-E) \leq \sum_{n=1}^{\infty} m^*(V_n - E_n)$

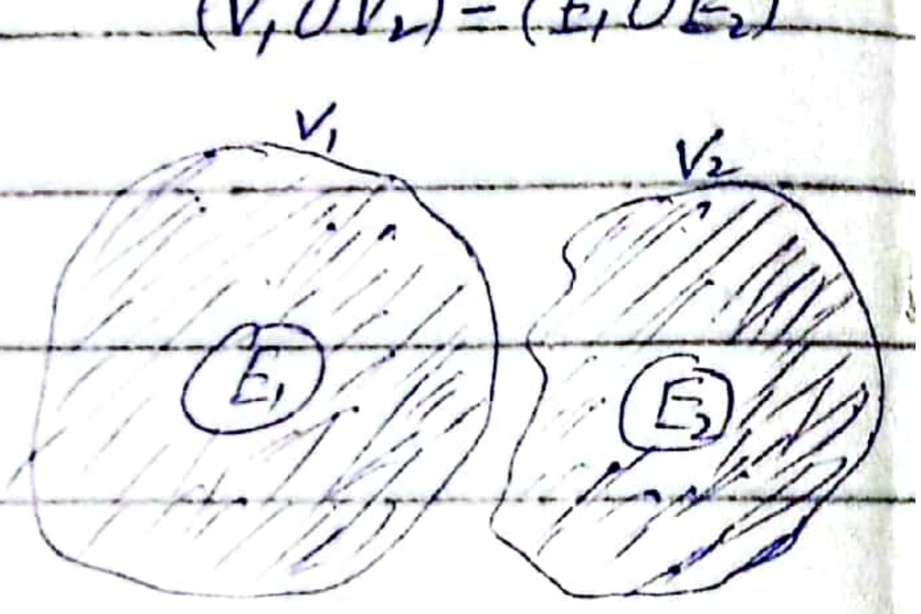


From (b) $\Rightarrow m^*(V-E) \leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n}$

$m^*(V-E) \leq \varepsilon\left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right) = \varepsilon(1)$

$m^*(V-E) \leq \varepsilon$

$(V_1 \cup V_2) = (E_1 \cup E_2)$



$(V_1 - E_1) \cup (V_2 - E_2)$

Conversely, let $\varepsilon = \frac{1}{n}$ (say) and

$\exists$ an open set $O_n \supseteq E$ such that $E \subseteq O_n \ \forall\, n \in \mathbb{N}$

and $m^*(O_n - E) < \varepsilon = \frac{1}{n} \longrightarrow (A)$

We will prove "E" is measurable.

put $G_1 = \bigcap_{n=1}^{\infty} O_n \Rightarrow$ "G_1" is measurable being countable intersection of measurable open set.

$E \subseteq O_n \ \forall\, n$ (given)

and $E \subseteq \bigcap_{n=1}^{\infty} O_n = G_1$

$G_1 - E \subseteq O_n - E$

$\Rightarrow m^*(G_1 - E) \leq m^*(O_n - E) < \frac{1}{n} = \varepsilon$

$\because \varepsilon \to 0$ arbitrary $m^*(G_1 - E) = 0$.

$\Rightarrow G_1 - E$ is measurable $\Rightarrow (G_1 - E)'$ is also measurable.

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$\therefore E = G_1 - (G_1 - E) = G_1 \cap (G_1 - E)'$ is measurable being intersection of two measurable set.

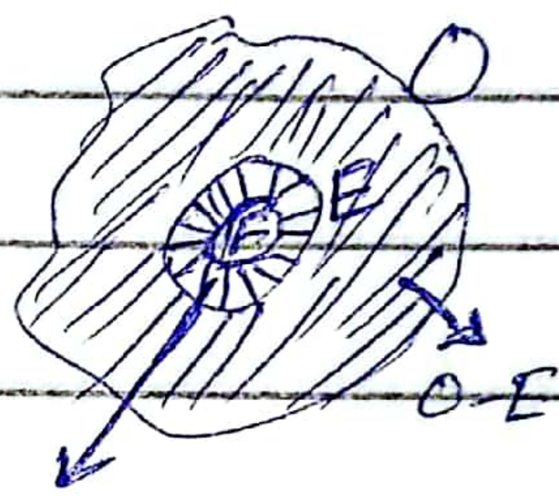
Topic #50

Theorem: For a subset "E" of R and given $\varepsilon > 0$ $\exists$ an open set $O \supseteq E$ and $m^*(O-E) < \varepsilon \Leftrightarrow \exists$ a closed set $F \subseteq E$ and $m^*(E-F) < \varepsilon$.

proof: For $\phi \neq E \subseteq R \Rightarrow E' \subseteq R$ $\exists$ for given $\varepsilon > 0$

$\exists$ an open set $O \supseteq E$ and

$m^*(O-E) < \varepsilon \longrightarrow (1)$



$O \supseteq E' \Rightarrow E'' \supseteq O' = F$ (say)

i.e. closed set

$\Rightarrow E \supseteq F \Rightarrow m^*(E-F) = m^*(E \cap F')$

$m^*(E-F) = m^*(F' \cap E)$

$m^*(E-F) = m^*(O-E') < \varepsilon$ from (1)

$\Rightarrow m^*(E-F) < \varepsilon$

Conversely for the set $E' \subseteq R$, $\exists$ a closed set "F" and $\varepsilon > 0$ such that $F \subseteq E'$ and $m^*(E'-F) < \varepsilon \longrightarrow (2)$ (given)

$F \subseteq E' \Rightarrow E = E'' \subseteq F' = O$ (open set) say.

$\Rightarrow m^*(O-E) = m^*(O \cap E') = m^*(E' \cap O)$

$\Rightarrow m^*(O-E) = m^*(E'-O') = m^*(E'-F) < \varepsilon$ from (2)

$m^*(O-E) < \varepsilon$ (Hence proved)

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Topic #52

Theorem: Let $\phi \neq E \subset R$, then the following statements are equivalent

(i) There is a G_δ set "G" with $E \subset G$, $m^*(G-E)=0$ → give

(ii) There is a F_σ set "F" with $F \subset E$, $m^*(E-F)=0$ → prove

proof: (i) $\Rightarrow$ (ii) $\because \phi \neq E \subset R \Rightarrow \phi \neq E' \subseteq R$

$\therefore$ For $\phi \neq E'$ given $E' \subseteq G$, $m^*(G-E')=0$

$\therefore \quad 0 = m^*(G-E') = m^*(G \cap (E')') = m^*(G \cap E)$

$= m^*(E \cap G) = m^*(E \cap (G')') = m^*(E-G')$

$= m^*(E-F)$, where $G_\delta = G = \bigcap_{n=1}^{\infty} G_n$

$\Rightarrow G_\delta' = G' = \left(\bigcap_{n=1}^{\infty} G_n\right)'$

$\Rightarrow G_\delta' = \bigcup_{n=1}^{\infty} G_n'$

$\Rightarrow G_\delta' = \bigcup_{n=1}^{\infty} F_n' = F = F_\sigma$

(ii) $\Rightarrow$ (i)

Conversly, again for $E' \subset R$, with $F \subset E'$

$m^*(E'-F)=0$ (given).

$\therefore \quad 0 = m^*(E'-E) = m^*(E' \cap F') = m^*(F' \cap E')$

$= m^*(F'-E) = m^*(G-E)$

where $F_\sigma' = F' = \left(\bigcap_{k=1}^{\infty} F_k\right)' = \bigcup_{k=1}^{\infty} F_k' = \bigcup_{k=1}^{\infty} G_n = G = G_\delta$

(proved)

Topic #52

Lebesgue Measure:

Let "M" be the class of measurable sets, which is σ-algebra, Then the Lebesgue measure of any set $E \subseteq M$ is defined to be its lebesgue outer measure.

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$$m^* \equiv M ; \left\{ A,\ m^*(A) = m^*(A \cap E) + m^*(A \cap E') \right.$$

Topic #53

Theorem: let $\{E_i\}_{i=1}^{n}$ be a sequence a in measurable space "M", then m is

(i) countably subadditive

(ii) finitely additive provided that $E_i \cap E_j = \phi, \forall\ i \neq j$.

(iii) countably additive if E_i's are pairwise disjoint.

(iv) "m" is monotonic.

proof: (i) All E_i's are measurable.

$\Rightarrow$ $\bigcup_{i=1}^{n} E_i$ is also measurable, $\therefore$ "M" is space of lebesgue measurable set i

$$\Rightarrow m\left(\bigcup_{i=1}^{n} E_i\right) = m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i) = \sum_{i=1}^{\infty} m(E_i)$$

(ii) $\forall\ \phi \neq A \subseteq R$ and for the sequence $\{E_i\}_{n=1}^{n}$ of pairwise disjoint measurable sets ;

$$m^*\left(A \cap \left(\bigcup_{i=1}^{\infty} E_i\right)\right) = \sum_{i=1}^{\infty} m^*(A \cap E_i)$$

put $A = R \Rightarrow A\left(\bigcup_{i=1}^{\infty} E_i\right) = \bigcup_{i=1}^{\infty} E_i$ $\quad A \cap E_i = R \cap E_i$

$$\Rightarrow m^*\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m^*(E_i)$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$$

(iii) $m^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m^*(E_i)$

$\Rightarrow m\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} m(E_i)$ finitly additive.

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iv Monotonicity : let $\phi \neq A, B \in M$, such that $B \subseteq A$.

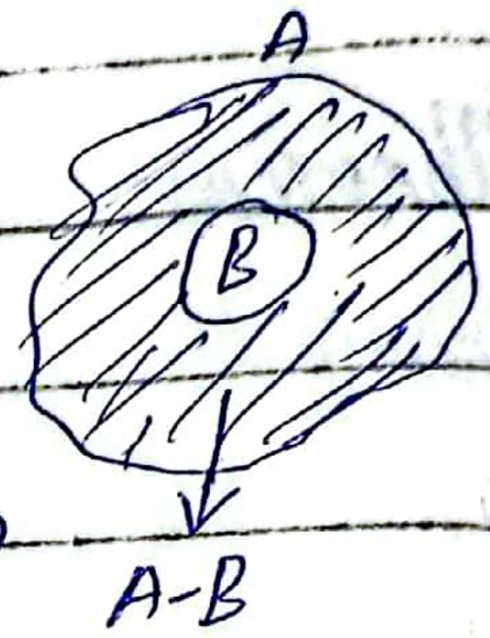
$\Rightarrow A = B \cup (A-B)$

$\therefore m^*$ is finitely additive,

$\Rightarrow m^*(A) = m^*(B) + m^*(A-B)$

$\Rightarrow m^*(A) \geq m^*(B)$ $\because m^*(A-B) \geq 0$

$\Rightarrow m^*(B) \leq m^*(A)$



$\because B \cap (A-B) \neq \phi$

Topic # 54

Theorem: Lebesgue measure "m" is translation invariant.

proof: $\because$ Given that say $\phi \neq E \in M$ is measurable.

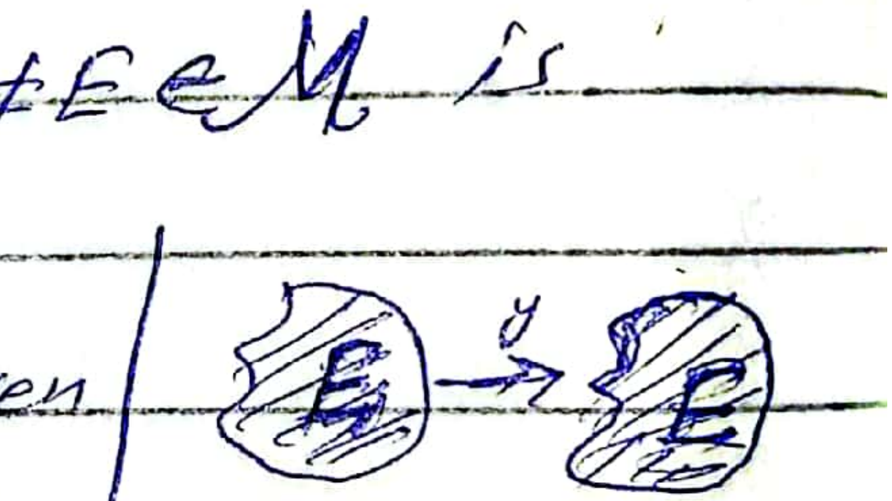
$\because$ "E" is measurable $\Leftrightarrow$ for given $\varepsilon > 0$, $\exists$ an open set $U \supseteq E$, such that $m^*(U-E) < \varepsilon$

"U" is open set in $\mathbb{R} \Rightarrow$ "U+y" is also open in $\mathbb{R}$, $\forall\ y \in \mathbb{R}$

if $E \subseteq U \Rightarrow E+y \subseteq U+y$

$\Rightarrow (U+y)-(E+y) = (U-E)+y$

$\Rightarrow m^*[(U+y)-(E+y)] = m^*[(U-E)+y]$



$\because$ Lebesgue outer measure is translation invariant.

$\Rightarrow m^*(U-E) < \varepsilon$

But $m^* = m \Rightarrow m[(U+y)-(E+y)] = m(U-E) < \varepsilon$

$\Rightarrow$ "E+y" is also measurable $\Rightarrow$ "m" is translation invariant.

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Topic # 55

Theorem: Suppose E_1 and E_2 are two measurable sets, then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

proof: If $m(E_1) = m(E_2) = \infty$, then the result is obvious.

Case I: If $m(E_1) < \infty$, $m(E_2) < \infty$

$\therefore$ if E_1 is measurable, then for any $\phi \neq A \subseteq R$

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1') \longrightarrow (1)$$

For particularly if $A = E_2$

$$(1) \Rightarrow m^*(E_2) = m^*(E_2 \cap E_1) + m^*(E_2 \cap E_1') \longrightarrow (2)$$

Similarly $E_2 \longleftrightarrow E_1$

$$m^*(E_1) = m^*(E_1 \cap E_2) + m^*(E_1 \cap E_2') \longrightarrow (3)$$

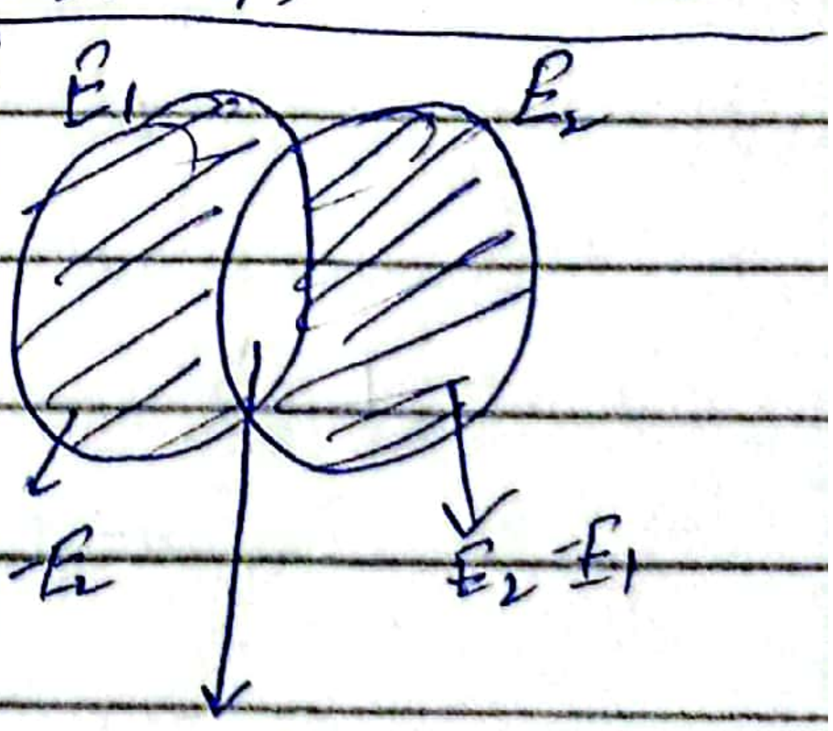
Adding (2) and (3): $(m^* \equiv m)$

$$m(E_1) + m(E_2) = m(E_1 \cap E_2) + m(E_1 \cap E_2') + m(E_2 \cap E_1) + m(E_2 \cap E_1') \longrightarrow (A)$$

$$E_1 \cup E_2 = (E_1 - E_2) \cup (E_2 - E_1) \cup E_1 \cap E_2$$

$$= (E_1 \cap E_2') \cup (E_1 \cap E_1') \cup (E_1 \cap E_2)$$

$$\Rightarrow m(E_1 \cup E_2) = m[(E_1 \cap E_2') \cup (E_2 \cap E_1') \cup (E_1 \cap E_2)]$$



"m" is finitely addable:

$$m(E_1 \cup E_2) = m(E_1 \cap E_2') + m(E_2 \cap E_1') + m(E_1 \cap E_2)$$

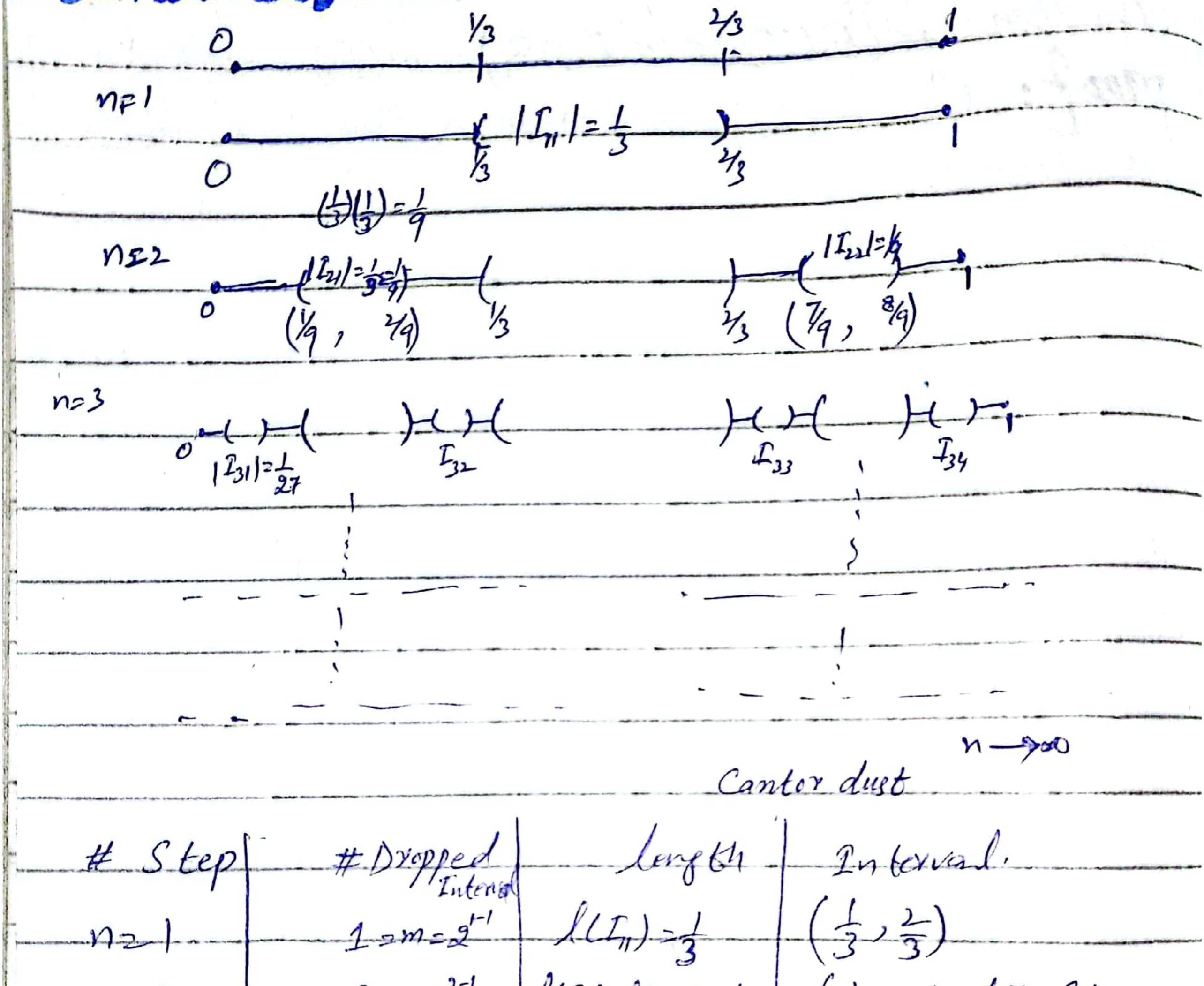
$$(A) \Rightarrow m(E_1) + m(E_2) = m(E_1 \cup E_2) + m(E_1 \cap E_2)$$

proved

Topic # 56

$R = R^2 = R^3$

Cantor Set:



Cantor dust

# Step	# Dropped Interval	length	Interval
n = 1	$1 = m = 2^{1-1}$	$l(I_{11}) = \frac{1}{3}$	$(\frac{1}{3}, \frac{2}{3})$
n = 2	$2 = m = 2^{2-1}$	$l(I_{21}) = l(I_{22}) = \frac{1}{3^2}$	$(\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9})$
n = 3	$4 = m = 2^{3-1}$	$l(I_{31}) = \dots = \frac{1}{3^3}$	
n	$m = 2^{n-1}$	$l(I_{n.}) = \dots = \frac{1}{3^n}$	

Dropped Interval = $G_1 = \underbrace{I_{11}}_{n=1}, \underbrace{I_{21}, I_{22}}_{n=2}, \underbrace{I_{31}, I_{32}, I_{33}, I_{34}, \dots}_{n=3}$

$I_{n1}, I_{n2}, \dots I_{nm}, \dots$

$n \to \infty, m = 2^{n-1}$

$\therefore G_1 = \{I_{11}\}, \{I_{21}, I_{22}\}, \{I_{31}, I_{32}, I_{33}, I_{34}\}, \dots$

$n \to \infty, m = 2^{n-1}$

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$$\Rightarrow \quad G_1 = \bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}$$

↗ universal set.

$\therefore$ Cantor Set $= C = [0,1] - G_1 = G_1'$

↘ $l = 1$

Topic #57

Measure of Cantor Set:

$\because$ "G_1" is open, being countable union of disjoint open intervals $\Rightarrow$ "G_1" is measurable.

$\because C = G_1'$ = closed and is also measurable.

$$m(G_1) = m\left(\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{2^{n-1}} I_{nm}\right) \quad \because \text{"}m\text{" is countably additive}$$

$$= \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} m(I_{nm}) = \sum_{n=1}^{\infty} \sum_{m=1}^{2^{n-1}} l(I_{nm})$$

expanding firstly by "n"

$$= \sum_{m=1}^{2^{1-1}} l(I_{1m}) + \sum_{m=1}^{2^{2-1}} l(I_{2m}) + \sum_{m=1}^{2^{3-1}} l(I_{3m}) + \cdots\cdots$$

$$= \underbrace{l(I_{11})}_{} + \underbrace{l(I_{21}) + l(I_{22})}_{} + \underbrace{l(I_{31}) + l(I_{32}) + l(I_{33}) + l(I_{34})}_{} + \cdots\cdots$$

$$= \frac{1}{3} + \left(\frac{1}{3^2} + \frac{1}{3^2}\right) + \left(\frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3} + \frac{1}{3^3}\right) + \cdots\cdots$$

$$m(G_1) = \frac{1}{3} + \frac{2}{3^2} + \frac{4}{3^3} + \cdots\cdots \quad \begin{cases} a = \frac{1}{3}, & r = \frac{2}{3} \end{cases}$$

$$S_\infty = \frac{a}{1-r} = \frac{1/3}{1-2/3} = 1$$

$\boxed{m(G_1) = 1}$ $\because [0,1] = G \cup C \Rightarrow m[0,1] = m(G_1) + m(C) \Rightarrow \boxed{m(C) = 0}$

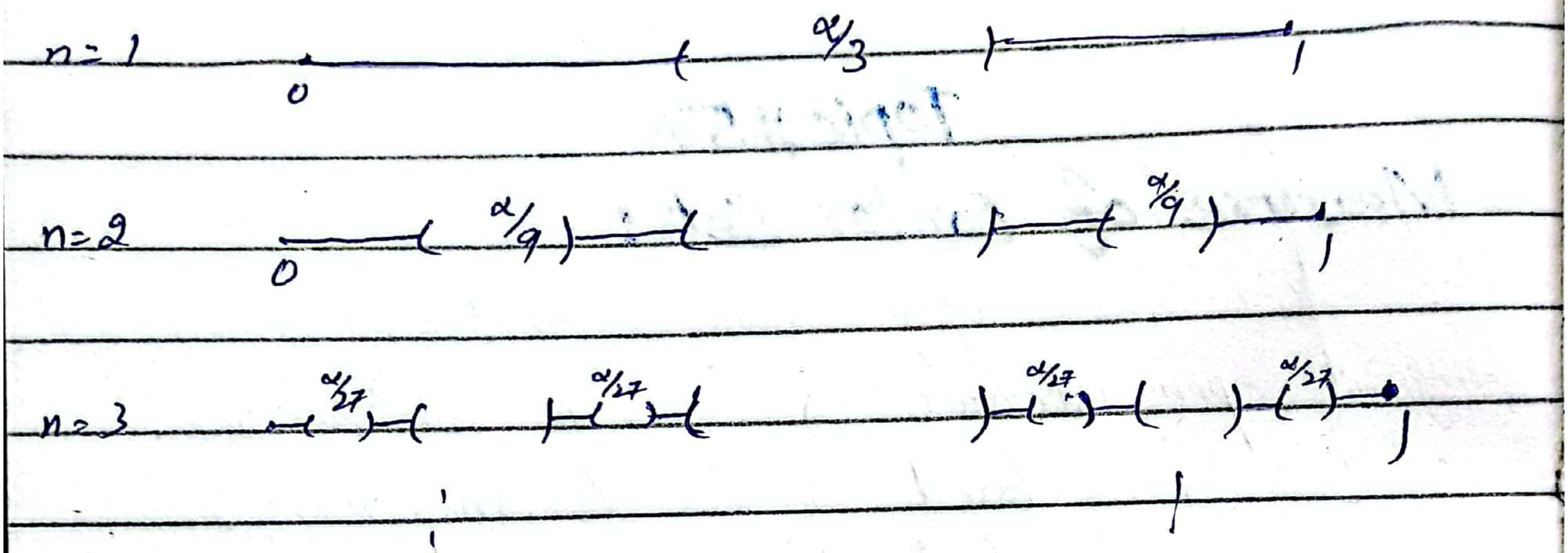
↑ disjoint

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Topic #58

Generalized Cantor Set and its Measure:





$$m(G_1) = \frac{\alpha}{3} + \left(\frac{\alpha}{9} + \frac{\alpha}{9}\right) + \left(\frac{\alpha}{27} + \frac{\alpha}{27} + \frac{\alpha}{27} + \frac{\alpha}{27}\right) + \cdots$$

$$m(G_1) = \frac{\alpha}{3} + \frac{2\alpha}{3^2} + \frac{4\alpha}{3^3} + \cdots$$

$$a = \alpha/3, \quad r = 2/3$$

$$S_\infty = \frac{a}{1-r} = \frac{\alpha/3}{1-2/3} = \frac{\alpha/3}{1/3} = \alpha$$

$$\boxed{m(G_1) = \alpha}$$

$\because [0,1] = G_1 \cup C$ (disjoint)

$$\Rightarrow m[0,1] = m(G_1) + m(C)$$

$$1 = \alpha + m(C)$$

$$\Rightarrow \boxed{m(C) = 1 - \alpha}$$

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Topic # 59

Theorem: Let $\{E_n \uparrow\}_{n=1}^{\infty}$ be an increasing sequence of measurable sets. Then

$$m^*\left(\bigcup_{n=1}^{\infty} E_n\right) = m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$$

Proof: $\because$ Given that $(E_n, \uparrow)$, such that

$\Rightarrow E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$ of measurable sets

Defining $F_1 = E_1$, $F_2 = E_2 - E_1$, $F_3 = E_3 - E_2$, $\cdots$

$F_n = E_n - E_{n-1}$, where $\{F_n\}_{n=1}^{\infty}$ is a seq. of disjoint measurable sets.

$$\Rightarrow \bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n} E_k \Rightarrow \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k.$$

$$\therefore m^*\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} F_k\right) \quad \text{(disjoint + measurable)}$$

$$= \sum_{k=1}^{\infty} m(F_k) = \lim_{n\to\infty} \sum_{k=1}^{\infty} m(F_k)$$

$$= \lim_{n\to\infty} m\left(\bigcup_{k=1}^{\infty} F_k\right) = \lim_{n\to\infty} m\left(\bigcup_{k=1}^{\infty} E_k\right)$$

$$= \lim_{n\to\infty} m^*(E_n) \qquad \text{(proved)}$$

Topic # 60

Theorem: Let $\{E_n \downarrow\}_{n=1}^{\infty}$ be a decreasing sequence of measurable sets such that $m(E_k) < \infty$ for some $k \in \mathbb{N}$, Then

$$m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n)$$

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proof: Let $m(E_1) < \infty$, Say (without lose of generality)

$\because \langle E_n, \downarrow \rangle \Rightarrow E_1 \supseteq E_2 \supseteq E_3 \supseteq \cdots$

$\Rightarrow \phi = E_1 - E_1 \subseteq E_1 - E_2 \subseteq E_1 - E_3 \subseteq \cdots E_1 - E_n \subseteq \cdots$

$E_1 \subseteq E_n \subseteq \cdots$

$$\Rightarrow m^*\left(\bigcup_{n=1}^{\infty}(E_1 - E_n)\right) = \lim_{n\to\infty} m^*(E_1 - E_n)$$

$$\bigcup_{n=1}^{\infty}(E_1 - E_n) = \bigcup_{n=1}^{\infty}(E_1 \cap E_n')$$

$$= (E_1 \cap E_1') \cup (E_1 \cap E_2') \cup \cdots$$

$$= E_1 \cap [E_1' \cup E_2' \cup \cdots]$$

$$= E_1 \cap [E_1 \cap E_2 \cap \cdots]'$$

$$= E_1 \cap \left(\bigcap_{n=1}^{\infty} E_n\right)'$$

$$= E_1 - \left(\bigcap_{n=1}^{\infty} E_n\right)$$

Sets Relation ↑

$$m^*\left(E_1 - \bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_1 - E_n)$$

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_1) - \lim_{n\to\infty} m^*(E_n)$$

$$m^*(E_1) - m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = m^*(E_1) - \lim_{n\to\infty} m^*(E_n)$$

$$\Rightarrow m^*\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m^*(E_n) \quad \text{(proved)}.$$

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Topic #61

Borel Set:

e.g. $G_\delta = \bigcap_{n=1}^{\infty} U_n$; $F_\sigma = \bigcup_{n=1}^{\infty} F_n$

Definition: A set "E" is said to be a Borel set if it can be expressed as countable union or intersection of open or closed sets.

e.g. $\{x\} = \bigcap_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ → G_δ Borel sets.

Topic #62

Borel Measure Space:

Restriction of Lebesgue measure "m" to the class of Borel set "B" is said to be Borel measure space. $(\mathbb{R}, B, m)$

Theorem: Every Borel set is measurable.

proof: Every open or closed set is measurable. ⇒ Countable union or intersection of closed or open set is also measurable.

∴ Borel set is also measurable being union or intersection of closed or open sets

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Topic #63

Theorem: Countable set are Borel sets with zero measure.

Proof: Case I: Singleton:

$\forall\ a \in \mathbb{R}: \{a\} = \bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$; $n \in \mathbb{N}$

Countable
↓
finite ⟷ infinite.
$\mathbb{Z}, \mathbb{N}, \mathbb{R}, \mathbb{Q}$ ---

$\therefore \left\{\left\{a - \frac{1}{n}, a + \frac{1}{n}\right\}, \downarrow\right\}$ is decreasing sequence.

with $m(E_n) = \frac{2}{n} < \infty$

where $E_n = \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$

$$\Rightarrow m\{a\} = m\left\{\bigcap_{n=1}^{\infty} E_n\right\} = \lim_{n \to \infty} m(E_n)$$

$$= \lim_{n \to \infty} \ell(E_n)$$

$$= \lim_{n \to \infty} \left(\frac{2}{n}\right) = \frac{2}{\infty}$$

$$= 0$$

Finite Set

$$\{x_1, x_2, x_3, \dots\dots x_n\} = \bigcup_{i=1}^{n} \{x_i\}$$

$$\Rightarrow m\left(\bigcup_{i=1}^{n} \{x_i\}\right) = \sum_{i=1}^{n} m\{x_i\} = 0$$

Infinite Set:

$$\{x_1, x_2, x_3, \dots\dots\dots\} = \bigcup_{i=1}^{\infty} \{x_i\}$$

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} \{x_i\}\right) = \sum_{i=1}^{\infty} m\{x_i\} = 0$$

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Topic #64

Sum Modulo 1:

Let $x, y \in A = [0,1)$ and we define sum modulo "1" of x and y as

$$x \hat{+} y = \begin{cases} x+y & \text{if } x+y<1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

Translation Modulo 1:

Let $E \subseteq A = [0,1)$ and $y \in A$, we define a translation modulo "1" of "E" as.

$$E \hat{+} y = \{ x \hat{+} y, x \in E \}.$$

$\downarrow I$

Set of all "Sum Modulo 1"

Topic #65

Theorem: Let $E \subset A = [0,1)$ be a measurable set, Then for ~~any~~ each $y \in A$. The set $E \hat{+} y$ is measurable and $m(E) = m(E \hat{+} y)$.

proof: Suppose $E_1 = E \cap [0, 1-y)$ and

$E_2 = E \cap [1-y, 1)$

$\therefore E_1 [0, 1-y), [1-y, 0)$ are measurable

$\Rightarrow E_1$ and E_2 are also measurable being the intersection of two measurable sets.

Also $E_1 \cap E_2 = \{E \cap [1-y, 1)\} \cap \{E \cap [0, 1-y]\}$

$= E \cap \{[0, 1-y) \cap [1-y, 1)\}$

$= E \cap \phi = E$

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and $E_1 \cup E_2 = E \cup \phi = E$

$\Rightarrow m(E) = m(E_1) + m(E_2) \longrightarrow (1)$

"m" is finitely additive.

Let $y_1 \in A$, if $y_1 \in E_1$

$\Rightarrow y_1 < 1-y \Rightarrow y_1 + y < 1 \Rightarrow y_1 \hat{+} y = y_1 + y$

$E_1 + y$

$\Rightarrow E_1 \hat{+} y = E_1 + y \longrightarrow (a)$

if $y_2 \in E_2 \Rightarrow y_2 \geq 1-y \Rightarrow y_1 + y \geq 1$

$\Rightarrow y_2 \hat{+} y = y_2 + y - 1$

$\Rightarrow E_2 \hat{+} y = E_2 + (y-1) \longrightarrow (b)$

$\because$ "m" is translation invariant.

$\Rightarrow E_1 + y$ and $E_2 + (y-1)$ are measurable.

$\Rightarrow E_1 \hat{+} y$ and $E_2 \hat{+} (y-1)$ are measurable.

$\Rightarrow m(E_1 \hat{+} y) = m(E_1 + y) = m(E_1)$

Similarly

$$m(E_2 \hat{+} y) = m(E_2 + y - 1) = m(E_2)$$

$\because E_1 \cup E_2 = E$, $E_1 \cap E_2 = \phi$

$\Rightarrow E \hat{+} y = (E_1 + y) \cup (E_2 + y)$ and $(E_1 + y) \cap (E_2 + y) = \phi$

"$E \hat{+} y$" being union of two disjoint measurables sets its measurable.

$\Rightarrow m(E \hat{+} y) = m(E_1 \hat{+} y) + m(E_2 \hat{+} y)$

$= m(E_1 + y) + m(E_2 + y)$

$= m(E_1) + m(E_2) = m(E)$

$\Rightarrow$ "m" is also invariant w.r.t translation modulo "1".

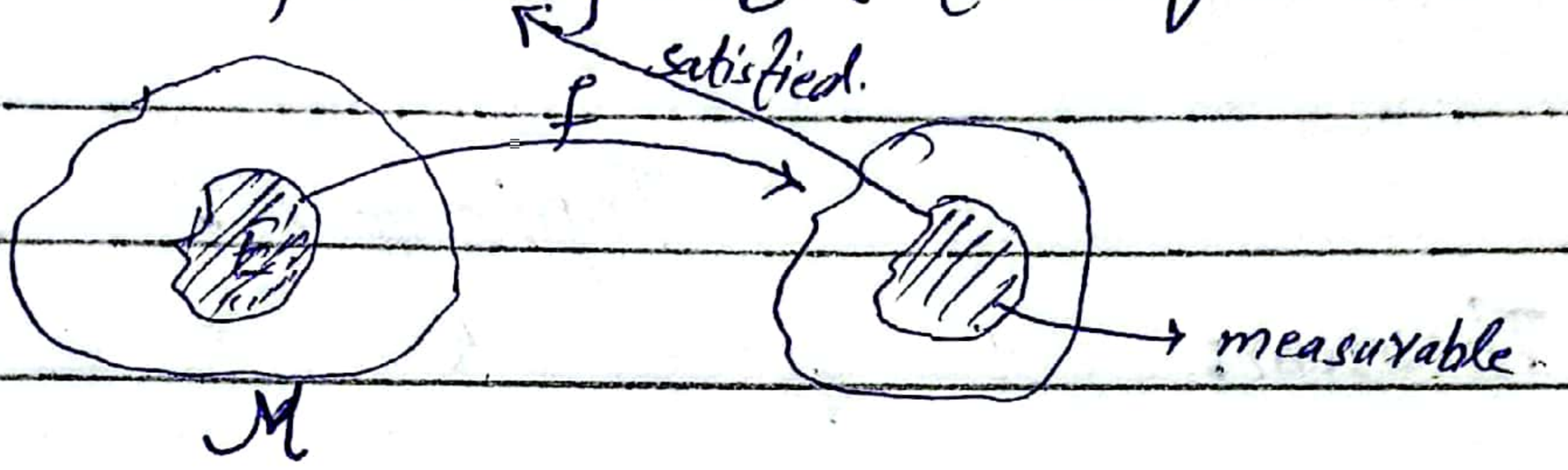
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Topic # 66

Measurable Function:

Let "M" be the measurable space (σ-algebra of measurable sets). a function $f: E \to \mathbb{R}$ is Lebesgue measurable function on $\phi \neq E \in M$.

if $\{x \in E : f(x) > a\} \in M$ for every $a \in \mathbb{R}$.



For Example (1): Suppose "f" is real valued identically constant function $f(x) = \beta$, Then:

$$\{x : f(x) > a\} = \begin{cases} \mathbb{R} & \text{if } \beta > a \\ \phi \in M \end{cases}$$

$\therefore \mathbb{R}, \phi \in M$

$\Rightarrow$ "f" is measurable.

For Example (2):

$$f(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

$$\{x : f(x) > a\} = \begin{cases} \phi \\ A \neq X \\ X = A \end{cases}$$

$\phi, A, X \in M \Rightarrow$ "f" is measurable.

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Example (3): $(R, B, m) \to$ (Borel σ-algebra)

$f(x) = x$

$\Rightarrow \{x : f(x) > a\} = \begin{cases} (a, +\infty) \in B \\ \text{i.e always measurable.} \end{cases}$

$\Rightarrow$ "f" is measurable.

Topic # 67

Theorem: Let $f : E \to R \cup \{\pm\infty\}$, $\phi \neq E \in M$, Then.

(i) "f" is measurable $\Leftrightarrow$ (ii) $\{x \in E : f(x) \geq a\}$ is measurable for all $a \in R \Leftrightarrow$ (iii) $\{x \in E : f(x) < a\}$ is measurable $\forall\ a \in R \Leftrightarrow$ (iv) $\{x \in E : f(x) \leq a\}$ is measurable $\forall\ a \in R$.

Proof: (i) $\to$ (ii) "f" is measurable $\to \{x \in E : f(x) > a\} \in M$

$$\{x \in E : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$$

Let $y \in \{x \in E : f(x) \geq a\} \Rightarrow f(y) \geq a, \forall\ a \in R$

$\Rightarrow f(y) > a - \frac{1}{n}, \forall\ n \in N$

$\Rightarrow y \in \{x \in E : f(x) > a - \frac{1}{n}\}, \forall\ n \in N$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$

$\{x \in E : f(x) \geq a\} \subset \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$ $\rightarrow$ (a)

Let conversely $t \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$

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$\Rightarrow f(t) > a - \frac{1}{n}, \forall\ n \in \mathbb{N}$

$\Rightarrow f(t) \geq a, \forall\ a \in \mathbb{R}$ ~~$\Rightarrow t \in$~~

$\Rightarrow t \in \{x \in E : f(x) \geq a\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\} \subseteq \{x : f(x) \geq a\}$ → (b)

from (a) and (b)

$\Rightarrow \{x \in E : f(x) \geq a\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$

$\because \{x \in E : f(x) > a - \frac{1}{n}\}$ is measurable.

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) > a - \frac{1}{n}\}$ is also measurable.

$\forall\ n \in \mathbb{N}$

$\Rightarrow \{x \in E : f(x) \geq a\}$ is also measurable.

Topic #68

(ii) $\Rightarrow$ (iii) $\{x \in E : f(x) \geq a\} \in M \Rightarrow \{x \in E : f(x) < a\} \in M$.

$\{x \in E : f(x) < a\} = \{x \in E : f(x) \geq a\}^c \quad \forall\ a \in \mathbb{R}$

$\because \{x \in E : f(x) \geq a\} \in M \Rightarrow \{x \in E : f(x) \geq a\}^c \in M$

$\Rightarrow \{x \in E : f(x) < a\} \in M$ i.e. measurable.

Topic #69

(iii) $\Rightarrow$ (iv) $\underbrace{\{x \in E : f(x) < a\}}_{\in M} \Rightarrow \{x \in E : f(x) < a\} \in M$.

ple show that $\{x \in E : f(x) \leq a\} =$

$= \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$

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Let $y \in \{x \in E : f(x) \leq a\}$ ——— (a)

$\Rightarrow f(y) \leq a$, $\forall\ a \in R$

$\Rightarrow f(y) < a + \frac{1}{n}$, $\forall\ n \in N$

$\Rightarrow y \in \{x \in E : f(x) < a + \frac{1}{n}\}$, $\forall\ n \in N$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$ ——— (B)

From (a) and (B) $\Rightarrow \{x \in E : f(x) \leq a\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$

Let $z \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\}$

$\Rightarrow z \in \{x \in E : f(x) < a + \frac{1}{n}\} \Rightarrow f(z) < a + \frac{1}{n}, \forall n \in N$

$\Rightarrow f(z) \leq a\ \forall\ a \in R \Rightarrow z \in \{x \in E : f(x) \leq a\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} \subseteq \{x \in E : f(x) \leq a\}$

$\therefore \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} = \{x \in E : f(x) \leq a\}$

Given that $\{x \in E : f(x) < a + \frac{1}{n}\} \in M$.

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) < a + \frac{1}{n}\} \in M$

$\Rightarrow \{x \in E : f(x) \leq a\} \in M$. i.e is measurable.

Topic # 70

(iv) $\Rightarrow$ (i) ; $\{x \in E : f(x) \leq a\} \in M$, $\Rightarrow$ "f" is measurable.

$f : E \to R \cup \{+\infty\}$, $\{x \in E : f(x) < a\} \in M$,

$\{x \in E : f(x) \leq a\} \in M$, $\forall\ a \in R$

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$= \{x \in E : f(x) \leq a\}^c = \{x \in E : f(x) > a\} \in M$

$\Rightarrow$ "f" is measurable.

Topic # 71

Theorem: Let $\phi \neq E \in M$, $f : E \to R \cup \{\pm\infty\}$, then $\{x \in E : f(x) = a\} \in M$, $\forall a \in R$.

Proof: Case I: $a \in R$

$\Rightarrow \{x \in E : f(x) = a\} = \{x \in E : f(x) \geq a\} \cap \{x \in E : f(x) \leq a\}$

$\therefore \{x \in E : f(x) \geq a\}$ and $\{x \in E : f(x) \leq a\} \in M$.

$\Rightarrow \{x \in E : f(x) \geq a\} \cap \{x \in E : f(x) \leq a\} \in M$.

$\Rightarrow \{x \in E : f(x) = a\} \in M$ (measurable)

Case II: If $a = \infty$, Then $\{x \in E : f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\}, \forall n \in N$

Let $y \in \{x \in E : f(x) = \infty\} \Rightarrow y \in \{x \in E : f(x) > n\}$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\}, \forall n \in N$

$\Rightarrow x \in E : f(x) = \infty \subset \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\}$ ——— (a)

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Now let $t \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\}, \forall n \in \mathbb{N}$

$\Rightarrow t \in \{x \in E : f(x) > n\}, \forall n \in \mathbb{N}$

$\Rightarrow f(t) > n, \forall n \in \mathbb{N}$

$\Rightarrow t \in \{x \in E : f(x) = \infty\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\} \subseteq \{x \in E : f(x) = \infty\} \longrightarrow (b)$

From (a) and (b)

So, $\{x \in E; f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) > n\}$

Case III: If $a = -\infty$, Then $\{x \in E : f(x) = -\infty\} =$

$$\bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}$$

Let $z \in \{x \in E : f(x) = -\infty\} \Rightarrow z \in \{x \in E : f(x) < -n\}$

$\Rightarrow z \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}, \forall n \in \mathbb{N}$

$\Rightarrow \{x \in E : f(x) = -\infty\} \subseteq \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\} \longrightarrow (a)$

Now let $s \in \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}, \forall n \in \mathbb{N}$

$\Rightarrow s \in \{x \in E : f(x) < -n\}, \forall n \in \mathbb{N}$

$\Rightarrow f(s) < -n, \forall n \in \mathbb{N}$

$\Rightarrow s \in \{x \in E : f(x) = -\infty\}$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\} \subseteq \{x \in E : f(x) = -\infty\} \longrightarrow (b)$

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From (a) and (b)

So, $\{x \in E : f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x \in E : f(x) < -n\}$

Topic # 72

Positive anel Negative parts of Extented Real Valued Function:

$$|f(x)| = \begin{cases} +f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

$$f^+ = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{if } f(x) < 0 \end{cases}$$

$$= \max\{f(x), 0\} = f \vee 0$$

$$f^- = \begin{cases} 0 & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

$$= \max\{-f(x), 0\} = -f \vee 0$$

where $f : A \to \mathbb{R} \cup \{\pm\infty\}$, $A \neq \phi$, $\forall x \in A$

Topic # 73

Theorem: Show that

(i) $f^+ = \frac{f + |f|}{2}$, (ii) $f^- = \frac{|f| - f}{2}$, (iii) $f = f^+ - f^-$

(iv) $|f| = f^+ + f^-$, (v) $\max(f, g) = \frac{(f+g) + |f-g|}{2}$

(vi) $\min(f, g) = \frac{(f+g) - |f-g|}{2}$

proof:

(i) If $f(x) \geq 0 \Rightarrow f^+(x) = f(x) \Rightarrow f^+ = f$

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$$|f(x)| = |f|(x) = f(x)$$

$$\therefore \frac{f+|f|}{2}(x) = \frac{f(x)+|f|(x)}{2} = \frac{f(x)+f(x)}{2} = f(x) = f^+(x)$$

If $f(x) < 0 \Rightarrow f^+(x) = 0$, $|f|(x) = |f(x)| = -f(x)$

$$\therefore \frac{f+|f|}{2}(x) = \frac{f(x)+|f|(x)}{2} = \frac{f(x)-f(x)}{2} = 0 = f^+(x)$$

$$\Rightarrow \frac{f+|f|}{2} = f^+ \quad \text{(proved)}$$

(ii) If $f(x) > 0 \Rightarrow f^-(x) = 0$, $|f(x)| = |f|(x) = +f(x)$

$$\therefore \frac{|f|-f}{2}(x) = \frac{|f|(x)-f(x)}{2} = \frac{+f(x)-f(x)}{2} = 0 = f^-(x)$$

If $f(x) < 0 \Rightarrow f^-(x) = -f(x)$, $|f(x)| = |f|(x) = -f(x)$

$$\therefore \frac{|f|-f}{2}(x) = \frac{|f|(x)-f(x)}{2} = \frac{-f(x)-f(x)}{2} = -f(x) = f^-(x)$$

$$\Rightarrow \frac{|f|-f}{2} = f^- \quad \text{(proved)}$$

(iii) $f = f^+ - f^-$

If $f(x) \geq 0 \Rightarrow f^+(x) = f(x)$, $f^-(x) = 0$

$$\therefore (f^+ - f^-)(x) = f^+(x) - f^-(x) = f(x) - 0 = f(x) = f$$

$$\Rightarrow f^+ - f^- = f \quad \text{(proved)}.$$

(iv) $|f| = f^+ + f^-$

If $f(x) \geq 0 \Rightarrow f^+(x) = f(x)$, $f^-(x) = 0$

$$\therefore (f^+ + f^-)(x) = f^+(x) + f^-(x) = f(x) + 0 = f(x) = |f(x)|$$

$$\Rightarrow f^+ + f^- = |f| \quad \text{(proved)}.$$

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(V) $\max(f,g) = \dfrac{(f+g)+|f-g|}{2}$

$$|f-g| = \begin{cases} +(f-g), & \text{if } \geq 0 \\ -(f-g), & \text{if } < 0 \end{cases}$$

$$|f-g|(x) = |f(x)-g(x)| = \begin{cases} f(x)-g(x), & f(x) \geq g(x) \\ g(x)-f(x), & f(x) < g(x) \end{cases}$$

Case I: If $f(x) \geq g(x) \Rightarrow f(x)-g(x) \geq 0$

$\Rightarrow |f(x)-g(x)| = f(x)-g(x)$

$\therefore \dfrac{(f+g)+|f-g|}{2}(x) = \dfrac{f(x)+g(x)+|f(x)-g(x)|}{2}$

$= \dfrac{f(x)+g(x)+f(x)-g(x)}{2} = f(x)$

Note: $f(x) \geq g(x)$ then $\max(f,g) = f(x)$

$\Rightarrow \dfrac{(f+g)+|f-g|}{2} = \max(f,g)$

Case II: If $f(x) < g(x) \Rightarrow f(x)-g(x) < 0$

$\Rightarrow |f(x)-g(x)| = -(f(x)-g(x)) = g(x)-f(x)$

and if $f(x) < g(x)$ then $\max(f,g) = g(x)$

$\therefore \dfrac{(f+g)+|f-g|}{2}(x) = \dfrac{f(x)+g(x)+|f(x)-g(x)|}{2}$

$= \dfrac{f(x)+g(x)+g(x)-f(x)}{2} = g(x)$

$\Rightarrow \dfrac{(f+g)+|f-g|}{2} = \max(f,g)$ (proved)

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(vi) $\min(f,g) = \dfrac{(f+g)-|f-g|}{2}$

$$|f-g|(x) = |f(x)-g(x)| = \begin{cases} f(x)-g(x) & ; f(x) \geq g(x) \\ g(x)-f(x) & ; f(x) < g(x) \end{cases}$$

Case I: If $f(x) \geq g(x) \Rightarrow f(x)-g(x) \geq 0$

$\Rightarrow |f(x)-g(x)| = f(x)-g(x)$ and

If $f(x) \geq g(x) \Rightarrow \min(f,g) = g(x)$

$\therefore \dfrac{(f+g)-|f-g|}{2}(x) = \dfrac{f(x)+g(x)-|f(x)-g(x)|}{2}$

$= \dfrac{f(x)+g(x)-(f(x)-g(x))}{2}$

$= \dfrac{f(x)+g(x)-f(x)+g(x)}{2} = g(x)$

$\Rightarrow \dfrac{(f+g)-|f-g|}{2} = \min(f,g)$

Case II: If $f(x) < g(x) \Rightarrow f(x)-g(x) < 0$

$\Rightarrow |f(x)-g(x)| = g(x)-f(x)$ and

If $f(x) < g(x) \Rightarrow \min(f,g) = f(x)$

$\therefore \dfrac{(f+g)-|f-g|}{2}(x) = \dfrac{f(x)+g(x)-|f(x)-g(x)|}{2}$

$= \dfrac{f(x)+g(x)-g(x)+f(x)}{2} = f(x)$

$\Rightarrow \dfrac{(f+g)-|f-g|}{2} = \min(f,g)$ (Proved).

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Topic # 74

Characteristic Function:

Let $\phi \neq X$ and $A \subseteq X$, then characteristic function $\chi_A : X \to R$ is defined and given by:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

If $A = \phi \Rightarrow \chi_A(x) = 0 = 0(x)$, $\forall x \in X$ (zero function)

$\Rightarrow \chi_\phi = 0$

And if $A = X \Rightarrow \chi_A(x) = 1 = 1(x)$ $\forall x \in X$ (Identity function)

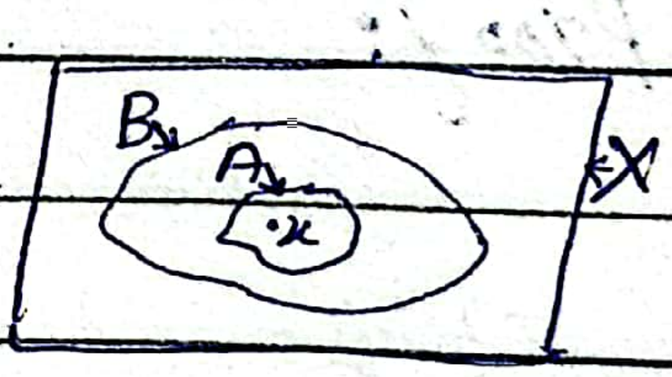
$\Rightarrow \chi_X = 1$

Topic # 75

Theorem: Prove that if $A \subseteq B$, then $\chi_A \leq \chi_B$.

Proof: Case I; If $x \in A$ and $A \subseteq B \Rightarrow x \in B$

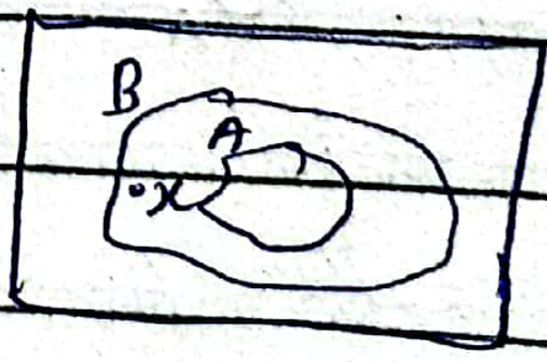
$\Rightarrow \chi_A = 1 = \chi_B \Rightarrow \chi_A \leq \chi_B$



Case II; If $x \in B$ but $x \notin A$

$\Rightarrow \chi_A = 0$ and $\chi_B = 1$

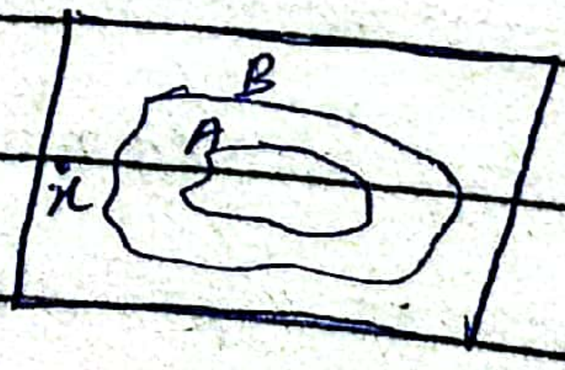
$\Rightarrow \chi_A \leq \chi_B$



Case III; If $x \notin A$ nor $x \notin B$

$\Rightarrow \chi_A = 0 = \chi_B \Rightarrow \chi_A \leq \chi_B$

(proved)



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Topic # 76

Theorem: Show that $\chi_{A\cap B} = \chi_A \cdot \chi_B$ 1

Proof: If $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow \chi_{A\cap B} = 1 = \chi_A = \chi_B = 1 \cdot 1 = \chi_A \cdot \chi_B$

If $x \notin A\cap B \Rightarrow$ {
- $x \in A$ but $x \notin B \Rightarrow \chi_A = 1, \chi_B = 0$
- $x \in B$ but $x \notin A \Rightarrow \chi_B = 1, \chi_A = 0$
- $x \notin A$ and $x \notin B \Rightarrow \chi_A = \chi_B = 0$

$\therefore \chi_A \cdot \chi_B = 0 = O(x)$ }

$x \notin A\cap B \Rightarrow \chi_{A\cap B} = 0$

For all, $x \notin A\cap B$; $\chi_{A\cap B} = 0 = O(x) = \chi_A \cdot \chi_B$

$\chi_{A\cap B} = \chi_A \cdot \chi_B$ (proved)

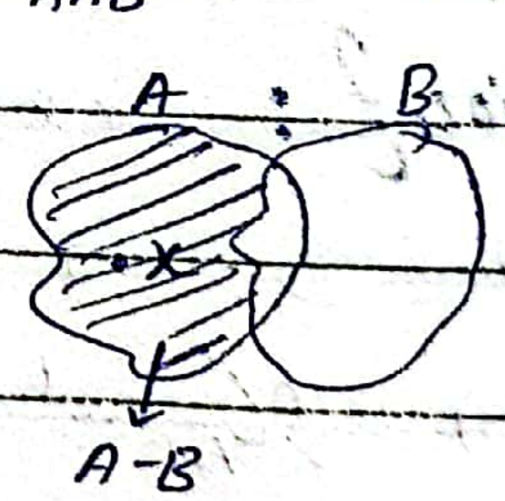
Topic # 77

Theorem: Show that $\chi_{A-B} = \chi_A - \chi_{A\cap B}$.

Proof: Case I: If $x \in A - B$

$\Rightarrow x \in A$ but $x \notin B, x \notin A\cap B$

$\Rightarrow \chi_{A-B} = 1$



$\chi_A - \chi_{A\cap B} = 1 - 0 = 1 = \chi_{A-B}$

Case II If $x \notin A - B \Rightarrow$ {
- $x \notin A, x \notin B, x \notin A\cap B \longrightarrow$ (a)
- $x \notin A, x \in B, x \notin A\cap B \longrightarrow$ (b)
- $x \in A, x \in B, x \in A\cap B \longrightarrow$ (c)

$\Rightarrow \chi_{A-B} = 0 - O(x)$

$\chi_A - \chi_{A\cap B} =$ {
- (a) $\Rightarrow 0 - 0 = 0$
- (b) $\Rightarrow 0 - 0 = 0$
- (c) $\Rightarrow 1 - 1 = 0$

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$\therefore$ For all cases; $\chi_{A-B} = \chi_A - \chi_{A\cap B}$ (proved)

Topic #78

Theorem: Show that $\chi_{A\times B} = \chi_A \cdot \chi_B$

Proof: $A\times B = \{(x,y) : x\in A, y\in B\}$

Case I:

$(x,y) \in A\times B \Rightarrow \begin{cases} x\in A \text{ and } x\in B \\ \Rightarrow \chi_A = 1 = \chi_B \end{cases}$

$\Rightarrow \chi_{A\times B}(x,y) = 1$

$\therefore \chi_{A\times B} = \chi_A \cdot \chi_B$

Case II: $\begin{cases} x\in A \text{ but } y\notin B \rightarrow (a) \\ x\notin A \text{ but } y\in B \rightarrow (b) \\ x\notin A \text{ and } y\notin B \rightarrow (c) \end{cases}$

If $(x,y) \notin A\times B \Rightarrow$

$\Rightarrow \chi_{A\times B}(x,y) = 0$

Also: $\chi_A \cdot \chi_B = \begin{cases} (a) \Rightarrow 1\cdot 0 = 0 \\ (b) \Rightarrow 0\cdot 1 = 0 \\ (c) \Rightarrow 0\cdot 0 = 0 \end{cases}$

In all cases: $\chi_{A\times B} = \chi_A \cdot \chi_B$ (proved)

Topic #79

Theorem: If $A = \bigcup_{n=1}^{\infty} A_n$ and $\{A_n\}_{n=1}^{\infty}$ is a sequence of pairwise disjoint subsets of X. Then show that $\chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

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Case I:

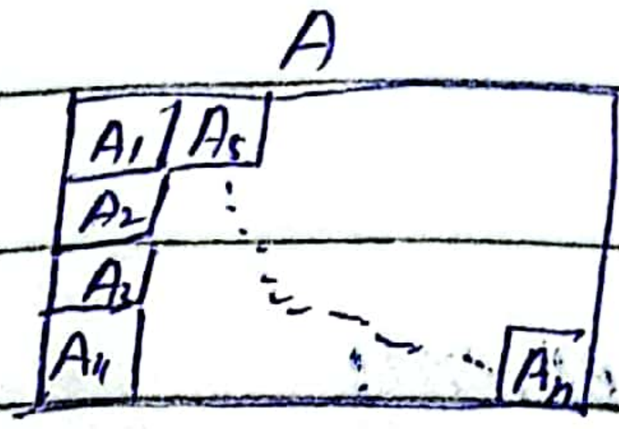
Proof: If $x \in A \Rightarrow x \in \bigcup_{n=1}^{\infty} A_n$

$\Rightarrow x \in A_k$ exactly for $n = k$

$x \notin \bigcup_{n=1}^{\infty} A_n - \{A_k\}$

$\Rightarrow \chi_{A_k}(x) = 1, \chi_{A_n}(x) = 0, \forall n \neq k$

$x \in A \Rightarrow \chi_A(x) = 1$



$\therefore \chi_A(x) = 1 = 0 + 0 + \cdots + \underset{n=k}{1} + 0 + 0 + \cdots$

$\chi_A(x) = 1 = \sum_{n=1}^{\infty} \chi_{A_n}(x) \Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$

Case II: $x \notin A = \bigcup_{n=1}^{\infty} A_n \Rightarrow x \notin A_n, \forall n \in \mathbb{N}$

$\Rightarrow \chi_A(x) = 0$ $\left\{ \Rightarrow \chi_{A_n}(x) = 0, \forall n \in \mathbb{N} \right.$

$= 0 + 0 + 0 + \cdots + 0 + \cdots = \sum_{n=1}^{\infty} \chi_{A_n}(x)$

$\Rightarrow \chi_A = \sum_{n=1}^{\infty} \chi_{A_n}$ (proved)

Topic # 80

Theorem: Let $\phi \neq A \subseteq X$ and for characteristic function $\chi_A : X \to \mathbb{R}$, χ_A is measurable $\Leftrightarrow$ "A" is measurable.

Proof: Suppose "χ_A" is measurable (function)

$\Rightarrow \forall a \in \mathbb{R}$ is set; $\{x : \chi_A(x) > a\}$ is measurable.

In particular for $a = 0$; $\{x : \chi_A(x) > 0\} = A$

By definition of χ_A.

$\therefore A \in M$ (measurable)

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Conversely, let $A \in M$ (measurable) and we prove "χ_A" is measurable.
In other words we want to prove the $\{x : \chi_A(x) \geq a\} \in M$, $\forall\ a \in R$

$$\{x : \chi_A(x) \geq a\} = \begin{cases} \phi & \text{if } a \geq 1 \\ A & \text{if } 0 \leq a < 1 \\ A \cup A^c & \text{if } a < 0 \end{cases}$$

(i) $a \geq 1$, $\chi_A(x) \geq 1 \Rightarrow$ there does not exist $x \in A$ such that $\chi_A(x) \geq 1$

(ii) $0 \leq a < 1$, $\chi_A(x) > 0.9$ (say)

(iii) $a < 0$ (say -2), $\chi_A(x) > -2$

In all cases; if $A \in M$ (measurable)

$\Rightarrow A^c \in M$ (measurable)

$\Rightarrow A \cup A^c \in M$ (measurable)

$\phi \in M$ (measurable)

$\Rightarrow \{x : \chi_A(x) \geq a\} \in M$ (measurable)

So "χ_A" is measurable. (Proved)

Topic # 81

Algebra of Measurable Functions:

If "f" and "g" are two extended real valued and measurable functions defined on same domain $\phi \neq D$ then (i) $f + k$, (ii) kf, (iii) $f \pm g$, (iv) fg, (v) $\frac{f}{g}, g \neq 0$, (vi) $|f|$, (vii) $f \vee g$ (viii) $f \wedge g$ are measurable functions.

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Topic # 82 & 83

Proof: is "$f+k$" is measurable, $k \in R$

$\because$ "f" is measurable $\Rightarrow \{x : f(x) > a\} \in M, \forall a \in R$

$\Rightarrow \{x : f(x) > a-k\} \in M$ (measurable)

$\Rightarrow \{x : f(x)+k > a\} \in M, \forall a \in R$

$\Rightarrow$ "$f+k$" is measurable.

83 (ii) "kf" is measurable, $\forall k \in R$ | $f \to kf$ Scaler composition

proof: $\because$ "f" is measurable.

$\Rightarrow \{x : f(x) > a\} \in M, \forall a \in R$

(a) If $k > 0$: $\because f(x) > a, \forall a \in R$

$\Rightarrow f(x) > \frac{a}{k} \Rightarrow kf(x) > a, \forall a \in R$

$\therefore \{x : kf(x) > a\} \in M \ \forall a \in R$

$\Rightarrow$ "kf" is measurable.

(b) If $k < 0$; $f(x) > a, \forall a \in R$ (given)

$\Rightarrow f(x) > \frac{a}{k} \Rightarrow kf(x) < a \quad (\because k < 0)$

$\Rightarrow \{x : kf(x) < a\} \in M$

$\Rightarrow$ "kf" is measurable.

(c) If $k = 0 \Rightarrow \{x : kf(x) > a\}, \forall a \in R$

$= \{x : 0 f(x) > a\} \ \because \forall a \in R$

$= \{x : 0 > a\}, \forall a \in R$ ... (1)

$\{x : 0 > a\} = \begin{cases} D & ; \ a < 0 \ (\because D : \exists x : f(x) > a) \\ \phi & ; \ a \geq 0 \end{cases}$

"D" is measurable (given) and ϕ is measurable.

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$\therefore \{x : |kf(x)| > a\}$ is measurable.

$\Rightarrow$ "kf" is measurable.

Topic # 84

(iii) "$f+g$" is measurable.

proof: We have to proof "$f+g$" is measurable.

$\Rightarrow \{x : (f+g)(x) > a\} \in M$ (To prove)

$\{x : (f+g)(x) > a\} = \{x : f(x) + g(x) > a\}$

$= \{x : f(x) > a - g(x)\}$, $\forall\, a \in R$

$\because \{f(x), a - g(x) \in R \Rightarrow \exists\, p \in Q$ such that

$f(x) > p > a - g(x)\}$

because "Q" is dense in R : $\bar{Q} = R$

$\therefore \{x : (f+g)(x) > a\} = \bigcap_{p \in Q} \{x : f(x) > p\} \cap \{x : a - g(x) < p\}$

i.e. our claim.

Let $y \in \{x : (f+g)(x) > a\} \Rightarrow y \in \{x : f(x) + g(x) > a\}$

$\Rightarrow f(y) + g(y) > a \Rightarrow f(y) > a - g(y)$

$\Rightarrow$ there exist($\exists$) $p \in Q$ such that $f(y) > p > a - g(y)$

$Q = R$

$\Rightarrow f(y) > p$ and $a - g(y) < p$

$\Rightarrow y \in \{x : f(x) > p\}$ and $y \in \{x : a - g(x) < p\}$

$\Rightarrow y \in \{x : f(x) > p\} \cap \{x : a - g(x) < p\}$

$\Rightarrow y \in \bigcup_{p \in Q} [\{x : f(x) > p\} \cap \{x : a - g(x) < p\}]$

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$\Rightarrow \{x : (f+g)(x) > a\} \subseteq \bigcup_{p \in \mathbb{Q}} \left[\{x : f(x) > p\} \cap \{x : a - g(x) < p\}\right]$ ——→ (A)

Now Let $t \in \bigcup_{p \in \mathbb{Q}} \left[\{x : f(x) > p\} \cap \{x : a - g(x) < p\}\right]$

$\Rightarrow t \in \{x : f(x) > p\} \cap \{x : a - g(x) < p\}$ for some $p \in \mathbb{Q}$

$\Rightarrow t \in \{x : f(x) > p\}$ and $t \in \{x : a - g(x) < p\}$

$\Rightarrow f(t) > p$ and $a - g(t) < p \Rightarrow p > a - g(t)$

$\Rightarrow f(t) > p > a - g(t) \Rightarrow f(t) > a - g(t)$

$\Rightarrow f(t) + g(t) > a \rightarrow (f+g)(t) > a$

$\Rightarrow t \in \{x : (f+g)(x) > a\}$

$\Rightarrow \bigcup_{p \in \mathbb{Q}} \left[\{x : f(x) > p\} \cap \{x : a - g(x) < p\}\right] \subseteq \{x : (f+g)(x) > a\}$ ——→ (B)

From (A) and (B) $\rightarrow \{x : (f+g)(x) > a\} =$

$$= \bigcup_{p \in \mathbb{Q}} \left[\{x : f(x) > p\} \cap \{x : a - g(x) < p\}\right]$$

$\therefore$ $\begin{cases} \text{"}f\text{" is measurable} \Rightarrow \{x : f(x) > p\} \in M \\ \text{"}g\text{" is measurable} \Rightarrow \{x : a - g(x) < p\} \in M \end{cases}$

$\Rightarrow \{x : f(x) > p\} \cap \{x : a - g(x) < p\} \in M$, for $p \in \mathbb{Q}$

$\Rightarrow \bigcup_{p \in \mathbb{Q}} \left[\{x : f(x) > p\} \cap \{x : a - g(x) < p\}\right] \in M$

$\Rightarrow \{x : (f+g)(x) > a\} \in M$ (measurable)

$\Rightarrow$ "$f+g$" is measurable function.

Now we have to prove "f-g" is measurable.

⇒ As g is measurable ⇒ (-1)g = -g is also M.

⇒ f + (-1)g = f - g is also measurable.

(proved)

Topic#85

(iv) "fg" is measurable.

proof: $fg = \frac{1}{4}[(f+g)^2 - (f-g)^2]$

Firstly we prove "f^2" is measurable.

⇒ we need to prove $\{x : f^2(x) > a\} \in M, \forall a \in R$.

$y = x^2$ (Say function)

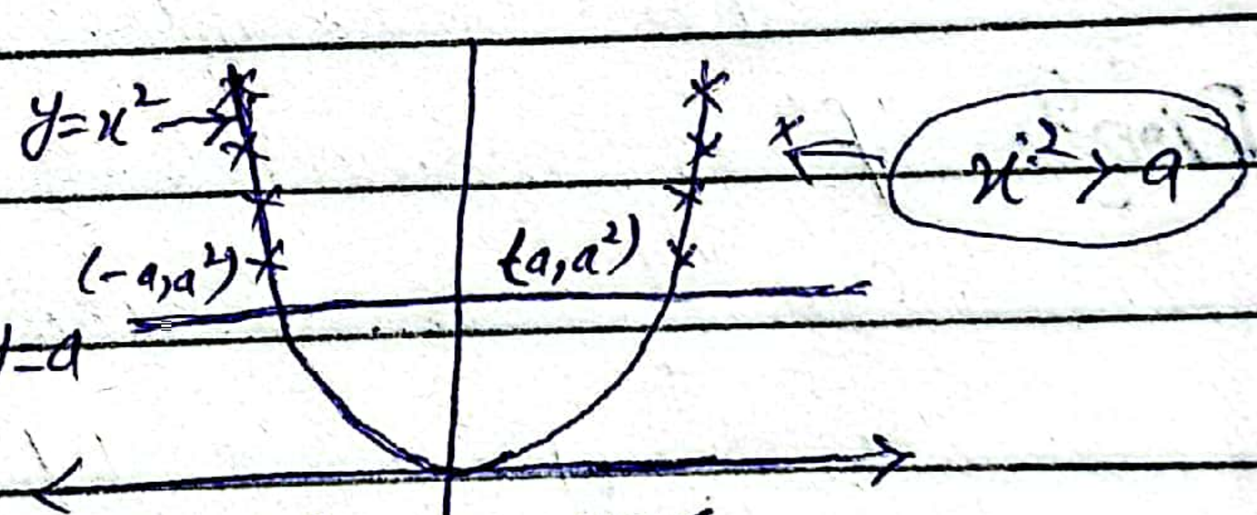
$y = a$ (line)

$\Rightarrow y^2 = a^2$

$y = x^2$

$a^2 = x^2$

$\Rightarrow x = \pm a$



Case 2: $f^2(x) > a$

$\Rightarrow f(x) > a$, $f(x) < -a$

$\{x : f^2(x) > a\} = \begin{cases} \{x : f(x) > \sqrt{a}\} \cup \{x : f(x) < -\sqrt{a}\}, & a \geq 0 \\ \text{All Domain} \cdot (D) & , a < 0 \end{cases}$

∴ "f" is measurable ⇒ $\{x : f(x) > \sqrt{a}\} \in M$

and $\{x : f(x) < -\sqrt{a}\} \in M$

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$\Rightarrow \{x: f(x) > \sqrt{q}\} \cup \{x: f(x) < -\sqrt{q}\} \in M$

and "Domain" is given to be measurable. i.e $D \in M$

$\therefore \{x: f^2(x) < q\} \in M \Rightarrow$ "f^2" is measurable.

$\because f, g$ is measurable $\Rightarrow (f \pm g)$ is also measurable.

$\Rightarrow (f \pm g)^2$ is measurable.

$\Rightarrow \frac{1}{4}[(f+g)^2 - (f-g)^2] = fg$ is also measurable.

($\because Kf$ is also measurable) $\forall K \in R$.

Topic # 86

prove that if "f" is measurable function, then "$\frac{1}{f}$" is also measurable.

Proof: "f" is measurable $\Rightarrow \{x: f(x) > a\} \in M, \forall a \in R$.

To prove "$\frac{1}{f}$" is measurable, we show that

$\{x: \frac{1}{f}(x) > a\} \in M, \forall a \in R$.

$$\frac{1}{f}(x) = \frac{1}{f(x)} > a$$

$$\left\{x: \frac{1}{f}(x) > a\right\} = \begin{cases} \{x: f(x) < \frac{1}{a}\} \cap \{x: f(x) > 0\}; & a > 0 \\ \{x \in D: f(x) > 0\}; & a = 0 \\ \{x: f(x) > 0\} \cup [\{x: f(x) < \frac{1}{a}\} \cap \{x: f(x) < a\}] & ; a < 0 \\ \phi \end{cases}$$

All the sets on R.H.S are measurable.

$\Rightarrow \{x: \frac{1}{f}(x) > a\}$ is also measurable, $\forall a \in R$.

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$\Rightarrow$ $\frac{1}{f}$ is measurable.

(v) $\frac{f}{g}$, $g \neq 0$, is measurable.

proof: $\frac{f}{g}$ is also measurable because $\frac{f}{g} = \frac{1}{g} \cdot f$ is measurable being product of two measurable functions.

Topic # 87

(vi) $|f|$ is measurable.

proof: $\because$ "f" is measurable $\Rightarrow \{x : f(x) > a\} \in M, \forall a \in R$

To prove $|f|$ is measurable, we need to show:

$\{x : |f|(x) > a\} \in M, \forall a \in R$

$$\{x : |f|(x) > a\} = \{x : |f(x)| > a\}$$

$$= \begin{cases} \underbrace{\{x : f(x) > a\}}_{\in M} \cup \underbrace{\{x : f(x) < -a\}}_{\in M}, & \forall a > 0 \\ \{x : |f|(x) > a\} = \\ D \in M, & a < 0 \end{cases}$$

In all cases on R.H.S, we get a measurable set. $\Rightarrow \{x : |f|(x) > a\} \in M$ (measurable), $\forall a \in R$

$\Rightarrow |f|$ is also measurable.

(vii) $f \vee g = \max(f, g)$ is measurable.

proof: $\because f \vee g = \max(f, g) = \frac{1}{2}[(f+g) + |f-g|]$

So $f \vee g$ is measurable because $(f+g)$ and $|f-g|$ is measurable and "kf" is measurable, $\forall\ k \in \mathbb{R}$.

(viii) $f \wedge g = \min(f, g)$ is measurable.

proof: $\because f \wedge g = \min(f, g) = \frac{1}{2}[(f+g) - |f-g|]$

So $f \wedge g$ is measurable because $(f+g)$ and $|f-g|$ is measurable and "kf" is measurable, $\forall\ k \in \mathbb{R}$.

Topic # 88

Restriction of a function:

Let $f: X \to \mathbb{Y}$ be a function and $A \subset X$, Then $g: A \to \mathbb{Y}$ defined as $g(x) = f(x)$, $x \in A$ is called the restriction of "f" to $A \subset X$ and given by $f_A = g$.

Theorem: Let "f" be an extended real valued measurable function on "D". Let $A \subset D$, Then restriction of "f" to "A" is also measurable and $A \in \mathcal{M}$.

Proof: $\because$ "f" is measurable $\Rightarrow \{x \in D : f(x) > a\} \in \mathcal{M}$. $\forall\ a \in \mathbb{R}$

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For any $a \in \mathbb{R}$, To prove "f_A" is measurable we have $\{x \in A : f_A(x) > a\} \in M$

Note: $\{x \in A : f_A(x) > a\} = \{x \in A : f(x) > a\} \cap A$

On R.H.S both the sets are measurable.

$\Rightarrow \{x \in A : f_A(x) > a\} \in M$

$\therefore$ Intersection of two measurable sets is measurable.

$\Rightarrow$ "f_A" is measurable.

Topic # 89

Theorem: An extended real valued function defined on a set of measure zero is measurable.

Proof: Let $\phi \neq E \subset \mathbb{R}$, such that $m^*(E) = 0$ and $f : E \to \mathbb{R} \cup \{\pm\infty\}$, To prove "$f$" is measurable, we need to show that:

$\{x \in E : f(x) > a\} \in M$, $\forall\ a \in \mathbb{R}$.

$\therefore \{x \in E : f(x) > a\} \subset E$

$\Rightarrow m^*\{x \in E : f(x) > a\} \leq m^*(E)$

$\because$ By monotonicity of m^*.

$\Rightarrow m^*\{x \in E : f(x) > a\} \leq 0$ ..... (given)

But by non-negativity of "m^*"

$\Rightarrow m^*\{x \in E : f(x) > a\} \geq 0$

$\Rightarrow m^*\{x \in E : f(x) > a\} = 0$; $\forall\ a \in \mathbb{R}$.

$\Rightarrow \{x \in E : f(x) > a\} \in M \; ; \forall \; a \in \mathbb{R}$

So "f" is measurable.

Topic # 90

Theorem: Let $\{f_n\}$ be a sequence of extended real valued measurable function on "D", then.

(i) $\sup_{n \in \mathbb{N}} f_n$, $\inf_{n \in \mathbb{N}} f_n$ are measurable.

(ii) $\max_{1 \le i \le n} (f_i)$, $\min_{1 \le i \le n} (f_i)$ are measurable.

(iii) $\overline{\lim_{n \in \mathbb{N}}} f_n$, $\underline{\lim_{n \in \mathbb{N}}} f_n$ are measurable.

(iv) If $\lim_{n \to \infty} f_n = f$, Then "f" is measurable.

Proof: Let $f = \inf_{n \in \mathbb{N}} f_n$ and To prove f is measurable.

Then we have to shows $\{x : f(x) > a\} \in M, \forall a \in \mathbb{R}$

We claim : $\{x : f(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\}$

Let $y \in \{x : f(x) > a\} \Rightarrow f(y) > a$

$\because f = \inf f_n \Rightarrow f(y) \le f_n(y), \forall \; n \in \mathbb{N}$

$\because f(y) > a \Rightarrow a < f(y) \le f_n(y), \forall \; n \in \mathbb{N}$

$\Rightarrow f_n(y) > a \; \forall \; n \in \mathbb{N} \Rightarrow y \in \{x : f_n(x) > a\} \; \forall \; n \in \mathbb{N}$

$\Rightarrow y \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\}$

$\Rightarrow \{x : f(x) > a\} \subseteq \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\} \longrightarrow (A)$

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Now let $t \in \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\}$

$\Rightarrow t \in \{x : f_n(x) > a\}, \forall a \in R, \forall n \in N$

$\Rightarrow f_n(t) > a, \forall n \in N$

$\Rightarrow a < f_n(t), \forall n \in N$

$\Rightarrow$ "a" is lower bound $\{f_1(t), f_2(t), \cdots\}$

But $f = \inf_{n \in N} f_n$ is also its lower bound.

$\Rightarrow a < \inf_{n \in N} f_n, \forall n \in N$

$\Rightarrow$ ~~$f_n(t) > a$~~ $\Rightarrow \inf_{n \in N} f_n > a \Rightarrow f(t) > a$

$f(t) > a \Rightarrow t \in \{x : f(t) > a\} \longrightarrow (A)$

$\Rightarrow \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\} \subset \{x : f(x) > a\} \longrightarrow (B)$

From (A) and (B) $\{x : f(x) > a\} = \bigcap_{n=1}^{\infty} \{x : f_n(x) > a\}$

So $f = \inf_{n \in N} f_n$ is measurable.

Now let $h = \sup_{n \in N} f_n$

... The set $\{x : h(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f_n(x) > a\}$ is measurable. So $h = \sup_{n \in N} f_n$ is measurable.

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Topic# 9

$\max_{1\le i\le n} f_i$, $\min_{1\le i\le n} f_i$ are measurable.

$\{f_n\}_{n=1}^{\infty}$ is seq of measurable functions.

Proof: Let $f = \max_{1\le i\le n} f_i$, To prove "f" is measurable, we need to show that $\{x : f(x) > a\} \in M$ and we claim

$$\{x : f(x) > a\} = \bigcup_{k=1}^{n} \{x : f_k(x) > a\} \quad \forall a \in \mathbb{R}.$$

let $y \in \{x : f(x) > a\} \Rightarrow f(y) > a$

$\therefore f(x) = \max_{1\le i\le n} f_i$ So there exist $J = \{1, 2, \dots, n\}$

$\Rightarrow f(x) = f_j(x) \Rightarrow f(y) = f_j(y) > a$

$\Rightarrow y \in \{x : f_j(x) > a\}$, for some $j = \{1, 2, \dots, n\}$

Hence $y \in \bigcup_{i=1}^{n} \{x : f_i(x) > a\}$

Now let $y \in \bigcup_{i=1}^{n} \{x : f_i(x) > a\}$

Then $y \in \{x : f_k(x) > a\}$ for some $k = \{1, 2, \dots, n\}$

$\Rightarrow f_k(y) > a$, By definition $f_i(y) \le \max f_i(y) = f(y)$ $\forall i = 1, 2, \dots, n.$

and $f_k(y) \le f(y) \Rightarrow h(y) > a$

$\Rightarrow y \in \{x : f(x) > a\}$.

The countable union of measurable sets is measurable. So $f = \max_{1\le i\le n} f_i$ is measurable.

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$\overline{\lim_{n\in\mathbb{N}}}$ → Limit superior

$\underline{\lim_{n\in\mathbb{N}}}$ → Limit inferior

Let $g = \min_{1\le i\le n} f_i$, To prove "g" is measurable we need to show that $\{x : g(x) > a\} \in M \quad \forall a \in \mathbb{R}$

and we claim $\{x : g(x) > a\} = \bigcap_{i=1}^{n} \{x : f_i(x) > a\}$

The countable intersection of measurable sets is measurable. So $g = \min_{1\le i\le n} f_i$ is measurable.

Topic #92

Show that $\overline{\lim_{n\in\mathbb{N}}} f_n, \underline{\lim_{n\in\mathbb{N}}} f_n$ are measurables.

Proof:

$$\overline{\lim_{n\in\mathbb{N}}} f_n = \inf_{n\in\mathbb{N}} \left(\sup_{k\ge n} f_k \right) \rightarrow (9)$$

$$\sup_{k\ge n} f_k = \sup\{f_k, f_{k+1}, f_{k+2}, \dots\}, \quad k = 1, 2, 3, \dots$$

$$= \begin{cases} \sup\{f_1, f_2, f_3, \dots\} = F_1 \text{ is measurable being "sup" of measurable functions.} \\ \sup\{f_2, f_3, f_4, \dots\} = F_2, \dots \end{cases}$$

$\Rightarrow \{F_n\}_{n=1}^{\infty}$ is a sequence of measurable functions

$\Rightarrow \inf_{n\in\mathbb{N}} \{F_n\}_{n=1}^{\infty}$ is also measurable

$\Rightarrow \overline{\lim_{n\in\mathbb{N}}} f_n = \inf_{n\in\mathbb{N}} \{f_n\} = \inf_{n\in\mathbb{N}} \left(\sup_{k\ge n} f_k \right)$ is also measurable

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Similarly $\underline{\lim}_{n\in\mathbb{N}} f_n = \sup_{n\in\mathbb{N}} (\inf_{k\geq n} f)$ is measurable.

Topic # 93

If $\lim_{n\to\infty} f_n = f$, then "f" is measurable.

Proof: $\because \lim_{n\to\infty} f_n = \overline{\lim}_{n\in\mathbb{N}} f_n = \underline{\lim}_{n\in\mathbb{N}} f_n = f$ $\{\{f_n\}_{n=1}^{\infty}$ ↓ Limit (given)

But $\overline{\lim}_{n\in\mathbb{N}} f_n$ and $\underline{\lim}_{n\in\mathbb{N}} f_n$ are measurable.

$\Rightarrow$ "f" is also measurable.

MTH644 - Measure Theory Handouts

Written by Mahar Afaq Safdar Muhammadi

MSc Math (Scholar), MA Islamic Studies

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